A Hennessy-Milner Property for Many-Valued Modal Logics

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Abstract

A Hennessy-Milner property, relating modal equivalence and bisimulations, is defined for many-valued modal logics that combine a local semantics based on a complete MTL-chain (a linearly ordered commutative integral residuated lattice) with crisp Kripke frames. A necessary and sufficient algebraic condition is then provided for the class of image-finite models of these logics to admit the Hennessy-Milner property. Complete characterizations are obtained in the case of many-valued modal logics based on BL-chains (divisible MTL-chains) that are finite or have universe [0,1], including crisp Łukasiewicz, Gödel, and product modal logics.

Keywords: Modal Logics, Many-Valued Logics, Hennessy-Milner Property.

1 Introduction

Many-valued modal logics combine the Kripke semantics of modal logics with a local many-valued semantics to model epistemic, spatio-temporal, and other modalities in the presence of vagueness or uncertainty (including, e.g., fuzzy belief [19,14], fuzzy similarity measures [15], many-valued tense logics [10], and spatial reasoning with vague predicates [24]). As in the classical setting, fuzzy description logics may also be interpreted as many-valued multi-modal logics (see [25,18]). General approaches to many-valued modal logics are described in [12,13,23,5]. Here, for convenience, we follow [5] in assuming that the underlying many-valued algebras of the logics are complete MTL-chains: that is,

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complete linearly ordered integral commutative residuated lattices. This framework spans, in particular, the families of Gödel and Łukasiewicz modal logics studied in [22,8,7] and [20], respectively. However, we restrict our attention in this paper to crisp many-valued modal logics where accessibility is a binary relation, leaving for future work, the case where accessibility is interpreted as a binary map from states to values of the algebra.

Theoretical studies of many-valued modal logics have concentrated to date mostly on issues of axiomatization, decidability, and complexity. Other topics from the rich theory of modal logics, such as first-order correspondence theory, canonical models, etc. have not as yet received much attention. In particular, the general question of the expressivity of many-valued modal logics has largely been ignored (although, see [2] for a coalgebraic approach to this topic). For classical modal logic, Van Benthem's theorem tells us that the modal logic K may be viewed as the bisimulation-invariant fragment of first-order logic, and it may be asked if similar results hold in the many-valued setting. A less demanding but still interesting question is whether analogues of the Hennessy-Milner property (modal equivalence coincides with bisimilarity) hold for image-finite models of many-valued modal logics. Modal equivalence between two states means in this context that each formula takes the same value in both states; the definition of a bisimulation matches the classical notion except that variables must take the same value in bisimilar states. Informally, our goal is to determine whether the language is expressive enough to distinguish image-finite models of many-valued modal logics.

More concretely, we define a (crisp) many-valued modal logic $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ based on one complete MTL-chain $\mathbf{A}$. The first main result of this paper is a necessary and sufficient algebraic condition on $\mathbf{A}$ for the class of image-finite models of $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ to admit the Hennessy-Milner property. We then obtain a complete classification when $\mathbf{A}$ is a divisible MTL-chain, called a BL-chain, that is finite or has universe $[0,1]$. In both cases, the property holds exactly when $\mathbf{A}$ is an MV-chain (a BL-chain where the negation is involutive) or the ordinal sum of two MV-chains. This means in particular that the class of image-finite models for (crisp) Łukasiewicz modal logics and the (crisp) three-valued Gödel modal logic admit the Hennessy-Milner property, but not (crisp) product modal logic or the (crisp) Gödel modal logics with more than three truth values.

Let us note finally that the approach to bisimulations and Hennessy-Milner properties for Heyting algebra based modal logics described by Eleftheriou et al. in [11] differs from the approach reported in this paper in several significant respects. Not only are (not necessarily linearly ordered) Heyting algebras considered, rather than the broad family of linearly ordered algebras investigated in this paper, but the Kripke frames are many-valued rather than crisp. This more general setting requires substantially different notions of bisimulation. Moreover, in order to obtain suitable Hennessy-Milner properties, it is assumed that the language contains constants for every element of the algebra, an assumption that would trivialize the approach taken here.

2 Many-Valued Modal Logics

For convenience, we follow [5] and restrict our attention to many-valued modal logics defined over *commutative integral residuated lattices*, algebraic structures

$$\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, \bot, \top \rangle$$

satisfying

(i) $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice where $a \leq b$ if and only if $a \wedge b = a$.

(ii) $\langle A, \cdot, \top \rangle$ is a commutative monoid.

(iii) $a \cdot b \leq c$ if and only if $a \leq b \to c$ for all $a, b, c \in A$.

We will be particularly interested in the case where $\mathbf{A}$ is both a *chain* (i.e., $\leq$ is a linear order on A) and *complete* (i.e., $\bigwedge B$ and $\bigvee B$ exist in A for all $B \subseteq A$). Such an algebra satisfies the prelinearity identity $\top \approx (x \to y) \vee (y \to x)$ and is called a complete *MTL-chain* (where MTL stands for monoidal t-norm logic).

Example 2.1 If the universe of $\mathbf{A}$ is the real unit interval $[0, 1]$, then the monoidal operation $\cdot$ is a t-norm with unit 1 and residual $\to$. Most notably, such algebras provide standard semantics for Łukasiewicz logic, Gödel logic, and product logic when $\cdot$ is the Łukasiewicz t-norm $\max(0, x+y-1)$, the minimum t-norm $\min(x, y)$, or the product t-norm xy (multiplication), respectively (see [17] for further details). Many-valued modal logics based on these and other algebras based on continuous t-norms will be considered in some detail in Section 5.

Our many-valued modal logics will be defined based on a language consisting of binary connectives $\to$, $\wedge$, $\vee$, constants $\bot$, $\top$, and unary (modal) connectives $\Box$ and $\Diamond$. The set of *formulas* $\mathrm{Fm}_{\Box\Diamond}$ of this language, with arbitrary members denoted $\varphi, \psi, \chi, \ldots$ is defined inductively over a fixed countably infinite set Var of (propositional) variables, denoted $p, q, \ldots$. We also denote the set of (purely) *propositional formulas* by Fm. *Subformulas* are defined in the usual way, and we let $\neg\varphi = \varphi \to \bot$, $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$, $\varphi^0 = \top$, and $\varphi^{n+1} = \varphi \cdot \varphi^n$ for $n \in \mathbb{N}$. We fix the *length* $\ell(\varphi)$ of $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ to be the number of symbols occurring in φ.

The many-valued modal logic $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ is defined over a fixed complete MTL-chain $\mathbf{A}$ as follows. A *(crisp) frame* is a pair $\langle W, R \rangle$ where W is a non-empty set of *states* and $R \subseteq W \times W$ is a binary *accessibility relation* on W.

A $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-*model* is a triple $\mathfrak{M} = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is a frame and $V \colon \mathrm{Var} \times \mathrm{W} \to \mathrm{A}$ is a mapping, called a *valuation.* The valuation V is extended to $V \colon \mathrm{Fm}_{\Box\Diamond} \times \mathrm{W} \to \mathrm{A}$ by

$$\begin{aligned}
V(\bot, w) &= \bot & V(\top, w) &= \top \\
V(\varphi \wedge \psi, w) &= V(\varphi, w) \wedge V(\psi, w) & V(\varphi \vee \psi, w) &= V(\varphi, w) \vee V(\psi, w) \\
V(\varphi \cdot \psi, w) &= V(\varphi, w) \cdot V(\psi, w) & V(\varphi \to \psi, w) &= V(\varphi, w) \to V(\psi, w) \\
V(\Box\varphi, w) &= \bigwedge\{V(\varphi, v) : Rwv\} & V(\Diamond\varphi, w) &= \bigvee\{V(\varphi, v) : Rwv\}.
\end{aligned}$$

A formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ is *valid* in a $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, R, V\rangle$ if $V(\varphi, w) = \top$ for all $w \in W$. If φ is valid in all $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models, then φ is said to be $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-*valid*. (Note, however, that $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-validity will not play any role in the remainder of this paper; we consider the values taken by formulas at states without giving $\top$ any special importance.)

Let us fix some useful notation. Given a frame $\langle W, R\rangle$, we define $R[w] = \{v \in W : Rwv\}$. We call a $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, R, V\rangle$ a *tree-model* with *root* w and *height* n if $\langle W, R\rangle$ is a tree with root w and height n. We also write $\boldsymbol{a}$ for $a_1, \ldots, a_n \in A^n$ and given a propositional formula $\varphi(p_1, \ldots, p_n)$, we write $\varphi[\boldsymbol{a}]$ for the value of φ (understood as a term function) at $\boldsymbol{a}$ in $\mathbf{A}$. Given $\psi(p_1, \ldots, p_n) \in \mathrm{Fm}$ and $\varphi_1, \ldots, \varphi_n \in \mathrm{Fm}_{\Box\Diamond}$, the formula $\psi[\varphi_1/p_1, \ldots, \varphi_n/p_n]$ is obtained by replacing all occurrences of p_i in ψ with φ_i for $i \in \{1, \ldots, n\}$. The following useful lemma is then proved by a straightforward induction on formula length.

Lemma 2.2 *Let $\psi(p_1, \ldots, p_n) \in \mathrm{Fm}$ and $\varphi_1, \ldots, \varphi_n \in \mathrm{Fm}_{\Box\Diamond}$. Then for any $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$:*

$$V(\psi[\varphi_1/p_1, \ldots, \varphi_n/p_n], w) = \psi[V(\varphi_1, w), \ldots, V(\varphi_n, w)].$$

As suggested by the superscript $^{\mathsf{C}}$, we can also define, more generally, $\mathsf{K}(\mathbf{A})$-models based on "$\mathbf{A}$-frames" where R is an "A-valued" accessibility relation $R \colon W \times W \to A$ and the above clauses for $\Box$ and $\Diamond$ are revised accordingly (see [5] for details). However, this requires also a significant revision of the definition of a bisimulation and such cases are therefore left for future work. (See also the concluding remarks.)

3 Modal Equivalence and Bisimulations

Let us fix again a complete MTL-chain $\mathbf{A}$ and consider two $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$. We will say that $w \in W$ and $w' \in W'$ are *modally equivalent*, written $w \rightsquigarrow w'$, if $V(\varphi, w) = V'(\varphi, w')$ for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$.

A non-empty binary relation $Z \subseteq W \times W'$ will be called a *bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions are satisfied:

(i) If wZw', then $V(p, w) = V'(p, w')$ for all $p \in \mathrm{Var}$.

(ii) If wZw' and Rwv, then there exists $v' \in W'$ such that vZv' and $R'w'v'$ (the forth condition).

(iii) If wZw' and $R'w'v'$, then there exists $v \in W$ such that vZv' and Rwv (the back condition).

We say that $w \in W$ and $w' \in W'$ are *bisimilar*, written $w \equiv w'$, if there exists a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{M}'$ such that wZw'.

Observe that the notions of modal equivalence and bisimulation defined here follow very closely the standard classical notions, the only distinguishing detail being that agreement of propositional variables in bisimilar states and formulas in modally equivalent states means that they take exactly the same values in these states. Note, moreover, that the proof that bisimilarity implies modal equivalence, is very similar to the classical proof (see, e.g., [4]).

Lemma 3.1 *Let $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$ be $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models. If $w \in W$ and $w' \in W'$ are bisimilar, then they are modally equivalent.*

Proof. We prove that for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $w \in W$, and $w' \in W'$, it holds that $w \equiv w'$ implies $V(\varphi, w) = V'(\varphi, w')$, proceeding by induction on $\ell(\varphi)$. For the case $\varphi \in \mathrm{Var}$, the claim follows directly from the definition of a bisimulation. The case where φ is a constant is immediate, and the cases of the propositional connectives follow immediately using the induction hypothesis.

Now let $\varphi = \Diamond\psi$, the case $\varphi = \Box\psi$ being very similar. Using $w \equiv w'$, it follows by the forth condition that for each $v \in R[w]$, there exists $v' \in R'[w']$ such that $v \equiv v'$ and, by the induction hypothesis, $V(\psi, v) = V'(\psi, v')$. So $V(\Diamond\varphi, w) \leq V'(\Diamond\varphi, w')$. But also by the back condition, for each $v' \in R'[w']$, there exists $v \in R[w]$ such that $v \equiv v'$ and, by the induction hypothesis, $V(\psi, v) = V'(\psi, v')$. So $V(\Diamond\psi, w) \geq V'(\Diamond\psi, w')$. Hence $V(\Diamond\psi, w) = V'(\Diamond\psi, w')$ as required. $\Box$

Of course modal equivalence does not in general imply the existence of a bisimulation even in the classical case. Rather we may consider certain classes of models for which this implication holds. Let us say that a class $\mathcal{K}$ of $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the *Hennessy-Milner property* if for any models $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$ in $\mathcal{K}$, whenever the states $w \in W$ and $w' \in W'$ are modally equivalent, they are bisimilar.

We call a $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model *image-finite* if $R[w]$ is finite for each $w \in W$. A central aim of this paper will be to investigate when exactly (i.e., for which $\mathbf{A}$) the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the Hennessy-Milner property.

Example 3.2 Consider the family of (crisp) Gödel modal logics $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ where A is a complete subset of $[0, 1]$ containing 0 and 1 and

$$\mathbf{A} = \langle A, \min, \max, \min, \to_{\mathsf{G}}, 0, 1\rangle$$

with $x \to_{\mathsf{G}} y = y$ if $x > y$ and 1 otherwise. Suppose that $|A| > 3$ where $0 < a < b < c$. Consider the $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$ displayed in Fig. 1 with $W = \{w_0, w_1, w_2, w_3\}$ and $W' = \{v_0, v_1, v_2\}$, R and R' as indicated by the arrows, and V and V' with the given values of p and all other values 1. Then it is easily shown (e.g., by considering the non-equivalent one-variable formulas) that w_0 and v_0 are modally equivalent. However, they are clearly not bisimilar, as there is no state in W' corresponding to w_2. So the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models does not have the Hennessy-Milner property.

The standard proof that classical modal logic K has the Hennessy-Milner property for image-finite Kripke models proceeds (roughly) as follows (see [4]). Suppose for a contradiction that there are image-finite Kripke models $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$ such that modal equivalence is not a bisimulation. Assume (without loss of generality) that the forth condition does not hold. Then there are $w, v \in W$ and $w' \in W'$ such that $w \leftrightsquigarrow w'$ and Rwv, but for each $v'_i \in R'[w'] = \{v'_1, \ldots, v'_n\}$, there is a formula φ_i satisfying $V(\varphi_i, v) = 0$

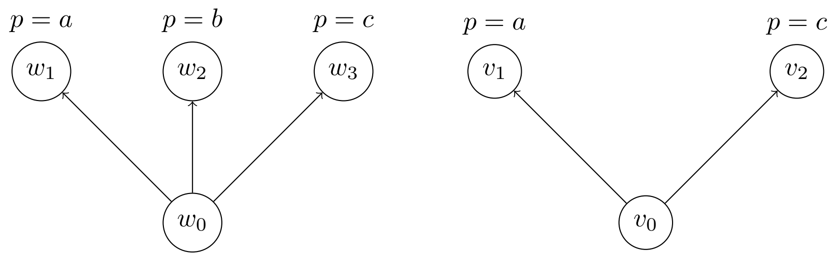

Fig. 1. Failure of the Hennessy-Milner property

but $V'(\varphi_i, v_i') = 1$. But then defining $\varphi = \Box(\varphi_1 \vee \ldots \vee \varphi_n)$, we have $V(\varphi, w) = 0$ and $V'(\varphi, w') = 1$, contradicting $w \leftrightsquigarrow w'$.

The above proof can be carried through for any logic $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ extended with additional constants for each $a \in A$. However, a more general condition suffices.

Lemma 3.3 *Suppose that for any distinct $a, b \in A$, there is a one-variable propositional formula $\psi_{a,b}(p) \in \mathrm{Fm}$ such that $\psi_{a,b}[a] = \top$ and $\psi_{a,b}[b] \neq \top$. Then the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the Hennessy-Milner property.*

Proof. We revisit the proof for the classical case. Suppose again for a contradiction that there are image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$ such that modal equivalence is not a bisimulation. Assuming (without loss of generality) that the forth condition does not hold, there are $w, v \in W$ and $w' \in W'$ such that $w \leftrightsquigarrow w'$ and Rwv, but for each $v_i' \in R'[w'] = \{v_1', \ldots, v_n'\}$, there exists $\varphi_i \in \mathrm{Fm}_{\Box\Diamond}$ satisfying $V(\varphi_i, v) \neq V'(\varphi_i, v_i')$. Let $a_i = V'(\varphi_i, v_i')$ and $b_i = V(\varphi_i, v)$ for $i \in \{1, \ldots, n\}$. Then, by assumption, there is a one-variable propositional formula $\psi_i(p) \in \mathrm{Fm}$ for each $i \in \{1, \ldots, n\}$ such that $\psi_i[a_i] = \top$ and $\psi_i[b_i] \neq \top$. We define

$$\varphi = \Box(\psi_1[\varphi_1/p] \vee \ldots \vee \psi_n[\varphi_n/p]).$$

Then, using Lemma 2.2 and the linearity of $\mathbf{A}$,

$$\begin{aligned} V(\varphi, w) &\leq V(\psi_1[\varphi_1/p], v) \vee \ldots \vee V(\psi_n[\varphi_n/p], v) \\ &= \psi_1[b_1] \vee \ldots \vee \psi_n[b_n] \\ &< \top, \end{aligned}$$

but also

$$\begin{aligned} V'(\varphi, w') &= \textstyle\bigwedge_{i=1}^n V'(\psi_1[\varphi_1/p] \vee \ldots \vee \psi_n[\varphi_n/p], v_i') \\ &\geq \textstyle\bigwedge_{i=1}^n V'(\psi_i[\varphi_i/p], v_i') \\ &= \textstyle\bigwedge_{i=1}^n \psi_i[a_i] \\ &= \top. \end{aligned}$$

This contradicts $w \leftrightsquigarrow w'$. □

Example 3.4 Consider the three-valued Gödel modal logic $\mathsf{K}(\mathbf{G_3})^{\mathsf{C}}$ where

$$\mathbf{G_3} = \langle \{0, \tfrac{1}{2}, 1\}, \min, \max, \min, \to_{\mathsf{G}}, 0, 1\rangle$$

with $x \to_{\mathsf{G}} y = y$ if $x > y$ and 1 otherwise. Then we use the following formulas to distinguish values in $\{0, \frac{1}{2}, 1\}$:

$$\psi_{1,0} = \psi_{1,\frac{1}{2}} = (p \leftrightarrow \top), \qquad \psi_{0,\frac{1}{2}} = \psi_{0,1} = (p \leftrightarrow \bot),$$
$$\psi_{\frac{1}{2},0} = \neg(p \leftrightarrow \bot), \qquad \psi_{\frac{1}{2},1} = \neg(p \leftrightarrow \top).$$

So, by Lemma 3.3, the class of image-finite $\mathsf{K}(\mathbf{G_3})^{\mathsf{C}}$-models has the Hennessy-Milner property.

Finding formulas $\psi_{a,b}$, as described in Lemma 3.3, that distinguish distinct elements $a, b \in A$ is sufficient to establish that $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ has the Hennessy-Milner property for the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models, but does not appear to be necessary. To obtain a complete characterization, we introduce a more complicated but still purely algebraic condition for $\mathbf{A}$.

Let $\boldsymbol{a} \in A^n$ and $\boldsymbol{C} = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_n) \in A^{n \times n}$. We call a propositional formula $\psi(p_1, \dots, p_n) \in \mathrm{Fm}$ an *$\boldsymbol{a}/\boldsymbol{C}$-distinguishing formula* if

$$\text{either} \quad \psi[\boldsymbol{a}] < \psi[\boldsymbol{c}_i] \quad \text{for } i \in \{1, \dots, n\} \qquad \text{or} \quad \psi[\boldsymbol{a}] > \psi[\boldsymbol{c}_i] \quad \text{for } i = \{1, \dots, n\}.$$

We say that $\mathbf{A}$ has the *distinguishing formula property* if for all $n \in \mathbb{N}$, $\boldsymbol{a} \in A^n$, and $\boldsymbol{C} = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_n) \in A^{n \times n}$ such that $\boldsymbol{a} \neq \boldsymbol{c}_i$ for $i \in \{1, \dots, n\}$, there is an $\boldsymbol{a}/\boldsymbol{C}$-distinguishing formula.

In the next section, we establish the following characterization.

Theorem 3.5 *The following are equivalent for any complete MTL-chain* $\mathbf{A}$*:*

(1) *The class of image-finite* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-models has the Hennessy-Milner property.*

(2) $\mathbf{A}$ *has the distinguishing formula property.*

We note that this theorem also holds (with an almost identical proof) for the box and diamond fragments of $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$, restricting the distinguishing formula property to the first (either) and second (or) condition, respectively.

4 Proof of Theorem 3.5

We first prove a useful lemma.

Lemma 4.1 *Let* $\mathfrak{M} = \langle W, R, V \rangle$ *and* $\mathfrak{M}' = \langle W', R', V' \rangle$ *be* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-tree-models of height one with roots* w *and* w'*, respectively. Suppose that* $V(p, w) = V'(p, w')$ *for all* $p \in \mathrm{Var}$ *and* $V(\Box\varphi, w) = V'(\Box\varphi, w')$ *and* $V(\Diamond\varphi, w) = V'(\Diamond\varphi, w')$ *for all* $\varphi \in \mathrm{Fm}$*. Then* w *and* w' *are modally equivalent.*

Proof. We prove the claim by induction on $\ell(\varphi)$. The base case is immediate and for the inductive step, the cases for the propositional connectives follow easily using the induction hypothesis. Suppose now that $\varphi = \Box\psi$, the case $\varphi = \Diamond\psi$ being very similar. Let ψ^* be the propositional formula obtained from ψ by replacing (iteratively) all subformulas of the form $\Box\psi'$ by $\top$ and all subformulas of the form $\Diamond\psi'$ by $\bot$. Then, using the fact that $\mathfrak{M}$ and $\mathfrak{M}'$ are tree-models of height one, it follows by an easy induction that $V(\psi, v) = V(\psi^*, v)$

for all $v \in W$ such that Rwv, and $V'(\psi, v') = V'(\psi^*, v')$ for all $v' \in W'$ such that $R'w'v'$. But then $V(\Box\psi, w) = V(\Box\psi^*, w) = V'(\Box\psi^*, w') = V'(\Box\psi, w')$ as required. $\Box$

Now we establish the implication (2) $\Rightarrow$ (1) of Theorem 3.5. Assume that $\mathbf{A}$ has the distinguishing formula property and suppose for a contradiction that there are two image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$ such that modal equivalence is not a bisimulation. If $w \leftrightsquigarrow w'$ for $w \in W$ and $w' \in W'$, then by definition $V(p, w) = V'(p, w')$ for all $p \in \mathrm{Var}$, so the back condition or forth condition must be violated.

Let us suppose that the forth condition fails, the back condition being very similar. Then there exist $w, v \in W$ and $w' \in W'$ such that

(i) $w \leftrightsquigarrow w'$ and Rwv.

(ii) No $v' \in W'$ satisfies both $R'w'v'$ and $v \leftrightsquigarrow v'$.

If $R'[w'] = \emptyset$, then consider $\Diamond\top$. We have $V(\Diamond\top, w) = \top$, but $V'(\Diamond\top, w') = \bigvee\emptyset = \bot$, which contradicts $w \leftrightsquigarrow w'$. Suppose then that $R'[w']$ is non-empty and (because of image-finiteness) finite, say $R'[w'] = \{v'_1, \ldots, v'_n\}$. So there are formulas φ_i such that $V(\varphi_i, v) \neq V'(\varphi_i, v'_i)$ for $i \in \{1, \ldots, n\}$.

We define $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{C} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n)$ with $\boldsymbol{c}_i = c_{i,1}, \ldots, c_{i,n}$ as follows:

$$a_i = V(\varphi_i, v) \quad \text{and} \quad c_{i,j} = V'(\varphi_j, v'_i) \quad \text{for } 1 \leq i, j \leq n.$$

Note that $\boldsymbol{a} \neq \boldsymbol{c_i}$ for $i \in \{1, \ldots, n\}$ (because $a_i \neq c_{i,i}$).

By the distinguishing formula property, there exists an $\boldsymbol{a}/\boldsymbol{C}$-distinguishing propositional formula $\psi(p_1, \ldots, p_n) \in \mathrm{Fm}$. Suppose that

$$\psi[\boldsymbol{a}] < \psi[\boldsymbol{c}_i] \text{ for each } i \in \{1, \ldots, n\},$$

the case where $\psi[\boldsymbol{a}] > \psi[\boldsymbol{c}_i]$ for $i \in \{1, \ldots, n\}$ being very similar. Now define

$$\varphi = \Box\psi[\varphi_1/p_1, \ldots, \varphi_n/p_n].$$

Then using Lemma 2.2 and the linearity of $\mathbf{A}$,

$$\begin{aligned} V(\varphi, w) &\leq V(\psi[\varphi_1/p_1, \ldots, \varphi_n/p_n], v) \\ &= \psi[V(\varphi_1, v), \ldots, V(\varphi_n, v)] \\ &= \psi[\boldsymbol{a}] \\ &< \textstyle\bigwedge_{i=1}^n \psi[\boldsymbol{c}_i] \\ &= \textstyle\bigwedge_{i=1}^n \psi[V'(\varphi_1, v'_i), \ldots, V'(\varphi_n, v'_i)] \\ &= \textstyle\bigwedge_{i=1}^n V'(\psi[\varphi_1/p_1, \ldots, \varphi_n/p_n], v'_i) \\ &= V'(\varphi, w'). \end{aligned}$$

This contradicts $w \leftrightsquigarrow w'$.

We turn our attention now to the implication (1) $\Rightarrow$ (2). Let us assume that the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the Hennessy-Milner property. Given $n \in \mathbb{N}$, $\boldsymbol{a} \in A^n$, and $\boldsymbol{C} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n) \in A^{n \times n}$ such that $\boldsymbol{a} \neq \boldsymbol{c}_i$ for $i \in \{1, \ldots, n\}$, we seek an $\boldsymbol{a}/\boldsymbol{C}$-distinguishing formula $\psi(p_1, \ldots, p_n)$.

Consider the two image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}' = \langle W', R', V' \rangle$ satisfying

(i) $W = \{w, v_1, \ldots, v_n, v\}$ and $W' = \{w', v'_1, \ldots, v'_n\}$

(ii) $R = \{(w, v_i) : 1 \leq i \leq n\} \cup \{(w, v)\}$ and $R' = \{(w', v'_i) : 1 \leq i \leq n\}$

(iii) $V(p_j, v_i) = V'(p_j, v'_i) = c_{i,j}$ and $V(p_j, v) = a_j$ for $1 \leq i, j \leq n$ (all other variables and all variables at w take value $\top$).

We observe first that w and w' are not bisimilar: there is no state in W' that is accessible from w and agrees with v on all propositional variables because $\boldsymbol{a} \neq \boldsymbol{c}_i$ for $i \in \{1, \ldots, n\}$.

Hence by the Hennessy-Milner property for image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models, w and w' are not modally equivalent. By Lemma 4.1, it follows that $V(\varphi, w) \neq V'(\varphi, w')$ for some formula $\varphi = \Box\psi$ or $\varphi = \Diamond\psi$ where $\psi \in \mathrm{Fm}$. Moreover, we may assume that ψ contains only the variables $p_1, \ldots, p_n$ as all other variables take the value $\top$.

Suppose that $\varphi = \Box\psi$ where $\psi(p_1, \ldots, p_n) \in \mathrm{Fm}$, the case $\varphi = \Diamond\psi$ being very similar. Clearly $V(\Box\psi, w) \leq V'(\Box\psi, w')$, so for each $i \in \{1, \ldots, n\}$,

$$\psi[\boldsymbol{a}] = V(\psi, v) = V(\Box\psi, w) < V'(\Box\psi, w') \leq V'(\psi, v'_i) = \psi[\boldsymbol{c}_i].$$

That is, ψ is the required $\boldsymbol{a}/\boldsymbol{C}$-distinguishing formula.

5 Divisible Chain Based Modal Logics

A commutative integral residuated lattice $\mathbf{A}$ is called *divisible* if for all $a, b \in A$ and $a \leq b$, there exists $c \in A$ such that $b \cdot c = a$. Equivalently, $\mathbf{A}$ is divisible if and only if $a \cdot (a \to b) = b \cdot (b \to a)$ for all $a, b \in A$. Divisible MTL-chains are also known (up to term equivalence) in the mathematical fuzzy logic literature as *BL-chains* (see [17,1,6]). In the case where $A = [0, 1]$, the monoidal operation $\cdot$ is a *continuous t-norm* and $\mathbf{A}$ is called a *standard BL-chain*. For convenience, in this section, we exploit the fact that $a \wedge b = a \cdot (a \to b)$ and $a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a)$ for all $a, b \in A$, and restrict to the (traditional) language with operation symbols $\cdot, \to, \bot, \top$.

Our goal in this section is to obtain a complete characterization of the finite and standard BL-chains $\mathbf{A}$ for which the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the Hennessy-Milner property. First, we consider the special case where $\mathbf{A}$ is an *MV-chain*, defined here (up to term equivalence) as a BL-chain satisfying the involution property $\neg\neg a = a$ for all $a \in A$. Consider in particular the MV-chains

$$\begin{aligned}
\mathbf{Ł_{n+1}} &= \langle \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \cdot_{\mathrm{Ł}}, \to_{\mathrm{Ł}}, 0, 1 \rangle \quad (n \in \mathbb{Z}^+) \\
\mathbf{Ł}_\infty &= \langle [0, 1], \cdot_{\mathrm{Ł}}, \to_{\mathrm{Ł}}, 0, 1 \rangle
\end{aligned}$$

where $x \cdot_{\text{Ł}} y = \min(1, x + y - 1)$ and $x \to_{\text{Ł}} y = \max(0, 1 - x + y)$. Also, for convenience, let $\mathbf{Ł_1}$ be the trivial MV-chain with one element. Then every finite MV-chain $\mathbf{A}$ is isomorphic to $\mathbf{Ł}_{|A|}$ and every standard MV-chain is isomorphic to $\mathbf{Ł}_\infty$ (see [9] for proofs and a wealth of further information on MV-algebras).

We show here that for each $\alpha \in \mathbb{Z}^+ \cup \{\infty\}$, the class of image-finite $\mathsf{K}(\mathbf{Ł}_\alpha)^\mathsf{C}$-models has the Hennessy-Milner property, and hence that the same holds for any finite or standard MV-chain.

Example 5.1 Consider the algebra $\mathbf{Ł_3}$ for three-valued Łukasiewicz logic. We can distinguish values using the following formulas:

$$\psi_{1,0} = (p \leftrightarrow \top), \qquad \psi_{1,\frac{1}{2}} = (p \cdot p), \qquad \psi_{\frac{1}{2},0} = (\neg p \to p)$$
$$\psi_{0,1} = (p \leftrightarrow \bot), \qquad \psi_{\frac{1}{2},1} = (p \to \neg p), \qquad \psi_{0,\frac{1}{2}} = (\neg p \cdot \neg p).$$

Hence, by Lemma 3.3, the class of image-finite $\mathsf{K}(\mathbf{Ł_3})^\mathsf{C}$-models has the Hennessy-Milner property.

More generally, we may distinguish between rational values in $[0,1]$ using unary *McNaughton functions*: that is, continuous functions $f\colon [0,1] \to [0,1]$ with the property that there exist linear functions $g_1, \ldots, g_k$ with integer coefficients such that for any $x \in [0,1]$, there is an $i \in \{1, \ldots, k\}$ satisfying $f(x) = g_i(x)$.

Theorem 5.2 (McNaughton [21]) *The free one-generated MV-algebra is isomorphic to the algebra of unary McNaughton functions equipped with pointwise defined operations.*

Lemma 5.3 *For each $\alpha \in \mathbb{Z}^+ \cup \{\infty\}$, the class of image-finite $\mathsf{K}(\mathbf{Ł}_\alpha)^\mathsf{C}$-models has the Hennessy-Milner property.*

Proof. Consider distinct $a, b \in [0,1]$. Then we can define a McNaughton function f such that $f(a) = 1$ and $f(b) = 0$. Suppose that $a < b$, the case $a > b$ being very similar. Then there exist $c, d \in \mathbb{Q}$ such that $a < c < d < b$ and we can define f to be 1 on the interval $[0, c]$, 0 on the interval $[d, 1]$, and linear on (c, d). Using Theorem 5.2, there exists a propositional formula $\psi_{a,b}(p)$ such that in the algebra $\mathbf{Ł}_\infty$, we have $\psi[x] = f(x)$ for all $x \in [0,1]$. Because $\mathbf{Ł}_\alpha$ is a subalgebra of $\mathbf{Ł}_\infty$ for each $\alpha \in \mathbb{Z}^+ \cup \{\infty\}$, it follows that $\psi_{a,b}[a] = 1$ and $\psi_{a,b}[b] \neq 1$ whenever a, b are distinct elements of the algebra. Hence, by Lemma 3.3, $\mathbf{Ł}_\alpha$ has the Hennessy-Milner property. $\square$

We now turn our attention to BL-chains, recalling a useful characterization of these algebras in terms of linearly ordered hoops (referring to [1,6] for further details). A *hoop* is an algebraic structure $\mathbf{H} = \langle H, \cdot, \to, \top \rangle$ such that $\langle H, \cdot, \top \rangle$ is a commutative monoid and for all $a, b, c \in H$:

(i) $a \to a = \top$.

(ii) $a \cdot (a \to b) = b \cdot (b \to a)$.

(iii) $a \to (b \to c) = (a \cdot b) \to c$.

Defining $a \leq b$ if and only if $a \to b = \top$ provides a semilattice order with meet operation $a \wedge b = a \cdot (a \to b)$ such that $\cdot$ and $\to$ are a residuated pair; i.e., $a \leq b \to c$ if and only if $a \cdot b \leq c$. If the order is linear, then $\mathbf{H}$ is called a *linearly ordered hoop* (*o-hoop* for short). Again, an o-hoop is *standard* if $H = [0, 1]$.

Now consider a linearly ordered set I with bottom element i_0 and suppose that $\mathbf{A}_i = \langle A_i, \cdot_i, \to_i, \top\rangle$ is a non-trivial o-hoop for each $i \in I$. Suppose, moreover, that $A_i \cap A_j = \{\top\}$ for distinct $i, j \in I$ and that $\mathbf{A}_{i_0}$ has a bottom element $\bot$. Then the *(bounded) ordinal sum* of $(\mathbf{A}_i)_{i\in I}$ is defined as

$$\bigoplus_{i\in I} \mathbf{A}_i = \langle \bigcup_{i\in I} A_i, \cdot, \to, \bot, \top\rangle$$

with operations

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if } x, y \in A_i \\ x & \text{if } x \in A_i \setminus \{\top\},\ y \in A_j, \text{ and } i < j \\ y & \text{if } y \in A_i \setminus \{\top\},\ x \in A_j, \text{ and } i < j \end{cases}$$

$$x \to y = \begin{cases} \top & \text{if } x \in A_i \setminus \{\top\},\ y \in A_j, \text{ and } i < j \\ x \to_i y & \text{if } x, y \in A_i \\ y & \text{if } y \in A_i,\ x \in A_j, \text{ and } i < j. \end{cases}$$

We also write $\mathbf{A}_1 \oplus \ldots \oplus \mathbf{A}_n$ when $I = \{1, \ldots, n\}$ has the usual order.

Every ordinal sum of o-hoops is a BL-chain. Moreover, each "irreducible BL-chain" $\mathbf{A}$ (those that cannot be expressed as proper ordinal sums of o-hoops) are of exactly two types: either $\mathbf{A}$ is the *hoop reduct* $\langle A, \cdot, \to, \top\rangle$ of an MV-chain $\langle A, \cdot, \to, \bot, \top\rangle$, or $\mathbf{A}$ satisfies $a \to (a \cdot b) = b$ for all $a, b \in A$ and is called a *cancellative o-hoop.* Note that there are no finite cancellative o-hoops and that every standard cancellative o-hoop is isomorphic to the o-hoop

$$\mathbf{C} = \langle (0, 1], \cdot_{\mathsf{C}}, \to_{\mathsf{C}}, 1\rangle$$

where $x \cdot_{\mathsf{C}} y = xy$ (multiplication) and $x \to_{\mathsf{C}} y$ is $\frac{y}{x}$ for $x > y$ and 1 otherwise.

Theorem 5.4 (Aglianò and Montagna [1]) *Every non-trivial BL-chain is the unique ordinal sum of a family of o-hoops each of which is either the hoop reduct of an MV-chain or a cancellative o-hoop.*

The next two lemmas identify ordinal sums $\mathbf{A}$ of o-hoops such that the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models does or does not have the Hennessy-Milner property. For convenience, we let $\mathbf{A}^{\mathbf{h}}$ denote the hoop reduct of a BL-chain $\mathbf{A}$.

Lemma 5.5 *Suppose that* $\mathbf{A}$ *is the ordinal sum of a family of (non-trivial) o-hoops* $(\mathbf{A}_i)_{i\in I}$. *If* $|I| \geq 3$ *or* $\mathbf{A}_i$ *is cancellative for some* $i \in I$, *then the class of image-finite* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-models does not have the Hennessy-Milner property.*

Proof. Suppose first that $\mathbf{A}_i$ is cancellative for some $i \in I$. Choose any element $c \in \mathbf{A}_i$ such that $c \neq \top$ and let $b = c \cdot c$ and $a = c \cdot c \cdot c$, noting that, by cancellativity, $a < b < c < \top$. Consider the $\mathsf{K}(\mathbf{A}_{\mathsf{i}})^{\mathsf{C}}$-models $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$ displayed in Fig. 1 with $W = \{w_0, w_1, w_2, w_3\}$ and $W' = \{v_0, v_1, v_2\}$, R and R' as indicated by the arrows, and V and V' with the given values of p and all other values $\top$. Clearly, w_0 and v_0 are not bisimilar. To see that they are modally equivalent, consider any propositional formula $\psi(p) \in \mathrm{Fm}$. An easy induction on $\ell(\psi)$ establishes that ψ restricted to A_i is equivalent to $\perp$ or p^k for some $k \in \mathbb{N}$. But then $V(\Box\psi, w_0) = V'(\Box\psi, v_0)$ and $V(\Diamond\psi, w_0) = V'(\Diamond\psi, v_0)$ for all $\psi \in \mathrm{Fm}$. By Lemma 4.1, w_0 and v_0 are modally equivalent. So the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models does not have the Hennessy-Milner property.

Now consider the case where $|I| \geq 3$ and no $\mathbf{A}_i$ is cancellative for $i \in I$. Then, by Theorem 5.4, each $\mathbf{A}_i$ is an MV-chain and has a bottom element distinct from the top element of $\mathbf{A}$ for $i \in I$. But these bottom elements and also the top element are idempotents (i.e., $a \cdot a = a$) and hence $\mathbf{A}$ contains as a subalgebra, a Gödel algebra with more than three elements. The failure of the Hennessy-Milner property then follows exactly as described in Example 3.2. $\Box$

Lemma 5.5 establishes the failure of the Hennessy-Milner property for the class of image-finite models of the product modal logic $\mathsf{K}(\mathbf{P})^{\mathsf{C}}$ for $\mathbf{P} = \langle [0,1], \cdot_{\mathsf{C}}, \to_{\mathsf{C}}, 0, 1\rangle$ where $x \cdot_{\mathsf{C}} y = xy$ (multiplication) and $x \to_{\mathsf{C}} y$ is $\frac{y}{x}$ for $x > y$ and 1 otherwise. Just observe that $\mathbf{P}$ is isomorphic to the ordinal sum of $\mathbf{Ł}^{\mathbf{h}}_{\mathbf{2}}$ and $\mathbf{C}$.

Lemma 5.6 *Let $\mathbf{A} = \mathbf{Ł}^{\mathbf{h}}_{\alpha} \oplus \mathbf{Ł}^{\mathbf{h}}_{\beta}$ with $\alpha, \beta \in \mathbb{Z}^+ \cup \{\infty\}$. Then the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the Hennessy-Milner property.*

Proof. We use Lemma 3.3. Consider distinct elements $a, b \in A$. There are three cases. Suppose first that $a, b \in \mathrm{Ł}_{\alpha}$. We consider $\mathbf{Ł}_{\alpha}$ as an MV-chain with $\perp$ added to the language. As in the proof of Lemma 5.3, we obtain a distinguishing formula $\psi_{a,b}(p)$. Now suppose that $a, b \in \mathrm{Ł}_{\beta}$. We consider $\mathbf{Ł}_{\beta}$ as an MV-chain with $\perp'$ added to the language. Again we obtain a distinguishing formula $\psi_{a,b}(p)$ as in the proof of Lemma 5.3; however, in this case we must also replace $\perp'$ in $\psi_{a,b}(p)$ by p^k where $k \in \mathbb{Z}$ is sufficiently large that $p^k[a] = p^k[b] = \perp$. For the final case, consider $a \in \mathrm{Ł}_{\beta}$ and $b \in \mathrm{Ł}_{\alpha}$ (the converse is very similar). We fix $\psi_{a,b}(p) = \neg\neg p$ and observe that $\psi_{a,b}[a] = 1$ and $\psi_{a,b}[b] = b$. That is, $\psi_{a,b}$ is the required distinguishing formula. $\Box$

Combining these two lemmas with Theorem 5.4, we obtain the following characterization theorems for many-valued modal logics based on finite and standard BL-chains.

Theorem 5.7 *The following are equivalent for any finite BL-chain $\mathbf{A}$:*

(1) *The class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the Hennessy-Milner property.*

(2) *$\mathbf{A}$ is isomorphic to $\mathbf{Ł}^{\mathbf{h}}_{\mathbf{n+1}}$ or $\mathbf{Ł}^{\mathbf{h}}_{\mathbf{n+1}} \oplus \mathbf{Ł}^{\mathbf{h}}_{\mathbf{m+1}}$ for some $m, n \in \mathbb{N}$.*

Theorem 5.8 *The following are equivalent for any standard BL-chain $\mathbf{A}$:*

(1) *The class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the Hennessy-Milner property.*

(2) *$\mathbf{A}$ is isomorphic to $\mathbf{Ł}^{\mathbf{h}}$ or $\mathbf{Ł}^{\mathbf{h}} \oplus \mathbf{Ł}^{\mathbf{h}}$.*

Note that Lemma 5.5 and Theorem 5.4 actually tell us a little more: namely, that MV-chains and ordinal sums of two hoop reducts of MV-chains are the only candidates for BL-chains whose classes of image-finite models has the Hennessy-Milner property. We know by Lemmas 5.3 and 5.6 that this is the case if the MV-chain or hoop reducts of MV-chains are finite or standard, but not what happens for other (hoop reducts of) MV-chains.

6 Concluding Remarks

We have provided here a purely algebraic necessary and sufficient condition for the class of image-finite models of a many-valued modal logic based on an MTL-chain to have the Hennessy-Milner property. This result can be extended in a number of directions. Note first that from an algebraic perspective, there is no particular reason (other than convenience and readability) to limit our attention to residuated lattices. A similar characterization may be obtained for many-valued modal logics based on complete chains with extra operations, although for this general case, valid equations rather than formulas should be considered. In fact, alternative quantifiers (e.g., expressing the average truth value at accessible worlds) may also be considered by adding corresponding conditions to the characterization. A more challenging problem, as our proofs rely at certain crucial steps on linearity, is to extend the approach beyond chains to arbitrary complete lattices with additional operations. We may also seek to establish Hennessy-Milner properties for broader classes of models: in particular, for models admitting some version of the modal saturation property used in the classical setting.

The many-valued modal logics investigated in this paper are all based on crisp Kripke frames, but useful many-valued modal logics are also considered (e.g., in connection with fuzzy description logics) that are based on many-valued Kripke frames where the accessibility relation is replaced by a binary map from worlds to elements of the algebra. Extending our approach to this family of logics clearly requires alternative and appropriate definitions of bisimulation. Notably, the paper [11] considers two different notions of a bisimulation in the context of many-valued modal logics based on Heyting algebras extended with additional constants for elements of the algebra. Similarly, we expect that to obtain Hennessy-Milner properties for the models of many-valued modal logics with many-valued accessibility relations, we will also require constants or additional modal operators in the language.

Finally, let us remark that bisimulations and Hennessy-Milner properties have been investigated extensively in the setting of coalgebra and coalgebraic modal logics (see, e.g, [16]). We might therefore hope or expect that recent advances towards defining many-valued coalgebraic logics [3,2] will allow methods and theorems developed in the coalgebraic setting to be used also in studying many-valued modal logics.

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