Advances in the theory of $\mu\text{Ł}\Pi$ Algebras

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Abstract

Recently an expansion of $\text{Ł}\Pi\frac{1}{2}$ logic with fixed points has been considered [23]. In the present work we study the algebraic semantics of this logic, namely $\mu\text{Ł}\Pi$ algebras, from algebraic, model theoretic and computational standpoints.

We provide a characterisation of free $\mu\text{Ł}\Pi$ algebras as a family of particular functions from $[0,1]^n$ to $[0,1]$. We show that the first-order theory of linearly ordered $\mu\text{Ł}\Pi$ algebras enjoys quantifier elimination, being, more precisely, the model completion of the theory of linearly ordered $\text{Ł}\Pi\frac{1}{2}$ algebras. Furthermore, we give a functional representation of any $\text{Ł}\Pi\frac{1}{2}$ algebra in the style of Di Nola Theorem for MV-algebras and finally we prove that the equational theory of $\mu\text{Ł}\Pi$ algebras is in PSPACE.

Keywords: $\mu\text{Ł}\Pi$ algebras, free algebras, real closed fields, computational complexity

1 Introduction

The aim of this work is to present some advances on the theory of $\mu\text{Ł}\Pi$ algebras. Such algebraic structures were first introduced in [23], where their strict connection with real closed fields is shown. By exploiting this connection, we will give several algebraic, model-theoretic, and computational results.

The paper is organized as follows. The next section provides the necessary background on algebras related to continuous t-norm based logics and, in particular, on $\text{Ł}\Pi\frac{1}{2}$ and $\mu\text{Ł}\Pi$ algebras. Section 3 focuses on a geometric characterization of free $\mu\text{Ł}\Pi$ algebras. Section 4 studies linearly ordered $\mu\text{Ł}\Pi$ algebras as models of a first-order theory, providing model-theoretic results, such as model-completeness, quantifier elimination and a functional representation. Finally, Section 5 deals with the computational complexity of the equational theory of $\mu\text{Ł}\Pi$ algebras, showing that it is in PSPACE.

2 Preliminaries

As witnessed by the large amount of literature on the subject, the foundational study of fuzzy logic, in the spirit of classical logic, is possible and has been remarkably successful. The objects bridging the gap between mathematical logic and the engineering

tools of fuzzy logic are *triangular norms*.

Triangular norms (t-norms for short) are commutative, associative, non-decreasing binary operations, defined over the real unit interval $[0,1]$ and having 1 as a neutral element (see [13]). For a t-norm $*$, the further requirement of (left-)continuity, as a function on $[0,1]$, guarantees the existence of a unique binary operation $\Rightarrow$, called *residuum*, such that for all $x, y, z \in [0,1]$, $x * y \leq z$ iff $x \leq y \Rightarrow z$. A continuous t-norm and its residuum provide a natural semantic interpretation for many-valued conjunction and implication. For this reason, t-norms are pivotal tools in fuzzy logic.

Given a continuous[1] t-norm $*$ and a propositional language L with set of connectives $\{\&, \rightarrow, \vee, \wedge, \overline{0}, \overline{1}\}$, one can define a $*$-*evaluation* v, as a homomorphism from the algebra of formulas of L into the algebra $[0,1]_* = \langle [0,1], *, \Rightarrow, \max, \min, 0, 1 \rangle$. A value in the real unit interval $[0,1]$ is assigned to each formula, $\&$ is interpreted as the left-continuous t-norm $*$, and $\rightarrow$ is interpreted as the residuum $\Rightarrow$. In this way it is possible to associate to a continuous t-norm a set $\mathcal{L}_*$ of formulas, called *the logic of the t-norm* $*$, defined as the set of all formulas φ such that for every $*$-evaluation v, $v(\varphi) = 1$. Similarly, it is possible to associate a logic $\mathcal{L}_\mathcal{K}$ to a class $\mathcal{K}$ of continuous t-norms, defined as the intersection of all $\mathcal{L}_*$ with $* \in \mathcal{K}$.

This approach was first suggested by Hájek in [8], where he introduced the Basic Logic BL as an attempt to provide an axiomatisation of the tautologies common to all continuous t-norms. BL was indeed shown to be the logic of continuous t-norms and their residua [9, 6]. Important many-valued logics, previously studied in mathematical logic in independent settings, such as the Łukasiewicz infinitely-valued logic or the Gödel-Dummett logic, were then proven to be extensions of BL. The Łukasiewicz infinitely-valued logic and the Gödel-Dummett logic are, in fact, based on two of the three most important continuous t-norms, i.e. the Łukasiewicz t-norm $x *_L y = \max\{x + y - 1, 0\}$, and the Gödel t-norm $x *_G y = \min\{x, y\}$. A third remarkable example of a continuous t-norm is given by the Product t-norm $x *_\Pi y = xy$, on which the Product logic is based [8].

The importance of the above t-norms is given by the following fact:

THEOREM 2.1 ([21])
Every continuous t-norm is locally isomorphic to either the Łukasiewicz, Product or Gödel t-norm.

Logics based on continuous t-norms have an algebraic semantics based on *residuated lattices* [25, 12]. Recall that a **commutative residuated lattice** is a structure $\mathcal{L} = \langle L, \vee, \wedge, \cdot, \Rightarrow, 1 \rangle$ such that: $\langle L, \cdot, 1 \rangle$ is a commutative monoid, $\langle L, \vee, \wedge \rangle$ is a lattice and $\Rightarrow$ is the residuum of $\cdot$, i.e.: $y \leq x \Rightarrow z$ if, and only if, $x \cdot y \leq z$, for all $x, y, z \in L$.

A residuated lattice is called **integral** if 1 is the top element, **bounded** if it has a bottom element (often denoted by 0).

A **BL algebra** is a bounded commutative integral residuated lattice satisfying the conditions:

$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1, \tag{lin}$$

$$x \cdot (x \Rightarrow y) = x \wedge y. \tag{div}$$

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[1] In fact, the existence of the residuum of a t-norm $*$ is equivalent to the left-continuity of $*$, so the same construction can be carried out for any left-continuous t-norm.

The divisibility condition implies the continuity of the monoidal operation with respect to the order topology: if $\mathcal{A}$ is a BL algebra, if $X, Y \subseteq \mathcal{A}$ and $\inf Y$ and $\sup X$ exists, then for all $a \in \mathcal{A}$, $a \cdot \inf Y = \inf(a \cdot Y)$, and $a \cdot \sup X = \sup(a \cdot X)$. The class of BL algebras clearly forms a variety.

Let $\neg x$ be an abbreviation for $x \Rightarrow 0$:

  i) An **MV algebra** is a BL algebra satisfying $\neg\neg x = x$[2];
 ii) A **Gödel algebra** is a BL algebra satisfying: $x \cdot x = x$;
iii) A **Product algebra** is a BL algebra satisfying $\neg x \vee ((x \Rightarrow (x \cdot y)) \Rightarrow y)$.

The above classes of algebras constitute the three main subvarieties of BL algebras, and are the equivalent algebraic semantics of the Łukasiewicz infinitely-valued logic, the Gödel-Dummett logic, and the Product logic (see [3, 5, 8]).

ŁΠ algebras were defined in [7] as a combination of both MV algebras and Product algebras. An **ŁΠ algebra** is a structure $\langle L, \oplus, \neg_L, \Rightarrow_\Pi, \cdot_\Pi, 0, 1 \rangle$ where:

 (i) $\langle L, \oplus, \neg_L, 0 \rangle$ is an MV algebra;
 (ii) $\langle L, \Rightarrow_\Pi, \cdot_\Pi, 0, 1 \rangle$ is a Product algebra;
(iii) $x \cdot_\Pi (y \oplus \neg_L z) = (x \cdot_\Pi y) \oplus \neg_L(x \cdot_\Pi z)$;
(iv) $\Delta(x \Rightarrow_L y) \leq (x \Rightarrow_\Pi y)$,

where $\Delta(x)$ stands for $\neg_L x \Rightarrow_\Pi 0$, and the symbol $\leq$ represents a partial order definable as $x \leq y$ if and only if $x \Rightarrow_L y = 1$. Notice that (iv) just states that the order defined by the Łukasiewicz implication is the same as the one obtained from the product implication. When the order defined above is linear we will call the algebra a **chain**. $\text{Ł}\Pi\frac{1}{2}$ algebras are an expansion of ŁΠ algebras with a constant $\frac{1}{2}$, satisfying the axiom $\frac{1}{2} = \neg_L \frac{1}{2}$.

In the rest of the paper, in order to simplify the notation, we will often write $\Rightarrow$ and $\neg$ for, respectively, $\Rightarrow_L$ and $\neg_L$.

EXAMPLE 2.2
The algebra $[0,1]_{\text{Ł}\Pi\frac{1}{2}} = \langle [0,1], \oplus, \neg, \cdot, \Rightarrow_\Pi, 0, 1, \frac{1}{2} \rangle$, where the operations are defined as

$$x \oplus y = \min\{x+y, 1\} \qquad\qquad \neg x = 1 - x$$

$$x \cdot y = xy \text{ (ordinary product)} \qquad x \Rightarrow_\Pi y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$$

is an $\text{Ł}\Pi\frac{1}{2}$ algebra. Moreover, $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$ generates the variety of $\text{Ł}\Pi\frac{1}{2}$ algebras (see [7]).

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[2] MV algebras are commonly presented as structures $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ satisfying the following equations:
   (MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,   (MV2) $x \oplus y = y \oplus x$,
   (MV3) $x \oplus 0 = x$,   (MV4) $\neg\neg x = x$,
   (MV5) $x \oplus \neg 0 = 0$,   (MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.
The presentation of MV algebras in that signature is term-wise equivalent to the presentation as residuated lattices, by the following definitions (see [5]):

$$x \cdot y = \neg(\neg x \oplus \neg y) \quad x \Rightarrow y = \neg x \oplus y.$$

Notice that every ŁΠ chain is either the two-element Boolean algebra (where $\cdot$, $\odot$, and $\wedge$ coincide) or it is infinite and contains an element x such that $x = \neg x$. Therefore, modulo an expansion of the language, every infinite ŁΠ chain is an ŁΠ$\frac{1}{2}$ algebra.

As mentioned above, ŁΠ$\frac{1}{2}$ algebras expand both MV and Product algebras, and they are easily seen to expand Gödel algebras as well. Moreover, as shown in [16], the equational theory of any variety generated by an algebra based on a continuous t-norm $*$ is definable in the equational theory of ŁΠ$\frac{1}{2}$ algebras if and only if $*$ is representable (up to isomorphism) as s finite ordinal sum of copies of the Gödel, Łukasiewicz, and Product t-norms.

We introduce now the algebraic structures we are going to deal with: namely, ŁΠ algebras with fixed points. Let us call **CTerm** the set of ŁΠ terms in which the symbol $\Rightarrow_\Pi$ does not appear. μŁΠ algebras are structures of type

$$\mathcal{A} = \langle A, \oplus, \neg, \cdot, \Rightarrow_\Pi, 0, 1, \{\mu x_{t(x,\bar{y})}\}_{t(x,\bar{y}) \in CTerm} \rangle,$$

where $\langle A, \oplus, \neg, \cdot, \Rightarrow_\Pi, 0, 1 \rangle$ is an ŁΠ algebra, and the following conditions are satisfied:

(i) $\mu x_{t(x,\bar{y})}(\bar{y}) = t(\mu x_{t(x,\bar{y})}(\bar{y}), \bar{y})$,
(ii) If $t(s(\bar{y}), \bar{y}) = s(\bar{y})$ then $\mu x_{t(x,\bar{y})}(\bar{y}) \le s(\bar{y})$,
(iii) $\bigwedge_{i \le n} \Delta(p_i \Leftrightarrow_L q_i) \le (\mu x_{t(x,\bar{y})}(p_1, ..., p_n) \Leftrightarrow_L \mu x_{t(x,\bar{y})}(q_1, ..., q_n))$[3].

It is readily seen that the first axiom states that $\mu x_{t(x,\bar{y})}$ is the fixed point of $t(x, \bar{y})$, the second forces $\mu x_{t(x,\bar{y})}$ to be the least among the fixed points of $t(x, \bar{y})$ and finally the third entails a good behaviour of μ with regards to substitutions. Obviously, the above axiomatization is not finite. Still, it is worth noticing that μŁΠ algebras form a variety.

EXAMPLE 2.3
Consider the algebra

$$[0,1]_{\mu\text{ŁΠ}} = \langle [0,1], \oplus, \neg, \cdot, \Rightarrow_\Pi, 0, 1, \{\mu x_{t(x,\bar{y})}(\bar{y})\}_{t(x,\bar{y}) \in CTerm} \rangle,$$

where the operations $\oplus, \neg, \cdot$ and $\Rightarrow_\Pi$ are defined as in Example 2.2 and, for any $t(x, \bar{y}) \in CTerm$, the value of $\mu x_{t(x)}(\bar{a})$ is given by the minimum fixed point of the term $t(x, \bar{a})$ seen as a real function from $[0,1]$ to $[0,1]$, for every $\bar{a} \in [0,1]^n$. Then, such a structure is a μŁΠ algebra. In other words, the role of the operator $\mu x_{t(x,\bar{y})}$, for any term $t(x, \bar{y}) \in$ **CTerm**, is to give as an output, on input $\bar{a}$, the minimum b such that $t(b, \bar{a}) = b$ (whose existence is guaranteed by Brower's Theorem[4]).

THEOREM 2.4 ([23])
The μŁΠ algebra $[0,1]_{\mu\text{ŁΠ}}$ generates the variety of μŁΠ algebras.

It is trivial to see that not every ŁΠ algebra is the reduct of a μŁΠ algebra. On the other hand note that the ŁΠ reduct of a μŁΠ algebra can be always expanded to an ŁΠ$\frac{1}{2}$ algebra. Indeed, the existence of an element x such that $\neg x = x$ is guaranteed by the fact that the equation $\neg \mu x_{\neg x} \Leftrightarrow_L \mu x_{\neg x} = 1$ holds in $[0,1]_{\mu\text{ŁΠ}}$, and, consequently,

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[3]Recall that $x \Leftrightarrow_L y$ is defined as $(x \Rightarrow_L y) \odot (y \Rightarrow_L x))$.

[4]Recall that Brower's Theorem states that every continuous function from the closed unit ball D^n to itself has at least one fixed point.

in every μŁΠ algebra. Also, note that the MV reduct of a μŁΠ algebra can always be expanded to a *divisible* MV algebra (see [24] and references therein).

The following result is the restriction to the linearly ordered case of a theorem in [23]:

THEOREM 2.5
There exists a categorical equivalence between real closed fields and μŁΠ chains.

In particular every μŁΠ chain is isomorphic to the linearly ordered μŁΠ algebra defined over the unit interval of a real closed field (up to isomorphism).

We close this section showing some tight connections between μŁΠ and ŁΠ algebras.

PROPOSITION 2.6
The category of μŁΠ algebras with their homomorphisms is a full subcategory of ŁΠ algebras with ŁΠ homomorphisms

PROOF. It is sufficient to note that being a fixed points is a property which can be expressed equationally, hence a homomorphism between the two ŁΠ reducts naturally extends to the whole structure of μŁΠ algebra. ■

PROPOSITION 2.7
Any ŁΠ$\frac{1}{2}$ chain is the subreduct of a unique μŁΠ chain, up to isomorphisms,

PROOF. Let A be an ŁΠ$\frac{1}{2}$ algebra, let F be the ordered field associated to it as in [19]. By Artin-Schreier Theorem F has an extension to a real closed field R. Clearly A embeds in the interval algebra of R, which is a μŁΠ algebra (for more details on the correspondence between μŁΠ algebras and real closed fields see [23]). Suppose now that there exist two, non-isomorphic, μŁΠ algebras B_1 and B_2 in which A embeds. Let R_1 and R_2 be the two real closed fields associated to B_1 and B_2. R_1 and R_2 are not isomorphic, and so we have two non-isomorphic real closed fields in which F can be embedded, contradicting Artin-Schreier Theorem. ■

3 Free μŁΠ algebras

We give now a functional description of the free μŁΠ algebra on an arbitrary number of generators. In order to characterise such functions we will need a few concepts from the theory of real closed fields and basic Galois theory (the reader may consult [14]).

The free μŁΠ algebra over κ generators will be denoted by $\mathcal{F}_\kappa(\mu$ŁΠ$)$. By a general result of Universal Algebra, $\mathcal{F}_\kappa(\mu$ŁΠ$)$ is the subalgebra of the algebra of all functions from $[0,1]^\kappa$ to $[0,1]$ generated by the projections under the closure for the μŁΠ operations defined point-wise. Giving a description of the $\mathcal{F}_\kappa(\mu$ŁΠ$)$ amounts to finding out which functions can be generated by the projections. For MV algebras, the answer comes from the classical McNaughton Theorem: the functions generated by the projections are exactly the continuous piecewise linear functions with integer coefficients.

It is clear that, in our description of free algebras, we can limit ourselves to the case in which κ is finite. This is not restrictive, since every element of $\mathcal{F}_\kappa(\mu$ŁΠ$)$ is generated by finitely many projections and hence belongs to $\mathcal{F}_n(\mu$ŁΠ$)$ for some finite n. In equivalent algebraic terms, $\mathcal{F}_\kappa(\mu$ŁΠ$)$ is the limit of a direct system of its subalgebras, each one isomorphic to a free μŁΠ algebra over finitely many generators.

6 Advances in the theory of μŁΠ Algebras

We start with $\mathcal{F}_0(\mu$ŁΠ$)$ showing that it is isomorphic to the interval algebra of $\mathbb{R}^{\mathrm{alg}}$, i.e. the real algebraic closure of the rational numbers. Indeed we will prove something stronger.

PROPOSITION 3.1
The μŁΠ algebra on the $[0,1]$ interval of $\mathbb{R}^{\mathrm{alg}}$ can be embedded into every μŁΠ chain.

PROOF. Every μŁΠ algebra is also an ŁΠ$\frac{1}{2}$ algebra. It is known [19, Lemma 4.3] that the ŁΠ$\frac{1}{2}$ algebra on $\mathbb{Q} \cap [0,1]$ can be embedded into every ŁΠ$\frac{1}{2}$ chain. In particular this means that $\mathbb{Q}$ can be embedded into every ordered field generated by an ŁΠ$\frac{1}{2}$ chain. This implies that the real closed field generated by a μŁΠ algebra must contain $\mathbb{Q}$. Since $\mathbb{R}^{\mathrm{alg}}$ is the smallest real closed field containing $\mathbb{Q}$, we have that any μŁΠ chain must contain an isomorphic copy of the μŁΠ algebra over $\mathbb{R}^{\mathrm{alg}} \cap [0,1]$. ∎

Clearly $\mathcal{F}_0(\mu$ŁΠ$)$ is linearly ordered since it contains only constant functions.

Notice that there is a strong connection between fixed points and roots of polynomials. Indeed given a polynomial $P(x)$ each of its solutions is a fixed point of the polynomial $P(x)+x$. Viceversa, each of its fixed points is a solution of the polynomial $P(x)-x$. This correspondence is preserved also when we restrict to the $[0,1]$ interval.

Let us denote by $\mathbb{Z}[x_1,\ldots,x_n]$ the domain of polynomials in n variables and integer coefficients and by $\mathbb{Q}(x_1,\ldots,x_n)$ its fraction field. With an abuse of notation, we will use the same symbols to denote both polynomials and their associated functions from $\mathbb{R}^n$ to $\mathbb{R}$. When we write $P/Q \in \mathbb{Q}(x_1,\ldots,x_n)$ we implicitly mean that $P,Q \in \mathbb{Z}[x_1,\ldots,x_n]$ and that P and Q do not have common factors.

DEFINITION 3.2
For any given $n \in \mathbb{N}$ we call **root function** every function $f(y_1,\ldots,y_n)$ such that if $P(x) = \sum_{i \leq n} a_i x^i \in \mathbb{R}^{\mathrm{alg}}[x]$ then $f(a_1,\ldots,a_n) = r$ if, and only if, r is the minimum value such that $P(r) = 0$.
We will call **super algebraic** every function which is

- a root function, or
- a rational polynomial function $P/Q \in \mathbb{Q}(x_1,\ldots,x_n)$, or
- a composition of the previous two kinds of function.

Note that root functions are not enough to characterise $\mathcal{F}_\kappa(\mu$ŁΠ$)$, since an element of $t \in CTerm$ can be represented as a member of $\mathbb{Z}[x_1,\ldots,x_n]$. Hence we need a different root function for every function represented by an element of $\mathbb{Z}[x_1,\ldots,x_n]$.

DEFINITION 3.3
Let $\mathcal{R}$ be the set of all functions f_R such that given $f \in \mathbb{Z}[x,x_1,\ldots,x_n]$

$$f_R[a_1,\ldots,a_n] = r$$
$$\text{iff}$$
$$r \text{ is the minimum value for which } f(r,a_1,\ldots,a_n) = 0.$$

LEMMA 3.4
$\mathcal{R}$ is the set of super algebraic functions.

PROOF. Notice that a rational polynomial function $P/Q \in \mathbb{Q}(x_1, \ldots, x_n)$ is the function in $\mathcal{R}$ associated to the polynomial $P(x_1, \ldots, x_n) - xQ(x_1, \ldots, x_n)$.
For the other direction notice that composing a root function with a suitable projection gives any desired function in $\mathcal{R}$. ∎

THEOREM 3.5 (Galois 1832)
A polynomial is **solvable by radicals** if, and only if, the group of automorphisms of the field of its solutions which fix the field of coefficients is solvable.

In particular there exist polynomials that are not solvable by radicals. Hence the set of super algebraic functions is strictly larger than the set of algebraic functions. What needs to be added to algebraic functions to obtain super algebraic functions is *unknown*.

Let $\{P > 0\}$ denote the set $\{v \in [0,1]^n \mid P(v) > 0\}$.

DEFINITION 3.6
A subset S of $[0,1]^n$ is a $\mathbb{Q}$-**semialgebraic** if it is a boolean combination of sets of the form $\{P > 0\}$ for some $P \in \mathbb{Z}[x_1, \ldots, x_n]$.
A subset S of $[0,1]^n$ is **semialgebraic** if it is a boolean combination of sets of the form $\{P > 0\}$ for some $P \in \mathbb{R}^{\mathrm{alg}}[x_1, \ldots, x_n]$.

DEFINITION 3.7
A LΠ-**hat** is a function $h : [0,1]^n \longrightarrow [0,1]$ for which there exist a $\mathbb{Q}$-semialgebraic set S and a function $f = P/Q \in \mathbb{Q}(x_1, \ldots, x_n)$ such that:

- Q has no zero in S,
- if $x \in S$ then $h(x) = f(x)$,
- if $x \notin S$ then $h(x) = 0$;

in this case we denote h by $\langle S, f \rangle$.
   A μ-**hat** is a function $h : [0,1]^n \longrightarrow [0,1]$ such that there exist a semialgebraic set S and a super algebraic function f such that if $x \in S$ then $h(x) = f(x)$ and if $x \notin S$ then $h(x) = 0$.
Also in this case we denote h by $\langle S, f \rangle$.

DEFINITION 3.8
A **basic LΠ-function** and a **basic μ-function** over $[0,1]^n$ are, respectively, a finite sum of LΠ-hats and a finite sum of μ-hats

$$\langle S_2, f_1 \rangle + \langle S_2, f_2 \rangle + \ldots + \langle S_k, f_k \rangle$$

such that $S_i \cap S_j = \emptyset$ for any $i \neq j$.
We denote by $\text{LΠ}B_n$ and B_n, respectively, the sets of LΠ-basic functions over $[0,1]^n$ and μ-basic functions over $[0,1]^n$

We have developed all the necessary definitions to describe the algebra $\mathcal{F}_\kappa(\mu\text{LΠ})$. Basically, the idea we follow to achieve the characterisation we are looking for is an extension of the technique used in [20], where, among others, the following result is proved.

THEOREM 3.9
ŁΠB_n is the free ŁΠ algebra over n generators.

LEMMA 3.10
B_n contains the projection functions and is a μŁΠ algebra under point-wise operations.

PROOF. An easy adaptation of the proof of the previous theorem shows that B_n is closed under ŁΠ operations. Given a term in $t \in Cterm$ let

$$g = \langle S_1, P_1 \rangle + ... + \langle S_r, P_r \rangle$$

be its associated function. Then we claim

$$\mu x_{t(x,\bar{y})} = \langle T_1, Q_1 \rangle + ... + \langle T_r, Q_r \rangle$$

where each T_i is a semialgebraic set and each Q_i is a μ-hat. Indeed, if we associate to any P_i a new polynomial $P_i' = P_i - x$ and call $R_{P_i'}$ the functions which give the minimum root of the polynomial P_i', then it is easy to check that:

$$\mu x_{t(x,\bar{y})} = \langle \{\bar{y} \mid \exists z \, (P_1(z, \bar{y}) = z \wedge (z, \bar{y}) \in S_1)\}, R_{P_1'} \rangle +$$

$$\vdots$$

$$+ \langle \{\bar{y} \mid \exists z \, (P_r(z, \bar{y}) = z \wedge (z, \bar{y}) \in S_r)\}, R_{P_r'} \rangle$$

Now it is sufficient to note that, for every $1 \leq i \leq r$, the set

$$\{\bar{y} \mid \exists z \, (P_i(x, \bar{y}) = x \wedge (x, \bar{y}) \in S_i)\}$$

is a projection of a semialgebraic set, which, by the Tarski-Seidenberg theorem, is, again, a semialgebraic set. Moreover all such sets are disjoint, since so are the sets S_i, and $R_{P_i'}(\bar{y}) \neq 0$ as $(x, \bar{y}) \in S_i$. From this follows that if $f_1, ..., f_n$ are functions in B_n then $\mu x_{t(x,\bar{y})}(f_1, ..., f_n)$ is also in B_n. ∎

LEMMA 3.11
Let $P \in \mathbb{R}^{\mathrm{alg}}[x_1, \ldots, x_n]$ and let $P^\sharp(\bar{v}) = \min\{\max\{P(\bar{v}), 0\}, 1\}$ for any $\bar{v} \in [0,1]^n$. Then $P^\sharp$ belongs to $\mathcal{F}_\kappa(\mu\text{ŁΠ})$.

PROOF. Let $P = \sum_{i \in I} k_i x_1^{j_{i1}} \cdot \ldots \cdot x_n^{j_{in}}$. We prove the claim by induction on $j = \sum_{i \in I} \sum_{m \leq n} j_{im}$. If $j = 0$ then P is constant and $P^\sharp \in \mathbb{R} \cap [0,1]$; as seen in Proposition 3.1 each of those constant belongs to $\mathcal{F}_\kappa(\mu\text{ŁΠ})$. Assume $j \geq 1$, then there exists a real algebraic number r and a polynomial $Q \in \mathbb{R}^{\mathrm{alg}}[x_1, \ldots, x_n]$ such that $P = Q \cdot (x - r)$. Now $(x - r)^\sharp$ is obviously in $\mathcal{F}_\kappa(\mu\text{ŁΠ})$ and $Q^\sharp$ is in $\mathcal{F}_\kappa(\mu\text{ŁΠ})$ by induction. ∎

COROLLARY 3.12
The characteristic function of every semialgebraic set is in $\mathcal{F}_\kappa(\mu\text{ŁΠ})$

PROOF. Since $\mathcal{F}_\kappa(\mu\text{ŁΠ})$ is closed under the boolean operator it suffices to prove that the characteristic functions of the sets of the form $\{P > 0\}$ are in $\mathcal{F}_\kappa(\mu\text{ŁΠ})$. But such a function is just $\neg\Delta(\neg P^\sharp)$ ∎

THEOREM 3.13
$\mathcal{F}_\kappa(\mu\text{ŁΠ})$ is the algebra of piecewise super algebraic functions from $[0,1]^\kappa$ to $[0,1]$

PROOF. We need to show that every basic function is in $\mathcal{F}_\kappa(\mu\text{ŁΠ})$. Since the semi-algebraic sets appearing in the definition of a basic function are pairwise disjoint we can substitute every $+$ with $\oplus$. Hence it is sufficient to show that every μ-hat is in $\mathcal{F}_\kappa(\mu\text{ŁΠ})$, which easily comes from the definition. ∎

4   Model-Theoretic Results

In this section we investigate some model-theoretic properties of μŁΠ chains. As one may expect, results heavily depend on the direct connection with real closed fields.

The first-order theory $\text{Th}(\text{ŁΠ}\frac{1}{2})$ of ŁΠ$\frac{1}{2}$ chains in the language $\langle\oplus,\neg,\cdot,\Rightarrow_\Pi,0,1,\frac{1}{2},<\rangle$ is axiomatized by the universal closure of the equations defining the variety of ŁΠ$\frac{1}{2}$ algebras plus sentences defining the linearity of the order $<$. Notice that $\text{Th}(\text{ŁΠ}\frac{1}{2})$ does not admit quantifier elimination in the given language. Indeed, the ŁΠ$\frac{1}{2}$ algebra of the rational numbers is not an elementary substructure of the ŁΠ$\frac{1}{2}$ algebra of the real numbers, and, consequently, $\text{Th}(\text{ŁΠ}\frac{1}{2})$ is not model-complete. As an example, the formula $\exists x\ (x\cdot x=\frac{1}{2})$ clearly does not hold over the rational numbers but does hold over the real algebraic numbers.

Henceforth we use $\sqcap$, $\sqcup$ and $\Longrightarrow$ for classical conjunction, disjunction and implication, respectively.

We define the first-order theory of μŁΠ chains as an extension of $\text{Th}(\text{ŁΠ}\frac{1}{2})$, in the language

$$\langle\oplus,\neg,\cdot,\Rightarrow_\Pi,0,1,\frac{1}{2},<\rangle,$$

with the following sentences, for each $t(x,\overline{y})\in CTerm$:

$$\forall\overline{y}\exists x\ t(x,\overline{y})=x,$$
$$\forall\overline{y}\forall z\exists x\ (t(z,\overline{y})=z)\Rightarrow(t(x,\overline{y})=x\sqcap(x\leq z)).$$

The first sentence states that each $t(x,\overline{y})\in CTerm$ has a fixed point, while the second one states that $t(x,\overline{y})$ also has a minimum fixed point. Notice that we are not using in the language the operators $\mu_{t(x,\overline{y})}$, which can be obviously given a first-order definition.

It is easily seen that every model of $\text{Th}(\mu\text{ŁΠ})$ is an ŁΠ$\frac{1}{2}$ chain isomorphic to the interval algebra of a real closed field, and is the reduct of a μŁΠ chain. The class of models of $\text{Th}(\mu\text{ŁΠ})$ then corresponds to the subclass of ŁΠ$\frac{1}{2}$ chains called *real closed* ŁΠ$\frac{1}{2}$ *chains* in [15]. The theory of real closed ŁΠ$\frac{1}{2}$ chains was shown to have quantifier elimination (see [15]), but no explicit axiomatization was given. Since the above set of sentences obviously axiomatizes the theory of real closed ŁΠ$\frac{1}{2}$ chains, the quantifier elimination result from [15] immediately applies to $\text{Th}(\mu\text{ŁΠ})$. However, we give here a slightly different (and more explicit) proof relying on definability of semialgebraic sets within ŁΠ$\frac{1}{2}$ algebras over real closed fields.

Let $\mathcal{L}$ be a signature of the form $\langle<,f_1,\ldots,f_n,c_1,\ldots,c_m\rangle$, where each f_i is a function symbol and each c_j is a constant symbol. $\mathcal{L}$ will be assumed to include no relation symbol but $<$ (and, of course, $=$). By an **unnested atomic formula** in $\mathcal{L}$

we mean one of the following formulas:

$$x = y, \quad x < y; \tag{i}$$

$$x = c, \quad c = x, \quad x < c, \quad c < x \qquad \text{for some constant symbol } c \in \mathcal{L}; \tag{ii}$$

$$f(\overline{x}) = y, \quad y = f(\overline{x}), \quad f(\overline{x}) < y, \quad y < f(\overline{x}) \quad \text{for some function symbol } f \in \mathcal{L}. \tag{iii}$$

A formula is called unnested if all its atomic subformulas are unnested. Then, it is easy to see:

LEMMA 4.1 ([11])
For a first-order language $\mathcal{L} = \langle <, f_1, \ldots, f_n, c_1, \ldots, c_m \rangle$, every formula is equivalent to an unnested formula.

DEFINITION 4.2
Let T_1 and T_2 be two theories in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. T_1 is **interpretable** in T_2 if there exists an $\mathcal{L}_2$-formula $\chi(z)$ and for every model $\mathcal{M} \models T_1$ there exists a unique (up to isomorphism) model $\mathcal{M}^\star \models T_2$, called the **complementary model of** $\mathcal{M}$ such that:

(i) there exists a bijection $h_{\mathcal{M}} : M \to \{a \mid \mathcal{M}^\star \models \chi(a)\}$ from the domain of $\mathcal{M}$ into the set defined by $\chi(z)$ over the domain of $\mathcal{M}^\star$;

(ii) for each unnested atomic $\mathcal{L}_1$-formula $\varphi(\overline{x})$, there exists an $\mathcal{L}_2$-formula $\varphi^\sharp(\overline{x})$ such that, for all $\overline{b} \in M$

$$\mathcal{M} \models \varphi(\overline{b}) \text{ if and only if } \mathcal{M}^\star \models \varphi^\sharp(h_{\mathcal{M}}(\overline{b})).$$

The above definition together with Lemma 4.1 yields that the interpretation of T_1 into T_2 can be extended to arbitrary formulas.

LEMMA 4.3 (See Theorem 5.3.2 in [11])
Let T_1 and T_2 be two theories in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Suppose that T_1 is interpretable in T_2. Then for every $\mathcal{M} \models T_1$ and for each $\mathcal{L}_1$-formula $\varphi(\overline{x})$, there exists an $\mathcal{L}_2$-formula $\varphi^\sharp(\overline{x})$ so that for all $\overline{b} \in M$

$$\mathcal{M} \models \varphi(\overline{b}) \text{ if and only if } \mathcal{M}^\star \models \varphi^\sharp(h_{\mathcal{M}}(\overline{b})),$$

where $\mathcal{M}^\star$ is the complementary model of $\mathcal{M}$, and $h_{\mathcal{M}}$ is the bijection defined in Definition 4.2(i).

Recall that (up to isomorphism) every μLΠ chain is isomorphic to the interval algebra of one real closed field. Thus there is a one-to-one connection between μLΠ chains and real closed fields. Moreover, any unnested atomic formula of Th(μLΠ) can be translated into a formula in the language of ordered fields as follows (the translation of inequalities is similar):

- $x \cdot y = z \ \mapsto \ x \cdot y = z$;
- $x \Rightarrow_\Pi y = z \ \mapsto \ ((x \leq y) \sqcap (z = 1)) \sqcup ((x > y) \sqcap (y = z \cdot x))$;
- $x \wedge y = z \ \mapsto \ ((x \leq z) \sqcap (x = z)) \sqcup ((y \leq x) \sqcap (y = z))$;
- $x \oplus y = z \ \mapsto \ ((x + y \leq 1) \sqcap (x + y = z)) \sqcup ((x + y \geq z) \sqcap (z = 1))$;

- $\neg(x) = y \;\mapsto\; 1 - x = y.$

This leads to:

LEMMA 4.4
Th(μŁΠ) is interpretable into RCF.

PROOF. Let $\varphi(\bar{x})$ be a formula in the language $\langle \oplus, \neg, \cdot, \Rightarrow_\Pi, 0, 1, \frac{1}{2}, < \rangle$, let $\mathcal{A}$ be a μŁΠ chain and $\mathcal{F}_\mathcal{A}$ its associated real closed field. By Lemma 4.3 and the above translation it is readily seen that there exists a formula $\varphi^\sharp(\bar{x})$ in the language of ordered fields such that, for all $\bar{a} \in \mathcal{A}$:

$$\mathcal{A} \models \varphi(\bar{a}) \text{ if and only if } \mathcal{F}_\mathcal{A} \models \varphi^\sharp(\bar{a})$$

∎

We show that Th(μŁΠ) admits quantifier elimination. Notice that, in spite of the deep connection with real closed fields, this result is not trivial being quantifier elimination sensible to the language.

THEOREM 4.5
Th(μŁΠ) admits quantifier elimination in the language

$$\langle \oplus, \neg, \cdot, \Rightarrow_\Pi, 0, 1, \frac{1}{2}, < \rangle.$$

PROOF. Let $\varphi(\bar{x})$ be a formula in the language of $\langle \oplus, \neg, \cdot, \Rightarrow_\Pi, 0, 1, \frac{1}{2}, < \rangle$, and $\varphi'(\bar{x})$ be its translation in the language of ordered fields. For any model $\mathcal{A}$ of Th(μŁΠ) for all $\bar{a} \in A$, $\mathcal{A} \models \varphi(\bar{a})$ if and only if $\mathcal{F}_\mathcal{A} \models \varphi'(\bar{a})$. $\varphi'(\bar{x})$ is equivalent to a quantifier-free formula $\psi(\bar{x})$ in the language of ordered fields. Clearly $\{\bar{a} \mid \mathcal{F}_\mathcal{A} \models \psi(\bar{a})\}$ is a semialgebraic subset of $[0,1]^n_{\mathcal{F}_\mathcal{A}}$. As seen in 3.9 the characteristic function of every semialgebraic set over $[0,1]^n$ is definable in the ŁΠ$\frac{1}{2}$ algebra over the reals. This fact obviously generalises to semialgebraic sets over any real closed field and its associated ŁΠ$\frac{1}{2}$ algebra. Hence the claim follows. ∎

A class $\mathbb{K}$ of structures in the same signature $\mathcal{L}$ is said to have the **amalgamation property** if for every tuple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, f, g)$ such that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ belong to $\mathbb{K}$, and $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{A} \to \mathcal{C}$ are embeddings, there exist a structure $\mathcal{D} \in \mathbb{K}$ and embeddings $f' : \mathcal{B} \to \mathcal{D}$, $g' : \mathcal{C} \to \mathcal{D}$ such that $f' \circ f = g' \circ g$. In this case $\langle \mathcal{D}, f', g' \rangle$ is called an **amalgam** for $(\mathcal{A}, \mathcal{B}, \mathcal{C}, f, g)$. A class $\mathbb{K}$ of structures in the same signature $\mathcal{L}$ is said to have the **strong amalgamation property**, if it has the amalgamation property and, moreover, $f'[\mathcal{B}] \cap g'[\mathcal{C}] = (f' \circ f)[\mathcal{A}] = (g' \circ g)[\mathcal{A}]$, where for any set X and function h on X, $h[X] = \{h(x) \mid x \in X\}$.

From the above quantifier-elimination result, we easily obtain the following consequences, easily derivable from general results in Model Theory:

COROLLARY 4.6

(i) The class of μŁΠ chains enjoys the strong amalgamation property.
(ii) The class of ŁΠ$\frac{1}{2}$ chains enjoys the amalgamation property.

Now, we want to show that the whole variety of ŁΠ$\frac{1}{2}$ algebras has the amalgamation property. In order to do so, we rely on the following result by Metcalfe, Montagna and Tsinakis.

THEOREM 4.7 ([18])
Let $\mathbb{V}$ be a variety, and let $\mathbb{S} \subseteq \mathbb{V}$ be such that:

1. Every subdirectly irreducible member of $\mathbb{V}$ is in $\mathbb{S}$.
2. $\mathbb{S}$ is closed under (isomorphic images and) subalgebras.
3. For any $\mathcal{A} \subseteq \mathcal{B}$ in $\mathbb{V}$ and for any $\theta \in \mathbf{Con}(\mathcal{A})$ such that $\mathcal{A}/\theta \in \mathbb{S}$, there is $\vartheta \in \mathbf{Con}(\mathcal{B})$ such that $\vartheta \cap (\mathcal{A})^2 = \theta$ and $\mathcal{B}/\vartheta \in \mathbb{S}$.
4. $\mathbb{S}$ has the amalgamation property with respect to $\mathbb{V}$.

Then $\mathbb{V}$ has the amalgamation property.

Then, we can show:

THEOREM 4.8
The variety of $Ł\Pi\frac{1}{2}$ algebras enjoys the amalgamation property.

PROOF. Let $\mathbb{S}$ be the class of linearly ordered $Ł\Pi\frac{1}{2}$ algebras. Every $Ł\Pi\frac{1}{2}$ algebra is subdirectly irreducible if and only if it is linearly ordered [7]. So, it is easily seen that $\mathbb{S}$ satisfies $(1), (2),$ and (4) (the latter comes form the above corollary).

Let $\mathcal{A} \subseteq \mathcal{B}$ be $Ł\Pi\frac{1}{2}$ algebras, and θ be a congruence of $\mathcal{A}$ such that $\mathcal{A}/\theta$ is linearly ordered. Recall that any $Ł\Pi\frac{1}{2}$ algebra is subdirectly irreducible iff it is linearly ordered iff it is simple ([7]). Let ϑ be the congruence generated in $\mathcal{B}$ by θ. It is easy to see that $\theta = \vartheta \cap A^2$. We show that $\mathcal{B}/\vartheta$ is linearly ordered. Suppose $\mathcal{B}/\vartheta$ is not linearly ordered, then it is not subdirecty irreducible nor simple. This implies that the lattice $[\vartheta, \mathbf{1}_B]$ contains another congruence ω between ϑ and $\mathbf{1}_B$. It is easy to see that $\omega \cap A^2$ is a congruence for $\mathcal{A}$ that extends θ. But this means that $[\theta, \mathbf{1}_A]$ contains more than two elements, which obviously contradicts the fact that $\mathcal{A}/\theta$ is simple.

Therefore, (4) holds for the class of linearly ordered $Ł\Pi\frac{1}{2}$ algebras, and so the whole variety has the amalgamation property. ∎

From the above we get:

THEOREM 4.9
$\mathrm{Th}(\mu Ł\Pi)$ is the model completion of $\mathrm{Th}(Ł\Pi\frac{1}{2})$.

PROOF. $\mathrm{Th}(\mu Ł\Pi)$ clearly is a model companion of $\mathrm{Th}(Ł\Pi\frac{1}{2})$. Indeed, $\mathrm{Th}(\mu Ł\Pi)$ is model-complete, every model of $\mathrm{Th}(\mu Ł\Pi)$ also is a model of $\mathrm{Th}(Ł\Pi\frac{1}{2})$, and every model $\mathcal{M}$ of $\mathrm{Th}(Ł\Pi\frac{1}{2})$ has an extension that is a model of $\mathrm{Th}(\mu Ł\Pi)$ by Proposition 2.7.

The fact that $\mathrm{Th}(Ł\Pi\frac{1}{2})$ has the amalgamation property and the fact that $\mathrm{Th}(\mu Ł\Pi)$ is a model companion of $\mathrm{Th}(Ł\Pi\frac{1}{2})$ imply that $\mathrm{Th}(\mu Ł\Pi)$ is the model completion of $\mathrm{Th}(Ł\Pi\frac{1}{2})$ (see [11]). ∎

Finally, we prove a representation theorem for $Ł\Pi\frac{1}{2}$ algebras in terms of ultrapowers.

THEOREM 4.10
Every $Ł\Pi\frac{1}{2}$ algebra $\mathcal{A}$ is an algebra of $[0,1]^*$-valued functions over some set, where $[0,1]^*$ is an ultrapower of $[0,1]$.

PROOF. Let $\mathcal{A}$ be a $Ł\Pi\frac{1}{2}$ algebra, and let $P(\mathcal{A})$ be the set of its prime filters. $\mathcal{A}$ is embeddable into the product $\prod\{\mathcal{A}/F_i \mid F_i \in P(\mathcal{A})\}$. Each $\mathcal{A}/F_i$ is embeddable into

an $L\Pi\frac{1}{2}$ algebra $\mathcal{B}_i$ with fixed points. By the joint embedding property, along with the fact that all the $\mathcal{B}_i$'s are elementarily equivalent to each other, there exists an $L\Pi\frac{1}{2}$ algebra $\mathcal{D}$ over a real closed field in which each $\mathcal{B}_i$ can be embedded. The algebra $\mathcal{D}$ is elementarily equivalent to the $L\Pi\frac{1}{2}$ algebra over the reals and so, by Freyne's Theorem [4], there exists an elementary embedding of $\mathcal{D}$ into some ultrapower $[0,1]^*$ of $[0,1]$. ∎

5   Decidability and Complexity

We give here a characterization of the computational complexity of the equational theory of $\mu L\Pi$ algebras. We will take advantage, again, of the connection with real closed fields.

THEOREM 5.1
The equational theory of $\mu L\Pi$ algebras is in PSPACE.

PROOF. As shown by Canny [2] the universal theory of real closed fields is in PSPACE. We will show that the equational theory of $\mu L\Pi$ algebras can be translated in polynomial time into the universal theory of the reals. We follow [10] and [16].

Let ϕ be any equation in the language $\langle \oplus, \neg, \cdot, \Rightarrow_\Pi, 0, 1, \{\mu_{t(x,\bar{y})}\} \rangle$. Without any loss of generality we can assume that ϕ is of the form $t(\bar{x}) = 1$. The first step is to eliminate all the occurrences of terms of the form $\mu_{t(x,\bar{y})}$. Then, let us associate to each occurrence of a term of type $\mu_{t(x,\bar{y})}$ the formula

$$\forall z(t(x,\bar{y}) = x \sqcap (t(z,\bar{y}) = z \implies x \le z)). \tag{5.1}$$

Let ϕ' be the equation obtained from ϕ by substituting each term $\mu_{t(x,\bar{y})}$ by its related variable x, and let $\phi_{\mu_1}, \ldots, \phi_{\mu_m}$ be the formulas of the form (5.1).

Now, the next step is to translate $\phi', \phi_{\mu_1}, \ldots, \phi_{\mu_m}$ into a universal formula in the language of ordered fields. Let $S = \{t_1, \ldots, t_n\}$ be the set of all subterms of $\phi', \phi_{\mu_1}, \ldots, \phi_{\mu_m}$. Notice that the cardinality of S is linear in the length of $\phi', \phi_{\mu_1}, \ldots, \phi_{\mu_m}$.

We assign to each subterm a new variable w_i (different variables for different subterms) and, for each subterm t_i, we define a formula γ_i in the following way. If t_i is a variable x, let γ_i be $w_i = x$; if t_i is a constant c, let γ_i be $w_i = c$; if t_i is $f(t_j, \ldots, t_k)$ for some function symbol f and for some subterms $t_j, \ldots, t_k$, then let γ_i be $w_i = f(w_j, \ldots, w_k)$. For each ϕ_{μ_j}, let $\chi_{\mu_{ij}}$ be the conjunction of all the formulas γ_i. Each ϕ_{μ_j} is equivalent to the universal closure of

$$\chi_{\mu_j} \implies ((w_{t(x,\bar{y})} = w_x \sqcap w_{t(z,\bar{y})} = w_z) \implies w_x \le w_z)),$$

where each w_l is the variable associated to the corresponding term.

Now, let χ' be the conjunction of all the formulas γ_i built from ϕ'. It is easy to see that ϕ is equivalent to the universal closure of the formula:

$$\left(\chi' \sqcap \prod_{j=1}^{m} \left(\chi_{\mu_j} \implies \left((w_{t(x,\bar{y})} = w_x \sqcap w_{t(z,\bar{y})} = w_z) \implies w_x \le w_z \right) \right) \right) \implies w = 1,$$

where, assuming ϕ is of the form $t(\bar{x}) = 1$, w si the variable associated to the term $t(\bar{x})$ in ϕ.

The above is an unnested formula in the language $\langle \oplus, \neg, \cdot, \Rightarrow_\Pi, 0, 1 \rangle$. To obtain a universal formula in the language of ordered fields, it is sufficient to substitute each unnested formula with the corresponding formula (in the same variables) in the language $\langle +, -, \cdot, <, 0, 1 \rangle$, as shown above. A simple inspection of the algorithm sketched above shows that such a translation can be performed in polynomial time. ∎

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