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CATEGORICAL FOUNDATIONS OF MATHEMATICS
Or how to provide foundations for *abstract* mathematics

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Abstract. Feferman's argument presented in 1977 seemed to block any possibility for category theory to become a serious contender in the foundational game. According to Feferman, two obstacles stand in the way: one logical and the other psychological. We address both obstacles in this paper, arguing that although Feferman's argument is indeed convincing in a certain context, it can be dissolved entirely by modifying the context appropriately.

§1. Introduction: Feferman's challenge in 1977.

> To avoid misunderstanding, let me repeat that I am *not* arguing for accepting current set-theoretical foundations of mathematics. Rather, it is that on the platonist point of view of mathematics something like present systems of set theory must be prior to any categorical foundations. More generally, on *any view of abstract mathematics* priority must lie with notions of operation and collection.(Feferman, 1977, p. 154) [I emphasize the second part.]

My goal in this paper is to argue that indeed, on any view of *abstract* mathematics, priority must lie with notions of operation and collection, but that, contrary to what Feferman claimed, categories play an indispensable role in such a foundational framework. Put differently, I claim that categories are at the core of an alternative formalization of the naive conception of collections and operations.[1]

Here is the passage that expresses clearly and directly Feferman's argument:

> The point is simply that *when explaining* the general notion of structure and of particular kinds of structure such as groups, rings, categories, etc. we implicitly *presume as understood* the idea of *operation* and *collection*; ... Thus at each step we must make use of unstructured notions of operation and collection to explain the structural notions to be studied. The *logical* and psychological priority if not primacy of the notions of operation and collection is thus evident. (Feferman, 1977, p. 150)

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[1] Notice that Feferman argues that on the *platonist* point of view, there must be something prior to category theory. It is far from clear that a categorical standpoint is wedded to a platonist point of view. Indeed, it is entirely possible to develop constructive foundations of mathematics in a categorical framework. I will ignore this point in this paper and rather focus on the more central issue raised by Feferman.

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A series of remarks is in order.

- First, notice that Feferman places the notion of category alongside those of groups, rings, etc., in other words alongside algebraic notions. Although this is at first sight perfectly reasonable, since, on the one hand, the concept of category can be seen as a generalization of the concept of monoid (and the notion of preorder), and, on the other hand, the notion of category is, in a precise technical sense of categorical logic, an algebraic notion, it is debatable in the context of foundational research. Indeed, in that context, one has to consider not an individual category,[2] but categories of categories and the latter turns out to be a rather different beast, a rich mixture of algebra and geometry.
- Second, the argument rests on two interrelated understandings of priority: the logically prior and the psychologically prior. The first argument is that the notions of operation and collection are logically prior to any structured concept and since the notion of category is a structured concept, it cannot be used to define the former notions. I will not challenge this argument directly. I do want to underline, however, that it is more complex and subtle than it appears. At the time when Feferman wrote his paper and even today, a lot of implicit theoretical developments were and still are automatically attached to the argument: standard ZF set theory, first-order logic, model theory, and the development of modern mathematics. Nothing in all this tells us precisely what is meant by abstract mathematics and its epistemology. If the expression "abstract mathematics" merely denotes set-based mathematics and the latter can only be captured by ZF set theory, then, clearly, the argument is begging the question. One of the larger themes I will touch upon in this paper is precisely the fact that abstract mathematics, like the concept of mathematical structure, is open, in the sense that what it denotes changes with respect to the theoretical tools used to interpret and illustrate the concept. For instance, if abstract mathematics is obtained from a process of mathematical abstraction and if this latter process goes from the complex to the simple, it is entirely possible that in order to explain certain abstract notions, one has to start from complex, structured contexts, in sharp contrast with what Feferman is here claiming.
- Third, even if the latter possibility might not seem palatable from a logical point of view—after all, the process of abstraction has been considered suspicious by logicians from Frege onwards—it might still be defensible from a psychological point of view. It might be entirely reasonable to claim that, at least during the period before someone has fully assimilated logic and logical notions, that a process of abstraction plays an essential role in our understanding of important abstract mathematical concepts. One might even go as far as claiming that understanding formal logic itself is an illustration of a process of abstraction, that is, going from a rich, complex context, to a more simple but polysemous context.

Be that as it may, Feferman is essentially correct. The main problem is that in an abstract framework, one has to start with a notion of *abstract* set and operation on those and I claim that the standard formal set theories, be it ZF, NBG, or Morse–Kelley, simply do not refer to abstract sets.

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[2] I should immediately point out that even in the practice of category theory, one usually considers categories and functors between them. This is inescapable.

Furthermore, I also believe that the logical dimension has to be separated from the psychological dimension. Mostly because, as Feferman has underlined, we are dealing with abstract mathematics. Although I won't be able to argue for this claim in the present paper, I do believe that as far as abstract mathematics is concerned, Feferman has the order of presentation backwards. We simply do not *explain* group theory to students by starting with sets and operations. We don't teach group theory in grade 10. We teach geometry, algebra, and, later, number theory. Then, once we believe that students have grasped the concepts involved in these contexts, we introduce the idea of a group and give the abstract *definition*. I thus separate the order of definition from the order of explanation, which very often has more to do with the function the concept plays in the overall system of mathematics. From the abstract point of view, one can then clarify various results, shed some light on various theories and theorems and even reconstruct a field completely from that perspective. The "explanation"—as Feferman calls it, whatever this expression means in this context—is not provided by the abstract definition, but rather by the various instances of the concepts in different mathematical contexts. In general, the relationships between the abstract and the "concrete" are rather subtle. One should take a careful look at the rise of abstract group theory as well as the role of representation theory—which we take to be the opposite of the process of abstraction—in the latter to see how they interact and contribute to one another. (See, for instance, Curtis, 1999; Wussing, 1984.) Finally, and I think this aspect has to be kept in mind also, it is entirely possible that various concepts come to be understood in parallel so to speak, by an interaction between them. Thus, it is theoretically plausible that our understanding of set theory and first-order logic, in fact, goes hand in hand in some sort of feedback mechanism. I will leave these empirical speculations aside. However, I hope the moral is clear: thinking in terms of priority issues might be considerably subtle and complex when we are dealing with understanding.

To recapitulate: I want to show that it is possible to develop an alternative conception of the notions of collections and operations such that categories play a crucial conceptual role in this conception. For this conception to be developed theoretically, one has to modify certain aspects of first-order logic, the conception of sets and also the notion of categories. The main elements of the motivation and of the required modifications are taken from Makkai (1998). I will then move to the issue of psychological priority and make a series of remarks. My goal in these latter sections is merely to make room for alternatives that seem to be discarded by Feferman.

§2. Logical priority. In the 1960s and 1970s, when one thought of sets and categories, one thought of classical first-order logic, ZF(C), and the usual definition of category in terms of Hom-sets. If these specific notions are the only ones available, then Feferman wins the argument and this paper is over, or almost. It would seem that the only issue left in these circumstances is the question of size: can we extend standard ZF(C) to make room for large categories in a "natural" way? Although this is certainly an interesting and important issue, I will simply ignore it altogether, for I believe that one has now to attend to more fundamental *conceptual* aspects of the situation. After all, it is far from clear that ZF(C) is an appropriate framework for *abstract* mathematics, that is, that it is faithful to the nature of abstract mathematics.[3]

[3] In fact, from an historical point of view, one can claim that ZF(C) was developed in parallel to the abstract approach to mathematics. See, for instance, Corry (1996). Sets played a crucial role

I assume that we all have a naive, preformal, and rudimentary notion of set and that we also have a naive, preformal, and rudimentary notion of operation. This is in fact an empirical claim that seems to be reasonably well supported by recent inquiries in the new field of cognitive mathematics. (See, for instance, Feigenson, 2011.) ZF(C) is a sophisticated theoretical development of a specific encoding of the combinatorial aspects of that notion, with many virtues but also with numerous drawbacks.[4]

Thus, my claim is that there is an alternative theoretical development of the naive notion in which categories are inescapable. I am not claiming that category theory *as it is* provides such a theoretical development, but that when one attends to specific features of a conception of *abstract* sets, the universe of mathematics thus arising comprises categories in its bones.

How does one go about developing an alternative *theoretical* framework of the naive conception of collection and operation? I submit that in order to have a theoretically useful and valuable framework, one needs to develop three interrelated components, namely:

1. A logic with an explicit syntax;
2. A semantic universe;
3. A theory presented by some axioms written in the logical system or the deduction system; the axioms should have, but it is certainly not required, some sort of immediate plausibility with respect to the semantic universe.[5]

I assume that Feferman would agree with these desiderata and that what he has in mind when he speaks of the set-theoretical foundations (on the platonist point of view) is the usual combination of 1) classical first-order logic with its usual syntax; 2) the cumulative hierarchy; and 3) the axioms of ZF(C). I believe that as I write this paper, researchers in categorical foundations would agree that any categorical framework has to rest on three similar components. Thus, I believe that any category theorist who is sensitive to foundational issues would claim that the underlying logic has to be (more or less) modified, that the universe of interpretation has to have a different structure than the cumulative hierarchy, and that the axioms presented have to be based on a different notion of set than the one usually assumed by ZF(C). Then, one has to argue that these modifications are still playing the same role as that played by classical first-order logic, the axioms of ZF, and the cumulative hierarchy. I am *not* claiming that category theorists would all agree on how these modifications have to be made. Indeed, there might be unending debates

in the development of the later and I am not denying that. I am claiming that ZF(C), the specific theory, is not faithful to the abstract approach.

[4] One of the most interesting properties of ZF is in fact revealed by a categorical analysis of its features as a certain algebra: it turns out to be *universal* among a class of algebras. See Joyal & Moerdijk (1995). I believe that this is one of the strengths of category theory: the notion of universal morphisms should certainly be at the center of any cognitive analysis of mathematical concepts and even cognition in general. See Macnamara & Reyes (1994).

[5] Although I won't argue for this claim here, I think that there are very good reasons why these three components are present. They are the result of the development of foundational research in the twentieth century and should now be seen as *norms* for mathematical knowledge. I would have to argue for each one of these components and show what its contribution as a norm amounts to. I will leave this part to another paper and merely assume that there is today an implicit consensus on these norms, at least when they are considered from a sufficiently general angle and from a foundational point of view.

regarding various details but also with respect to fundamental aspects of the larger or global framework.[6]

I will argue here for one such series of modifications. In the framework that I will articulate, one can stick to first-order logic, but with a slightly different syntax, one can start with a certain notion of abstract set and introduce categorical universes naturally. Notice, however, that I will not present axioms, that is, a specific theory. I will merely sketch the conceptual motivation underlying such a program. Its implementation might in fact vary somewhat according to some decisions regarding specific aspects of the syntax and the semantics. What I will try to underline is the fact that when one attends to the *abstract* nature of mathematics—and I would be tempted to say *on any conception of abstract mathematics*—as exhibited by category theory, one has to slightly modify the underlying logic, the notion of set and the notion of category.

2.1. Logic. Here is a platitude: first-order logic has a history.[7] But that platitude is relevant to our story. It took some times before the precise formulation of first-order logic (FOL) crystallized and found its niche in the foundational landscape.

> Presenting in a nutshell the results of our quick historical overview, we can say that around 1900 logic was conceived as a theory of sentences, sets and relations; after World War I and as late as 1930 the exemplar of modern logic was a higher-order system, simple type theory; and only around 1940-1950 did the community of logicians as a whole come to agree that the paradigm logical system is FOL. (Ferreirós, 2001, p. 448)

We tend to forget that the development and status of FOL is intimately linked to the formalization of set theory and the crystallization of foundational studies in the framework of ZF. It is not entirely absurd to affirm that FOL and ZF were two sides of the same constitutive movement. FOL was enough to formalize set theories like ZF and the latter was enough to express and prove basic properties of FOL. Furthermore, modern mathematics could be written and developed rigorously in this framework.

> The earliest systems formalized in an *elementary* way, within FOL, were actually axiom systems for set theory — the Zermelo system with the work of Skolem, and Von Neumann's system in his own work. During the 1930s several authors emphasized the fact that an axiomatization of set theory, and therefore the foundations of abstract mathematics, only required the FOL system. This was the case with Tarski . . . , Quine . . . , Bernays . . . , Gödel, ... (Ferreirós, 2001, p. 448)

Notice that logicians of that period, like Feferman, assimilated the axiomatization of set theory with the foundations of abstract mathematics. I believe that this assimilation made perfect sense in the 1930s and that many mathematicians identified abstract or modern

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[6] For a brief overview of some of the views defended within the community, see Landry & Marquis (2005). I should also mention the fact that many category theorists would probably be wiling to endorse some form of pluralism with respect to the foundations of mathematics. I will also ignore these important issues here. For more on pluralism in the foundations of mathematics, see for instance Hellman & Bell (2006).

[7] Some standard references are Moore (1987, 1988).

mathematics with the set-based mathematics developed at that time. There was, however, a hiatus introduced between the practice of abstract mathematics and the foundational research. In some sense, the latter still had deep roots in nineteenth century mathematics via the works of Frege, Peano, and Russell & Whitehead. This divergence increased considerably with the introduction of categorical methods in algebraic geometry, homological algebra, algebraic topology, and homotopy theory in the 1960s, allowing various mathematicians to realize, with hindsight, that set theoretical foundations were not faithful to the abstract character of modern mathematics.

As Ferreiros rightly emphasizes and is well-known, sets have been for a long period of time considered as being a part of logic. The roots of this identification are, again according to Ferreiros, probably linguistic.

> This strong tendency to regard classes or sets as a part of logic, from about 1850, can be explained briefly as follows. The copula has three different meanings, carefully analyzed by Peano and Frege. As Peano said, we have the meaning of identity, the meaning of membership, and the meaning of inclusion, and one should very carefully differentiate them (otherwise one gets into contradictions). The first meaning justifies considering the theory of identity as a part of elementary logic; even today we frequently employ FOL with identity as a logical framework. On the other hand, membership and inclusion are basic relations of set theory; quite naturally, then, set theory belongs to logic. This standpoint was still defended or at least represented by Quine and Tarski as late as 1940. (Ferreirós, 2001, p. 464)

Category theory introduces a new, original linguistic component in mathematics: the language of arrows, commutative diagrams and, in the case of monoidal categories and in the higher-dimensional case, of geometric transformations of diagrams. Although these were for a long time considered to be merely linguistic devices, they should be seen as genuine conceptual ingredients of mathematical knowledge which capture conceptual dependencies. Thus, in the same way that the theoretical development of the combinatorial conception of set rested on the formal distinctions between identity, membership, and inclusion, the theoretical development of an abstract notion of set forces us to modify in subtle but crucial manners the formalizations of identity, membership, and inclusion. Category theory requires a completely original articulation of identities between mathematical entities of various kinds and this articulation has to be clarified and implemented in the syntax right from the start. Thus, in the same way that FOL and standard formulations of set theory have developed together to give us our current foundational framework, one has to devise the modified logic together with the modified set theory, make the necessary adjustments to both of them to find the appropriate fit between them.

Thus, four components of logic have to be modified to capture the abstract character of contemporary mathematics. First, logic has to be multisorted or typed; second, one has to introduce the notion of a context (for terms and formulas); third, the notion of dependent sort or of conceptual dependence must be incorporated in the logical framework right from the start; and fourth, the notion of identity has to receive special attention, since it is articulated differently in a categorical framework and, I should add, in a way that appears to be faithful to the abstract nature of contemporary mathematics. I will not give full technical expositions of these four components. Rather, I will try to motivate these modifications and give pointers in the literature for more technical presentations.

2.1.1. Multisorted logic. As soon as one tries to develop logic in a categorical framework, one has to introduce sorts or types, which we will denote by $X, Y, Z, \ldots$. Traditional first-order logic becomes a special case in which there is but one sort, the usual domain of interpretation. This is thus a natural generalization of the standard Tarskian semantics. Although sorts arise naturally in a categorical framework, there is nonetheless another motivation to work in such a framework. Indeed, in a context of abstract mathematics, one considers entities of different kinds, each kind having its own criterion of identity and the sorts reflect this situation.

A multisorted language has a slightly different syntax and, in this case, certain grammatical choices and rules are introduced. Thus, one has right from the start to declare the variables, that is, to stipulate that a given variable x is of a given sort X. This is usually written $x : X$. These declarations are *not* propositions, they are part of the grammar of the language. In other words, one does not assert $x : X$. It follows that one cannot write $\neg(x : X)$, for only a proposition can be negated. The latter is simply not grammatical, and therefore meaningless. This simple fact underlines the basic differences with the usual syntax of set theory with the membership relation. Notice also that since we are still in first-order logic, variables are for individuals. There are no variables for higher-order types or sorts.

2.1.2. Free logic and Contexts. This is also a slight modification of the logic that arises naturally when logic is interpreted in a categorical framework. Indeed, in a category one can have empty sorts, whereas the latter is always ignored in a classical setting. There are of course numerous ways of dealing with empty sorts, but category theorists essentially retained two options. The first option rests on the introduction of an existence predicate **E** and one reads **E**x as "x exists." Axioms relating the predicate to the other aspects of the language are then introduced. (See Fourman, 1977; Fourman & Scott, 1979; Scott, 1979.) The second option consists in making a slight modification to the syntax and the entailment relation and is somewhat more general than the first. (See Marquis & Reyes, 2011 for some of the motivation and the history.)

The definition is very simple and straightforward: a *context* is a finite list (or set) $\vec{x} = x_1, \ldots, x_n$ of distinct variables, where $n = 0$ is admitted. Given any formula ϕ of our language, the free variables of ϕ, usually denoted by $\mathrm{Var}(\phi)$, is a context.

A term t and a formula ϕ have to be interpreted in a context $\vec{x}$. This is spelled out by recursion on the complexity of the term and of the formula ϕ. The notion of logical consequence is also defined with respect to a context $\vec{x}$. A sequent is a sequence of symbols of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are formulas and $\vec{x}$ is a context containing all the free variables of ϕ and ψ. As I have already mentioned, the interpretation of these ingredients in categories then becomes almost automatic. (See, for instance, Makkai & Reyes, 1977 or Johnstone, 2002 for details.)

In practice, contexts introduce certain constraints in the grammar and the logic of the system which turn out to be important, especially when they are used in a system of dependent sorts, to which we now turn.

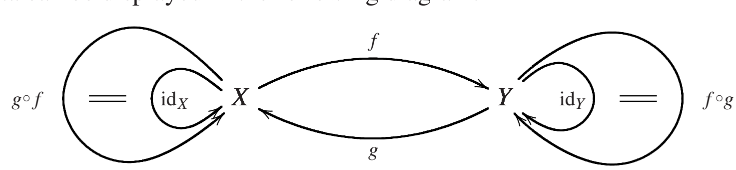
2.1.3. Dependent sorts. Considering sorts and contexts arise naturally and are natural adjustments to the standard framework to do logic in categories. The notion of dependent sorts, although as natural, has a different flavor and is not an intrinsic part of categorical logic. The main, simple, idea is that some sorts depend upon other sorts. One standard example in mathematical practice is the notion of an indexed family of sets or structures of some kind. Thus, it is very common to consider a family $\{X_i\}_{i \in I}$ of objects of a certain kind, which can easily seen to be another way of writing that there is a morphism $X \to I$, which indicates the dependence. In the terminology of dependent sorts, one says that the family of sorts X_i depends upon the sort I. Dependent sorts thus underly basic mathematical constructions. What we are doing here is to push the dependence relations in the syntax. It turns out to have an important impact on the definition of various concepts and brings to the forth (logical) priority issues!

Let us illustrate some of the simple dependences that arise. As Feferman himself has underlined, operations are fundamental in mathematics and they are usually represented by functions between collections: $f : X \to Y$. In the language of dependent sorts, one would say that f is of the sort $Mor(X, Y)$. However, there is an implicit order in this declaration. It has to be specified that $X, Y : Object$ and then that $f : Mor(X, Y)$. Thus in certain approaches, this would yield an explicit presentation of the dependence between the various concepts involved in the foundational framework.

2.1.4. Identity. This is perhaps the most delicate and subtle issue. Identity is usually added to first-order logic as a primitive and logical relation satisfying the usual properties of reflexivity, symmetry, and transitivity. When one turns to sets in ZF, identity is captured by the axiom of extensionality. This is precisely where certain aspects of abstract mathematics escape the standard analysis or explication in terms of ZF-sets, or any other notion of set based on extensionality. (For more, see Marquis, 2011.) I will give a specific example in the next section, but it seems to me that the failure of Feferman's argument rests on this specific aspect of the situation. Feferman failed to notice that, on any view of abstract mathematics, the notion of identity has a rich, complex structure which is not prior to the abstract objects present. I will start by giving certain general remarks about identities in categories to illustrate what I have in mind.

Recall that the notion of isomorphism is directly defined in the language of categories: two objects X and Y of a category $\mathbb{C}$ are said to be *isomorphic*, denoted as usual by $X \cong Y$, if there is an isomorphism $f : X \to Y$ between them, that is, a morphism $f : X \to Y$ and a morphism $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

This data can be displayed in the following diagram:



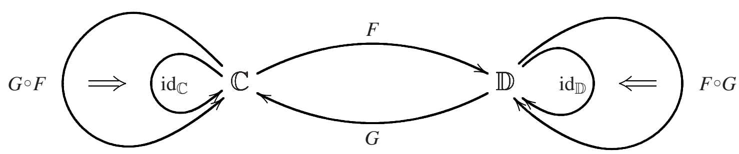
Very often in an abstract framework, results are proved "up to isomorphism," that is, if something is proved of a given object X, then it holds for any other object Y isomorphic to X. More formally, this can be written thus: given a property P of the objects under study, if P(X) and $X \cong Y$, then P(Y). This is simply a form of Leibniz's principle. Two elements immediately stand out in this formulation: first, one has to have a way of determining what is a meaningful property P for the objects of the given sort; second, the proper notion of isomorphism for objects of the given sort has to be determined too.[8] In fact, it might be wise to replace the term "isomorphism" by a more neutral term that evokes a type of identity. Notice that one could stipulate that once the proper criterion of identity has been discovered, then the meaningful properties are precisely those that satisfy Leibniz's principle. I would even dare suggest that the latter is a key property of what it means to be

[8] This is not as trivial as it might sound. For instance, it took a surprisingly long time for mathematicians to carve out the notion of homeomorphism between topological spaces. See Moore (2007) for an historical exposition.

abstract for mathematical objects.[9] If, moreover, this criterion of identity can be determined on a logical basis, then one can indeed determine the meaningful properties in a systematic manner. We will come back to this important point in a later section.

For many mathematicians, being isomorphic is precisely what being abstract amounts to. This means that X and Y are, from an abstract point of view, essentially the same. In many cases, however, one does not want to "identify" isomorphic objects, but rather to keep the isomorphism as indicating one of the ways that the objects can be considered "the same." For the isomorphism very often contain relevant and revealing information about the objects. In fact, in many cases, one wants to consider all the isomorphisms between two objects, and more globally, the groupoid of isomorphism between objects of the same sort.

As is well-known, the set-theoretical notion of isomorphism fails as a criterion of identity for categories themselves. As was pointed out for the first time by Grothendieck in the 1950s, the proper notion of identity for categories is the notion of *equivalence* of categories.[10] Recall the definition: an *equivalence* between categories $\mathbb{C}$ and $\mathbb{D}$ is a pair of functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$, together with two natural *isomorphisms* $\alpha : id_{\mathbb{C}} \to G \circ F$ and $\beta : F \circ G \to id_{\mathbb{D}}$. Notice that we have replaced the equalities in the notion of isomorphism by isomorphisms! Thus, the foregoing diagram for isomorphisms can be slightly modified to represent the situation thusly:



Equalities have not entirely disappeared from the picture however: there are new identities in this diagram, although they are not exhibited. Indeed, since α and β are (natural, but I won't explain what this mean here) isomorphisms, one has to have $\alpha \circ \alpha^{-1} = id_{G \circ F}$, $\alpha^{-1} \circ \alpha = id_{id_{\mathbb{C}}}$, $\beta^{-1} \circ \beta = id_{F \circ G}$, and $\beta \circ \beta^{-1} = id_{id_{\mathbb{D}}}$. Thus, the equalities reappear at a "higher-level," so to speak and the identities below this highest level are isomorphisms.

And the process does not step there: the equalities between the natural isomorphisms could themselves be replaced by natural isomorphisms, thus going one level up, and so on. Although this idea seems to be a pointless game at first, in turns out to be conceptually well-motivated, as I will indicate in a later section.

What this shows is that a foundational framework adequate for abstract mathematics has to make room for these various notions of identity in one way or another. Put differently, there is no unique, global, and universal relation of identity for abstract objects. But as we will indicate later, there is a uniformity between these various criteria of identity, a fundamental feature of abstract mathematics that the universe of ZF-sets does not capture directly at the level of the principle of extensionality.

2.2. Set. All the foregoing considerations are necessary adjustments that have to be made to logic if it has to cover the abstract nature of mathematics, in particular the notion of abstract set. This is clearly the key component of our argument. I claim that Feferman had something like the standard ZF notion of set in mind when he articulated his argument

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[9] I will develop this idea elsewhere. As it should be obvious by now, I think that a proper analysis of what it is to be abstract for mathematical objects is long overdue.

[10] See Krömer (2007) for more details on this point.

and failed to see that category theory rests on a different notion of set, namely the abstract notion.

Let me first illustrate why the standard notion of set is inadequate to capture the abstract nature of contemporary mathematics.[11] Consider two abstract groups G and H. Suppose we want to define an operation $G \boxtimes H$ on groups and suppose we want the underlying set of $G \boxtimes H$ to be $G \cup H$. The main problem, from the abstract point of view, is that given this definition the new operation $G \boxtimes H$ cannot be invariant under isomorphism. Indeed, to be invariant in this case means that

$$G \cong G' \quad \text{and} \quad H \cong H' \Rightarrow G \boxtimes H \cong G' \boxtimes H'.$$

But clearly, with standard sets, the union operation will not provide the proper answer for the respective underlying sets. In other words, the operation cannot satisfy this condition for cardinality reasons. In fact, the traditional union operation does not even make sense for abstract sets! This can be seen at the level of the grammar of the language already. The usual union is defined thus:

$$x \in X \cup Y \Leftrightarrow x \in X \vee x \in Y.$$

However, as we have seen, this proposition cannot be written down in a multisorted formal system, for the variable x is untyped in this sentence. We should immediately point out that for similar reasons the usual operation of intersection cannot be defined either. Thus, the standard set-theoretical operations do not respect isomorphism of sets. If the latter is taken as a necessary condition in the characterization of what it is to be abstract, then one could conclude that standard set theory fails for *abstract* mathematical notions!

However, the product of two abstract sets as well as the disjoint union make perfect sense when they are defined categorically. In other words, the categorical language simply reflects the abstract character of sets.

Feferman explicitly mentions abstract mathematics. I submit that underlying abstract mathematics, one can find a notion of abstract set, a notion that appeared around the same time as Zermelo's axiomatization of the extensional notion of set. Some of the first explicit mention of this notion can be found in Fréchet's writings. I will here give a quote from a rather late paper, but it essentially goes back to his early work on metric spaces and this conception is probably closer to Cantor's conception of set:[12]

> In modern times it has been recognized that is is possible to elaborate full mathematical theories dealing with elements of which the nature is not specified, that is, with abstract elements. A collection of these elements will be called an *abstract set*. (Fréchet, 1951, p. 147)

Thus, an abstract set is a set in which the elements are abstract. This is taken to mean that the nature of the elements is not specified. In other words, the elements have no other properties than those that they have in virtue of being elements of that specific set. This

[11] This argument is taken from Makkai (1998).

[12] Cantor did explicitly mentioned abstraction in his papers: recall how he defined ordinals $\bar{M}$ and cardinals $\bar{\bar{M}}$ with his bar denoting a process of abstraction. What he meant by that process is, of course, rather nebulous and is not what we mean here. Unfortunately, I cannot give more details in the present paper.

obscure claim will be clarified in a short while. Clearly, the sets of ZF are *not* abstract in that sense.[13]

The abstract notion of set was explicitly endorsed by Lawvere in the 1970s. To wit:

> These considerations lead one to formulate the following "purified" concept of (constant) *abstract set* as the one actually used in naive set-theoretic practice of modern mathematics: An abstract set X has elements each of which has no internal structure whatsoever; X has no internal structure except for equality and inequality of pairs of elements, and has no external properties save its cardinality: still an abstract set is more refined (less abstract) than a cardinal number in that it does have elements while a cardinal number does not. (Lawvere, 1976, p. 119)

This conception is very close to Fréchet's conception: nothing is known about the elements of a set X except that there is an internal identity relation between the elements.

How do we know these sets? On the standard, extensional, conception of set, one looks at the elements of the sets. Since the latter are fully identifiable individuals, they can be known and knowing a set amounts to knowing these elements. Here, we take a different route: to know a set and its components, including its elements since these become a special case of the more general notion of component, one looks at it "geometrically," that is by considering morphisms going in the set and morphisms going out of the set. This is the key epistemological or methodological component of the picture unraveling before us: objects cannot be broken up into atoms, rather they can be known only by investigating how they project and transform into other objects in a given conceptual environment.[14] We can therefore postulate that there are functions between abstract sets, denoted by $f : X \to Y$. Such a function is informally a projection or an image of X into Y. By considering these images, one can know properties of X and properties of Y. Moreover, one can combine these functions to define operations on the abstract sets.

If one considers the properties that these functions have to satisfy, one quickly comes to the conclusion that the collection of abstract sets is simply a category, in the standard sense of that expression: it has as objects abstract sets and morphisms, functions between these sets and the latter satisfy the usual data of a category, namely compositions of morphisms exist whenever it is possible, it is associative and composition has a left and a right unit.

What, then, is the identity between abstract sets? In the extensional conception of sets, it is captured by the axiom of extensionality:

$$X = Y \Leftrightarrow \forall x(x \in X \Leftrightarrow x \in Y).$$

Again, as with the union operation, this sentence is simply meaningless in our logical framework. The variable x is untyped or, put differently, it would have to be untied for the sentence to make sense. Putting the types in, we would get the sentence:

$$X = Y \Leftrightarrow (\forall x : X)(\exists y : Y)(y = x) \wedge (\forall y : Y)(\exists x : X)(x = y),$$

which is different indeed.

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[13] In fact, they are abstract only in a very weak ontological sense, that is in the same sense that any concept is abstract, which is usually identified as being not concrete. This fails to reflect the nature of mathematical abstractness, a crucial issue in this context.

[14] For more on this point of view, see Marquis (2009).

How do we express identity of abstract sets? It is not a relation, but rather a *structure*. What can be declared in the given language is that abstract sets can be *isomorphic*. But we do not identify isomorphic sets, unless it is useful for some specific mathematical purpose.

If the criterion of identity for abstract sets is given by the notion of isomorphism of sets, what can we say of the totality of these abstract sets? At this point, the natural answer is that this totality constitutes a category. It is precisely at this juncture that the notion of category comes into play. All we had to start with were the notions of (abstract) collection and operation. When we put those together and want to consider in a first-order multisorted logic the various constructions available on these sets and operations, the fact that it forms a category seems inescapable and more than useful. Indeed, turning to the language of category theory, one can define products, coproducts, equalizers, coequalizers, pull-backs, push-outs, etc., that is, various limits and colimits and prove various properties of the universe of abstract sets. Notice that the universe of abstract sets is not a set nor a class: it is a category.[15] Thus, from the point of view of abstract mathematics, category theory is inescapable. The real surprise is probably not that the notion of category is essential to articulate and develop a totality of abstract collections and operations, but rather that it is not enough. Let us now see how the third notion in the picture, namely the notion of category itself, has to be developed from the point of view of abstract mathematics.

2.3. Category. In mathematical practice, one needs to consider, at one point or another, more than one category and the latter usually have a structure of some kind, for example, finite products, finite limits, monoidal, etc. In fact, in the practice of category theory, one has to construct, define, and prove properties of functors, and especially structure preserving functors, between categories, some of which are very often functor categories themselves, as it so often occurs when the Yoneda lemma is used. In the same way that collections and operations are inescapable, totality of categories with structures are just as inescapable as soon as one admits that they are mathematically indispensable. So, what is the totality or a totality of categories, for example, the totality of categories with finite products or the totality of cartesian closed categories?

Before we answer this question, we must pause and examine what it means for a functor $F : \mathbb{C} \to \mathbb{D}$ to preserve the structure of a category. This is important since many results in category theory amounts to the claim that certain functors preserve limits or colimits, for example, if a functor $U : \mathbb{C} \to \mathbb{D}$ has a left-adjoint $F : \mathbb{D} \to \mathbb{C}$, then F preserves whatever colimits exist in $\mathbb{D}$. So, what does it mean for a functor to preserve, say products from $\mathbb{C}$ to $\mathbb{D}$? Given a product $X \times Y$ in $\mathbb{C}$, F preserves products whenever, for all X and Y, $F(X \times Y) \cong F(X) \times F(Y)$. The important point to notice here is that $F(X \times Y)$ has to be *isomorphic*, not equal, to $F(X) \times F(Y)$, which is a product of $F(X)$ and $F(Y)$ in $\mathbb{D}$.[16] In fact, as is well known, any isomorphic object will do.[17]

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[15] This is not to mean that issues of sizes are ignored. One still has to postulate that there is an infinite set, etc. Other issues related to the standard paradoxes are still lurking in this picture, at least at this stage of the game.

[16] Recall that products are defined up to (a unique) isomorphism. Recall also that the notion of product can itself be analyzed as a functor $(-,-) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

[17] This raises a question as to the very definition of functor: usually, functors are defined with the usual equality, whereas the codomain of a functor evaluated at an object of the domain could very well be defined up to isomorphism. How this should be done precisely is a rather subtle technical issue which we will leave aside. We will come back to this question is Section 3.3.

Why does this matter? We are now considering totalities like the totality **CAT** of all categories, or the totality **Cart** of all cartesian categories, etc. Consider the latter. From the ordinary categorical point of view, **Cart** does not have an initial object.[18] Here is the argument: suppose that **Cart** has an initial object, the category denoted by $\mathbb{o}$. By definition, the category $\mathbb{o}$ must have at least one object, since it has a terminal object. Consider now the category $\mathbb{2}$ which contains two objects and an isomorphism between them (together with the usual trivial morphisms). It can easily be verified that the category $\mathbb{2}$ is cartesian closed, thus an object of **Cart**. It is as easy to verify that there are two constant functors from $\mathbb{o}$ to $\mathbb{2}$ in **Cart**, contradicting the fact that $\mathbb{o}$ is initial. But, there is a sense in which **Cart** does have an initial object![19] The usual definition of an initial object in a category cannot be transferred directly to a totality of categories. This is not merely a technical point, for it is an indication that a totality of categories, with or without a certain structure, cannot be a mere category. The latter theory simply lacks the concepts to capture crucial features of the situation.

For similar reasons, one can claim that the totality of categories with a given structure cannot be a set or a class: the criterion of identity for sets is totally inadequate. Can it be a category? Recall that the proper criterion of identity within a category $\mathbb{C}$ is the notion of isomorphism for the objects of $\mathbb{C}$. Thus, if a totality of category is a category, this means that the criterion of identity for categories would be the notion of isomorphism of categories, which, as we have indicated, is not only inadequate from the point of view of the practice of abstract mathematics, but also from the logical point of view we are now articulating.

Abstract objects are of different sorts and this should mean, almost by definition, that there is no global, universal notion of identity for sorts.[20] Each sort X is equipped with an internal identity relation but there is no identity relation that would apply to all sorts. As we have seen, for abstract sets themselves, the identity structure is given by the notion of isomorphism. This is logically possible for the identity within sorts is used to determine whether two functions are identical or not. When we try to do the same for categories, it fails to be meaningful. Indeed, in this case we would have two functors $F : \mathbb{C} \to \mathbb{D}$, $G : \mathbb{D} \to \mathbb{C}$ such that, and here are the equalities, $G \circ F = id_{\mathbb{C}}$ and $F \circ G = id_{\mathbb{D}}$. But we are dealing here with functors and, therefore, what we have is that for all objects X of $\mathbb{C}$, $id_{\mathbb{C}}(X) = GF(X)$, that is $X = GF(X)$. But this is now an identity *between* objects, that is, between sorts and this is simply not given a priori in our language. We are thus lead, for purely conceptual reasons, to the notion of equivalence of categories defined above. In that definition, we simply mentioned *natural isomorphisms* between functors, that is, we used the notion of natural transformation, morphisms between functors. This is an additional structure and lead to the notion of a 2-category. More precisely, but still informally, a 2-category **C** has objects $\mathbb{X}, \mathbb{Y}, \ldots$, and for all objects $\mathbb{X}$ and $\mathbb{Y}$, a *category*

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18 Recall that an object 0 is a *initial object* of a category $\mathbb{C}$ if there is a unique morphism $0 \to X$ for all objects X of $\mathbb{C}$.

19 Needless to say, this is directly related to the foregoing question regarding functors preserving structure. For the existence of an initial object in a category amounts to the existence of an adjoint functor to a trivial functor. The reader can certainly guess at this stage that the notion of adjunction also has to be modified for a totality of categories.

20 Notice that this is also confirmed by mathematical practice: each sort of abstract entity, for example, monoid, group, ring, field, topological space, partial order, etc., has its criterion of identity. It is certainly a nice feature of category theory that it provides a unified analysis of these criteria of identity as being isomorphisms in the appropriate category.

$\mathbf{C}(\mathbb{X}, \mathbb{Y})$ of morphisms, with the usual composition of functors and the usual three axioms of a category, namely associativity of composition and the left and right unit laws. An ordinary category is usually defined has having, for all objects X, Y a *set* $Hom(X, Y)$ of morphisms with a partial composition, etc.

Thus, the short answer is that a totality of categories is not a category, but a 2-category. And in the same way that the notion of category organizes a totality of sets with operations, the notion of 2-category organizes a totality of categories and functors.

It is possible to define various constructions for 2-categories, for example, 2-functors, 2-natural transformations, 2-adjunctions, etc. In other words, 2-categories may have a structure... It is therefore natural and inescapable to consider a totality of 2-categories (with a given structure, say). What kind of mathematical object is such a totality of 2-categories? It would be fairly easy to adapt the foregoing argument to conclude that a totality of 2-categories cannot be a 2-category. It has to be a 3-category. And we could go on like this and introduce n-categories, for an arbitrary n.

This is very far from being the whole story. One must also define bicategories and double categories. And then n-categories come in two versions: there are *strict* n-categories and *weak* n-categories and both play a role in the foundational environment.[21] Our only concern here is to convince the reader that one cannot simply stop at categories. When one considers totality of categories, then *new* kinds of abstract entities arise.

§3. Putting the pieces together. There are at least three different ways of putting the foregoing components together. Needless to say, I won't give precise, complete formal presentations of these systems, especially since two of these systems are still undergoing important and fundamental developments. These sections will be rather short and sketchy.

3.1. ETCS. The Elementary Theory of the Category of Sets (ETCS) was introduced by Lawvere as early as 1964. (See Lawvere, 1964, 2005.) In this paper, Lawvere proposed an axiomatization of set theory in a categorical framework. The axiomatization was later subsumed under the notion of an elementary topos where it became a special type of topos. This is probably the combination of the three components mentioned above that is the closest to the standard set theoretical framework. It is developed in the standard first-order logic without dependent sorts. One starts with the (first-order) axioms for an elementary topos and add axioms to get as close as possible to ZFC. To be more precise, one adds to the axioms of an elementary topos, the existence of a Natural Number Object (NNO), that the terminal object 1 is a generator and that every epimorphism splits (an equivalent version of the axiom of choice in the context of toposes). It is well-known that such toposes are necessarily Boolean. It can be shown that such a topos is equivalent to Bounded Zermelo set theory with choice (BZC). As such, the axiom of replacement is absent. But there are natural ways to add it to the theory. Thus, one gets a somewhat weaker theory than the usual ZFC and the question is whether it is enough for the usual mathematical purposes. (See also McLarty, 1992 and Shulman, 2008 for details.)

It is well-known that an elementary topos is equivalent, in a precise technical sense, to a type theory and, in that case, it is not a dependent type theory and it is higher order, for

[21] Thus, I am deliberately being technically vague at this point. There are, at the moment, various versions of the notion of n-categories and it is still unclear which one is preferable for foundational purposes. Only time and more research will tell.

example, given two types X and Y, one can construct the type Y^X. (See Boileau & Joyal, 1981 or Lambek & Scott, 1986.) This illustrates how the logic can vary in a categorical context. We nonetheless have a logical framework, specific axioms, and a universe.

It should be emphasized that this has always been presented by Lawvere as being a *set* theory and that Lawevere's claim was meant to cover any elementary topos, not only ETCS. As we have seen, the theory elementary toposes is, according to him, the theory of abstract-constant sets. An arbitrary elementary topos ought to be seen, according to Lawvere, as a universe of variable sets. And, indeed, it is possible to build a translation between elementary toposes (with a natural number object) and (intuitionistic and bounded) set theories in general, also called Basic Intuitionistic Set Theory (BIST) in the literature. (See Awodey *et al*., 2007; Awodey, 2008; van den Berg & Moerdijk, 2009.) In particular, BIST contains the full axiom of replacement and the logic is the usual first-order logic, that is, the quantifiers are not bounded.

Quite a lot of mathematics can be developed in an arbitrary topos, and most of nineteenth century and early twentieth century classical mathematics can be reconstructed in ETCS. But as we have already indicated, when one is considering a category with structure, like a topos, one quickly has to consider functors between such categories, in particular a category of toposes and the latter is a 2-category. When one gets to that point, one is no longer in the framework of elementary topos theory. It is at least a 2-topos and important results in categorical logic, for example, completeness theorems, arise naturally in that context. We are now moving towards the picture I want to discuss.

3.2. Makkai's FOLDS and higher-dimensional categories. Michael Makkai has been developing a categorical foundational framework which is explicitly based on the notion of abstract sets since the mid 1990s.[22] In contrast with the previous approach, Makkai's approach is based on a general purpose formal system, namely First-Order Logic with Dependent Sorts (FOLDS). FOLDS is a general purpose formal system which could have the same status as first-order logic in foundational studies. As Makkai has himself pointed out, FOLDS was inspired by Martin-Löf type theory and, in the same way that first-order logic can be considered to be a proper part of a type theory, in can be considered to be a proper part of Martin-Löf's type theory. Thus, the first, syntactical, component is first-order logic with dependent sorts.

I will not give a detailed technical description of FOLDS.[23] I will limit myself to some of its salient features that are relevant to this discussion.

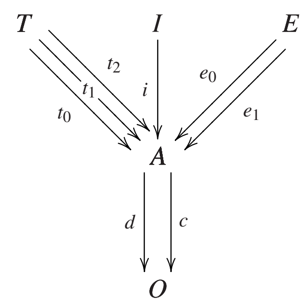
The most striking feature of FOLDS is that identity is no longer a primitive notion, although it is still a logical notion. The proper identity relation is extracted from given data by a purely logical manipulation and the identity relation thus extracted automatically satisfies Leibniz's principle presented above. Hence, in the context of FOLDS, whatever one writes down about a given type of mathematical object in the given language will make sense. In other words, only meaningful properties of the given objects can be expressed in the language.

[22] The basic reference here is certainly Makkai (1998). It contains a rather detailed account of what we are about to sketch here. See also Makkai (1999) for a more general reflection on the type of structuralism Makkai is defending. However, there is much more on his Web site. See, for instance, http://www.math.mcgill.ca/makkai/, in particular the link to the foundation seminar.

[23] A detailed description can be found on Makkai's Web site under the title "First Order Logic with Dependent Sorts, with Applications to Category Theory."

To do this, one starts with the notion of a FOLDS signature $\mathcal{L}$. The latter can be given a slick characterization by saying that a FOLDS signature is a category $\mathcal{L}$ that satisfies two simple conditions: i) it is reverse well-founded, that is, there are no infinite paths $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \cdots, n < \omega$, such that for all n, $f_n \neq id_{X_n}$; it is finite fan-out, that is for every object X, there are finitely many arrows with domain X. This is it.

Here is a simple illustration of a FOLDS signature, the signature $\mathcal{L}_{Cat}$ for categories. A presentation of this signature is given by the following diagram:



with the following identities:

$$d \circ t_0 = d \circ t_2; \tag{1}$$

$$c \circ t_0 = d \circ t_1; \tag{2}$$

$$c \circ t_1 = c \circ t_2; \tag{3}$$

$$d \circ i = c \circ i; \tag{4}$$

$$d \circ e_0 = d \circ e_1; \tag{5}$$

$$c \circ e_0 = c \circ e_1. \tag{6}$$

The presentation exhibits the dependencies between the sorts. They are of course chosen with a specific interpretation in mind, which we will see in a few paragraph. At this stage, one should start using the signature to write down sentences of theory and define various obvious notions, for example, the notion of products.

Two remarks are in order. First, although the notion of a FOLDS signature is expressed in the language of category theory, it is *not* a *categorical* notion, for the given properties are not preserved under equivalences of categories. This is an interesting case where the categorical symbolism is used for syntactical purposes. Second, there is an equivalent definition of a FOLDS signature along that standard syntactical lines, for example, by defining the syntactical elements explicitly. But it is considerably more intricate than the one given in terms of categories.

Given a FOLDS signature $\mathcal{L}$, an $\mathcal{L}$-structure is a functor $M : \mathcal{L} \longrightarrow \mathbf{Set}$.[24] In our foregoing example, an $\mathcal{L}_{Cat}$-structure $M : \mathcal{L}_{Cat} \longrightarrow \mathbf{Set}$ yields a bona fide category. $M(O)$ is the set of objects of the category, $M(A)$ is the set of arrows, the two arrows become functions, namely the domain function and the codomain function. The set $M(I)$ is the set of identities, $M(T)$ is the set of triangles, and the set $M(E)$ is the set of equality of

[24] Perhaps this should be called a **Set**-$\mathcal{L}$-structure, but it is rather inelegant. The point is that the functor does not have to be in **Set**, it could be any category $\mathbb{C}$ with the appropriate amount of structure, for example, a regular category.

parallel arrows. The interpretation of the arrows and the various identities between them should be obvious. Given the notion of $\mathcal{L}$-structure, one can define the category $\mathsf{Str}_{\mathbf{Set}}(\mathcal{L})$ of $\mathcal{L}$-structures in the category of sets in the obvious way and it would be possible at this stage to develop a model theory for theories in languages with a FOLDS signature. Let us turn to its use in the foundations of mathematics instead.

The key innovation is the notion of $\mathcal{L}$-equivalence. The technical definition is as follows: let M and N be $\mathcal{L}$-structures. $M \simeq_{\mathcal{L}} N$ is an *$\mathcal{L}$-equivalence* if and only if there exists an $\mathcal{L}$-structure P together with morphisms of $\mathcal{L}$-structures $m : P \longrightarrow M$ and $n : P \longrightarrow N$ such that m and n are fiberwise surjective.[25] (We refer the reader to Makkai's papers for the technical details. Suffice it to say that a morphism is fiberwise surjective if it has the right lifting property with respect to all injective morphisms.) It can be verified that this yields an equivalence relation. This definition allows one to prove the following main theorem: for all FOLDS sentence ϕ, if $M \simeq_{\mathcal{L}} N$ and $M \models \phi$, then $N \models \phi$, where $M \models \phi$ is the usual model theoretical notion of truth in a structure. Makkai has shown that when one gives the appropriate signatures for various notions, then one obtains the "right" notions of equivalence, that is, the signature of categories yields the usual notion of equivalence, the signature for bicategories yields the usual notion of biequivalence, etc.[26] In contrast with the usual notion of elementary equivalence of first-order logic, Makkai's notion is more sensible since it captures more natural—in the sense of naturally arising in the practice of mathematics—notions of equivalence. FOLDS strikes a balance between expressiveness and invariance.[27]

Makkai therefore proposes a general purpose formal system which has a general semantics (in categories, not just in sets), just like first-order logic and incorporates all the foregoing elements mentioned above since we are firmly in categorical logic.

When one considers the standard foundations of mathematics, one introduces the universe of sets, usually via the cumulative hierarchy. The latter is replaced by Makkai by higher-dimensional categories, more specifically by what he calls the multitopic universe. (See Hermida *et al.*, 2000, 2001, 2002.) The informal picture should be immediately clear from what I have said already: we start with abstract sets, which are our objects at level 0. They are related by maps and the criterion of identity for abstract sets is given by a structure: the notion of isomorphism of sets. Collecting sets together, say sets with a structure, we get categories, or, more precisely, 1-categories (thus, sets should be thought of as 0-categories), since these objects are now at level 1 and their criterion of identity is given by the notion of equivalence of 1-categories. Collecting 1-categories together, we get 2-categories, etc., and so on where n-categories have as objects $(n-1)$-categories with morphisms between them. Slightly more formally, meaning here that I will use formal symbols to express some of the ideas involved, a n-category, $n \geq 0$, consists of

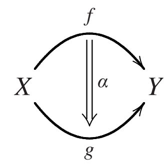
- *0-cells* or *objects* $X, Y, \ldots$;
- *1-cells* or *morphisms* $f, g, h, \ldots$ depicted by the usual arrow notation $f : X \to Y$;

25 The category of $\mathcal{L}$-structures $\mathsf{Str}_{\mathbf{Set}}(\mathcal{L})$ has an internal notion of isomorphism. It can be shown that two isomorphic structures are $\mathcal{L}$-equivalent, as one would expect.

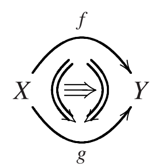
26 Makkai gives a nice example for Kan complexes. He defines a subcategory of the category of simplicial sets that is a FOLDS signature and proves that $\mathcal{L}$-equivalent structures for this signature are precisely homotopy equivalent.

27 I want to thank an anonymous referee for suggesting this formulation.

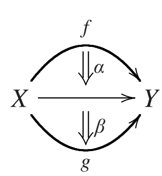
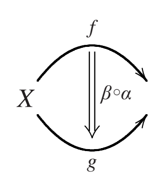
- *2-cells* or *morphisms between morphisms* $\alpha, \beta, \ldots$, depicted by

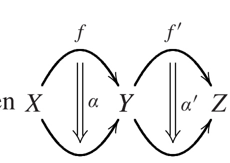
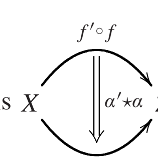


- *3-cells* $\Gamma, \Delta, \ldots$, depicted (and here you have to visualize the diagram in three dimensions) by



- …
- *n-cells*;
- These cells have to *compose* in various ways and satisfy various *composition laws*. For instance, 1-cells compose in the usual manner, that is given $X \xrightarrow{f} Y$ and $Y \xrightarrow{z} Z$, one gets the usual $X \xrightarrow{g\circ f} Z$. Things get rather tricky with higher dimensional cells. For one thing, they can compose with cells of the same dimension and one has to determine what happens with cells of lower dimensions. Thus, 2-cells compose in two ways, vertically and horizontally so to speak. These compositions are illustrated by these diagrams:

Given  , the vertical composition is  .

Given  , the horizontal composition is  .

- And various identities, that is specific i-cells, $0 \leq i \leq n$, together with laws for them have to be introduced.

Of course, this is not a mathematical definition but it gives a rough idea of the data that one has to piece together. Furthermore, one would really want to give a definition of ω-categories, that is, categories of n-categories for all n, and not just up to a fixed n. This can now be done in a surprisingly compact way, provided various preliminary geometric and combinatorial constructions have been given. For instance, Makkai has given the definition of a *multitopic ω-category* in four lines.[28] But in order to do so, one has to define multitopic sets and multitopes and these require, at least for now, a rather long presentation. It should be noted that FOLDS comes in the description of these latter concepts. Be that as it may, as I write this, there is still no consensus on what should be the right mathematical definition. There are, in fact, many different definitions in the literature. (See Leinster, 2002.) Whether they are equivalent or not is a delicate issue and, in fact, even proving the equivalence between these definitions is a subtle mathematical and conceptual problem. (See, for instance, Makkai & Zawadowski, 2001.)

This universe might seem absurdly complicated and difficult to understand at first and, thus, unsuitable as a foundational universe. We are back to the argument based on some sort of simplicity or immediacy of foundational notions. I think that if one gets to that point, it is because one has forgotten what we were trying to do in the first place: to provide a foundational framework for *abstract* mathematics and that latter expression does not merely means for abstract entities in the ontological sense of that expression.

I haven't presented a system of axioms for the latter universe. These have still to be provided. Be that as it may, it should be clear that we have, in the last two examples, frameworks in which an abstract notion of sets is at work and and that are developed in a way that the abstract nature of the objects involved is reflected down in the syntax and the logic of the systems.

3.3. Homotopy type theory. This is a recent proposal undergoing a rapid and vigorous development and I will not provide any technical detail about this framework. It is based on Martin-Löf type theory, introduced in the early 1970s to provide a formal framework for *constructive* mathematics. (See Martin-Löf, 1984 or Nordström *et al.*, 2000.) Thus it might be surprising to see it appear in the context of *abstract* mathematics. But as we have said, Makkai's FOLDS was directly inspired by Martin-Löf type theory and, not surprisingly, it shares many of the key properties of Makkai's theory, or at least the so-called intensional theory does. Another indication that the theory is relevant to abstract mathematics is that it has direct links to homotopy theory, in particular to homotopy types. These are not only connected to abstract sets, indeed homotopy 0-types can be thought of as abstract sets,[29] but they constitute in my opinion the perfect example of mathematical abstract entities that cannot be captured by the purely extensional point of view of traditional set-theory. Notice, and this is an important remark, that the extensional point of view can be recovered from the more abstract point of view.

Homotopy type theory is a genuine type theory, not a first-order logic. Originally, all the variations on Martin-Löf type theory were motivated by the well-known Curry–Howard correspondence, that is, the propositions-as-types interpretation. The correspondence is given by the following simple rule: a term x of a type X can be thought of as a proof of the corresponding proposition. This interpretation is still useful, even when the formal system is interpreted in the context of homotopy theory. (See Awodey & Warren, 2009; Awodey, 2012; Voevodsky, 2010.) It will be enough to underline two features of the theory.

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[28] J. Baez and J. Dolan who were the first to give such a compact definition of a n-category have formulated a definition of opetopic ω-category in two lines. See Baez & Dolan (1998). Of course, to understand these two lines, one has to go through the whole paper carefully.

[29] I am using the traditional notation where a homotopy n-type is exhibited by a topological space whose homotopy groups π_m, $m > n$ are all trivial. For reasons I will not discuss here, in homotopy type theory, these are sometimes denoted by homotopy 2-types.

1. The original feature of homotopy type theory is that it has an identity type $\mathrm{Id}(X, Y)$, which under the Curry–Howard interpretation can be read as the type of proofs that X is identical to Y. But this type also has a geometric interpretation in homotopy theory: it becomes a path object in that framework.
2. The theory satisfies the invariance principle which constitutes the core element of the approach to abstract mathematics: for any property P definable in the language, if $\mathrm{P}(X)$ and $X \simeq Y$, then $\mathrm{P}(Y)$. And here too the notion of equivalence is context sensitive.

The main moral should be clear: we now have at the very least two formal systems together with a semantics that can be used as frameworks to develop foundations for abstract mathematics and in both cases, categories play a crucial role.

§4. Psychological priority. All the foregoing considerations should be sufficient to convince anyone that priority issues are far from being trivial and transparent, especially when dealing with abstract mathematics.[30] Should logical priority reflect psychological priority? How? Do we have a theory of psychological priority of mathematical concepts? Do we have empirical studies upon which one could determine a proper order of understanding? How about historical priority, that is, the actual historical development of mathematical concepts? Should this be taken into account in foundational studies? In many cases, the simplest concepts come very late in the historical development, for example, the concept of set itself.[31]

The issue of psychological priority is particularly delicate: Feferman ties it with *understanding* and not only concept acquisition. Understanding is certainly a thorny concept and it might come in degrees, pretty much like our reading ability. We learn to read in stages and some of us never get to the "higher" stages of reading which is ironically linked to understanding. The parallel might be interesting to some extent. Some people learn to read mathematics like children have learn to read in third or fourth grade and never go beyond this stage. Others come to a deep understanding of mathematics. But there might also be important differences between reading and learning mathematics. Indeed, as recent research in cognitive neuroscience tend to indicate, mathematics seem to be based on nonverbal core systems—one for arithmetic and one for geometry—that we seem to have as a result of our evolutionary history. These core systems seem to play a key role in our subsequent learning of mathematics. Thus, in some sense, the psychological priority still seem to go to arithmetic *and* geometry. (See Dehaene & Brannon, 2011; Izard *et al.*, 2011; Spaepen *et al.*, 2011.) Logic clearly comes in at some point—much earlier than what Piaget had suggested—but it seems to rely more, although not exclusively, on language. (See Goel, 2007 for a review.) The psychological relationships between logic and mathematics still has to be elucidated. However, it appears to be implausible at this stage to claim that, from the psychological point of view, mathematics rests on logic.[32]

30 I have tried to clarify some of the issues involved in an earlier paper: see Marquis (1995).

31 Piaget, for one, thought that by looking at how children acquire mathematical concepts, thus the psychological order of acquisition, we might be able to understand or have a better idea of how humans, as a species, came to develop mathematical concepts. It should also be mentioned that Piaget thought that an abstraction process, the process of reflective abstraction in his terminology, was at the core of the acquisition of concepts and systems of concepts.

32 Should we conclude that some form of mathematical constructivism is vindicated by neuropsychology? This might rest on a confusion between different kinds of foundations.

(See Houdé & Tzourio-Mazoyer, 2003 for an excellent review.) But this is very far from *abstract* mathematics, at least in the sense that contemporary mathematics is abstract. When Feferman wrote his argument, some still thought that the teaching of mathematics should reflect its logical organization, that is, psychological priority should reflect logical priority. This is now seen as profoundly mistaken.

> Thus, bombarding the juvenile brain with abstract axioms is probably useless. A more reasonable strategy for teaching mathematics would appear to go through a progressive enrichment of children's intuitions, leaning heavily on their precocious understanding of quantitative manipulations and of counting. One should first arouse their curiosity with some amusing numerical puzzles and problems. Then, little by little, one may introduce them to the power of symbolic mathematical notation and the shortcuts it provides — but at this stage, great care should be taken never to divorce such symbolic knowledge from the child's quantitative intuitions. Eventually, formal axiomatic systems may by introduced. Even then, they should never be imposed on the child, but rather they should always be justified by a demand for greater simplicity and effectiveness. Ideally, each pupil should mentally, in condensed form, retrace the history of mathematics and its motivations. (Dehaene, 2011, p. 224)

And the history of mathematics is presently leaning heavily towards categories. But they should be taught at the end, not at the beginning. And this, despite the fact categories might provide a logical foundation for abstract mathematics. And despite the fact that category theory might constitute a genuine alternative to ZF(C) as a theoretical development of a naive and preformal understanding of the notions of collection and operation.

§5. Psychological and logical priority. I submit that, as far as abstract mathematics is concerned, the order of logical priority and historical priority are dual or go in opposite directions to each other. I also submit that the logical aspects, at least its purely *formal* expression, is developed in parallel with our understanding of what it is to be abstract mathematically. Thus, as we have seen, the very logic has somewhat to be modified, slightly but surely, to accommodate the abstract nature of contemporary mathematics. What is striking is the fact that conceptual dependencies have to be brought in right from the start. What is even more surprising is that the dependencies required have a strong *geometrical* flavor, reproducing, so to speak, the geometric dimensions of space. It is as if the concepts themselves would have to be organized in a spatial manner. This is a genuinely new component in the foundational landscape. Whereas the cumulative hierarchy is basically two-dimensional, the categorical universe is ω-dimensional or, if you prefer, ∞-dimensional. Thus, in this sense, the categorical universe is a generalization and an abstraction of the standard picture of sets.

§6. Conclusion. When Feferman formulated his arguments in the mid 1970s, they were certainly justified to a large extent. Category theory was still, in some sense, in its infancy. Although the language of categories, functors, and natural transformations was introduced in 1945 by Eilenberg and Mac Lane, the *theory* really took shape in the 1960s after the introduction of some key concepts like adjoint functors, representable functors, monads, Abelian categories, etc. Its successes in homological algebra, algebraic topology, and algebraic geometry were certainly an indication of its potential in mathematics and, I would add, in abstract mathematics. For the methods of category theory are tailored for abstract mathematics. It can be claimed that category theory introduces a new level of abstraction in mathematics. When Lawvere suggested that the category of categories could be taken as a foundation of mathematics, it was a bold and courageous claim to make. Lawvere saw immediately that his claim was aiming at contemporary abstract mathematics and I suppose that everyone thought that set theory had been devised precisely for that purpose. But that is not quite right. Standard set theory does not provide a foundational framework for abstract mathematics. I hope to have shown that on any conception of abstract mathematics, category theory will have to play a key role in its foundations. And sets and operations will still play a key role even in this framework. Thus, in some sense, Feferman was correct, but not in the way he envisaged.

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