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Combining Temporal Logic Systems

MARCELO FINGER and DOV GABBAY

Abstract This paper investigates modular combinations of temporal logic systems. Four combination methods are described and studied with respect to the transfer of logical properties from the component one-dimensional temporal logics to the resulting combined two-dimensional temporal logic. Three basic logical properties are analyzed, namely soundness, completeness, and decidability. Each combination method comprises three submethods that combine the languages, the inference systems, and the semantics of two one-dimensional temporal logic systems, generating families of two-dimensional temporal languages with varying expressivity and varying degrees of transfer of logical properties. The *temporalization method* and the *independent combination method* are shown to transfer all three basic logical properties. The method of *full join* of logic systems generates a considerably more expressive language but fails to transfer completeness and decidability in several cases. So a weaker method of *restricted join* is proposed and shown to transfer all three basic logical properties.

1 Introduction We are interested in describing systems in which two distinct temporal "points of view" coexist. Descriptions of temporal systems under a single point of view, i.e., one-dimensional temporal systems, abound in the literature. These one-dimensional temporal logics differ from each other in several ways. They differ on the approach, whether proof-theoretic, model-theoretic, or algebraic. They differ on the ontology of time adopted, whether time is represented as a set of points, intervals, or events. They can also differ on the properties assigned to flows of time, whether linear or branching time, discrete or dense, continuous or allowing for gaps. In this paper we contemplate both proof- and model-theoretic presentations of temporal logics on a point-based ontology. Most of the results presented assume that the flow of time is linear.

The motivation for the present work came from the study of applications of two-dimensional temporal logics by Finger [7]. We were aware of Venema's [22] negative results concerning the unaxiomatizability of two-dimensional temporal logics over the upper semi-plane of $\mathbb{N} \times \mathbb{N}$, $\mathbb{Z} \times \mathbb{Z}$, and $\mathbb{R} \times \mathbb{R}$ (see also Proposition 6.3 below). However, for our purposes then, the full expressivity of Venema's two-dimensional language was not required, and a weaker language provided the appropriated expressivity.

It then became clear that this weaker two-dimensional language could be generalized and a family of languages resulting from adding a (second) temporal dimension *externally* to a temporal logic system was thus obtained. This process was formalized by Finger and Gabbay [6], where it was called *temporalization*, and several results were obtained concerning the transfer of logical properties from the component logical system to the combined one. As a result, a family of temporalized logic systems was obtained, the properties of which can be derived from the properties of the component logic systems via the transfer results.

The next step, which we present in this work, comes from the observation that there may be several distinct ways in which two temporal logic systems can be combined, generating thus several families of combined two-dimensional temporal systems. Different combination methods may be presented by the distinct interactions between related parts of the two logic systems involved, leading to two-dimensional systems based on distinct languages with distinct semantical structure, expressive power, and other properties (that may be transferred or not from the component systems).

Several cases in which two distinct temporal dimensions (or temporal "points of view") can co-exist are described next, motivating several different methods for combining two temporal logics. We will also attempt to relate these methods to recent, mostly unpublished work on combining two generic (not necessarily temporal) logics systems, e.g., Gabbay [12],[11].

1.1 First case: external time One temporal point of view can be *external* to the other. The external point of view is seen as describing the temporal evolution of a system $\mathcal{S}$, when system $\mathcal{S}$ is itself a temporal description. Suppose $\mathcal{S}$ is described using a temporal logic T, and suppose that the external point of view is given in a possibly distinct logic $\overline{\mathsf{T}}$. For example, consider an agent *A*, whose temporal beliefs are expressed in logic $\overline{\mathsf{T}}$, that we want to allow to reason about the temporal beliefs of an agent *B*, which are expressed in a possibly distinct logic T. This is illustrated in Figure 1.

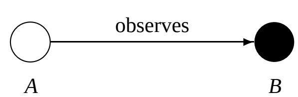

Figure 1: One agent externally observing the other

Agent *A*'s beliefs are external to agents *B*'s beliefs, so that $\overline{\mathsf{T}}$ is externally describing the evolution of T. The external temporal point of view $\overline{\mathsf{T}}$ is then applied to the internal system T, in a process called *temporalization* or *adding a temporal dimension to a logic system*, defined in [6]. The resulting combined logic system $\overline{\mathsf{T}}(\mathsf{T})$ is illustrated in Figure 2.

The temporalization associates every time point in $\overline{\mathsf{T}}$ with a temporal description in T, where those T-descriptions need not be all identical. Given the logical properties of T and $\overline{\mathsf{T}}$, what can be said about the logical properties of $\overline{\mathsf{T}}(\mathsf{T})$?

In terms of a generic combination of logics, the temporalization method can be matched with a process called *"fuzzling"* or *layering*, which is characterized by the fact that the formulas of system T can be substituted for the atoms of system $\overline{\mathsf{T}}$. In

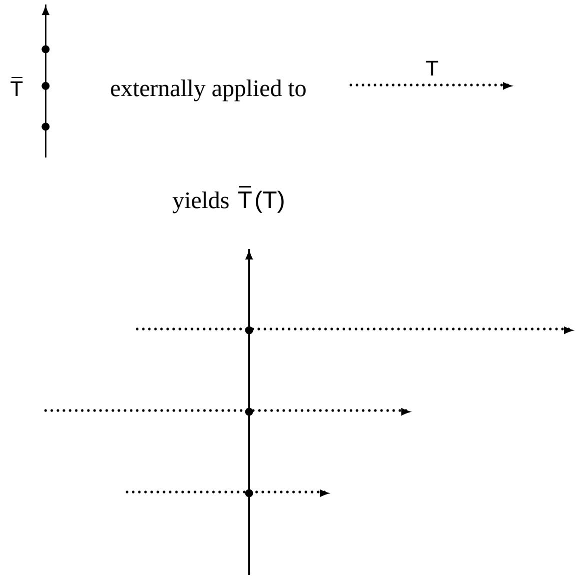

Figure 2: The combined flow of time resulting from temporalization

(Kripke-) semantical terms, this means that every possible world of $\overline{\mathsf{T}}$ is associated to a whole model of T; see [11].

1.2 Second case: independent agents Suppose now that agent A has the ability of referring to agent B's temporal beliefs and vice versa. The agents are therefore observing each other, as illustrated in Figure 3.

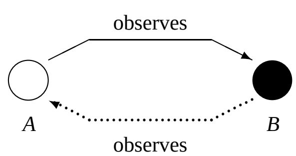

Figure 3: Independent interaction of agents

The agents' beliefs are then capable of interacting with each other through several levels of cross-reference, as in the sentence "A believes that B believes that A believes that…." A new combination method for T and $\overline{\mathsf{T}}$ is needed in order to represent such a sentence as a formula; which is called the *independent combination*, $\overline{\mathsf{T}} \oplus \mathsf{T}$. Since a formula of $\overline{\mathsf{T}} \oplus \mathsf{T}$ has a finite nature, it can be unravelled in a finite number of alternating temporalizations, as illustrated in Figure 4.

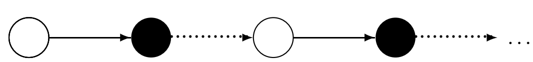

Figure 4: Unravelling the independent combination

Figure 4 suggests a way of analyzing the properties of the independent combination method using the temporalization method as an intermediary step. It will turn out that the independent combination method is the (infinite) union of all finite alternated temporalizations. An illustration of a possible independently combined flow of time is presented in Figure 5.

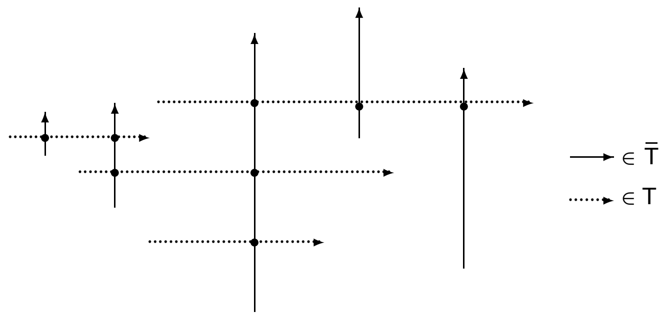

Figure 5: Independently combined flow of time

In terms of a generic combination of two logics, this process can be matched to the *dovetailing* process of [11], whereby atoms of T can be substituted by formulas of $\overline{\mathsf{T}}$ and vice-versa. The semantical counterpart is obtained by providing each possible world with two distinct accessibility relations, $<$ and $\overline{<}$, so that from every possible world it is possible to reach another possible world either via $<$ or via $\overline{<}$.

1.3 Third case: two-dimensional plane Yet another distinct situation can be found where we have the coexistence of two distinct temporal "points of view." This time a single agent with temporal reasoning capabilites is considered, and we want to be able to describe the evolution of his own beliefs. This is perhaps better illustrated by considering the agent as a temporal database where each piece of information is associated to a validity time (or interval). For example, consider the traditional database relation $employee(Name, Salary, Manager)$. Suppose the following is in the database at March 94.

Name	*Salary*	*Dept*	*Start*	*End*
Peter	1000	R&D	Apr 93	Mar 94

Where the attributes start and end represents the end points of the validity interval associated with the information. We assume that Peter's salary has not changed since Apr 93. Suppose in Apr 94 Peter receives a retroactive promotion dating back to the beginning of the year, increasing his salary to 2000. The whole database evolution is illustrated in Figure 6, where only the value of Peter's salary is indicated at each point.

If T represents valid-time and $\overline{\mathsf{T}}$ represents transaction-time, we have guaranteed a two-dimensional plane $\overline{\mathsf{T}} \times \mathsf{T}$ in order to represent the database evolution.

Another application of the two-dimensional plane (or its NW-semi-plane) is in the representation of intervals on a line, as presented in [22]. In Figure 7 we can see a line considered the diagonal of a two-dimensional plane and that an interval $[t_1, t_2]$ on that line is represented by the point (t_1, t_2) on the NW-semi-plane.

The combination of two temporal systems that generates a combined flow of time that is isomorphic to a two-dimensional plane is called the *join* of two logical systems. We have adopted the term *join of logics* here (instead of the previously used *interlacing of logics* in Finger [8]) to be in accordance to the concept as defined in

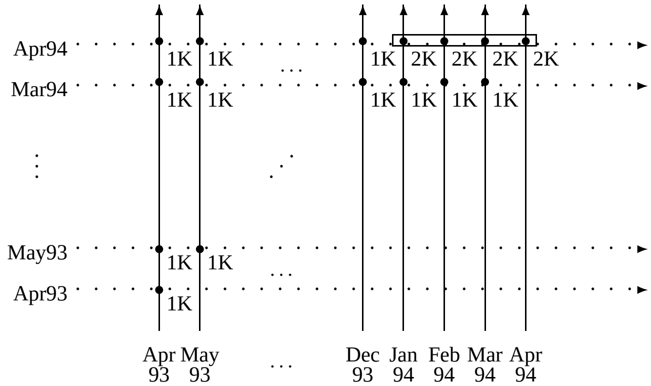

Figure 6: Two-dimensional temporal database evolution

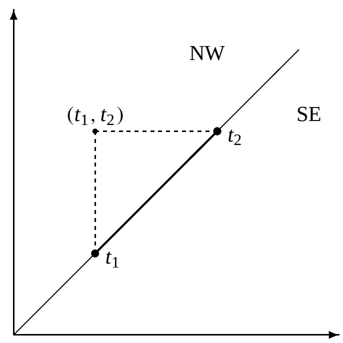

Figure 7: Two-dimensional representation of intervals

the larger context of generic combination of logics [11]. Although the language generated in this process is the same as that of independent combination (for the case of two temporal logics), the semantic interaction between T and $\overline{\mathsf{T}}$ is a lot stronger; this is due to the fact that the temporal operators of the two logics are commutative in the join. As it will be seen in Section 7, it is necessary to restrict this interaction to obtain the transfer of logical properties. The restiction will be applied to the type of operators allowed in one of the logics involved in the restricted join.

1.4 Aims of this paper In this paper we study three situations of coexistence of "two temporal points of view" that result from a combination of two linear, one-dimensional temporal logics. In this sense this paper is a continuation on the work started in [6] on the combination of temporal logics. There, a process for adding a temporal dimension to a logic system was described, in which a temporal logic T is *externally* applied to a generic logic system L, generating a combined logic $\mathsf{T}(\mathsf{L})$. We now explore several methods for systematically combining two temporal logics, T and $\overline{\mathsf{T}}$, thus generating for each method a new family of *two-dimensional temporal logics*.

A great number of (one-dimensional) temporal logics exist in the literature to deal with the great variety of properties one may wish to express about flows of time. When building two-dimensional temporal logics, the combination of two classes of flows of time generates an even greater number of possible systems to be studied. Furthermore, as we will see, there are several distinct classes of temporal logics that may be considered two-dimensional, each generated by a distinct combination method. It is, therefore, desirable to study whether it is possible to *transfer the properties* of long known and studied (one-dimensional) temporal logic systems to the two-dimensional case.

So the main goal of this paper is to study, for each combination method, the transfer of logical properties from component one-dimensional temporal systems to a combined two-dimensional one. We concentrate on the transfer of three basic properties of logic systems, namely soundness, completeness, and decidability. This by no means implies that those are the only properties whose transfer deserve to be studied, but, as has already been noted in [6] for the temporal case, and in Kracht and Wolter [16] and an unpublished paper by Fine and Schurtz, for the monomodal case, the transfer of completeness serves as a basis for the transfer of several other properties of logical systems.

We consider the following methods for combining two temporal logics.

1. The temporalization method, i.e., the external application of a temporal logic to another temporal system, also known as adding a temporal dimension to a logic system;
2. the independent combination of two temporal systems;
3. the full join of two temporal systems, where flows of time are considered over a two-dimensional plane;
4. the restricted join of two temporal system, a combination method that restricts the previous one but generates nice transfer results.

We proceed as follows. Section 2 presents the basic notions of one- and two-dimensional temporal logics. Section 3 discusses combinations of logics in general terms, so that in the rest of the paper we can present special cases of combination methods. Section 4 briefly examines the transfer results obtained for the temporalization method in [6]. Section 5 studies the independent combination method. Section 6 deals with the full join method and Section 7 with its restricted version. Section 8 analyzes the properties of a two-dimensional diagonal on the model generated by the full and restricted join methods. In Section 9 we discuss the results of this work.

2 Preliminaries For the purposes of this paper, a logic system is composed of three elements:

1. a language, normally given by a set of formation rules generating well formed formulas over a signature and a set of logical connectives.
2. An inference system, i.e., a relation $\vdash$ between sets of formulas, normally represented by upper case Greek letters Δ, Γ, Σ, Ψ, Φ and a single formula, normally represented by upper case letters $A, B, C, \ldots$; the fact that A is inferred from a set Δ is indicated by $\Delta \vdash A$. When Δ is a singleton, $\Delta = \{B\}$, the notation is abused and we write $B \vdash A$.
3. The semantics of formulas over a class $\mathcal{K}$ of model structures. The fact that a formula A is true of or holds at a model $\mathcal{M} \in \mathcal{K}$ is indicated by $\mathcal{M} \models A$.

In providing a method for combining two logics into a third one, it will be necessary to provide three sub-methods that combine the languages, inference systems, and semantics of the component logic systems. The component systems considered in this paper will be one-dimensional linear US-temporal logics. Their language is built from a countable signature of propositional letters $\mathcal{P} = \{p_1, p_2, \ldots\}$, the Boolean connectives $\wedge$ (conjunction) and $\neg$ (negation), the two-place temporal operation U (until) and S (since), possibly renamed, and the following formation rules:

- every propositional letter is a formulas
- if A and B are formulas, so are $\neg A$ and $A \wedge B$
- if A and B are formulas, so are $U(A, B)$ (reads "until A is true in the future, B will be true") and $S(A, B)$ (reads "since A was true in the past, B has been true").
- nothing else is a formula.

The *mirror image* of a formula is another temporal formula obtained by swapping all occurrences of U and S, e.g., the mirror image of $U(A, S(B, C))$ is $S(A, U(B, C))$.

The other Boolean connectives $\vee$ (disjunction), $\rightarrow$ (material implication), $\leftrightarrow$ (material bi-implication) and the constants $\bot$ (false) and $\top$ (true) can be derived in the standard way. Similarly, the one-place temporal operators F ("sometime in the future"), P ("sometime in the past"), G ("always in the future"), and H ("always in the past") can be defined in terms of U and S.

To provide the semantics of temporal formulas we have to consider a (one-dimensional) *flow of time*, $\mathcal{F} = (T, <)$, where T is a set of time points and $<$ is an order over T. A temporal valuation $h : T \to 2^{\mathcal{P}}$ associates every time point with a set of propositional letters, i.e., $h(t)$ is the set of propositions that are true at time t. (Equivalently, and perhaps more usually, a valuation could be defined as a function $h : \mathcal{P} \to 2^T$, associating every propositional letter to a set of time points in which it holds true; see Burgess [5], Gabbay [13].) A model structure $\mathcal{M} = (T, <, h)$ consists of a flow of time $(T, <)$ and a temporal assignment h, and for the purposes of combination of logics we consider a "current world" $t \in T$ as part of the model. $\mathcal{M}, t \models A$ reads "A is true at t over model $\mathcal{M}$." Classes of models are normally defined by restrictions over the order relation $<$ of the flow of time.

The semantics of temporal formulas is given by:

$\mathcal{M}, t \models p$	iff	$p \in \mathcal{P}$ such that $p \in h(t)$.
$\mathcal{M}, t \models \neg A$	iff	it is not the case that $\mathcal{M}, t \models A$.
$\mathcal{M}, t \models A \wedge B$	iff	$\mathcal{M}, t \models A$ and $\mathcal{M}, t \models B$.
$\mathcal{M}, t \models S(A, B)$	iff	there exists an $s \in T$ with $s < t$ and $\mathcal{M}, s \models A$ and for every $u \in T$, if $s < u < t$ then $\mathcal{M}, u \models B$.
$\mathcal{M}, t \models U(A, B)$	iff	there exists an $s \in T$ with $t < s$ and $\mathcal{M}, s \models A$ and for every $u \in T$, if $t < u < s$ then $\mathcal{M}, u \models B$.

The following restriction will be applied throughout this presentation. Flows of time will always be considered to have the properties:

1. irreflexivity: $\forall t \neg(t < t)$
2. transitivity: $\forall t, s, u(t < s \wedge s < u \rightarrow t < u)$
3. totality: $\forall t, s(t = s \vee t < s \vee s < t)$

The class of all flows respecting the restrictions above is the class $\mathcal{K}_{lin}$ of linear flows of time. We also represent the class of all models based on linear flows as $\mathcal{K}_{lin}$. Further restrictions can be applied to the nature of flows of time so that several other linear subclasses can be formed, e.g., the classes of dense ($\mathcal{K}_{dense}$), discrete ($\mathcal{K}_{dis}$), $\mathbb{Z}$-like, $\mathbb{Q}$-like, and $\mathbb{R}$-like flows of time. The linearity property allows for the definition of the "at all times" temporal connective $\Box$.

$$\Box A = A \wedge GA \wedge HA$$

In case of discrete flows of time, the operator "next time," $\bigcirc$, and "previous time," $\bullet$, are also defined.

$$\bigcirc A = U(A, \bot)$$
$$\bullet A = S(A, \bot)$$

The inference systems will be considered to be finite axiomatizations, i.e., a pair (Σ, I) where Σ is a finite set of formulas called *axioms* and I is a set of inference rules. Consider the Burgess-Xu [4], Xu [23] axiomatization for $\mathcal{K}_{lin}$ consisting of the following axioms:

A0 all classical tautologies
A1a $G(p \to q) \to (U(p, r) \to U(q, r))$
A2a $G(p \to q) \to (U(r, p) \to U(r, q))$
A3a $(p \wedge U(q, r)) \to U(q \wedge S(p, r), r)$
A4a $U(p, q) \to U(p, q \wedge U(p, q))$
A5a $U(q \wedge U(p, q), q) \to U(p, q)$
A6a $(U(p, q) \wedge U(r, s)) \to$
$(U(p \wedge r, q \wedge s) \vee U(p \wedge s, q \wedge s) \vee U(q \wedge r, q \wedge s))$

plus their mirror images (**b** axioms). The inference rules are:

Subst Uniform Substitution, i.e., let $A(q)$ be an axiom containing the propositional letter q and let B be any formula, then from $\vdash A(q)$ infer $\vdash A(q \backslash B)$ by substituting all appearances of q in A by B.
MP Modus Ponens: from $\vdash A$ and $\vdash A \to B$ infer $\vdash B$.
TG Temporal Generalization: from $\vdash A$ infer $\vdash HA$ and $\vdash GA$.

A formula A is deducible from the set of formulas Δ, $\Delta \vdash A$, if there exists a finite sequence of formulas $B_1, \ldots, B_n = A$ such that every B_i is either

(a) a formula in Δ; or
(b) an axiom; or
(c) obtained from previous formulas in the sequence through the use of an inference rule.

We write $\vdash A$ for $\varnothing \vdash A$, i.e., only items (b) and (c) above are used in the deduction of A, in which case A is said to be a *theorem*. A set of formulas Δ is *inconsistent* if $\Delta \vdash \bot$, otherwise it is *consistent*. A formula A is consistent if $\{A\}$ is consistent.

On the semantical side, a set of formulas Δ is *satisfiable* over a class of models $\mathcal{K}$ if there exists a model $\mathcal{M} \in \mathcal{K}$ with a $t \in T$ such that, for every $B \in \Delta$, $\mathcal{M}, t \models B$.

A formula A is *valid* over $\mathcal{K}$, $\mathcal{K} \models A$, if for every model $\mathcal{M} = (T, <, h) \in \mathcal{K}$ and every $t \in T$, $\mathcal{M}, t \models A$. The expression $\Delta \models A$ represents that every model satisfying Δ also satisfies A.

An inference system is *sound* with respect to a class of models $\mathcal{K}$ iff every theorem is a valid formula, i.e., $\vdash A$ implies $\mathcal{K} \models A$. An inference system is *(weakly)* complete over $\mathcal{K}$ if every theorem $\vdash A$ is valid, $\mathcal{K} \models A$, or equivalently if every consistent formula is satisfied over $\mathcal{K}$. Strong completeness states that whenever $\Delta \models A$ then $\Delta \vdash A$, for a possibly infinite Δ. Let $\mathsf{L} = \langle \mathcal{L}, \vdash, \models \rangle$ be a logic system with language $\mathcal{L}$, inference system $\vdash$, and semantics $\models$. We say that L is *decidable* if there exists an algorithm (decision procedure) that determines, for every $A \in \mathcal{L}$, whether A is a theorem or not. The *validity problem* for L is to determine whether some $A \in \mathcal{L}$ is a valid formula or not.

We have the following results.

Theorem 2.1 ([4],[23]) *The Burgess-Xu axiomatization presented above is sound and complete over the class $\mathcal{K}_{lin}$.*

Theorem 2.2 (Rabin [17]) *The logic $\mathsf{US} = \langle \mathcal{L}_{\mathsf{US}}, \vdash_{\mathsf{US}}, \models_{\mathsf{US}} \rangle$ is decidable over $\mathcal{K}_{lin}$.*

3 Combining logics As we have mentioned earlier, the combination of two one-dimensional temporal logics will generate a two-dimensional temporal logic. Throughout this presentation, we refer to one of the temporal dimensions as the *horizontal dimension* and the other one as the *vertical dimension*; the symbols related to the vertical dimension are normally obtained by putting a bar on top of the corresponding horizontal ones, e.g., T and $\overline{\mathsf{T}}$, $F$and $\overline{F}$, $<$ and $\overline{<}$.

There are two distinct criteria for defining a modal/temporal logic system as two-dimensional:

1. If the alphabet of the language contains two nonempty, disjoint sets of corresponding modal or temporal operators, Φ and $\overline{\Phi}$, each set associated with a distinct flow of time, $(T, <)$ and $(\overline{T}, \overline{<})$, then the system is two-dimensional.
2. If the truth value of a formula is evaluated with respect to two time points, then the system is two-dimensional. In this case, we even have the distinction between strong and weak interpretations of formulas that, as a consequence, generates different notions of valid formulas (a formula is valid if it holds in all models for all pairs of time points). Under the *strong interpretation*, the truth value of atoms depends on both dimensions, giving rise to the notion of *strongly valid formulas* when the evaluation of formulas is inductively extended to all connectives. In the *weak interpretation*, the truth value of atoms depends only on the one dimension, e.g., the horizontal dimension, giving rise to the notion of *weakly valid formulas*. Usually for this notion of two-dimensionality, both time points refer to the same flow of time, so we may also have the notion of (weak/strong) *diagonally valid* formulas by restricting validity to the case where both dimensions refer to the same point, i.e., A is diagonally valid iff $\mathcal{M}, t, t \models A$ for all $\mathcal{M}$ and t; see [13] for more details.

Criterion 1 will be called the *syntactic criterion* for two-dimensionality, although it is not completely syntactic, i.e., it depends on the semantic notion of flows of time; criterion 2 will be called the *semantic criterion* for two-dimensionality.

Note that both cases can yield, as an extreme case, one-dimensional temporal logic. In 1, this can be done by making $T = \overline{T}$ and $\overline{<} = (<)^{-1} = (>)$, i.e., by taking two flows with the same set of time points such that one order is the inverse of the other; the future operators $\Phi = \{F, G, U\}$ are associated with $(T, <)$ and the past operators $\overline{\Phi} = \{P, H, S\}$ are associated with $(T, >)$. In 2, this can be done by fixing one dimension to a single time point so that it becomes redundant.

These two distinct approaches to the two-dimensionality of a system are independent. In fact, we will see in Section 5 a system that contains two distinct sets of operators over two classes of flows of time, but its formulas are evaluated at a single point. On the other hand, there are several temporal logics in the literature satisfying 2 but not 1, containing a single set of temporal operators in which formulas are evaluated according to two or more time points in the same flow, e.g., [13], Aqvist [1], Kamp [15].

A logic system that respects both the syntactic and the semantic criteria for two-dimensionality is called *broadly two-dimensional*, and this will be the kind of system we will be aiming to achieve through combination methods; we consider in this work only strong evaluation and validity; the weak interpretation generates systems with the expressivity of only monadic first-order language [13], but for broadly two-dimensional systems we are interested in the expressivity of dyadic first-order language, although it is known that no set of temporal operators can be expressively complete over dyadic first-order language [22]. (A modal/temporal language is *expressively complete* over a class of first-order formulas if, for any first-order formula A in that class, there exists a modal/temporal formula B such that A is first-order equivalent to B^*, where B^* is the standard first-order translation of B; see [13].) Venema's [22] two-dimensional temporal logic, Segerberg's [19] two-dimensional modal logic, and the temporalization of a temporal logic are all broadly two-dimensional; so are the combined logics in Sections 6 and 7.

In the study of one-dimensional temporal logics (1DTLs) several classes of flows of time are taken into account. When we move to 2DTLs, the number of such classes increases considerably, and every pair of one-dimensional classes can be seen as generating a different two-dimensional class. The study of 2DTLs would benefit much if the properties known to hold for 1DTLs could be systematically transferred to 2DTLs, avoiding the repetition of much of the work that has been published in the literature. This is a strong motivation to consider methods of combination of 1DTLs into 2DTLs and studying the transfer of logical properties through each method. Also in favor of such an approach is the fact that the results concerning 2DTLs are then presented in a general, compact, and elegant form.

In providing a method to combine two 1DTLs $\overline{\mathsf{T}}$ and T we have to pay attention to the following points:

(a) A method for combining logics $\overline{\mathsf{T}}$ and T is composed of three sub-methods, namely a method for combining the languages of $\overline{\mathsf{T}}$ and T, a method for combining their inference systems, and a method for combining their semantics.

(b) We study the combined logic system with respect to the way certain logical properties of $\overline{\mathsf{T}}$ and T are transferred to the two-dimensional combination. We focus here on the properties of soundness, completeness, and decidability of the combined system given those of the component ones.

(c) The combined language should be able to express some properties of the interaction between the two-dimensions; otherwise the combination is just a partial one, and the two systems are not fully combined. For example, it is desirable to express formulas like $F\overline{F}A \leftrightarrow \overline{F}FA$ and $P\overline{F}A \leftrightarrow \overline{F}PA$ that are not in the temporalized language of $\overline{\mathsf{T}}(\mathsf{T})$.

(d) If we want to strengthen the interaction between the two systems, some properties of the interaction between the two-dimensions are expected to be theorems of the combined system, e.g., the commutativity of horizontal and vertical future operators such as $F\overline{F}A \leftrightarrow \overline{F}FA$ and $P\overline{F}A \leftrightarrow \overline{F}PA$. Those are called the *interaction axioms* in [11].

(e) We want the combination method to be as independent as possible from the underlying flows of time.

All methods of combination must comply with item (a). The method for combining the languages of $\overline{\mathsf{T}}$ and T includes the choice of which sublanguage of $\overline{\mathsf{T}}$ and T is going to be part of the combined two-dimensional language, as well as the way in which this combination is done; in this presentation we will work, in the most general case, with the standard languages of S and U, $\overline{S}$ and $\overline{U}$, but we also consider some sublanguages, e.g., the sublanguage generated by a set of derived operators, such as the vertical "previous" ($\bar{\bullet}$) and "next" ($\bar{\bigcirc}$) in Section 7. In combining the inference systems of $\overline{\mathsf{T}}$ and T, we will assume that they are both an extension of classical logic and that they are presented in the form of a regular, normal axiomatic system (Σ, I), where Σ is a set of axioms and I is a set of inference rules. One important requirement is that the combined system be a *conservative extension* of the two components. The conservativeness property states that if A is a formula in the language of L and L^* is a logic system extending L (i.e., the language of L is a sublanguage of the language of L^*), then A will be a theorem of L^* only if it is a theorem of L already; conservativeness guarantees that no new information about the original system L is present in the extended one L^*.

The combined semantics has to deal with the structure of the combined model, the evaluation of two-dimensional formulas over those structures and also with the combinations of classes of flows of time.

Items (b), (c), (d) and (e) may conflict with each other. In fact, the rest of this paper shows that this is the case, as we try to compromise between expressivity, independence of the underlying flow of time and the transfer of logical properties.

4 Temporalizing a logic The first of the combination methods, known as "adding a temporal dimension to a logic system" or simply "temporalizing a logic system," has been extensively discussed in [6].

Temporalization is a methodology whereby an arbitrary logic system L can be enriched with temporal features to create a new system $\mathsf{T}(\mathsf{L})$. The new system is constructed by combining L with a pure propositional temporal logic T (such as linear temporal logic with "Since" and "Until") in a special way.

Although we are interested here only in temporalizing an already temporal system, so as to generate a 2DTL, the original method is more general and is applicable to a generic logic L; L is constrained to be an extension of classical logic, i.e., all propositional tautologies must be valid in it, but such a constraint does not affect us, for we are assuming that both temporal systems T and L are extensions of $\mathsf{US}/\mathcal{K}_{lin}$. The language of a temporalized system is based on the US language and on a subset of the language of L, $\mathcal{L}_{\mathsf{L}}$. The set $\mathcal{L}_{\mathsf{L}}$ is partitioned in two sets, BC_{L} and ML_{L}. A formula $A \in \mathcal{L}_{\mathsf{L}}$ belongs to the set of *boolean combinations*, BC_{L}, iff it is built up from other formulas by the use of one of the boolean connectives $\neg$ or $\wedge$ or any other connective defined only in terms of those; it belongs to the set of *monolithic formula* ML_{L} otherwise.

The result of temporalizing over $\mathcal{K}$ the logic system L is the logic system $\mathsf{T(L)}/\mathcal{K}$. The alphabet of the temporalized language uses the alphabet of L plus the two-place operators S and U, if they are not part of the alphabet of L; otherwise, we use $\overline{S}$ and $\overline{U}$ or any other proper renaming.

Definition 4.1 (Temporalized formulas) The set $\mathcal{L}_{\mathsf{T(L)}}$ of formulas of the logic system L is the smallest set such that:

1. If $A \in ML_{\mathsf{L}}$, then $A \in \mathcal{L}_{\mathsf{T(L)}}$;
2. If $A, B \in \mathcal{L}_{\mathsf{T(L)}}$ then $\neg A \in \mathcal{L}_{\mathsf{T(L)}}$ and $(A \wedge B) \in \mathcal{L}_{\mathsf{T(L)}}$;
3. If $A, B \in \mathcal{L}_{\mathsf{T(L)}}$ then $S(A, B) \in \mathcal{L}_{\mathsf{T(L)}}$ and $U(A, B) \in \mathcal{L}_{\mathsf{T(L)}}$.

Note that, for instance, if $\blacksquare$ is an operator of the alphabet of L and A and B are two formulas in $\mathcal{L}_{\mathsf{L}}$, the formula $\blacksquare U(A, B)$ is *not* in $\mathcal{L}_{\mathsf{T(L)}}$. The language of $\mathsf{T(L)}$ is independent of the underlying flow of time, but its semantics and inference system are not, so we must fix a class $\mathcal{K}$ of flows of time over which the temporalization is defined; if $\mathcal{M}_{\mathsf{L}}$ is a model in the class of models of L, $\mathcal{K}_{\mathsf{L}}$, for every formula $A \in \mathcal{L}_{\mathsf{L}}$ we must have either $\mathcal{M}_{\mathsf{L}} \models A$ or $\mathcal{M}_{\mathsf{L}} \models \neg A$. In the case that L is a temporal logic we must consider a "current time" o as part of its model to achieve that condition.

Definition 4.2 (Semantics of the temporalized logic) Let $(T, <) \in \mathcal{K}$ be a flow of time and let $g : T \to \mathcal{K}_{\mathsf{L}}$ be a function mapping every time point in T to a model in the class of models of L. A model of $\mathsf{T(L)}$ is a triple $\mathcal{M}_{\mathsf{T(L)}} = (T, <, g)$ and the fact that A is true in $\mathcal{M}_{\mathsf{T(L)}}$ at time t is written as $\mathcal{M}_{\mathsf{T(L)}}, t \models A$ and defined as:

$\mathcal{M}_{\mathsf{T(L)}}, t \models A$, $A \in ML_{\mathsf{L}}$	iff	$g(t) = \mathcal{M}_{\mathsf{L}}$ and $\mathcal{M}_{\mathsf{L}} \models A$.
$\mathcal{M}_{\mathsf{T(L)}}, t \models \neg A$	iff	it is not the case that $\mathcal{M}_{\mathsf{T(L)}}, t \models A$.
$\mathcal{M}_{\mathsf{T(L)}}, t \models (A \wedge B)$	iff	$\mathcal{M}_{\mathsf{T(L)}}, t \models A$ and $\mathcal{M}_{\mathsf{T(L)}}, t \models B$.
$\mathcal{M}_{\mathsf{T(L)}}, t \models S(A, B)$	iff	there exists $s \in T$ such that $s < t$ and $\mathcal{M}_{\mathsf{T(L)}}, s \models A$ and for every $u \in T$, if $s < u < t$ then $\mathcal{M}_{\mathsf{T(L)}}, u \models B$.
$\mathcal{M}_{\mathsf{T(L)}}, t \models U(A, B)$	iff	there exists $s \in T$ such that $t < s$ and $\mathcal{M}_{\mathsf{T(L)}}, s \models A$ and for every $u \in T$, if $t < u < s$ then $\mathcal{M}_{\mathsf{T(L)}}, u \models B$.

Figure 2 illustrates a temporalized model. The inference system of $\mathsf{T(L)}/\mathcal{K}$ is given by the following.

Definition 4.3 (Axiomatization for $\mathsf{T(L)}$) An axiomatization for the temporalized logic $\mathsf{T(L)}$ is composed of:

- The axioms of $\mathsf{T}/\mathcal{K}$;

- The inference rules of $\mathsf{T}/\mathcal{K}$;
- For every formula A in $\mathcal{L}_{\mathsf{L}}$, if $\vdash_{\mathsf{L}} A$ then $\vdash_{\mathsf{T(L)}} A$, i.e., all theorems of L are theorems of $\mathsf{T(L)}$. This inference rule is called **Persist**.

Example 4.4 (Temporalizing propositional logic) Consider classic propositional logic $\mathsf{PL} = \langle \mathcal{L}_{\mathsf{PL}}, \vdash_{\mathsf{PL}}, \models_{\mathsf{PL}} \rangle$. Its temporalization generates the logic system $\mathsf{T(PL)} = \langle \mathcal{L}_{\mathsf{T(PL)}}, \vdash_{\mathsf{T(PL)}}, \models_{\mathsf{T(PL)}} \rangle$.

It is not difficult to see that $\mathcal{L}_{\mathsf{T(PL)}} = \mathcal{L}_{\mathsf{US}}$ and $\vdash_{\mathsf{T(PL)}} = \vdash_{\mathsf{US}}$, i.e., the temporalized version of PL over any $\mathcal{K}$ is actually the temporal logic $\mathsf{T} = \mathsf{US}/\mathcal{K}$. With respect to $\mathcal{M}_{\mathsf{T(L)}}$, the function g actually assigns, for every time point, a PL model.

Example 4.5 (Temporalizing US-temporal logic) If we temporalize over $\mathcal{K}$ the one-dimensional logic system $\mathsf{US}/\mathcal{K}$ we obtain the two-dimensional logic system $\mathsf{T(US)} = \langle \mathcal{L}_{\mathsf{T(US)}}, \vdash_{\mathsf{T(US)}}, \models_{\mathsf{T(US)}} \rangle = \mathsf{T}^2(\mathsf{PL})/\mathcal{K}$. In this case we have to rename the two-place operators S and U of the temporalized alphabet to, say, $\overline{S}$ and $\overline{U}$.

In order to obtain a model for $\mathsf{T(US)}$, we must fix a "current time," o, in $\mathcal{M}_{\mathsf{US}} = (T_1, <_1, g_1)$, so that we can construct the model $\mathcal{M}_{\mathsf{T(US)}} = (T_2, <_2, g_2)$ as previously described. Note that, in this case, the flows of time $(T_1, <_1)$ and $(T_2, <_2)$ need not to be the same. $(T_2, <_2)$ is the flow of time of the upper-level temporal system, whereas $(T_1, <_1)$ is the flow of time of the underlying logic which, in this case, happens to be a temporal logic. The logic system we obtain by temporalizing US-temporal logic is the two-dimensional temporal logic described in [7].

Example 4.6 (n-dimensional temporal logic) If we repeat the process started in the last two examples, we can construct an n-dimensional temporal logic $\mathsf{T}^n(\mathsf{PL})/\mathcal{K}$ (its alphabet including S_n and U_n) by temporalizing a $(n-1)$-dimensional temporal logic.

Every time we add a temporal dimension, we are able to describe changes in the underlying system. Temporalizing the system L once, we are creating a way of describing the history of L; temporalizing for the second time, we are describing how the history of L is viewed in different moments of time. We can go on indefinitely, although it is not clear what the purpose of doing so would be.

To present the transfer results we restrict the logic systems to $\mathsf{L} = \mathsf{US}/\mathcal{K}$ and $\mathsf{T} = \bar{\mathsf{U}}\bar{\mathsf{S}}/\overline{\mathcal{K}}$, where $\mathcal{K}, \overline{\mathcal{K}} \subseteq \mathcal{K}_{lin}$. We write $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ instead of $\mathsf{T(L)}$ and the generated class of models is referred to as $\overline{\mathcal{K}}(\mathcal{K})$. For this system, we enumerate a series of results that are proved in [6]. Those results will be useful for the discussion of the independent combination method.

Theorem 4.7 (Transfer via temporalization) *Let $\bar{\mathsf{U}}\bar{\mathsf{S}}/\overline{\mathcal{K}}$ and $\mathsf{US}/\mathcal{K}$ be two logic systems such that $\overline{\mathcal{K}}, \mathcal{K} \subseteq \mathcal{K}_{lin}$.*

(a) If $\bar{\mathsf{U}}\bar{\mathsf{S}}$ is sound with respect to $\overline{\mathcal{K}}$ and US is sound with respect to $\mathcal{K}$, then $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ is sound w.r.t. $\overline{\mathcal{K}}(\mathcal{K})$.

(b) If $\bar{\mathsf{U}}\bar{\mathsf{S}}$ is complete w.r.t. $\overline{\mathcal{K}}$ and US is complete w.r.t. $\mathcal{K}$ then $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ is complete w.r.t. $\overline{\mathcal{K}}(\mathcal{K})$.

(c) If $\bar{\mathsf{U}}\bar{\mathsf{S}}$ is complete w.r.t. $\mathcal{K}$, then $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ is a conservative extension of both $\bar{\mathsf{U}}\bar{\mathsf{S}}$ and US.

(d) If $\bar{\mathsf{U}}\bar{\mathsf{S}}$ is complete and is decidable over $\overline{\mathcal{K}}$ and US is complete and decidable over $\mathcal{K}$ then $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$ is decidable over $\overline{\mathcal{K}}(\mathcal{K})$.

5 Independent combination We have seen in the previous section how to add a temporal dimension to a logic system. In particular, if a temporal logic is itself temporalized we obtain a two-dimensional temporal logic. Such a logic system is, however, very weakly expressive; if US is the internal (horizontal) temporal logic in the temporalization process (F is derived in US), and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ is the external (vertical) one ($\overline{F}$ is defined in $\bar{\mathsf{U}}\bar{\mathsf{S}}$), we cannot express that vertical and horizontal future operators commute,

$$F\overline{F}A \leftrightarrow \overline{F}FA.$$

In fact, the subformula $F\overline{F}A$ is not even in the temporalized language of $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$, nor is the whole formula. In other words, the interplay between the two-dimensions is not expressible in the language of the temporalized $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US})$.

The idea is then to define a new method for combining logic systems that puts together all the expressivity of the two component logic systems in an independent way; for that we assume that the language of a system is given by a set of formation rules.

Definition 5.1 Let $Op(\mathsf{L})$ be the set of non-boolean operators of a generic logic L. Let $\overline{\mathsf{T}}$ and T be logic systems such that $Op(\mathsf{T}) \cap Op(\overline{\mathsf{T}}) = \varnothing$. The *fully combined language* of logic systems $\overline{\mathsf{T}}$ and T over the set of atomic propositions $\mathcal{P}$ is obtained by the union of the respective sets of connectives and the union of the formation rules of the languages of both logic systems.

Let the operators U and S be in the language of US and $\overline{U}$ and $\overline{S}$ be in that of $\bar{\mathsf{U}}\bar{\mathsf{S}}$. Note that the renaming of the temporal operator is done prior to the combination, so that the combined system contains the set of boolean operators $\{\neg, \wedge\}$ coming from both components, plus the set of temporal operators $\{U, S, \overline{U}, \overline{S}\}$. Their fully combined language over a set of atomic propositions $\mathcal{P}$ is given by

- every atomic proposition is in it;
- if A, B are in it, so are $\neg A$ and $A \wedge B$;
- if A, B are in it, so are $U(A, B)$ and $S(A, B)$.
- if A, B are in it, so are $\overline{U}(A, B)$ and $\overline{S}(A, B)$.

In general, we do not want any non-boolean operator to be shared between the two languages, for this may cause problems when combining their axiomatizations. For example (this example is due to Ian Hodkinson), if a generic operator $\Box$ belongs to both temporal logic systems such that T contains axiom $q \leftrightarrow \Box q$ and system $\overline{\mathsf{T}}$ contains axiom $\neg q \leftrightarrow \Box q$, the union of their axiomatizations will result in an inconsistent system even though each system might have been itself consistent. To avoid such behavior the restriction $Op(\mathsf{T}) \cap Op(\overline{\mathsf{T}}) = \varnothing$ is imposed on the fully combined language of $\overline{\mathsf{T}}$ and T.

This new combination method is called *independent* because it takes the independent union of the axiomatization of its two component systems, and it is based on their fully combined language.

Definition 5.2 Let US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ be two US-temporal logic systems defined over the same set $\mathcal{P}$ of propositional atoms such that their languages are independent. The *independent combination* $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ is given by the following:

- The fully combined language of US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$.
- If (Σ, I) is an axiomatization for US and $(\overline{\Sigma}, \overline{I})$ is an axiomatization for $\bar{\mathsf{U}}\bar{\mathsf{S}}$, then $(\Sigma \cup \overline{\Sigma}, I \cup \overline{I})$ is an axiomatization for $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$. Note that, apart from the classical tautologies, the set of axioms Σ and $\overline{\Sigma}$ are supposed to be disjoint, but not the inference rules.
- The class of independently combined flows of time is $\mathcal{K} \oplus \overline{\mathcal{K}}$ composed of biordered flows of the form $(\tilde{T}, <, \overline{<})$ where the connected components of $(\tilde{T}, <)$ are in $\mathcal{K}$ and the connected components of $(\tilde{T}, \overline{<})$ are in $\overline{\mathcal{K}}$, and $\tilde{T}$ is the (not necessarily disjoint) union of the sets of time points T and $\overline{T}$ that constitute each connected component; such a biordered flow of time has been discussed in [16] for the case of the independent combination of two mono-modal systems.

  A model structure for $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K} \oplus \overline{\mathcal{K}}$ is a 4-tuple $(\tilde{T}, <, \overline{<}, g)$, where $(\tilde{T}, <, \overline{<}) \in \mathcal{K} \oplus \overline{\mathcal{K}}$ and g is an assignment function $g : \tilde{T} \to 2^{\mathcal{P}}$. An independently combined model is illustrated in Figure 5.

  The semantics of a formula A in a model $\mathcal{M} = (\tilde{T}, <, \overline{<}, g)$ is defined as the union of the rules defining the semantics of $\mathsf{US}/\mathcal{K}$ and $\bar{\mathsf{U}}\bar{\mathsf{S}}/\overline{\mathcal{K}}$. The expression $\mathcal{M}, t \models A$ reads that the formula A is true in the (combined) model $\mathcal{M}$ at the point $t \in \tilde{T}$. The semantics of formulas is given by induction in the standard way:

$\mathcal{M}, t \models p$	iff	$p \in g(t)$ and $p \in \mathcal{P}$.
$\mathcal{M}, t \models \neg A$	iff	it is not the case that $\mathcal{M}, t \models A$.
$\mathcal{M}, t \models A \wedge B$	iff	$\mathcal{M}, t \models A$ and $\mathcal{M}, t \models B$.
$\mathcal{M}, t \models S(A, B)$	iff	there exists an $s \in \tilde{T}$ with $s < t$ and $\mathcal{M}, s \models A$ and for every $u \in \tilde{T}$, if $s < u < t$ then $\mathcal{M}, u \models B$.
$\mathcal{M}, t \models U(A, B)$	iff	there exists an $s \in \tilde{T}$ with $t < s$ and $\mathcal{M}, s \models A$ and for every $u \in \tilde{T}$, if $t < u < s$ then $\mathcal{M}, u \models B$.
$\mathcal{M}, t \models \overline{S}(A, B)$	iff	there exists an $s \in \tilde{T}$ with $s \overline{<} t$ and $\mathcal{M}, s \models A$ and for every $u \in \tilde{T}$, if $s \overline{<} u \overline{<} t$ then $\mathcal{M}, u \models B$.
$\mathcal{M}, t \models \overline{U}(A, B)$	iff	there exists an $s \in \tilde{T}$ with $t \overline{<} s$ and $\mathcal{M}, s \models A$ and for every $u \in \tilde{T}$, if $t \overline{<} u \overline{<} s$ then $\mathcal{M}, u \models B$.

Note that, despite the combination of two flows of time, formulas are evaluated according to a single point. The independent combination generates a system that is two-dimensional according to the first criterion but fails the second one, so it is not broadly two-dimensional.

The following result is due to Thomason [21] and is more general than the independent combination of two US-logics.

Proposition 5.3 *With respect to the validity of formulas, the independent combination of two modal logics is a conservative extension of the original ones.*

Note that we have previously defined conservative extension in proof-theoretic terms; completeness for the independently combined case will lead to the conservativeness with respect to derivable theorems.

As usual, we will assume that $\mathcal{K}, \overline{\mathcal{K}} \subseteq \mathcal{K}_{lin}$, so $<$ and $\overline{<}$ are transitive, irreflexive and total orders; similarly, we assume that the axiomatizations are extensions of

$\mathsf{US}/\mathcal{K}_{lin}$.

The temporalization process will be used as an inductive step to prove the transfer of soundness, completeness and decidability for $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K} \oplus \overline{\mathcal{K}}$. Let us first consider the *degree of alternation* of a $(\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}})$-formula A for US, $dg(A)$, and $\bar{\mathsf{U}}\bar{\mathsf{S}}$, $\overline{dg}(A)$.

$dg(p) = 0$	$\overline{dg}(p) = 0$
$dg(\neg A) = dg(A)$	$\overline{dg}(\neg A) = \overline{dg}(A)$
$dg(A \wedge B) = max\{dg(A), dg(B)\}$	$\overline{dg}(A \wedge B) = max\{\overline{dg}(A), \overline{dg}(B)\}$
$dg(S(A, B)) = max\{dg(A), dg(B)\}$	$\overline{dg}(\bar{S}(A, B)) = max\{\overline{dg}(A), \overline{dg}(B)\}$
$dg(U(A, B)) = max\{dg(A), dg(B)\}$	$\overline{dg}(\bar{U}(A, B)) = max\{\overline{dg}(A), \overline{dg}(B)\}$
$dg(\bar{S}(A, B)) = 1 + max\{\overline{dg}(A), \overline{dg}(B)\}$	$\overline{dg}(S(A, B)) = 1 + max\{dg(A), dg(B)\}$
$dg(\bar{U}(A, B)) = 1 + max\{\overline{dg}(A), \overline{dg}(B)\}$	$\overline{dg}(U(A, B)) = 1 + max\{dg(A), dg(B)\}$

Any formula A of $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ can be seen as a formula of some finite number of alternating temporalizations of the form $\mathsf{US}(\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US}(\ \dots\ )))$; more precisely, A can be seen as a formula of $\mathsf{US}(\mathsf{L}_n)$, where $dg(A) = n$, $\mathsf{US}(\mathsf{L}_0) = \mathsf{US}$, $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{L}_0) = \bar{\mathsf{U}}\bar{\mathsf{S}}$, and $\mathsf{L}_{n-2i} = \bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{L}_{n-2i-1})$, $\mathsf{L}_{n-2i-1} = \mathsf{US}(\mathsf{L}_{n-2i-2})$, for $i = 0, 1, \dots, \lceil \frac{n}{2} \rceil - 1$. This fact is illustrated in Figure 4. The following Lemma actually allows us to see the independent combination as the (infinite) union of a finite number of alternating temporalizations of US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$; it will also be used in the proof of the transfer of completeness and decidability (given completeness) for $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$.

Lemma 5.4 *Let US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ be two complete logic systems. Then, A is a theorem of $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ iff it is a theorem of $\mathsf{US}(\mathsf{L}_n)$, where $dg(A) = n$.*

Proof: If A is a theorem of $\mathsf{US}(\mathsf{L}_n)$, all the inferences in its deduction can be repeated in $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$, so it is a theorem of $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$.

Suppose A is a theorem of $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$; let $B_1, \dots, B_m = A$ be a deduction of A in $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ and let $n' = max\{dg(B_i)\}$, $n' \geq n$. We claim that each B_i is a theorem of $\mathsf{US}(\mathsf{L}_{n'})$. In fact, by induction on m, if B_i is obtained in the deduction by substituting into an axiom, the same substitution can be done in $\mathsf{US}(\mathsf{L}_{n'})$; if B_i is obtained by Temporal Generalization from B_j, $j < i$, then by the induction hypothesis, B_j is a theorem of $\mathsf{US}(\mathsf{L}_{n'})$ and so is B_i; if B_i is obtained by Modus Ponens from B_j and B_k, $j, k < i$, then by the induction hypothesis, B_j and B_k are theorems of $\mathsf{US}(\mathsf{L}_{n'})$ and so is B_i.

So A is a theorem of $\mathsf{US}(\mathsf{L}_{n'})$ and, since US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ are two complete logic systems, by Theorem 4.7 each of the alternating temporalizations in $\mathsf{US}(\mathsf{L}_{n'})$ is a conservative extension of the underlying logic; it follows that A is a theorem of $\mathsf{US}(\mathsf{L}_n)$, as desired. □

The transfer of soundness, completeness, and decidability follows directly from this result.

Theorem 5.5 (Independent Combination) *Let US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ be two sound and complete logic systems over the classes $\mathcal{K}$ and $\overline{\mathcal{K}}$, respectively. Then their independent combination $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ is sound and complete over the class $\mathcal{K} \oplus \overline{\mathcal{K}}$. If US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ are complete and decidable, so is $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$.*

Proof: Soundness follows immediately from the validity of axioms and inference rules. For completeness, suppose that A is a consistent formula in $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$; by Lemma 5.4, A is consistent in $\mathsf{US}(\mathsf{L}_n)$, so we construct a temporalized model for it, and we obtain a model $(\tilde{T}_1, <_1, g_1, o_1)$ over $\mathcal{K}(\overline{\mathcal{K}}(\mathcal{K}(\ldots)))$, where o_1 is the "current time" necessary for the successive temporalizations. We show now how it can be transformed into a model over $\mathcal{K} \oplus \overline{\mathcal{K}}$.

Without loss of generality, suppose that US is the outermost logic system in the multi-layered temporalized system $\mathsf{US}(\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US}(\ldots)))$, and let n be the number of alternations. The construction is recursive, starting with the outermost logic. Let $i \leq n$ denote the step of the construction; if i is odd, it is a US-temporalization, otherwise it is a $\bar{\mathsf{U}}\bar{\mathsf{S}}$-temporalization. At every step i we construct the sets $\tilde{T}_{i+1}$, $<_{i+1}$ and $\overline{<}_{i+1}$ and the function g_{i+1}.

We start the construction of the model at step $i = 0$ with the temporalized model $(\tilde{T}_1, <_1, g_1, o_1)$ such that $(\tilde{T}_1, <_1) \in \mathcal{K}$, and we take $\overline{<}_1 = \varnothing$. At step $i < n$, consider the current set of time points $\tilde{T}_i$; according to the construction, each $t \in \tilde{T}_i$ is associated to:

- a temporalized model $g_i(t) = (\tilde{T}^t_{i+1}, <^t_{i+1}, g^t_{i+1}, o^t_{i+1}) \in \mathcal{K}$ and take $\overline{<}^t_{i+1} = \varnothing$, if i is even; or
- a temporalized model $g_i(t) = (\tilde{T}^t_{i+1}, \overline{<}^t_{i+1}, g_{i+1}, o^t_{i+1}) \in \overline{\mathcal{K}}$ and take $<^t_{i+1} = \varnothing$, if i is odd.

The point t is made identical to $o^t_{i+1} \in \tilde{T}^t_{i+1}$, so as to add the new model to the current structure; note that this preserves the satisfiability of all formulas at t. Let $\tilde{T}_{i+1}$ be the (possibly infinite) union of all $\tilde{T}^t_{i+1}$ for $t \in \tilde{T}_i$; similarly, $<_{i+1}$ and $\overline{<}_{i+1}$ are generated. And finally, for every $t \in \tilde{T}_{i+1}$, the function g_{i+1} is constructed as the union of all g^t_{i+1} for $t \in \tilde{T}_i$.

Repeating this construction n times, we obtain a combined model over $\mathcal{K} \oplus \overline{\mathcal{K}}$, $\mathcal{M} = (\tilde{T}_n, <_n, \overline{<}_n, g_n)$, such that for all $t \in \tilde{T}_n$, $g_n(t) \subseteq \mathcal{P}$. Since satisfiability of formulas is preserved at each step, it follows that $\mathcal{M}$ is a model for A, and completeness is proved.

For decidability, suppose we want to decide whether a formula $A \in \mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ is a theorem. By Lemma 5.4, this is equivalent to deciding whether $A \in \mathsf{US}(\mathsf{L}_n)$ is a theorem, where $n = dg(A)$. Since $\mathsf{US}/\mathcal{K}$ and $\bar{\mathsf{U}}\bar{\mathsf{S}}/\overline{\mathcal{K}}$ are both complete and decidable, by successive applications of Theorem 4.7(b) and (d), it follows that the following logics are decidable: $\mathsf{US}(\bar{\mathsf{U}}\bar{\mathsf{S}})$, $\bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{US}(\bar{\mathsf{U}}\bar{\mathsf{S}})) = \bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{L}_2), \ldots, \bar{\mathsf{U}}\bar{\mathsf{S}}(\mathsf{L}_{n-1}) = \mathsf{L}_n$; so a last application of Theorem 4.7(b) and (d) yields that $\mathsf{US}(\mathsf{L}_n)$ is decidable. □

6 Full join With respect to the generation of two-dimensional systems, the method of independent combination has two main drawbacks. First, it generates logic systems whose formulas are evaluated at one single time point, not generating a broadly two-dimensional logic. Second, since the method independently combines the two component logic systems, no interaction between the dimensions is provided. As a consequence, although a formula like $F\overline{F}A \leftrightarrow \overline{F}FA$ is expressible in its language, it will not be valid, as can easily be verified, for it expresses an interplay between the dimensions. We therefore introduce the notion of a *two-dimensional plane model*.

Definition 6.1 Let $\mathcal{K}$ and $\overline{\mathcal{K}}$ be two classes of flow of time. A *two-dimensional plane model* over the *fully combined class* $\mathcal{K} \times \overline{\mathcal{K}}$ is a 5-tuple $\mathcal{M} = (T, <, \overline{T}, \overline{<}, g)$, where $(T, <) \in \mathcal{K}$, $(\overline{T}, \overline{<}) \in \overline{\mathcal{K}}$ and $g : T \times \overline{T} \to 2^{\mathcal{P}}$ is a two-dimensional assignment. The semantics of the horizontal and vertical operators are independent of each other.

$\mathcal{M}, t, x \models S(A, B)$	iff	there exists $s < t$ such that $\mathcal{M}, s, x \models A$ and for all u, $s < u < t$, $\mathcal{M}, u, x \models B$.
$\mathcal{M}, t, x \models \overline{S}(A, B)$	iff	there exists $y \overline{<} x$ such that $\mathcal{M}, t, y \models A$ and for all z, $y \overline{<} z \overline{<} x$, $\mathcal{M}, t, z \models B$.

Similarly for U and $\overline{U}$, the semantics of atoms and boolean connectives remaining the standard one. A formula A is (strongly) valid over $\mathcal{K} \times \overline{\mathcal{K}}$ if for all models $\mathcal{M} = (T, <, \overline{T}, \overline{<}, g)$, for all $t \in T$ and $x \in \overline{T}$ we have $\mathcal{M}, t, x \models A$.

With respect to the expressivity of fully combined two-dimensional languages, Venema [22] has shown that no finite set of two-dimensional temporal operators is expressively complete over the class of linear flows with respect to dyadic first-order logic — despite the fact that US-temporal logic is expressively complete with respect to monadic first-order logic over $\mathbb{N}$ and over $\mathbb{R}$, and that, with additional operators (the Stavi operators), we can get expressive completeness over $\mathbb{Q}$ and $\mathcal{K}_{lin}$ (see Gabbay [10]). So expressive completeness is transferred by neither full join nor any other combination method.

It is easy to verify that the following formulas expressing the commutativity of future and past operators between the two dimensions are valid formulas in two-dimensional plane models.

I1 $F\overline{F}A \leftrightarrow \overline{F}FA$

I2 $F\overline{P}A \leftrightarrow \overline{P}FA$

I3 $P\overline{F}A \leftrightarrow \overline{F}PA$

I4 $P\overline{P}A \leftrightarrow \overline{P}PA$

Therefore, if we want to satisfy both the syntactic and the semantic criteria for two-dimensionality, we may define the method of *full join* containing the fully combined language of US and $\overline{\text{US}}$ and their fully combined class of models. The question is whether there is a method for combining their axiomatizations so as to generate a *fully joined axiomatization* that transfers the properties of soundness, completeness, and decidability. The answer, however, is no, not in general. In some cases we can obtain the transfer of completeness, in some other cases it fails. To illustrate that, we consider completeness results over classes of the form $\mathcal{K} \times \mathcal{K}$.

We start by defining some useful abbreviations. Let p be a propositional atom, and define:

$$
\begin{aligned}
hor(p) &= \Box(p \wedge \overline{H}\neg p \wedge \overline{G}\neg p) \\
ver(p) &= \overline{\Box}(p \wedge H\neg p \wedge G\neg p).
\end{aligned}
$$

It is clear that $hor(p)$ makes p true along the horizontal line and false elsewhere; similarly for $ver(p)$ with respect to the vertical.

The axiomatization of $\mathsf{US} \times \bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K}_{lin} \times \mathcal{K}_{lin}$ extends that of $\mathsf{US} \oplus \bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K}_{lin} \oplus \mathcal{K}_{lin}$ by including the join axioms **I1**–**I4** and the following inference rules:

IR1 if $\vdash hor(p) \to A$ and p does not occur in A, then $\vdash A$

IR2 if $\vdash ver(p) \to A$ and p does not occur in A, then $\vdash A$.

IR1 and **IR2** are two-dimensional extensions of the irreflexivity inferrence rule (IRR) defined in Gabbay [9] for the one-dimensional case: if $\vdash p \wedge H\neg p \to A$ and p does not occur in A, then $\vdash A$.

Theorem 6.2 (2D-completeness) *There is a sound and complete axiomatization over the class of full two-dimensional temporal models over $\mathcal{K}_{lin} \times \mathcal{K}_{lin}$.*

The proof consists of a Henkin-style construction of a two-dimensional grid, where each point is a maximally consistent set. The basic step of the construction is the elimination of "defects" from the grid, i.e., adding new points to the grid for a semantic condition that fails for the grid. The final model is obtained as the (infinite) union of all steps, and the grid thus constructed is shown to be a $\mathcal{K}_{lin} \times \mathcal{K}_{lin}$ model for an original consistent formula. The full details of the proof can be found in [8], but due to space limitations (the full proof takes up to ten pages) we omit it here. If $\mathcal{K}_{dis}$ is the class of all linear and discrete flows, [8] also shows completeness results for the classes $\mathcal{K}_{dis} \times \mathcal{K}_{dis}$, $\mathbb{Q} \times \mathbb{Q}$, $\mathcal{K}_{lin} \times \mathcal{K}_{dis}$, $\mathcal{K}_{lin} \times \mathbb{Q}$, and $\mathbb{Q} \times \mathcal{K}_{dis}$.

The negative result is the following.

Proposition 6.3 (2D-unaxiomatizability) *There are no finite axiomatizations for the (strongly) valid two-dimensional formulas over the classes $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, and $\mathbb{R} \times \mathbb{R}$.*

This proposition follows directly from Venema's proof that the valid formulas over the upper half two-dimensional semi-plane are not enumerable for $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, or $\mathbb{R} \times \mathbb{R}$, which in turn was based on Halpern and Shoham [14]. Since there are sound, complete, and decidable US-temporal logics over $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$ (cf. [17], Reynolds [18], Büchi [2], Burgess and Gurevich [3]), the general conclusion on full join is the following.

Theorem 6.4 (Full Join) *Completeness and decidability do not transfer in general through full join.*

It has to be noted that two-dimensional temporal logics seem to behave like modal logics in the following sense. We can see the result of the independent combination of US and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ as generating a "minimal" combination of the logics, i.e., one without any interference between the dimensions. The addition of extra axioms, inference rules, or an extra condition on its models has to be studied on its own, just as adding a new axiom to a modal logic or imposing a new property on its accessibility relation has to be analyzed on its own.

The full join method illustrates the conflict between the generality of a method and its ability to achieve the transfer of logical properties. We next restrict the join method so as to recover the transfer of logical properties.

7 Restricted join The fact that the transfer of logical properties fails for the join of two US-temporal logics does not mean that the join of any two temporal logic systems fails to achieve this transfer. We restrict the vertical logic system to a temporal logic $\bar{\mathsf{N}}\bar{\mathsf{P}}$ with operators $\bar{\bigcirc}$ for Next time and $\bar{\bullet}$ for Previous time; the formation rules for the formulas of $\bar{\mathsf{N}}\bar{\mathsf{P}}$ are the standard ones. This restriction of the $\bar{\mathsf{U}}\bar{\mathsf{S}}$-language for $\bar{\bigcirc}$ and $\bar{\bullet}$ can be defined in terms of $\overline{U}$ and $\overline{S}$, namely by

$$\bar{\bigcirc} A =_{def} \overline{U}(A, \bot)$$
$$\bar{\bullet} A =_{def} \overline{S}(A, \bot)$$

Not only is the expressivity of the language reduced this way, but also the underlying flow of time is now restricted to a discrete one; in fact, we concentrate our attention on integer-like flows of time.

Let $h : \mathbb{Z} \to \mathcal{P}$ be a temporal assignment over the integers so that the semantics of $\bar{\mathsf{N}}\bar{\mathsf{P}}$ over the integers is the usual one for atoms and boolean operators and

$$(\mathbb{Z}, <, h), t \models \bar{\bigcirc} A \quad \text{iff} \quad (\mathbb{Z}, <, h), t+1 \models A$$
$$(\mathbb{Z}, <, h), t \models \bar{\bullet} A \quad \text{iff} \quad (\mathbb{Z}, <, h), t-1 \models A.$$

An axiomatization for $\mathsf{NP}/\mathbb{Z}$ is given by the classical tautologies plus

NP1 $\bar{\bigcirc}\bar{\bullet} p \to p$

NP2 $\bar{\bigcirc}\neg p \leftrightarrow \neg\bar{\bigcirc} p$

NP3 $\bar{\bigcirc}(p \wedge q) \to \bar{\bigcirc} p \wedge \bar{\bigcirc} q$

NP4 The mirror image of **NP1–3** obtained by swapping $\bar{\bigcirc}$ and $\bar{\bullet}$.

The rules of inference are the usual Substitution, Modus Ponens, and Temporal Generalization (from A infer $\bar{\bigcirc} A$ and $\bar{\bullet} A$).

The converse of each axiom can be straightforwardly derived, so the formulas on both sides of the $\to$-connective are actually equivalent. It follows that every $\bar{\mathsf{N}}\bar{\mathsf{P}}$-formula can be transformed into an equivalent one by "pushing in" the temporal operators, e.g., by following the arrows of the axioms, and by "cancelling" the occurrences of $\bar{\bigcirc}$ and $\bar{\bullet}$ in a string of temporal operators, e.g., $\bar{\bigcirc}\bar{\bullet}\bar{\bullet}\bar{\bigcirc}\bar{\bullet} p$ is equivalent to $\bar{\bullet} p$. The resulting $\bar{\mathsf{N}}\bar{\mathsf{P}}$-*normal form* formula is a boolean combination of formulas of the form $\bar{\bigcirc}^k p$ and $\bar{\bullet}^l q$, where p and q are atoms, $k, l \in \mathbb{N}$ and $\bar{\bigcirc}^k$ is a sequence of $\bar{\bigcirc}$-symbols of size k, similarly for $\bar{\bullet}^l$; it is useful sometimes to consider k negative or 0, so we define $\bar{\bigcirc}^{-k} A = \bar{\bullet}^k A$ and $\bar{\bigcirc}^0 A = A$. As an example, the formula $\bar{\bigcirc}\bar{\bigcirc}(\bar{\bullet}\bar{\bullet}\bar{\bullet}(p \wedge q) \vee p)$ has normal form $(\bar{\bullet} p \wedge \bar{\bullet} q) \vee \bar{\bigcirc}\bar{\bigcirc} p$. The existence of such normal form gives us very simple proofs for completeness and decidability of $\bar{\mathsf{N}}\bar{\mathsf{P}}/\mathbb{Z}$ that we outline next.

For completeness, let Σ be a possibly infinite consistent set of $\bar{\mathsf{N}}\bar{\mathsf{P}}$-formulas and assume all formulas in the set are in the normal form. Σ can be seen as a consistent set of propositional formulas where each maximal subformula of the form $\bar{\bigcirc}^k p$ is understood as a new propositional atom, so let h_0 be a propositional valuation assigning every extended atom into $\{true, false\}$. For $n \in \mathbb{Z}$, let $h(n) = \{p \in \mathcal{P} \mid h_0(\bar{\bigcirc}^n p) = true\}$. Clearly $(\mathbb{Z}, <, h)$ is a model for the original set.

For decidability, let A be a formula of $\bar{\mathsf{N}}\bar{\mathsf{P}}$ and let A^* be its normal form; clearly there exists an algorithm to transform A into A^*. By considering subformulas of the form $\bar{\bigcirc}^k p$ as new atoms, k possibly negative, we apply any decision procedure for propositional logic to A^*. A is a $\bar{\mathsf{N}}\bar{\mathsf{P}}$-valid formula iff A^* is a propositional tautology.

Definition 7.1 The *restricted join* of temporal logic systems $\mathsf{US}/\mathcal{K}$ and $\bar{\mathsf{N}}\bar{\mathsf{P}}/\mathbb{Z}$ is the two-dimensional temporal logic system $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$ given by:

- the fully combined language of US and $\bar{\mathsf{N}}\bar{\mathsf{P}}$;
- the two-dimensional plane model over $\mathcal{K} \times \mathbb{Z}$, equipped with the broadly two-dimensional semantics;
- the union of the axioms of $\mathsf{US}/\mathcal{K}$ and $\bar{\mathsf{N}}\bar{\mathsf{P}}/\mathbb{Z}$ plus the join axioms

  $$\bar{\bigcirc} U(p, q) \to U(\bar{\bigcirc} p, \bar{\bigcirc} q)$$
  $$\bar{\bigcirc} S(p, q) \to S(\bar{\bigcirc} p, \bar{\bigcirc} q)$$

  plus their duals obtained by swapping $\bar{\bigcirc}$ with $\bar{\bullet}$; the inference rules are just the union of the inference rules of both component systems.

What has therefore been restricted in the interlacing process is the expressivity of the language over the vertical dimension, which also restricted the underlying flow of time to a discrete one. The following gives us a normal form for $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$.

Lemma 7.2 *Let A be a formula of $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$. There exists a normal form formula A^* equivalent to A such that all the occurrences of $\bar{\bigcirc}$ and $\bar{\bullet}$ in it are in the form $\bar{\bigcirc}^k p$ and $\bar{\bullet}^l q$, where p and q are atoms.*

Proof: First we show that the converse of the join axioms are theorems too. For that, note that U and S respect the *congruence property*, i.e., if $A \leftrightarrow C$ and $B \leftrightarrow D$ then $U(A, B) \leftrightarrow U(C, D)$ and $S(A, B) \leftrightarrow S(C, D)$. Also note that

$$\textbf{(equiv)} \quad \vdash (p \leftrightarrow \bar{\bigcirc}\bar{\bullet} p) \text{ and } \vdash (p \leftrightarrow \bar{\bullet}\bar{\bigcirc} p).$$

The transitivity of the $\to$-operator connects the steps in the proof of the formula $U(\bar{\bigcirc} p, \bar{\bigcirc} q) \to \bar{\bigcirc} U(p, q)$ below:

$$\begin{array}{rll} U(\bar{\bigcirc} p, \bar{\bigcirc} q) & \to \bar{\bigcirc}\bar{\bullet} U(\bar{\bigcirc} p, \bar{\bigcirc} q) & \text{by } \textbf{equiv} \\ & \to \bar{\bigcirc} U(\bar{\bullet}\bar{\bigcirc} p, \bar{\bullet}\bar{\bigcirc} q) & \text{by join axiom} \\ & \to \bar{\bigcirc} U(p, q) & \text{by } \textbf{equiv} \text{ and congruence} \end{array}$$

It follows that $U(\bar{\bigcirc} p, \bar{\bigcirc} q) \leftrightarrow \bar{\bigcirc} U(p, q)$. It is completely analogous to show the converse of other join axioms, so we omit the details.

Given A in the language of $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$, the equivalence between both sides of the join axioms allows for "pushing in" the vertical operators $\bar{\bigcirc}$ and $\bar{\bullet}$, so a simple induction on the number of nested temporal operators in A shows an algorithmic way to generate an equivalent formula A^* in the desired normal form. □

Theorem 7.3 (Completeness via restricted join) *Let US be a logic system complete over the class $\mathcal{K} \subseteq \mathcal{K}_{lin}$. Then the two-dimensional system $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$ is complete over $\mathcal{K} \times \mathbb{Z}$.*

Proof: Consider a $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$-consistent formula A and assume it is in the normal form. So we can see A as a US-formula over the extended set of atoms $\bar{\bigcirc}^k$, k possibly negative or 0. From the completeness of $\mathsf{US}/\mathcal{K}$ there exists a one-dimensional model

$(T, <, h_{US})$ for A at a point $o \in T$, where $(T, <) \in \mathcal{K}$. Define the two-dimensional assignment

$$h(k, t) = \{p \in \mathcal{P} \mid \overline{\bigcirc}^k p \in h_{US}(t)\}.$$

Clearly, $(T, <, \mathbb{Z}, <_{\mathbb{Z}}, h)$ is a two-dimensional plane $\mathsf{US}\times\bar{\mathsf{N}}\bar{\mathsf{P}}$-model for A at $(o, 0)$. □

Corollary 7.4 *If* $\mathsf{US}/\mathcal{K}$ *is strongly complete, so is* $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}/\ \mathcal{K} \times \mathbb{Z}$*.*

Theorem 7.5 (Decidability via restricted join) *If the logic system* US *is decidable over* $\mathcal{K}$*, so is* $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$ *over* $\mathcal{K} \times \mathbb{Z}$*.*

Proof: The argument of the proof is the same as that of the decidability of NP. All we have to do is note that there exists an algorithmic way to convert a combined two-dimensional formula into its normal form, so it can be seen as a US-formula, and we can apply the US-decision procedure to it. □

So by restricting the expressivity and the underlying class of flows of time, we can obtain the transfer of the basic logical properties via restricted join. It should not be difficult to extend these results to $\mathbb{N}$ instead of $\mathbb{Z}$, although we do not explore this possibility here.

It is also worth noting that the restricted join method answers a conjecture posed by Venema [22] on the existence of some expressively limited two-dimensional temporal logic over $\mathbb{Z} \times \mathbb{Z}$ that was "well behaved" in the sense of having the completeness and decidability properties.

8 The two-dimensional diagonal We now study some properties of the diagonal in two-dimensional plane models. The diagonal is a privileged line in the two-dimensional model intended to represent the sequence of time points we call "now," i.e., the time points on which an historical observer is expected to traverse. The observer is, therefore, on the diagonal when he or she poses a query (i.e., evaluates the truth value of a formula) on a two-dimensional model. The diagonal is illustrated in Figure 8.

So let δ be a special atom and consider the formulas:

D1 $\Diamond\delta \wedge \overline{\Diamond}\delta$

D2 $\delta \to (G\neg\delta \wedge H\neg\delta \wedge \overline{G}\neg\delta \wedge \overline{H}\neg\delta)$

D3 $\delta \to (\overline{H}G\neg\delta \wedge \overline{G}H\neg\delta)$

Let $Diag = \Box\overline{\Box}(\mathbf{D1} \wedge \mathbf{D2} \wedge \mathbf{D3})$. The intuition behind *Diag* is the following. **D1** implies that the two-dimensional diagonal can always be reached in both vertical and horizontal directions; **D2** implies that there are no two diagonal points on the same horizontal line and on the same vertical line, and **D3** implies that the diagonal goes in the direction SW–NE. We say that *Diag* characterizes a two-dimensional diagonal in the following sense.

Proposition 8.1 *Let* $\mathcal{M} = (T, <, \overline{T}, \overline{<}, g)$ *be a full two-dimensional model over* $\mathcal{K} \times \overline{\mathcal{K}}$*,* $\mathcal{K}, \overline{\mathcal{K}} \subseteq \mathcal{K}_{lin}$*, and let* δ *be a propositional letter. Then the following are equivalent.*

(a) $\mathcal{M}, t, x \models Diag$*, for some* $t \in T$ *and* $x \in \overline{T}$*.*

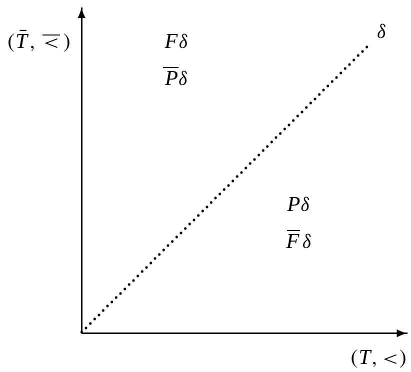

Figure 8: The two-dimensional diagonal

(b) $\mathcal{M}, t, x \models$ *Diag, for all* $t \in T$ *and* $x \in \overline{T}$.
(c) *There exists an isomorphism* $i : T \to \overline{T}$ *such that* $\mathcal{M}, t, x \models \delta$ *iff* $x = i(t)$.

Proof: It is straightforward to show that (a) $\Longleftrightarrow$ (b) and (c) $\Longrightarrow$ (a); we show only (b) $\Longrightarrow$ (c). So assume that $\mathcal{M}, t, x \models Diag$, for all $t \in T$ and $x \in \overline{T}$. Define

$$i = \{(t, x) \in T \times \overline{T} \mid \mathcal{M}, t, x \models \delta\}.$$

All we have to show is that i is an isomorphism.

- i, i^{-1} are functions such that $dom(i) = T$ and $dom(i^{-1}) = \overline{T}$. Suppose that both (t, x_1) and (t, x_2) are in i; then $\mathcal{M}, t, x_1 \models \delta$ and $\mathcal{M}, t, x_2 \models \delta$. By linearity of $\overline{T}$, $x_1 = x_2$, $x_1 \overline{<} x_2$ or $x_2 \overline{<} x_1$, but **D2** eliminates the latter two; **D1** gives us that $dom(i) = T$. Similarly, the linearity of T and **D2** gives us that i^{-1} is a function and **D1** gives us that $dom(i^{-1}) = \overline{T}$.
- $i(t) = x$ iff $i^{-1}(x) = t$ follows directly from the definition. So i is a bijection.
- i preserves ordering. Suppose $t_1 < t_2$; by the linearity of $\overline{T}$ we have three possibilities:
  - $i(t_1) = i(t_2)$ contradicts i is a bijection.
  - $i(t_2) \overline{<} i(t_1)$ contradicts **D3**.
  - $i(t_1) \overline{<} i(t_2)$ is the only possible option.

Therefore i is an isomorphism, which proves the result. □

This result shows that by adding **D1**–**D3** to the axiomatization over the two-dimensional plane $\mathcal{K}_{lin} \times \mathcal{K}_{lin}$ of Section 6 gives us completeness over the class of models of the form $(T, <, T, <, g)$, where $(T, <) \in \mathcal{K}_{lin}$. It follows from [14], however, that such a logic system is undecidable.

The diagonal is interpreted as the sequence of time points we call “now.” The diagonal divides the two-dimensional plane in two semi-planes. The semi-plane that is to the (horizontal) left of the diagonal is “the past,” and the formula $F\delta$ holds over all points of this semi-plane. Similarly, the semi-plane that is to the (horizontal) right of the diagonal is “the future,” and the formula $P\delta$ holds over all points of this semi-plane. Figure 8 puts this fact in evidence. If we assume that *Diag* holds over $\mathcal{M}$ such that i is the isomorphism defined in Lemma 8.1, $t < s$ iff $i(t) \overline{<} i(s)$, then

$$\begin{array}{lll} \mathcal{M}, t, x \models F\delta & \text{iff} & \text{exists } s > t \text{ such that } \mathcal{M}, s, x \models \delta \text{ and } i(s) = x \\ & \text{iff} & \text{exists } y = i(t) \overline{<} x \text{ such that } \mathcal{M}, t, y \models \delta \\ & \text{iff} & \mathcal{M}, t, x \models \overline{P}\delta. \end{array}$$

Similarly, it can be shown that:

$$\mathcal{M}, t, x \models P\delta \quad \text{iff} \quad \mathcal{M}, t, x \models \overline{F}\delta.$$

It follows that the following formula is valid for $\mathsf{US} \times \bar{\mathsf{U}}\bar{\mathsf{S}}$ over $\mathcal{K}_{lin} \times \mathcal{K}_{lin}$:

$$Diag \to \big((F\delta \leftrightarrow \overline{P}\delta) \wedge (P\delta \leftrightarrow \overline{F}\delta)\big).$$

As a consequence, $\overline{P}\delta$ holds over all points of the "past" semi-plane and $\overline{F}\delta$ holds over all points of the "future" semi-plane, as is indicated in Figure 8.

The formula *Diag* is in the language of $\mathsf{US} \times \bar{\mathsf{U}}\bar{\mathsf{S}}$but not in the language of $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$, for *Diag* contains the vertical temporal operators $\overline{G}$, $\overline{H}$, $\overline{\Box}$ and $\overline{\Diamond}$. To characterize a two-dimensional diagonal in $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$ we do the following. We say that a formula A *holds over* or *is valid over* a two-dimensional model $\mathcal{M}$ if for every $t \in T$ and every $x \in \overline{T}$, it is the case that $\mathcal{M}, t, x \models A$. Consider the formulas

d1 $\Diamond\delta$
d1 $\delta \to (G\neg\delta \wedge H\neg\delta)$
d1 $\delta \leftrightarrow \bar{\bigcirc}\bigcirc\delta$

where δ is a proposition. Those formulas are all in the language of $\mathsf{US} \times \bar{\mathsf{N}}\bar{\mathsf{P}}$ for *Diag* (so also in the language of $\mathsf{US} \times \bar{\mathsf{U}}\bar{\mathsf{S}}$), and they can characterize the two-dimensional diagonal due to the following property.

Proposition 8.2 *Let $\mathcal{M}$ be a two-dimensional plane model over $\mathbb{Z} \times \mathbb{Z}$. Then the formula* **D1** $\wedge$ **D2** $\wedge$ **D3** *holds over $\mathcal{M}$ iff* **d1** $\wedge$ **d2** $\wedge$ **d3** *holds over $\mathcal{M}$.*

Proof: From Proposition 8.1 we know that **D1** $\wedge$ **D2** $\wedge$ **D3** holds over $\mathcal{M}$ iff the relation i defined as

$$i = \{(t, x) \in \mathbb{Z} \times \mathbb{Z} \mid \mathcal{M}, t, x \models \delta\}$$

is an isomorphism in $\mathbb{Z}$. So all we have to do is to prove that i as defined above is an isomorphism iff **d1** $\wedge$ **d2** $\wedge$ **d3** holds over $\mathcal{M}$. The *only if* part is a straightforward verification that for all x and t in $\mathbb{Z}$, $\mathcal{M}, t, x \models$ **d1** $\wedge$ **d2** $\wedge$ **d3**.

For the *if* part, assume **d1** $\wedge$ **d2** $\wedge$ **d3** holds over $\mathcal{M}$. Then:

(a) **d1** gives us that for every x there exists a t such that $\mathcal{M}, t, x \models \delta$;
(b) **d2** gives us that for every $x, t, t', t \neq t'$, $\mathcal{M}, t, x \models \delta$ implies $\mathcal{M}, t', x \not\models \delta$;
(c) **d3** give us that for every x, t, $\mathcal{M}, t, x \models \delta$ iff $\mathcal{M}, t+1, x+1 \models \delta$ iff for every $n \in \mathbb{Z}$, $\mathcal{M}, t+n, x+n \models \delta$.

The first two items give us that $i^{-1} : \mathbb{Z} \to \mathbb{Z}$ is a function. To show that i is also a function, suppose that $(t, x_1), (t, x_2) \in i$. By linearity of $\mathbb{Z}$, it follows that either $x_1 < x_2$ or $x_2 < x_1$ or $x_1 = x_2$. Let $x_1 - x_2$ = m; then, by the third item above, $(t + m, x_2 + m = x_1) \in i$, so $t = (t + m)$ and $m = 0$. It follows that $x_1 = x_2$, so $i : \mathbb{Z} \to \mathbb{Z}$ is a function. Directly by the definition of i, it follows that i is a bijection.

By the third item above, if $i(t_1) = x_1$ and $i(t_2) = x_2$, then $t_1 - t_2 = x_1 - x_2$. It follows that i is order preserving and hence an isomorphism, which finishes the proof. □

It would be desirable to generalize the idea of a diagonal as the sequence of "now" moments to any pair of flows of time that are not necessarily isomorphic. For that, we would have to create an order between the points of the two flows, i.e., we would have to merge the flows.

So let $(T, <)$ and $(\overline{T}, \overline{<})$ be two flows of time such that T and $\overline{T}$ are disjoint. Then there always exists a flow $(T', <')$ and a mapping $f : T \cup \overline{T} \to T'$ such that f is one-to-one and order preserving. The *f-merge of $(T, <)$ and $(\overline{T}, \overline{<})$* is the flow of time consisting of the image of f ordered by the restriction of $<'$ to the image of f. An example of an f-merge is shown in Figure 9, where $f(y)$ is made equal, via merge, to $f(\bar{x})$ and on the merged flow the order is preserved, i.e., originally $x < y$ and $\bar{x} \overline{<} \bar{y}$ and on the f-merged flow $f(x) <' f(y) = f(\bar{x}) <' f(\bar{y})$.

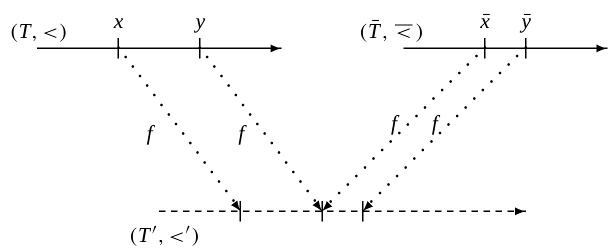

Figure 9: The f-merge

We can then construct a two-dimensional model with two copies of the f-merged flow, in which we can define a diagonal over $(T', <') \times (T', <')$ as shown in Figure 10. Another particularly interesting situation arises when the f-merged flow $(T', <')$ is identical to one of the component flows, e.g., $(\overline{T}, \overline{<})$, so that f is an *embedding* of $(T, <)$ into $(\overline{T}, \overline{<})$. In this case, the flow $(T, <)$ could be viewed as a more "abstract" representation of $(\overline{T}, \overline{<})$ wherein several details, i.e., pieces of information, points in time, are ignored.

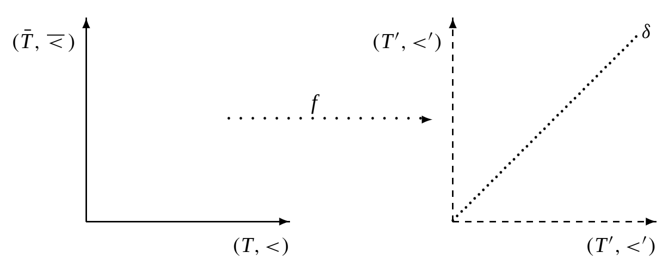

Figure 10: The diagonal of two distinct flows

The f-merge construction serves as motivation for another method of combining two one-dimensional temporal logics, this time generating another one-dimensional logic. This could be achieved over the class of all f-merges of its two-component flows of time or subclasses of it. We could then study the transfer of logical properties in the same way as we have done in this and the previous section, but those matters remain beyond the scope of this paper.

9 Conclusion This paper dealt with the combination of two logic systems in order to obtain a new logic system. The issues were:

- Several methods of combination of two logic systems were presented. Each combination involved at least one temporal logic system. Each method had a particular discipline for combining the language, the semantics, and the inference system of two logic systems. Each combination generated a single logic system.
- The study of the transfer of logical properties from the component systems into their combined form has been the major point in the analysis of combination methods. The basic logical properties whose transfer was analyzed were soundness, completeness, and decidability; for some combination methods, the transfer of other properties, such as conservativeness and the compactness property (in the form of strong completeness), was also investigated.
- The investigation of four basic methods has been accomplished. The temporalization method and the independent combination method were shown to transfer all basic properties, although they do not generate a sufficiently expressive system to be called fully two-dimensional. The full join method does generate a fully two-dimensional temporal system, but in many cases it fails to transfer even the completeness property. As a compromise, it was shown that a restricted join method, although generating two-dimensional temporal logic systems that were not as expressive and generic as the fully interlaced one, accomplishes the transfer of all basic logical properties.

Another contribution of our analysis was to answer a question raised by Venema [22] on the existence of a fragment of the two-dimensional plane temporal logic that, in his own words, was "better behaved" than the two-dimensional plane system with respect to completeness and decidability properties. We have shown that the two-dimensional temporal logic systems obtained by restricted join are an example of such fragments.

Another question raised by Venema in that work remains open, namely, whether there exists a complete axiomatization over the two-dimensional model using only canonical inference rules, i.e., without using the special inference rules **IR1** and **IR2**. This problem seems to be a very hard one. Nevertheless we succeeded in extending Venema's completeness result, which originally holds for only two-dimensional flows built from two identical one-dimensional flows, to any two-dimensional flow built from any flow in the classes $\mathcal{K}_{lin}$, $\mathcal{K}_{dis}$, $\mathcal{K}_{dense}$, and $\mathbb{Q}$.

9.1 Comparisons, extensions, and further work With respect to combination of logics, the works found in the literature that most closely approximate ours in spirit and aims are Kracht and Wolter [16] and an unpublished paper of Fine and Schurz. Both works concentrated on monomodal logics and investigated the transfer of logical properties for only the method we called here independent combination. However, their work investigated several paths that suggest that further work may be done in our studies. First, they analyzed the transfer of many other properties from two logic systems to its combined form, e.g., finite model property and interpolation. Second, both works did not concentrate only on linear systems, and they were able to extend their results to any class of underlying Kripke frames. Third, Fine and Schurz's work generalized the independent combination method to more than two monomodal logics.

Those two papers cited above therefore suggest several extensions to our work. Note, however, that the temporalization method was easily shown to be extensible to many temporal logic systems in Example 2.4. The focus on linear flows of time was due to database applications of two-dimensional temporal logics as in [6] and [8], but we believe that this restriction may be lifted without damaging the transfer results of the temporalization and independent combination methods. These have to be further investigated, and the transfer of any other logical property has to be analyzed on its own.

The generalization of combination methods other than the independent combination method to modal logics is another area for further work. As noted in [6], the temporalization process is directly extendable to monomodal logics. It may even be the case that, for monomodal logics, the full join method achieves transfer of completeness over several classes of fully two-dimensional Kripke frames using only canonical inference rules, as suggested by the results in [19].

The complexity class of the decision problem for the combined logic is another interesting subject for study. For the independent combination of monomodal logics, such a study was done by Spaan [20], and the conclusion was that the satisfiability problem of an independently combined logic is either reducible to that of one of the component logics, or it is PSPACE-hard or it is in NP. We believe a similar result can be obtained for the temporalization and the independent combination of temporal logics, although the details have not yet been worked out. The complexity of the full and restricted join methods still have to be studied.

All the systems dealt with in this paper were extensions of classical logic. It is possible that the temporalization process preserves its transfer properties even when the underlying system is not an extension of classical logic. What if the external temporal logic is nonclassical itself? The same question applies to other combination methods. Do they transfer logical properties when one or both of the combined temporal of modal logics is not classical? Gabbay [11] has recently posed that question in a very generic framework involving Labelled Deductive Systems (LDS) and found that in order to obtain the transfer of completeness we do not need the full power of classical logic but only some weaker form of monotonicity. He has also developed general methods of combination called *fibring* that depend on the choice of a fibring function. A fibring function maps the truth value of atoms in one logic's semantics to the semantics of formulas in another logic's semantics. Gabbay's *dovetailing* process, obtained with a certain class of fibring functions, is similar to the independent combination method extended to logics respecting those weaker conditions of monotonicity. More work on this area is needed to clarify exactly how fibring is related to existing combination methods.

There are also other possible types of combinations of one-dimensional temporal logics that may be explored. As pointed out in Section 8, two linear flows of time can be merged into another one; the question is then how to combine two one-dimensional temporal logics into another one-dimensional temporal logic over the merged flow.

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Departamento de Cienca da Computação
Instituto de Matemática e Estatística
Universidade de São Paulo
P. O. Box 66281
05389–970 Brazil
email: mfinger@ime.usp.br

Department of Computing
Imperial College
180 Queen's Gate
London SW7 2BZ
UK
email: dg@doc.ic.ac.uk