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Finite Basis Problems and Results for Quasivarieties*

Abstract. Let \mathcal{K} be a finite collection of finite algebras of finite signature such that $SP(\mathcal{K})$ has meet semi-distributive congruence lattices. We prove that there exists a finite collection \mathcal{K}_1 of finite algebras of the same signature, $\mathcal{K}_1 \supseteq \mathcal{K}$, such that $SP(\mathcal{K}_1)$ is finitely axiomatizable. We show also that if $HS(\mathcal{K}) \subseteq SP(\mathcal{K})$, then $SP(\mathcal{K})$ is finitely axiomatizable. We offer new proofs of two important finite basis theorems of D. Pigozzi and R. Willard. Our actual results are somewhat more general than this abstract indicates.

Keywords: quasivarieties, finite axiomatizability, pseudo-complemented congruence lattices, Willard terms.

1. Introduction

One of the deepest classical results in universal algebra is K. Baker's theorem stating that every finite algebra belonging to a congruence distributive variety of finite signature has a finite basis of equations. (This appeared in K. Baker [2], but was proved before 1970.) This means that for such an algebra \mathbf{A} , the variety generated by \mathbf{A} , i.e., the class $HSP(\mathbf{A})$, is finitely axiomatizable in first-order logic. In this paper, when we assert that a class \mathcal{K} is finitely axiomatizable, we always intend this to have the same meaning as in K. Baker's theorem, namely, that there is a sentence ϕ of first-order logic (in the appropriate language) such that $\mathcal{K} = \text{Mod}(\phi)$ (the class of models of ϕ).

Throughout this paper, we interpret the class-operators H, S, P in the inclusive sense, so that for example, $S(\mathcal{K})$ denotes the class of all algebras isomorphic to a subalgebra of some algebra in \mathcal{K} .

K. Baker's theorem has been significantly extended in three different directions. In 1987, R. McKenzie [21] proved that $HSP(\mathbf{A})$ is finitely axiomatizable whenever \mathbf{A} is a finite algebra of finite signature and $HSP(\mathbf{A})$ is congruence modular and residually small. About the same time, D. Pigozzi

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[30] proved that $SP(\mathbf{A})$ is finitely axiomatizable whenever \mathbf{A} is a finite algebra of finite signature and $\mathcal{K} = SP(\mathbf{A})$ is relatively distributive—i.e., the lattice $\mathbf{Con}_{\mathcal{K}}\mathbf{B}$ of all congruences θ of \mathbf{B} such that $\mathbf{B}/\theta \in \mathcal{K}$ is a distributive lattice, for every $\mathbf{B} \in \mathcal{K}$. Around 1998, R. Willard [36] proved that $HSP(\mathbf{A})$ is finitely axiomatizable whenever \mathbf{A} is a finite algebra of finite signature and $HSP(\mathbf{A})$ is congruence meet semi-distributive and has a finite residual bound. D. Pigozzi's finite basis theorem will be extended in this paper, where we prove (Corollary 6.4) that every finitely generated quasivariety of finite signature which has pseudo-complemented congruence lattices and satisfies the weak extension property is finitely axiomatizable. This result and its proof should be compared with the extension of D. Pigozzi's result in J. Czelakowski, W. Dziobiak [8], and the constructive proof of it in W. Dziobiak [10]. We extend R. Willard's finite basis theorem with Corollary 5.7, which asserts that $SP(\mathcal{K})$ is finitely axiomatizable whenever \mathcal{K} is a finite set of finite algebras of a finite signature, such that $SP(\mathcal{K})$ has pseudo-complemented congruence lattices and $HS(\mathcal{K}) \subseteq SP(\mathcal{K})$.

It has long been conjectured that some of the principal finite basis theorems might admit common generalizations. D. Pigozzi pointed out that a natural conjecture, extending both his finite basis theorem and R. McKenzie's finite basis theorem for congruence modular, residually small varieties, would be that every relatively modular quasivariety generated by a finite algebra of finite signature is finitely axiomatizable. K. Kearnes, R. McKenzie [14] developed a commutator theory for relative congruences in relatively modular quasivarieties, with the expectation that it can be used to prove D. Pigozzi's conjecture; but the truth of this conjecture remains an open question.

Our Corollaries 6.4 and 5.7, extending D. Pigozzi's and R. Willard's finite basis theorems, can be easily unified into one theorem (Theorem 7.1) which is stated but not proved here. The reader will judge whether Theorem 7.1 truly unifies or explains these theorems.

Each of the six finite basis theorems mentioned above can be viewed as asserting the finite axiomatizability of $SP(\mathcal{K})$, where \mathcal{K} is a finite set of finite algebras and some appropriate conditions, including properties of the congruence lattices or the relative congruence lattices, are imposed. In particular, every finitely generated variety whose finite axiomatizability follows from one of these theorems is actually finitely generated as a quasivariety, i.e., it has a finite residual bound. In this connection, R. E. Park has conjectured that $HSP(\mathbf{A})$ is finitely axiomatizable whenever \mathbf{A} is a finite algebra of finite signature and $HSP(\mathbf{A})$ has a finite residual bound. Park's conjecture is almost thirty years old, and is still un-resolved.

The best classical results about groups, rings and semigroups do not fall under this rubric. For example, $HSP(\mathbf{A})$ is finitely axiomatizable if \mathbf{A} is any finite group (S. Oates, M. B. Powell [26]), any finite associative ring (R. L. Kruse [15], I. V. L'vov [16]), any commutative semigroup (P. Perkins [28]) or idempotent semigroup (J. A. Gerhard [11]).

All of the results we have described would become uninteresting if in fact (as one might naively expect) every finite algebra of finite signature turned out to have a finite basis of equations and a finite basis of quasi-equations. This whole direction of research began with a series of surprising discoveries of non-finitely-based finite algebras. R. Lyndon's 7-element groupoid was the first example of a finite algebra \mathbf{A} with $HSP(\mathbf{A})$ non-finitely axiomatizable (R. Lyndon [18], 1951). Additional examples were discovered by V. V. Vishin [34], V. L. Murskiĭ [25], and P. Perkins [28]. Their algebras are groupoids (having precisely one binary operation) with 4, 3, and 6 elements, respectively. P. Perkins' example is a semigroup. V. P. Belkin [5] found finite lattices with non-finitely based quasi-equations. S. V. Polin [31] found a finite non-associative ring that has no finite basis of equations. R. Bryant [7] found a finite algebra $\langle G, \circ, g \rangle$, consisting of a group $\langle G, \circ \rangle$ with a distinguished constant $g \in G$, which has no finite basis of equations. These two last results make it seem very unlikely that there can exist any general universal algebraic theorem (with the flavor, say of K. Baker's theorem) which would explain the Oates-Powell theorem for finite groups.

A. Yu. Ol'shanskiĭ [27] proved that if \mathbf{G} is a finite group then $SP(\mathbf{G})$ is finitely axiomatizable iff $HSP(\mathbf{G}) = SP(\mathcal{K})$ for some finite set \mathcal{K} of finite groups. V. P. Belkin [5] proved the analogous result for finite associative rings. A plausible extension of these results to general algebras would be V. A. Gorbunov's conjecture that if a finite algebra belongs to a congruence modular variety and has a nilpotent subalgebra that is not Abelian, then its quasi-equational theory is not finitely axiomatizable. I. P. Bestsenyi [6] has proved this conjecture under certain additional hypotheses.

A finite algebra \mathbf{A} of finite signature is called inherently non-finitely-based for equations (quasi-equations) if \mathbf{A} belongs to no locally finite, finitely axiomatizable, variety (quasivariety). Several of the individual finite algebras mentioned above turned out to be inherently non-finitely-based for equations. J. Lawrence, R. Willard [17] found the first example of a finite algebra that is inherently non-finitely-based for quasi-equations. (It is an 18-element semigroup with one additional unary operation.) The Oates-Powell Theorem implies that no finite group is inherently non-finitely-based for quasi-equations. S. Margolis and M. V. Sapir [19] obtained the deep result that no finite semigroup is inherently non-finitely-based for quasi-equations.

Our Corollary 5.5 below (combined with Theorem 4.1) implies that any finite algebra that is inherently non-finitely-based for quasi-equations must have a subalgebra that possesses a non-trivial Abelian congruence. This result implies that no such algebra can be found belonging to any congruence meet semi-distributive (or relatively congruence meet semi-distributive) quasivariety.

Naturally, people are searching for ever more general finite basis results, and for general principles that could unify some of the known results and make them more understandable. For example, K. Baker and Ju Wang [3] have introduced a new concept of “definable principal subcongruences”. They prove that a variety of finite signature possessing definable principal subcongruences is finitely axiomatizable iff its class of subdirectly irreducible algebras is finitely axiomatizable. They prove that every finitely generated congruence distributive variety of finite signature possesses definable principal subcongruences. These results yield a remarkably direct and easy proof of K. Baker’s finite basis theorem from 1970. G. McNulty and Ju Wang [24] have proved that if \mathbf{G} is any finite group, then the class of all subdirectly irreducible groups in $HSP(\mathbf{G})$ has definable principal subcongruences; and they use this fact to establish that this class of subdirectly irreducible groups is finitely axiomatizable. They are searching for a universal algebraic explanation of the Oates-Powell theorem that could place it in a more general context. (When contemplating the examples of R. Park and S. V. Polin mentioned earlier, it seems a faint hope that such a universal algebraic explanation could exist.)

As all these results slowly accumulated, it became more and more clear that the discovery of which finite algebras are finitely based, for equations or for quasi-equations, is a very subtle problem. Might it be possible to prove, as Alfred Tarski had suggested in 1966, that there is no algorithm for deciding if a finite algebra has a finite basis? In 1993, the second author did just that. R. McKenzie [22], [23] constructs for every Turing machine \mathcal{T} , two finite algebras $\mathbf{A}(\mathcal{T})$ and $\mathbf{F}(\mathcal{T})$. It is proved that if \mathcal{T} halts then $HSP(\mathbf{F}(\mathcal{T}))$ is finitely axiomatizable and $HSP(\mathbf{A}(\mathcal{T}))$ has a finite residual bound, while if \mathcal{T} does not halt then neither of $HSP(\mathbf{F}(\mathcal{T}))$, $HSP(\mathbf{A}(\mathcal{T}))$ is included in any finitely axiomatizable locally finite variety, and the latter variety has an infinite subdirectly irreducible algebra. This showed (via standard results from logic) that for each of the following properties of a finite algebra \mathbf{A} , there can be no algorithm (in the sense of a recursive function) to determine correctly whether any finite algebra has the property: \mathbf{A} has a finite equational base, \mathbf{A} is equationally inherently non-finitely-based, $HSP(\mathbf{A})$ is residually finite, $HSP(\mathbf{A})$ has a finite residual bound.

All of these undecidability results were extended to groupoids through a result of R. McKenzie [20], and to three-element algebras in unpublished work of G. McNulty and R. Willard.

R. Willard [35] found that $\mathbf{A}(\mathcal{T})$ possesses the same properties stated for $\mathbf{F}(\mathcal{T})$, and thereby obtained a considerably shorter proof of the undecidability of the equational finite basis property. A little later, he proved his finite basis theorem for congruence meet semi-distributive varieties, one consequence of which is that the undecidability of the equational finite basis property now admitted a really short and almost painless proof. (The algebras $\mathbf{A}(\mathcal{T})$ of R. McKenzie [22] all have semilattices as reducts, thus generate congruence meet semi-distributive varieties.)

The proof that the equational finite basis property is undecidable has not killed the interest in finite basis problems, but it has generated an interest in other open problems of decidability. We obviously have yet a lot to learn. For instance, it is not known if there is an algorithm to determine if a finite semigroup \mathbf{S} has a finite basis of equations, or of quasi-equations. The approach of R. McKenzie [22], [23], if it could be used to address this problem, would apparently require major modifications, since, for example, it is known (M. V. Sapir [32]) that there is an algorithm to determine if \mathbf{S} is equationally inherently non-finitely-based, and no finite semigroup is inherently non-finitely-based for quasi-equations (S. Margolis, M. V. Sapir [19]).

It is not known if there is an algorithm to determine if an arbitrary finite algebra has a finite basis of quasi-equations. This problem has been attributed to M. V. Sapir. The expectation is certainly that there can be no such algorithm, but no-one has been able to prove it. This paper, written in August 2003, is really the outcome of extended (and unproductive) efforts over the past ten years to explore the possibility that some simple modification of the method of R. McKenzie [22] could be used to attack this decision problem for quasi-equations. A major breakthrough pointing to the results presented here came in July 2002 when the first author proved versions of Corollaries 5.6 and 5.7 for algebras that are semilattices with operators. J. Ježek, M. Maroti, R. McKenzie [13] is also a product of those efforts; it develops finite basis theorems for flat semilattices with operators satisfying some conditions.

2. Quasivarieties with Willard terms

By a quasivariety, we mean of course a class \mathcal{K} of algebras of one signature such that $S(\mathcal{K}) = P(\mathcal{K}) = P_u(\mathcal{K}) = \mathcal{K}$ where $P_u(\mathcal{K})$ denotes the class of

all algebras isomorphic to an ultraproduct of a system of algebras from \mathcal{K} . Since $P(\mathcal{K})$ contains a product of an empty system of algebras, a quasivariety contains the one-element algebras of its signature.

We shall now define nine congruence properties for quasivarieties, which will play some role in our study. These properties are all weakened forms of congruence-distributivity, or have an interesting interaction with some weak form of congruence-distributivity. The two properties most central to our work are the property of having pseudo-complemented congruence lattices and the weak extension property. We shall see that a quasivariety has pseudo-complemented congruence lattices if and only if it has Willard terms, and that a locally finite quasivariety has pseudo-complemented congruence lattices if and only if its algebras have no non-trivial Abelian congruences.

For any quasivariety \mathcal{K} and algebra $\mathbf{A} \in \mathcal{K}$, we have the congruence lattice, $\mathbf{Con} \mathbf{A}$, and the relative congruence lattice $\mathbf{Con}_{\mathcal{K}} \mathbf{A}$. The latter is composed of the lattice-ordered set of all congruences $\alpha \in \mathbf{Con} \mathbf{A}$ with $\mathbf{A}/\alpha \in \mathcal{K}$. There is a map $' : \mathbf{Con} \mathbf{A} \rightarrow \mathbf{Con}_{\mathcal{K}} \mathbf{A}$ defined by $\alpha' = \bigcap \{ \gamma \in \mathbf{Con}_{\mathcal{K}} \mathbf{A} : \alpha \leq \gamma \}$. The lattice operations in $\mathbf{Con} \mathbf{A}$ are denoted $\alpha \wedge \beta$ and $\alpha \vee \beta$. The lattice operations in $\mathbf{Con}_{\mathcal{K}} \mathbf{A}$ are $\alpha \wedge \beta = \alpha \cap \beta$ and $\alpha \vee^{\mathcal{K}} \beta = (\alpha \vee \beta)'$. The least and largest elements of these lattices are the same, denoted 0_A and 1_A . They are the identity relation and the universal relation over A .

Our nine properties are (where \mathcal{K} is any quasivariety):

(CD) We write $\mathcal{K} \models \text{CD}$ to denote that for all $\mathbf{A} \in \mathcal{K}$, $\mathbf{Con} \mathbf{A}$ is a distributive lattice.

(SD(\wedge)) We write $\mathcal{K} \models \text{SD}(\wedge)$ to denote that for all $\mathbf{A} \in \mathcal{K}$, $\mathbf{Con} \mathbf{A}$ is a meet semi-distributive lattice, i.e., for all congruences ϕ, θ, ψ of \mathbf{A} , $\phi \wedge \theta = \phi \wedge \psi$ implies $\phi \wedge \theta = \phi \wedge (\theta \vee \psi)$.

(PCC) We write $\mathcal{K} \models \text{PCC}$ to denote that for all $\mathbf{A} \in \mathcal{K}$, $\mathbf{Con} \mathbf{A}$ is a pseudo-complemented lattice, i.e., for every congruence α of \mathbf{A} , there is a largest congruence $\alpha^c = \delta$ (the pseudo-complement of α) with the property $\alpha \wedge \delta = 0_A$.

(\mathcal{K} -CD) We write $\models \mathcal{K}\text{-CD}$, and say that \mathcal{K} is relatively congruence distributive, if for all $\mathbf{A} \in \mathcal{K}$, $\mathbf{Con}_{\mathcal{K}} \mathbf{A}$ is a distributive lattice.

(\mathcal{K} -SD(\wedge)) We write $\models \mathcal{K}\text{-SD}(\wedge)$, and say that \mathcal{K} is relatively congruence meet semi-distributive, if for all $\mathbf{A} \in \mathcal{K}$, $\mathbf{Con}_{\mathcal{K}} \mathbf{A}$ is a meet semi-distributive lattice.

(\mathcal{K} -PCC) We write $\models \mathcal{K}\text{-PCC}$, and say that \mathcal{K} has pseudo-complemented relative congruence lattices, if for all $\mathbf{A} \in \mathcal{K}$, $\mathbf{Con}_{\mathcal{K}} \mathbf{A}$ is a pseudo-complemented lattice.

(EP) We write $\mathcal{K} \models \text{EP}$, and say that \mathcal{K} has the extension property, if for all $\mathbf{A} \in \mathcal{K}$, the \mathcal{K} -extension map $'$ is a lattice homomorphism of $\mathbf{Con} \mathbf{A}$ onto $\mathbf{Con}_{\mathcal{K}} \mathbf{A}$.

(WEP) We write $\mathcal{K} \models \text{WEP}$, and say that \mathcal{K} has the weak extension property, if for all $\mathbf{A} \in \mathcal{K}$, the \mathcal{K} -extension map $'$ is a lattice homomorphism locally at 0_A , by which we mean that whenever $\{\phi, \theta\} \subseteq \mathbf{Con} \mathbf{A}$ and $\phi \wedge \theta = 0_A$ then and $\phi' \wedge \theta' = 0_A$.

(W) We write $\mathcal{K} \models \text{W}$ to denote that \mathcal{K} has Willard terms (explained below).

Various characterizations of congruence meet semi-distributive varieties—i.e., varieties satisfying $\text{SD}(\wedge)$ —have been given by G. Czédli, K. Kearnes and Á. Szendrei, P. Lipparini, D. Hobby and R. McKenzie (for locally finite varieties, in [12]) and R. Willard. For many purposes, R. Willard’s result is most useful.

DEFINITION 2.1. Let \mathcal{K} be any class of algebras of one signature. A *set of Willard terms for \mathcal{K}* is a finite sequence $\{(f_i, g_i) : 0 \leq i \leq M\}$ of pairs of ternary terms in the language of \mathcal{K} such that the equations $f_i(x, y, x) \approx g_i(x, y, x)$ ($0 \leq i \leq M$) hold in \mathcal{K} and also the sentence

$$(\forall x, y) \left(x \neq y \rightarrow \bigvee_{0 \leq i \leq M} [f_i(x, x, y) = g_i(x, x, y) \leftrightarrow f_i(x, y, y) \neq g_i(x, y, y)] \right).$$

THEOREM 2.2. (R. Willard [36]) *For varieties, the properties $\text{SD}(\wedge)$ and W are equivalent.*

For a variety \mathcal{V} , the congruence lattices and the relative congruence lattices are the same, and we have $\text{CD} \Leftrightarrow \mathcal{V}\text{-CD}$, and $\text{SD}(\wedge) \Leftrightarrow \mathcal{V}\text{-SD}(\wedge) \Leftrightarrow \text{W} \Leftrightarrow \text{PCC} \Leftrightarrow \mathcal{V}\text{-PCC}$, and the extension property EP trivially holds. For a quasivariety \mathcal{K} , the implications among the nine properties that are known to us are:

$$\begin{aligned} \text{CD} &\Rightarrow \text{SD}(\wedge) \Rightarrow \text{PCC} \Leftrightarrow \text{W} \\ \mathcal{K}\text{-CD} &\Rightarrow \mathcal{K}\text{-SD}(\wedge) \Leftrightarrow \mathcal{K}\text{-PCC} \Rightarrow \text{W} \\ \mathcal{K}\text{-CD} &\Rightarrow \text{EP} \Rightarrow \text{WEP} \end{aligned}$$

We shall prove below that $\text{W} \Leftrightarrow \text{PCC}$ for quasivarieties, which gives Willard’s Theorem 2.2 as a corollary. We shall also prove the implications

$\text{SD}(\wedge) \Rightarrow \text{W}$ and $\mathcal{K}\text{-SD}(\wedge) \Rightarrow \text{W}$. The implication $\mathcal{K}\text{-CD} \Rightarrow \text{EP}$ is proved in K. Kearnes, R. McKenzie [14]. (Actually, they prove that every relatively congruence modular quasivariety has the extension property.) We shall reproduce, as Theorem 6.1, a result of [14] characterizing the quasivarieties with the property $\mathcal{K}\text{-CD}$, from which the implication $\mathcal{K}\text{-CD} \Rightarrow \text{W}$ again immediately follows. The remaining implications displayed above are trivial; and it is trivial that PCC and $\mathcal{K}\text{-PCC}$ are equivalent if $\mathcal{K} \models \text{WEP}$.

Finally, it follows from our Corollary 6.4 that the non-finitely axiomatizable quasivariety of V. P. Belkin, $SP(\mathbf{M}_{3,3})$ with $\mathbf{M}_{3,3}$ a ten-element lattice, does not satisfy WEP , although it does satisfy CD . Hence CD does not imply $\mathcal{K}\text{-CD}$ for quasivarieties. This example, and examples showing that $\mathcal{K}\text{-CD}$ does not imply CD , can be found in [14]. It is not hard to construct quasivarieties that satisfy $\text{W}\&\neg \text{SD}(\wedge)$.

Observe that each of the properties CD and $\text{SD}(\wedge)$ holds in a quasivariety \mathcal{K} iff it holds in the variety $H(\mathcal{K})$.

The following is a quasivarieties version of R. Willard's characterization of congruence meet semi-distributive varieties [36]. Our proof is a modification of his. Where \mathbf{A} is an algebra and $\{a, b\} \subseteq A$, we write $\theta_{\mathbf{A}}(a, b)$ for the (principal) congruence of \mathbf{A} generated by the pair (a, b) .

THEOREM 2.3. *For any quasivariety \mathcal{K} , the following are equivalent.*

- (1) \mathcal{K} has pseudo-complemented congruence lattices.
- (2) \mathcal{K} has a set of Willard terms.
- (3) If $\{\alpha, \beta, \gamma\} \subseteq \text{Con}_{\mathcal{K}}\mathbf{A}$, $\mathbf{A} \in \mathcal{K}$, then $\alpha \cap (\beta \circ \gamma) \subseteq \beta_{\infty}$ where β_n, γ_n are defined inductively for all $n < \omega$ by $\beta_0 = \beta$, $\gamma_0 = \gamma$, $\beta_{n+1} = \beta \vee^{\mathcal{K}} (\alpha \wedge \gamma_n)$, $\gamma_{n+1} = \gamma \vee^{\mathcal{K}} (\alpha \wedge \beta_n)$ and β_{∞} is $\bigcup_{n < \omega} \beta_n$.
- (4) In the free algebra $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(x, y, z)$, taking $\alpha = \theta_{\mathbf{F}}(x, z)$, $\beta = \theta_{\mathbf{F}}(x, y)$, $\gamma = \theta_{\mathbf{F}}(y, z)$ and computing $\beta_n, \gamma_n \in \text{Con}_{\mathcal{K}}\mathbf{F}$ as in (3), we have $(x, z) \in \beta_m$ for some m .

PROOF. We prove (1) \rightarrow (3) \rightarrow (4) \rightarrow (2) \rightarrow (1).

For (2) \Rightarrow (1), suppose that $\{(f_i, g_i)\}_{i \leq M}$ is a set of Willard terms for \mathcal{K} . We assume that $\mathbf{A} \in \mathcal{K}$ and $\phi, \theta, \psi \in \text{Con } \mathbf{A}$ with $\phi \wedge \theta = \phi \wedge \psi = 0_A$, and we prove that then $\phi \wedge (\theta \vee \psi) = 0_A$. This will imply that the pseudo-complement ϕ^c exists. Let $(a, b) \in \phi \wedge (\theta \vee \psi)$. We need to prove that $a = b$, equivalently, $(a, b) \in \phi \wedge \theta$.

Suppose that $a \neq b$. Choose $i \leq M$ with $f_i(a, a, b) = g_i(a, a, b) \leftrightarrow f_i(a, b, b) \neq g_i(a, b, b)$ (using a property of Willard terms). Choose a $\theta \vee \psi$

chain $a = b_0, b_1, \dots, b_m = b$ with $(b_j, b_{j+1}) \in \theta \cup \psi$ for all $j < m$. Clearly, there exists $j < m$ such that $f_i(a, b_j, b) = g_i(a, b_j, b) \leftrightarrow f_i(a, b_{j+1}, b) \neq g_i(a, b_{j+1}, b)$. Suppose, say, that $f_i(a, b_j, b) = g_i(a, b_j, b)$ and $f_i(a, b_{j+1}, b) \neq g_i(a, b_{j+1}, b)$. (If the equality/inequality pattern is reversed, the proof is the same.) Suppose that $(b_j, b_{j+1}) \in \theta$. (If $(b_j, b_{j+1}) \in \psi$ the proof is the same.) Then

$$f_i(a, b_{j+1}, b) \theta f_i(a, b_j, b) = g_i(a, b_j, b) \theta g_i(a, b_{j+1}, b)$$

showing that $f_i(a, b_{j+1}, b) \theta g_i(a, b_{j+1}, b)$. We also have

$$f_i(a, b_{j+1}, b) \phi f_i(a, b_{j+1}, a) = g_i(a, b_{j+1}, a) \phi g_i(a, b_{j+1}, b)$$

showing that $f_i(a, b_{j+1}, b) \phi g_i(a, b_{j+1}, b)$. Since $\phi \wedge \theta = 0_A$, then we have $f_i(a, b_{j+1}, b) = g_i(a, b_{j+1}, b)$, contradiction.

For (1) \Rightarrow (3), suppose that $\mathcal{K} \models \text{PCC}$. Let $\mathbf{A} \in \mathcal{K}$ and let $\{\alpha, \beta, \gamma\} \subseteq \text{Con } \mathbf{A}$, and define β_n, γ_n as in statement (3). Note that $\beta_n \leq \beta_{n+1}$ and $\gamma_n \leq \gamma_{n+1}$, $\alpha \wedge \beta_n \leq \gamma_{n+1}$, $\alpha \wedge \gamma_n \leq \beta_{n+1}$ hold for all n . Put

$$\beta_\infty = \bigcup_n \beta_n, \quad \gamma_\infty = \bigcup_n \gamma_n.$$

Then $\beta_\infty, \gamma_\infty$ belong to $\text{Con}_{\mathcal{K}} \mathbf{A}$ and $\alpha \wedge \beta_\infty = \alpha \wedge \gamma_\infty (= \delta, \text{ say})$. Since $\mathbf{A}/\delta \in \mathcal{K}$, its congruence α/δ has a pseudo-complement. This means that there is a largest congruence μ in \mathbf{A} with $\alpha \wedge \mu = \delta$, call it $\alpha \rightarrow \delta$. Of course, $(\alpha \rightarrow \delta)/\delta$ is the pseudo-complement of α/δ . Thus we have that $\beta_\infty \vee \gamma_\infty \leq \alpha \rightarrow \delta$. Then

$$\alpha \cap (\beta \circ \gamma) \subseteq \alpha \cap (\beta_\infty \vee \gamma_\infty) \subseteq \alpha \cap (\alpha \rightarrow \delta) = \delta,$$

giving $\alpha \cap (\beta \circ \gamma) \subseteq \beta_\infty$ as required.

For (3) \Rightarrow (4), note that for the given congruences of the free algebra $F_{\mathcal{K}}(x, y, z)$, one has $\{\alpha, \beta, \gamma\} \subseteq \text{Con}_{\mathcal{K}} \mathbf{F}$ and $(x, z) \in \alpha \cap (\beta \circ \gamma)$.

Finally, we prove (4) \Rightarrow (2). Let Φ be the set of all pairs of terms $(f(x, y, z), g(x, y, z))$ such that $(f^{\mathbf{F}}(x, y, z), g^{\mathbf{F}}(x, y, z)) \in \alpha$, i.e., such that $\mathcal{K} \models f(x, y, x) \approx g(x, y, x)$. We show that the infinite sentence

$$(\forall x, y) \left(x \neq y \rightarrow \bigvee_{(f,g) \in \Phi} (f(x, x, y) = g(x, x, y) \leftrightarrow f(x, y, y) \neq g(x, y, y)) \right)$$

is valid in \mathcal{K} . If this is true, then a compactness argument yields that the infinite disjunction above can be replaced by a finite disjunction, and thus some finite subset of Φ is a set of Willard terms for \mathcal{K} .

So suppose that $\mathbf{A} \in \mathcal{K}$, $\{a, b\} \subseteq A$, and for all $(f, g) \in \Phi$ we have $f^{\mathbf{A}}(a, a, b) = g^{\mathbf{A}}(a, a, b)$ iff $f^{\mathbf{A}}(a, b, b) = g^{\mathbf{A}}(a, b, b)$. This means that where π_1 and π_2 are the homomorphisms $\mathbf{F} \rightarrow \mathbf{A}$ mapping (x, y, z) respectively to (a, a, b) , (a, b, b) , and $\eta_i = \ker(\pi_i)$, we have

$$\mathbf{Con}_{\mathcal{K}}\mathbf{F} \models \alpha \wedge \eta_1 = \alpha \wedge \eta_2 = \alpha \wedge \eta_1 \wedge \eta_2.$$

Notice that $\beta \subseteq \eta_1$ and $\gamma \subseteq \eta_2$ and $\{\alpha, \beta, \gamma, \eta_1, \eta_2\} \subseteq \mathbf{Con}_{\mathcal{K}}\mathbf{F}$. Now, inductively, we can show that $\beta_n \leq \eta_1, \gamma_n \leq \eta_2$ for all n . Indeed, if this is true for n , then

$$\beta_{n+1} = \beta \vee^{\mathcal{K}} (\alpha \wedge \gamma_n) \leq \eta_1 \vee^{\mathcal{K}} (\alpha \wedge \eta_2) \leq \eta_1,$$

and similarly, $\gamma_{n+1} \leq \eta_2$. Now by (4), we have, say $(x, z) \in \beta_m$. Then, as we have seen, $(x, z) \in \eta_1$, i.e., $a = \pi_1(x) = \pi_1(z) = b$. This concludes our proof of the theorem. ■

The promised implications $\text{SD}(\wedge) \Rightarrow \text{W}$ and $\mathcal{K}\text{-SD}(\wedge) \Rightarrow \text{W}$ can be deduced from the fact that $\text{SD}(\wedge)$ implies statement (1) in Theorem 2.3 and $\mathcal{K}\text{-SD}(\wedge)$ implies statement (3) in Theorem 2.3.

The concept of a Willard variety is implicit in R. Willard [36]; and the theorem that follows the definition is proved in [36]. We offer a less constructive, more conceptual proof.

DEFINITION 2.4. By a *Willard variety*, we shall mean a variety \mathcal{W} with the following property: \mathcal{W} has a set of Willard terms $\{(f_i, g_i) : 0 \leq i \leq M\}$, such that where $T = \{f_i : i \leq M\} \cup \{g_i : i \leq M\}$, there is a finite set of equations Γ , with each member of Γ taking one of the forms (for some $t, t_0, t_1 \in T$)

$$t(x, x, x) \approx x, \quad t_0(x, y, x) \approx t_1(x, y, x), \quad t_0(x, x, y) \approx t_1(x, x, y),$$

$$\text{or } t_0(x, y, y) \approx t_1(x, y, y),$$

such that Γ includes all the equations $t(x, x, x) \approx x$ ($t \in T$) and $\mathcal{W} = \text{Mod}(\Gamma)$.

THEOREM 2.5. *A variety \mathcal{V} is congruence meet semi-distributive iff it is a subvariety of some Willard variety.*

PROOF. If $\mathcal{V} \subseteq \mathcal{W}$, and \mathcal{W} is a Willard variety, then \mathcal{V} has Willard terms. By Theorem 2.3, \mathcal{V} then has pseudo-complemented congruence lattices. Since it is a variety, it follows that $\mathcal{V} \models \text{SD}(\wedge)$.

Conversely, suppose that $\mathcal{V} \models \text{SD}(\wedge)$. Thus $\mathcal{V} \models \text{PCC}$ and statement (4) of Theorem 2.3 holds. We examine now our proof of (4) implies (2) in

Theorem 2.3. Since $\mathbf{Con F} = \mathbf{Con}_{\mathcal{V}}\mathbf{F}$, the occurrences of $\vee^{\mathcal{V}}$ in the definition of the sequences of congruences β_n, γ_n are ordinary equivalence relation joins.

Suppose that $(x, z) \in \beta_m$. Note that $\alpha \vee \beta = \alpha \vee \gamma = \beta \vee \gamma = \theta$, say. Let $I = x/\theta \subseteq F$. Then I is the set of all $t^{\mathbf{F}}(x, y, z) \in F$ such that $\mathcal{V} \models t(x, x, x) \approx x$. Since I is a union of blocks for each of α, β, γ , then there is a finite set $T' \subseteq I$, $\{x, y, z\} \subseteq T'$, such that where $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are the restrictions of these congruences to T' , we have $(x, z) \in \bar{\beta}_m$. (Note that $\bar{\beta}_m$ is not defined as the restriction of β_m to T' , but rather is the equivalence relation over T' calculated by the rules previously given, from the relations $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$.) Let T be a finite set of terms $t(x, y, z)$ such that $T' = \{t^{\mathbf{F}}(x, y, z) : t \in T\}$ and $\{x, y, z\} \subseteq T$.

Let Γ be the set of all equations of forms $t(x, x, x) \approx x, t(x, y, x) \approx s(x, y, x), t(x, x, y) \approx s(x, x, y), t(x, y, y) \approx s(x, y, y)$ ($\{s, t\} \subseteq T$) that hold in \mathcal{V} . Then $\mathcal{V} \subseteq \mathcal{W} = \text{Mod}(\Gamma)$. Also, where $\mathbf{F}' = \mathbf{F}_{\mathcal{W}}(x, y, z), \alpha' = \theta_{\mathbf{F}'}(x, z), \beta' = \theta_{\mathbf{F}'}(x, y), \gamma' = \theta_{\mathbf{F}'}(y, z)$ we have that $(x, z) \in \beta'_m$.

Now take $\{(f_i, g_i) : i \leq M\}$ to be a list of all the pairs $(f, g), (\{f, g\} \subseteq T)$ such that $\mathcal{W} \models f_i(x, y, x) \approx g_i(x, y, x)$. Our claim is that $\{(f_i, g_i)\}$ is a set of Willard terms for \mathcal{W} . If this is true, then \mathcal{W} is a Willard variety including \mathcal{V} .

To prove the claim, assume that $\mathbf{A} \in \mathcal{W}$ and $\{a, b\} \subseteq A$. In order to be able to re-use the notation above, we now make the harmless assumption that $\mathcal{W} = \mathcal{V}$. Let π_1, π_2 be, as before, the homomorphisms $\mathbf{F} \rightarrow \mathbf{A}$ mapping (x, y, z) to (a, a, b) , respectively (a, b, b) . Let η_i ($i \in \{1, 2\}$) denote the kernel of π_i . Suppose that for all $i \leq M, f_i(a, a, b) = g_i(a, a, b)$ iff $f_i(a, b, b) = g_i(a, b, b)$. We have to show that $a = b$. The hypothesis means that for all $\{f, g\} \subseteq T, (f^{\mathbf{F}}(x, y, z), g^{\mathbf{F}}(x, y, z)) \in \alpha \Rightarrow ((f^{\mathbf{F}}(x, y, z), g^{\mathbf{F}}(x, y, z)) \in \eta_1 \Leftrightarrow (f^{\mathbf{F}}(x, y, z), g^{\mathbf{F}}(x, y, z)) \in \eta_2)$. In other words, $\bar{\alpha} \cap \eta_1 = \bar{\alpha} \cap \eta_2 (= \bar{\alpha} \cap \eta_1 \cap \eta_2)$. As before, we conclude that for all $n, \bar{\beta}_n \subseteq \eta_1$ and $\bar{\gamma}_n \subseteq \eta_2$. Thus $(x, z) \in \bar{\beta}_m \subseteq \eta_1$, implying that $a = b$. ■

3. Pseudo-complemented congruences

DEFINITION 3.1. Let \mathbf{A} be any algebra. For $\{a, b, c, d\} \subseteq A$, we shall write $(a, b) \leq (c, d)$ to denote that $(a, b) \in \theta_{\mathbf{A}}(c, d)$, i.e., the elements a and b are congruent modulo the principal congruence generated by (c, d) . For $\{a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}\} \subseteq A$, we define

$$PCD_n(\bar{a}, \bar{b}) \leftrightarrow \mathbf{A} \models (\forall x, y) \left(\bigwedge_{i < n} (x, y) \leq (a_i, b_i) \rightarrow x = y \right).$$

We call this $2n$ -ary relation $PCD_n(\bar{x}, \bar{y})$ on A the *principal congruence n -disjointness relation over \mathbf{A}* .

Observe that the condition that **Con \mathbf{A}** is pseudo-complemented is equivalent to: $\eta_{x,y} = \{(u, v) : PCD_2(u, x, v, y)\}$ is a congruence, for all $\{x, y\} \subseteq A$; and also equivalent to: $\psi \wedge \theta = \psi \wedge \phi = 0_A \Rightarrow \psi \wedge (\theta \vee \psi) = 0_A$ for all $\{\psi, \theta, \phi\} \subseteq \text{Con } \mathbf{A}$.

Let ϕ be any sentence of the form

$$(\forall x_0, \dots, x_{n-1}) \bigvee_{i < m} \sigma_i(\bar{x}) \approx \tau_i(\bar{x})$$

where $\sigma_i(\bar{x}), \tau_i(\bar{x})$ are terms. K. Baker called such sentences UDE's. They are among the positive universal sentences of first-order logic; and every positive universal sentence in an algebraic first-order language is equivalent to a conjunction of UDE's. Given ϕ as above, we define $PCD(\phi)$ to be the condition

$$PCD(\phi) : (\forall \bar{x}) PCD_m(\sigma_0(\bar{x}), \dots, \sigma_{m-1}(\bar{x}), \tau_0(\bar{x}), \dots, \tau_{m-1}(\bar{x})).$$

Note that we cannot expect the relations PCD_n and the conditions $PCD(\phi)$ to be defined by first-order formulas for most algebras. Rather, they are naturally expressed (for algebras of countable signature) by formulas of an infinitary language $L_{\omega_1\omega}$.

The principal new ideas contributed by this paper are expressed in the statement and proof of the next theorem.

THEOREM 3.2. *Let \mathcal{K} be a quasivariety of countable signature with pseudo-complemented congruences, and let ϕ be a positive universal sentence in the language of \mathcal{K} , say $\vdash \phi \leftrightarrow \bigwedge_{i < N} \phi_i$ where $\{\phi_i\}_{i < N}$ is a set of UDE's. Then the quasivariety $\mathcal{K} \cap SP(\text{Mod}(\phi))$ is the class of all algebras $\mathbf{A} \in \mathcal{K}$ such that $\mathbf{A} \models \bigwedge_{i < N} PCD(\phi_i)$.*

PROOF. We can assume ϕ_i is the sentence

$$(\forall \bar{x}) \bigvee_{j < N_i} \sigma_{ij}(\bar{x}) \approx \tau_{ij}(\bar{x}).$$

First, assume that $\mathbf{A} \in SP(\text{Mod}(\phi))$. We can suppose that $\mathbf{A} \leq \prod_{t \in T} \mathbf{A}_t$ with $\mathbf{A}_t \models \phi$ for all $t \in T$. Let $i < N$, suppose that the quantifiers in ϕ_i are over x_0, \dots, x_{n-1} , and let $f_0, \dots, f_{n-1} \in A$. Assume that

$$(g, h) \in \bigcap_{j < N_i} \theta_{\mathbf{A}}(\sigma_{ij}(\bar{f}), \tau_{ij}(\bar{f})).$$

Choose any $t \in T$. Since $\mathbf{A}_t \models \phi$, there is $j < N_i$ such that the functions $\sigma_{ij}^{\mathbf{A}}(\bar{f}), \tau_{ij}^{\mathbf{A}}(\bar{f})$ agree at t . Since $(g, h) \in \theta_{\mathbf{A}}(\sigma_{ij}^{\mathbf{A}}(\bar{f}), \tau_{ij}^{\mathbf{A}}(\bar{f}))$, we must have $g(t) = h(t)$. Since t was arbitrary, then $g = h$. This argument shows that $\mathbf{A} \models PCD(\phi_i)$.

For the other direction, suppose that $\mathbf{A} \in \mathcal{K}$ and for all $i < N$, $\mathbf{A} \models PCD(\phi_i)$. To show that $\mathbf{A} \in SP(\text{Mod}(\phi))$, it will suffice to prove this claim.

Claim. Let \mathbf{B} be any countable subalgebra of \mathbf{A} and let $\{a, b\} \subseteq B, a \neq b$. There exists an algebra \mathbf{C} such that $\mathbf{C} \models \phi$ and there is a homomorphism $f : \mathbf{B} \rightarrow \mathbf{C}$ such that $f(a) \neq f(b)$.

We enumerate the set $B^{(2)}$ of all pairs $(x, y) \in B \times B, x \neq y$, as $\{(b_i, c_i) : i < \omega\}$ with $(b_0, c_0) = (a, b)$. Next, we inductively define $(e_j, f_j) \in B^{(2)}, j < \omega$. Put $(e_0, f_0) = (a, b)$. If (e_k, f_k) has been defined, set $(e_{k+1}, f_{k+1}) = (e_k, f_k)$ if $\theta_{\mathbf{B}}(b_k, c_k) \cap \theta_{\mathbf{B}}(e_k, f_k) = 0_B$. Otherwise, choose

$$(e_{k+1}, f_{k+1}) \in B^{(2)} \cap \theta_{\mathbf{B}}(b_k, c_k) \cap \theta_{\mathbf{B}}(e_k, f_k).$$

For each k , define θ_k to be the pseudo-complement of $\theta_{\mathbf{B}}(e_k, f_k)$.

By this construction, we obtain

$$(e_0, f_0) \geq (e_1, f_1) \geq \dots \geq (e_k, f_k) \geq \dots,$$

$$\theta_0 \leq \theta_1 \leq \dots \leq \theta_k \leq \dots$$

and we put $\theta = \bigcup_{k < \omega} \theta_k$. Clearly, θ is a congruence of \mathbf{B} and $(a, b) \notin \theta$. For every $x, y \in B$, either $(x, y) \in \theta$ —which is equivalent to $\theta_{\mathbf{B}}(x, y) \cap \theta_{\mathbf{B}}(e_k, f_k) = 0_B$ for large k —or else $(e_k, f_k) \leq (x, y)$ for large k . We have that $\mathbf{B}/\theta \models \phi$. For if not, say $\mathbf{B}/\theta \models \neg\phi_i$. Let the quantifiers in ϕ_i be over x_0, \dots, x_{n-1} . So there are $a_0, \dots, a_{n-1} \in B$ such that for all $j < N_i$, $(\sigma_{ij}^{\mathbf{B}}(\bar{a}), \tau_{ij}^{\mathbf{B}}(\bar{a})) \notin \theta$. Then there is k such that

$$(e_k, f_k) \in \bigcap_{j < N_i} \theta_{\mathbf{B}}(\sigma_{ij}^{\mathbf{B}}(\bar{a}), \tau_{ij}^{\mathbf{B}}(\bar{a})).$$

But this contradicts that $\mathbf{A} \models PCD(\phi_i)$. This concludes our proof of Theorem 3.2. ■

REMARK 3.3. The reader may have noticed that the congruence θ produced by the construction that proves Theorem 3.2 is independent of the chosen UDE ϕ such that $\mathbf{B} \models PCD(\phi)$. This construction seems to be of some independent interest, and can be used to obtain further results. In the proof above, the algebra \mathbf{B}/θ with $a/\theta \neq b/\theta$ turns out to satisfy (as shown by the

given argument), every UDE ϕ for which $\mathbf{B} \models PCD(\phi)$. Equivalently, by the theorem, \mathbf{B}/θ satisfies ϕ whenever ϕ is a UDE such that $\mathbf{B} \in SP(\text{Mod}(\phi))$. The obvious conclusion is that there is a smallest positive universal class \mathcal{K} with the property that $\mathbf{B} \in SP(\mathcal{K})$. With standard model-theoretic arguments, this conclusion can be extended to prove a theorem stating that for every class \mathcal{L} of algebras of some (not necessarily countable) signature, if \mathcal{L} has *PCC* then there is a unique smallest positive universal class \mathcal{K} (which might be called the *pseudo-root* of \mathcal{L}) with the property that $\mathcal{L} \subseteq SP(\mathcal{K})$.

4. Locally finite quasivarieties with PCC

Our two main results, Theorems 5.4 and 6.3, concern quasivarieties that possess pseudo-complemented congruence lattices and are included in some finitely generated quasivariety, and thus are locally finite quasivarieties with Willard terms. The theorem below gives another characterization of these quasivarieties, which conveys a much clearer understanding of what they are.

THEOREM 4.1. *A locally finite quasivariety \mathcal{W} satisfies PCC if and only if no algebra in \mathcal{W} has a non-trivial Abelian congruence. If \mathcal{K} is a finite set of finite algebras then $SP(\mathcal{K})$ satisfies PCC if and only if the members of $S(\mathcal{K})$ have no non-trivial Abelian congruences.*

PROOF. This proof relies on concepts and results from tame congruence theory, for which see D. Hobby, R. McKenzie [12].

First we show that if any algebra \mathbf{A} fails to have pseudo-complemented congruences, then \mathbf{A} has a non-trivial Abelian congruence. Thus, suppose that $\{\phi, \eta, \rho\} \subseteq \text{Con } \mathbf{A}$ and $\phi \wedge \eta = \phi \wedge \rho = 0_A$, while $\phi \wedge (\eta \vee \rho) = \delta > 0_A$. Then η and ρ each centralize ϕ , implying that $\eta \vee \rho$ centralizes ϕ . Since $\delta \leq \phi$ and $\delta \leq \eta \vee \rho$, then δ centralizes δ —i.e., δ is an Abelian congruence.

To see that a locally finite quasivariety satisfying PCC has no non-trivial Abelian congruences, we use the equivalence $\text{PCC} \Leftrightarrow \text{W}$ from Theorem 2.3.

So suppose that \mathcal{W} is a locally finite quasivariety with Willard terms $\{(f_i, g_i) : i \leq M\}$. To see that algebras in \mathcal{W} have no non-trivial Abelian congruences, it suffices to show that no finite algebra in \mathcal{W} has a minimal Abelian congruence. We argue by contradiction. Let \mathbf{A} be a finite algebra in \mathcal{W} and δ be a minimal Abelian congruence of \mathbf{A} . Then the congruence cover $(0_A, \delta)$ has either type **1** or type **2**. In both cases, we select a $(0_A, \delta)$ -minimal set U , and write $U = e(A)$ where $e(x)$ is some idempotent unary polynomial of \mathbf{A} . We select $(a, b) \in \delta$, with $a \neq b$, $\{a, b\} \subseteq U$. We choose $i \leq M$ so that,

say,

$$f_i(a, a, b) = g_i(a, a, b) \text{ while } f_i(a, b, b) \neq g_i(a, b, b).$$

The unequal elements $f_i(a, b, b)$ and $g_i(a, b, b)$ are δ -congruent, so there is a polynomial p such that where $q(x) = ep(x)$, we have $q(f_i(a, b, b)) \neq q(g_i(a, b, b))$. We write f'_i, g'_i , respectively, for the polynomials qf_i, qg_i . Thus U is closed under f'_i and g'_i and we have $f'_i(x, y, x) = g'_i(x, y, x)$ for all $\{x, y\} \subseteq U$, $f'_i(a, a, b) = g'_i(a, a, b)$, $f'_i(a, b, b) \neq g'_i(a, b, b)$.

Now if $(0_A, \delta)$ is of type **1**, then each of f'_i, g'_i when restricted to the set $\{a, b\}$ can depend on at most one variable. Then, by considering the various cases depending on which of the three variables each of the two operations depends on, one easily obtains a contradiction.

Finally, suppose that $(0_A, \delta)$ is of type **2**. We have that $\{a, b\}$ is contained in some $(0_A, \delta)$ -trace $N \subseteq U$. Since $f'_i(a, b, b) \delta g'_i(a, b, b)$, then there is some $(0_A, \delta)$ -trace $N' \subseteq U$ with $\{f'_i(a, b, b), g'_i(a, b, b)\} \subseteq N'$. We can assume that $N' = N$ (or else compose f'_i and g'_i with some polynomial permutation of U to achieve this situation). There is a vector space over a finite field whose set of vectors is N , and which has $a = 0$, the zero element, and whose operations are polynomials of \mathbf{A} restricted to the set N . Here, f'_i and g'_i restricted to N can be expressed as vector space polynomials,

$$\begin{aligned} f'_i(x, y, z) &= r_0x + r_1y + r_2z + c, \\ g'_i(x, y, z) &= s_0x + s_1y + s_2z + d \end{aligned}$$

for all $\{x, y, z\} \subseteq N$. Since $f'_i(0, 0, 0) = g'_i(0, 0, 0)$ then we have $c = d$. Since $f'_i(0, y, 0) = g'_i(0, y, 0)$, then $r_1y = s_1y$ for $y \in N$. Since $f'_i(0, 0, b) = g'_i(0, 0, b)$, we have $r_2b = s_2b$. Finally, the equalities just established yield

$$f'_i(0, b, b) = r_1b + r_2b + c = s_1b + s_2b + d = g'_i(0, b, b).$$

But this is a contradiction.

To prove the final assertion of the theorem, suppose that \mathcal{K} is a finite set of finite algebras and $\mathcal{W} = SP(\mathcal{K})$. This quasivariety is locally finite, so we already know that $\mathcal{W} \models \text{PCC}$ iff algebras in \mathcal{W} have no non-trivial Abelian congruences. It must be proved that if some algebra in \mathcal{W} has a non-trivial Abelian congruence, then some algebra in $S(\mathcal{K})$ has such a congruence.

So suppose that $\mathbf{A} \in \mathcal{W}$, δ is an Abelian congruence of \mathbf{A} , $0_A < \delta$. We can suppose that \mathbf{A} is a subdirect product, $\mathbf{A} \leq \prod_{t \in T} \mathbf{B}_t$ with $\{\mathbf{B}_t : t \in T\} \subseteq S(\mathcal{K})$. Thus \mathbf{A} has congruences η_t ($t \in T$) with $\mathbf{A}/\eta_t \cong \mathbf{B}_t$ and $\bigcap_{t \in T} \eta_t = 0_A$. We can choose $t \in T$ with $\delta \not\leq \eta_t$. Since δ is Abelian, then $\delta \vee \eta_t$ is solvably equivalent to η_t . Then choose for γ any congruence satisfying $\eta_t \prec \gamma \leq \delta \vee \eta_t$. The solvable cover (η_t, γ) is Abelian. Thus γ/η_t is a non-trivial Abelian congruence of $\mathbf{A}/\eta_t \cong \mathbf{B}_t$. ■

5. Principal congruence disjointness properties

Theorem 3.2 motivates our consideration of the relations $PCD_n(\bar{x}, \bar{y})$. Of course, we cannot expect that either of the relations $(x, y) \leq (u, v)$, or $PCD_n(\bar{x}, \bar{y})$ will be first-order definable over the algebra \mathbf{A} . However, we shall see that $PCD_n(\bar{x}, \bar{y})$ is first-order definable over any finitely generated quasivariety of finite signature that has Willard terms.

For the next theorem, \mathcal{W} denotes a quasivariety of finite signature and (f_i, g_i) , $0 \leq i < M$, is a set of Willard terms for \mathcal{W} . We write $\mathcal{V} = H(\mathcal{W})$, and define \mathcal{V}_n to be the class of all algebras in \mathcal{V} having at most n elements, for each positive integer n . For each n , we choose a finite list of terms $t_0^n(x, \bar{y}), \dots, t_{\ell_n}^n(x, \bar{y})$ which represent all the elements of the free algebra on $n + 1$ generators x, y_0, \dots, y_{n-1} in $SP(\mathcal{V}_n)$. Note that $t_0^n(x, \bar{y}), \dots, t_{\ell_n}^n(x, \bar{y})$ also represent all the elements of the free algebra over $SP(\mathcal{V}_k)$ for $1 \leq k \leq n$.

We define several first-order formulas. Let k, m, n be positive integers. First, $(x, y) \leq_n (u, v)$ is an abbreviation for

$$x = y \vee \bigvee_{i \leq \ell_n} (\exists \bar{z}) (\{t_i(u, \bar{z}), t_i(v, \bar{z})\} = \{x, y\}).$$

Then, $(x, y) \leq_{n,k} (u, v)$ is an abbreviation for

$$(\exists x_0, \dots, x_k) \left(x_0 = x \wedge x_k = y \wedge \bigwedge_{i < k} ((x_i, x_{i+1}) \leq_n (u, v)) \right).$$

Next, $\delta_{n,k}^m(x_0, \dots, x_{m-1}, y_0, \dots, y_{m-1})$ is the formula

$$(\forall x, y) \left(\left\{ \bigwedge_{i < m} (x, y) \leq_{n,k} (x_i, y_i) \right\} \rightarrow x = y \right).$$

For a positive integer m and any first-order formula $\delta = \delta(\bar{x}, \bar{y})$, where \bar{x}, \bar{y} are m -tuples of variables, we define $\Delta(\delta)$ to be the property of an algebra \mathbf{A} that holds iff for all $\bar{a}, \bar{b} \in A^m$, $\mathbf{A} \models \delta(\bar{a}, \bar{b})$ iff $\bigcap_{i < m} \theta_{\mathbf{A}}(a_i, b_i) = 0_{\mathbf{A}}$, equivalently,

$$\mathbf{A} \models \Delta(\delta) \text{ iff } \mathbf{A} \models (\forall \bar{x}, \bar{y}) (\delta(\bar{x}, \bar{y}) \leftrightarrow PCD_m(\bar{x}, \bar{y})).$$

A version of the next theorem, for congruence meet semi-distributive varieties with a finite residual bound, appears in a not yet published manuscript by K. Baker, G. McNulty and Ju Wang [4].

THEOREM 5.1. *Let \mathcal{W} be a quasivariety of finite signature with Willard terms as above, and let $\mathcal{V} = H(\mathcal{W})$. Then for all $n \geq 1, k \geq 1, \mathcal{W} \cap SP(\mathcal{V}_n)$ satisfies $\Delta(\delta_{n+2k, 2^k}^k)$.*

PROOF. We hold n and k fixed, and assume that $\mathbf{A} \in \mathcal{W} \cap SP(\mathcal{V}_n)$. We need to prove that for all $\bar{a}, \bar{b} \in A^k$, if $PCD_k(\bar{a}, \bar{b})$ fails, then there is a pair $(c, d), c \neq d$, of elements of \mathbf{A} such that $(c, d) \leq_{n+2k, 2^k} (a_i, b_i)$ for all $i < k$.

So suppose that $(a, b) \in \bigcap_{i < k} \theta_{\mathbf{A}}(a_i, b_i)$, and $a \neq b$. Choose a finite subalgebra \mathbf{F} of \mathbf{A} containing $\{a, b, a_0, \dots, b_{k-1}\}$ so that $(a, b) \in \bigcap_{i < k} \theta_{\mathbf{F}}(a_i, b_i)$. It will suffice to find $c \neq d$ in \mathbf{F} such that $\mathbf{F} \models (c, d) \leq_{n+2k, 2^k} (a_i, b_i)$ for all $i < k$. Replacing (a, b) by another pair, if necessary, we can assume that $\theta_{\mathbf{F}}(a, b) = \alpha$ is a minimal congruence of \mathbf{F} .

Since $\mathbf{F} \in SP(\mathcal{V}_n)$, it has a congruence θ such that $(a, b) \notin \theta$ and $|\mathbf{F}/\theta| \leq n$. The assumed minimality of α gives that $\alpha \cap \theta = 0_{\mathbf{F}}$.

Recall that $(f_i, g_i), i \leq M$, are Willard terms for \mathcal{W} .

LEMMA 5.2. *Suppose that $a', b', u, v \in F, (a', b') \in \alpha, a' \neq b'$, and $(a', b') \in \theta_{\mathbf{F}}(u, v)$. There is $i \leq M$ and a term $t(x, y_0, \dots, y_{n-1})$ and $\bar{c} \in F^n$ so that*

$$f_i(a', t(u, \bar{c}), b') = g_i(a', t(u, \bar{c}), b') \leftrightarrow f_i(a', t(v, \bar{c}), b') \neq g_i(a', t(v, \bar{c}), b').$$

PROOF OF LEMMA 5.2. We have $(a'/\theta, b'/\theta) \in \theta_{\mathbf{F}/\theta}(u/\theta, v/\theta)$. Since every polynomial function of \mathbf{F}/θ is of the form $t(x, \bar{c}/\theta)$ for some $n + 1$ -ary term $t(x, \bar{y})$ (as $|\mathbf{F}/\theta| \leq n$), there is a Maltsev chain $a'/\theta = q_0, q_1, \dots, q_m = b'/\theta$ in \mathbf{F}/θ such that $\{q_j, q_{j+1}\} = \{t_j(u/\theta, \bar{c}_j/\theta), t_j(v/\theta, \bar{c}_j/\theta)\}$ where $t_j(x, \bar{y})$ is an $n + 1$ -variable term, for each $j < m$. This gives a $\theta \vee \theta_{\mathbf{F}}(u, v)$ -chain $a' = p_0, p_1, \dots, p_{2m+1} = b'$ such that $(p_{2j}, p_{2j+1}) \in \theta$ for $j \leq m$, while $\{p_{2j+1}, p_{2j+2}\} = \{t_j(u, \bar{c}_j), t_j(v, \bar{c}_j)\}$ for $j < m$.

Since $a' \neq b'$, we can choose $i \leq M$ so that

$$f_i(a', a', b') = g_i(a', a', b') \leftrightarrow f_i(a', b', b') \neq g_i(a', b', b').$$

There must exist $\ell \leq 2m$ such that

$$f_i(a', p_\ell, b') = g_i(a', p_\ell, b') \leftrightarrow f_i(a', p_{\ell+1}, b') \neq g_i(a', p_{\ell+1}, b').$$

It is impossible that ℓ be even, because if, say $f_i(a', u', b') = g_i(a', u', b')$ and $(u', v') \in \theta$, then the elements $f_i(a', v', b')$ and $g_i(a', v', b')$ are congruent modulo both α and θ , and so are equal.

Thus ℓ is odd, say $\ell = 2j + 1$. Setting $t = t_j$ and $\bar{c} = \bar{c}_j$ gives the desired result. ■

Continuing with our proof of Theorem 5.1, we next find $c_1 \neq d_1$ with $(c_1, d_1) \in \alpha$ and $(c_1, d_1) \leq_{n+2,2} (a_1, b_1)$ in \mathbf{F} . By Lemma 5.2, applied to $(a', b') = (a, b)$ and $(u, v) = (a_1, b_1)$, there is $i \leq M$ and a term $t(x, \bar{y})$ and $\bar{c} \in F^n$ such that

$$f_i(a, t(a_1, \bar{c}), b) = g_i(a, t(a_1, \bar{c}), b) \leftrightarrow f_i(a, t(b_1, \bar{c}), b) \neq g_i(a, t(b_1, \bar{c}), b).$$

Rewriting this as

$$\begin{aligned} f_i(a, t(u, \bar{c}), b) &= g_i(a, t(u, \bar{c}), b); \\ f_i(a, t(v, \bar{c}), b) &\neq g_i(a, t(v, \bar{c}), b) \end{aligned}$$

with $\{u, v\} = \{a_1, b_1\}$, we can put $(c_1, d_1) = (f_i(a, t(v, \bar{c}), b), g_i(a, t(v, \bar{c}), b))$. It is clear that $(c_1, d_1) \in \alpha$. To see that $(c_1, d_1) \leq_{n+2,2} (a_1, b_1)$, choose $i_0, i_1 \leq \ell_{n+2}$ so that $\mathcal{V}_n \models t_{i_0}(x, \bar{y}, y_n, y_{n+1}) \approx f_i(y_n, t(x, \bar{y}), y_{n+1})$ and $\mathcal{V}_n \models t_{i_1}(x, \bar{y}, y_n, y_{n+1}) \approx g_i(y_n, t(x, \bar{y}), y_{n+1})$. Then we have

$$c_1 = t_{i_0}(v, \bar{c}, a, b), \quad t_{i_0}(u, \bar{c}, a, b) = t_{i_1}(u, \bar{c}, a, b), \quad t_{i_1}(v, \bar{c}, a, b) = d_1.$$

Thus $(c_1, d_1) \leq_{n+2,2} (a_1, b_1)$.

Now, inductively on i , for $1 \leq i \leq k$, we shall find $(c_i, d_i) \in \alpha$, $c_i \neq d_i$, so that we have

$$(c_i, d_i) \leq_{n+2(i-j+1), 2^{i-j+1}} (a_j, b_j), \quad \text{for } 1 \leq j \leq i.$$

Then (c_k, d_k) will witness that $\mathbf{F} \not\models \delta_{n+2k, 2^k}^k(\bar{a}, \bar{b})$, which implies of course that $\mathbf{A} \not\models \delta_{n+2k, 2^k}^k(\bar{a}, \bar{b})$ as well.

We already have the base case $i = 1$. For the inductive step, assume that $1 \leq i < k$ and (c_i, d_i) satisfy the required conditions. Taking $(a', b') = (c_i, d_i)$, $(u, v) = (a_{i+1}, b_{i+1})$ in Lemma 5.2, we find $i_0 \leq M$ and a term $t(x, \bar{y})$ and $\bar{c} \in F^n$, and $\{u', v'\} = \{a_{i+1}, b_{i+1}\}$ so that

$$\begin{aligned} f_{i_0}(c_i, t(u', \bar{c}), d_i) &= g_{i_0}(c_i, t(u', \bar{c}), d_i); \\ f_{i_0}(c_i, t(v', \bar{c}), d_i) &\neq g_{i_0}(c_i, t(v', \bar{c}), d_i). \end{aligned}$$

We take, of course,

$$(c_{i+1}, d_{i+1}) = (f_{i_0}(c_i, t(v', \bar{c}), d_i), g_{i_0}(c_i, t(v', \bar{c}), d_i)).$$

To see that this works, notice first that we clearly have $(c_{i+1}, d_{i+1}) \leq_{n+2,2} (a_{i+1}, b_{i+1})$. Now let $1 \leq j \leq i$. We need to verify that

$$(c_{i+1}, d_{i+1}) \leq_{n+2(i-j+2), 2^{i-j+2}} (a_j, b_j).$$

Taking $e = t(v', \bar{c})$, we have the chain

$$c_{i+1} = f_{i_0}(c_i, e, d_i), \quad f_{i_0}(c_i, e, c_i) = g_{i_0}(c_i, e, c_i), \quad g_{i_0}(c_i, e, d_i) = d_{i+1}.$$

Since $(c_i, d_i) \leq_{n+2(i-j+1), 2^{i-j+1}} (a_j, b_j)$, it is easy to calculate from the formulas displayed just above that, indeed, $(c_{i+1}, d_{i+1}) \leq_{n+2(i-j+2), 2^{i-j+2}} (a_j, b_j)$. This completes our proof of Theorem 5.1. \blacksquare

Now let \mathbf{A} be any algebra of finite signature with pseudo-complemented congruences, and let $\delta(\bar{x}, \bar{y})$ be a first-order formula in the language of \mathbf{A} where \bar{x}, \bar{y} are m -tuples of variables. Suppose that $\mathbf{A} \models \Delta(\delta)$. Let $\bar{a}, \bar{b} \in A^m$ with $\mathbf{A} \models \delta(\bar{a}, \bar{b})$. Put $\theta_0 = \bigcap_{1 \leq i < m} \theta_{\mathbf{A}}(a_i, b_i)$. Let ψ_0 be the largest congruence ψ satisfying $\theta_0 \cap \psi = 0_A$. Obviously, ψ_0 must be identical with the set of all pairs (x, y) with $\theta_0 \cap \theta_{\mathbf{A}}(x, y) = 0_A$ —i.e., ψ_0 is the set of all pairs $(x, y) \in A^2$ such that

$$\mathbf{A} \models \delta(x, a_1, \dots, a_{m-1}, y, b_1, \dots, b_{m-1}).$$

Inductively, for $1 \leq i \leq m-1$, define ψ_i to be the largest congruence disjoint from

$$\theta_i = \psi_0 \cap \dots \cap \psi_{i-1} \cap \theta_{\mathbf{A}}(a_{i+1}, b_{i+1}) \cap \dots \cap \theta_{\mathbf{A}}(a_{m-1}, b_{m-1}).$$

Thus ψ_i is the set of all pairs $(x, y) \in A^2$ satisfying: for all $(u, v) \in \psi_0 \cap \dots \cap \psi_{i-1}$,

$$\mathbf{A} \models \delta(u, \dots, u, x, a_{i+1}, \dots, a_{m-1}, v, \dots, v, y, b_{i+1}, \dots, b_{m-1}).$$

Evidently, it follows from these definitions that $(a_i, b_i) \in \psi_i$ for all $i < m$ and $\psi_0 \cap \dots \cap \psi_{m-1} = 0_A$. The point of these observations is that, given δ , there are first-order formulas $p_i(x, y, \bar{x}, \bar{y})$, $i < m$, (actually displayed above) so that, assuming that an algebra \mathbf{A} has pseudo-complemented congruences, and that $\mathbf{A} \models \Delta(\delta)$, and given $\bar{a}, \bar{b} \in A^m$, then

$$\psi_i = \{(x, y) \in A^2 : \mathbf{A} \models p_i(x, y, \bar{a}, \bar{b})\}$$

is a congruence relation and $(a_i, b_i) \in \psi_i$ for all $i < m$, and $\bigcap_{i < m} \psi_i = 0_A$.

Let $\gamma(\delta)$ be the sentence

$(\forall x_0, \dots, x_{m-1}, y_0, \dots, y_{m-1})$ “for all $i < m$, $\{(x, y) : p_i(x, y, \bar{x}, \bar{y})\}$ is a

congruence ψ_i , and $\bigcap_{i < m} \psi_i = 0$ ”.

This can be expressed as a first order sentence.

THEOREM 5.3. *Let \mathbf{A} be an algebra of finite signature, and $\delta(\bar{x}, \bar{y})$ be a first-order formula, as above. Assume that $\mathbf{A} \models PCD_m(\bar{x}, \bar{y}) \rightarrow \delta(\bar{x}, \bar{y})$. Then $\mathbf{A} \models \gamma(\delta)$ iff \mathbf{A} has pseudo-complemented congruences and $\delta(\bar{x}, \bar{y})$ defines the relation $PCD_m(\bar{x}, \bar{y})$ over \mathbf{A} .*

PROOF. We have seen that $\mathbf{A} \models \gamma(\delta)$ if \mathbf{A} has pseudo-complemented congruences and \mathbf{A} satisfies $\Delta(\delta)$.

Next, suppose that \mathbf{A} satisfies $\gamma(\delta)$ and $\mathbf{A} \models \delta(\bar{a}, \bar{b})$. We aim to prove that $\bigcap_{i < m} \theta_{\mathbf{A}}(a_i, b_i) = 0_A$. That $\mathbf{A} \models p_i(a_i, b_i, \bar{a}, \bar{b})$ for all i follows from the definitions of these formulas. Thus $(a_i, b_i) \in \psi_i$ for all i and $\bigcap_{i < m} \psi_i = 0_A$. Clearly, then, $\bigcap_{i < m} \theta_{\mathbf{A}}(a_i, b_i) = 0_A$.

Finally, assume that $\mathbf{A} \models \gamma(\delta)$ and $\{a, b\} \in A$. We want to show that $\theta_{a,b} = \{(x, y) : \delta(x, a, \dots, a, y, b, \dots, b)\}$ is the pseudo-complement of $\theta_{\mathbf{A}}(a, b)$. Since $\mathbf{A} \models \gamma(\delta)$, then $\theta_{a,b} = \psi_0$ is a congruence. We have that $(a, b) \in \bigcap_{1 \leq i < m} \psi_i$, so that $\theta_{a,b} \cap \theta_{\mathbf{A}}(a, b) = 0_A$. If $\theta_{\mathbf{A}}(x, y) \cap \theta_{\mathbf{A}}(a, b) = 0_A$, then we have $\delta(x, a, \dots, a, y, b, \dots, b)$ since $\mathbf{A} \models PCD_m(\bar{x}, \bar{y}) \rightarrow \delta(\bar{x}, \bar{y})$; thus $(x, y) \in \theta_{a,b}$. This concludes our proof. ■

Combining Theorems 3.2, 5.1, 5.3, we get our first finite basis theorem. For any class \mathcal{K} of algebras and positive integer n , we write \mathcal{K}_n for the class of members of \mathcal{K} having n or fewer elements.

THEOREM 5.4. *Let \mathcal{W} be a quasivariety of finite signature with pseudo-complemented congruence lattices. Let n be a positive integer, and put $\mathcal{V} = H(\mathcal{W})$. Let ϕ be a positive universal sentence in the language of \mathcal{W} . Then $\mathcal{W} \cap SP(\mathcal{V}_n) \cap SP(\text{Mod}(\phi))$ is finitely axiomatizable relative to \mathcal{W} .*

PROOF. Write β_n for the UDE

$$(\forall x_0, \dots, x_n) \bigvee_{i < j \leq n} x_i \approx x_j.$$

By Theorem 2.3, \mathcal{W} has Willard terms. By Theorems 5.1, 5.3, for every positive integer m , there is a formula pcd_m in $2m$ variables such that $\mathcal{W} \cap SP(\mathcal{V}_n)$ satisfies $\gamma(pcd_m)$ and $PCD_m(\bar{x}, \bar{y}) \rightarrow pcd_m(\bar{x}, \bar{y})$, and so pcd_m defines the relation PCD_m over any algebra in $\mathcal{W} \cap SP(\mathcal{V}_n)$.

For any UDE

$$\psi : (\forall \bar{x}) \bigvee_{i < M} \sigma_i(\bar{x}) \approx \tau_i(\bar{x}),$$

write $pcd(\psi)$ for the sentence

$$pcd(\psi) : (\forall \bar{x}) pcd_M(\sigma_0(\bar{x}), \dots, \sigma_{M-1}(\bar{x}), \tau_0(\bar{x}), \dots, \tau_{M-1}(\bar{x})).$$

We can assume that ϕ is the sentence $\bigwedge_{i < N} \phi_i$ where ϕ_i is the sentence

$$\phi_i : (\forall x_0, \dots, x_{c-1}) \bigvee_{j < N_i} \sigma_{ij}(\bar{x}) \approx \tau_{ij}(\bar{x}).$$

(Without loss of generality, we can assume that the number c of variables under the quantifiers is the same for all i .)

It follows from Theorems 3.2, 5.1, 5.3, that $\mathcal{W} \cap SP(\mathcal{V}_n)$ is the class of all algebras in \mathcal{W} that satisfy $\gamma(pcd_{\frac{n(n+1)}{2}})$ and $pcd(\beta_n)$; and that $\mathcal{W} \cap SP(\mathcal{V}_n) \cap SP(\text{Mod}(\phi))$ is the class of all algebras in \mathcal{W} that in addition satisfy $\gamma(pcd_{N_i})$ for all $i < N$ and satisfy all of the sentences $pcd(\phi_i)$, $i < N$. ■

COROLLARY 5.5. *Every quasivariety \mathcal{K} of finite signature, contained in a finitely generated quasivariety and having pseudo-complemented congruence lattices, is contained in a finitely axiomatizable, locally finite, quasivariety.*

PROOF. Let \mathcal{K} be such a quasivariety. Thus \mathcal{K} has Willard terms. For some n , we have $\mathcal{K} \subseteq SP(\mathcal{V}_n)$ where \mathcal{V} is the class of all algebras of the signature of \mathcal{K} . There is a finitely axiomatizable quasivariety $\mathcal{W} \supseteq \mathcal{K}$ with the same Willard terms as \mathcal{K} . Then $\mathcal{K} \subseteq \mathcal{W} \cap SP(H(\mathcal{W})_n) = \mathcal{W} \cap SP(H(\mathcal{W})_n) \cap SP(\mathcal{V}_n)$ which is finitely axiomatizable by Theorem 5.4, as \mathcal{V}_n is axiomatized by an obvious UDE. ■

COROLLARY 5.6. *Every finitely generated congruence meet semi-distributive quasivariety of finite signature is contained in a finitely axiomatizable, finitely generated, congruence meet semi-distributive quasivariety.*

PROOF. Let \mathcal{K} be such a quasivariety, say $\mathcal{K} = SP(\mathcal{K}_n)$ and $H(\mathcal{K})$ is a congruence meet semi-distributive variety. Let \mathcal{W} be a (finitely axiomatizable) Willard variety, $H(\mathcal{K}) \subseteq \mathcal{W}$. (See Theorem 2.5.) Now $\mathcal{K} \subseteq \mathcal{W} \cap SP(\mathcal{W}_n) = SP(\mathcal{W}_n)$ and the latter quasivariety is finitely axiomatizable by Theorem 5.4, since \mathcal{W}_n is axiomatized by a positive universal sentence. ■

COROLLARY 5.7. *Let \mathcal{K} be a finite set of finite algebras of the same finite signature such that $SP(\mathcal{K}) \models \text{PCC}$. If $HS(\mathcal{K}) \subseteq SP(\mathcal{K})$ then $SP(\mathcal{K})$ is finitely axiomatizable.*

PROOF. We can assume that $SP(\mathcal{K}) = SP(\mathcal{K}_1)$ where $HS(\mathcal{K}_1) = \mathcal{K}_1$, and \mathcal{K}_1 is, up to isomorphism, a finite set of algebras each having at most n elements. Now \mathcal{K}_1 is the class of all models of a positive universal sentence ϕ . Hence $SP(\mathcal{K}) = \mathcal{W} \cap SP(\mathcal{V}_n) \cap SP(\text{Mod}(\phi))$ where \mathcal{W} is any finitely axiomatizable quasivariety with Willard terms, $\mathcal{W} \supseteq \mathcal{K}$, and $\mathcal{V} = H(\mathcal{W})$. ■

The corollary below is the chief result of R. Willard [36].

COROLLARY 5.8. *Every congruence meet semi-distributive variety of finite signature with a finite residual bound is finitely axiomatizable.*

PROOF. We can write $\mathcal{V} = SP(\mathcal{K})$ where \mathcal{K} is a finite set of finite algebras (up to isomorphism, all the subdirectly irreducible algebras in \mathcal{V}). By Theorem 2.3, $\mathcal{V} \models \text{PCC}$. We have $HS(\mathcal{K}) \subseteq SP(\mathcal{K})$. Hence this corollary follows from Corollary 5.7. ■

6. Extending Pigozzi's theorem

We begin with a characterization of relatively congruence distributive quasivarieties, which is a part of Theorem 4.3 in K. Kearnes, R. McKenzie [14]. Recall the definition of the extension property EP from Section 2.

THEOREM 6.1. *A quasivariety \mathcal{K} satisfies \mathcal{K} -CD iff \mathcal{K} has the extension property, and there exists a finite set $\{(r_i, s_i, t_i) : 0 \leq i \leq M\}$ of triples of ternary terms in the language of \mathcal{K} such that: (i) if $i \leq M$ then the equations*

$$r_i(x, y, x) \approx s_i(x, y, x) \approx t_i(x, y, x),$$

$$r_i(x, x, y) \approx s_i(x, x, y), \quad s_i(x, y, y) \approx t_i(x, y, y)$$

are valid in \mathcal{K} ; (ii) the quasi-equation

$$\left(\bigwedge_{i \leq M} (r_i(x, y, z) \approx s_i(x, y, z) \wedge s_i(x, y, z) \approx t_i(x, y, z)) \right) \rightarrow x \approx z$$

is valid in \mathcal{K} .

A system of triples of terms $\{(r_i, s_i, t_i) : 0 \leq i \leq M\}$ satisfying (i) and (ii) over \mathcal{K} will be called a set of *quasi-Jónsson terms* for \mathcal{K} .

COROLLARY 6.2. *Every relatively congruence distributive quasivariety has Willard terms.*

PROOF. If $\{(r_i, s_i, t_i) : 0 \leq i \leq M\}$ is a set of quasi-Jónsson terms for \mathcal{K} then, clearly, the system of pairs $\{(r_i, s_i) : i \leq M\}$ is a set of Willard terms for \mathcal{K} . ■

Recall the definition of the weak extension property, WEP. Note that if \mathcal{K} has pseudo-complemented congruences, then WEP is equivalent to: for all $\mathbf{A} \in \mathcal{K}$ and $\theta \in \text{Con } \mathbf{A}$, the pseudo-complement of θ belongs to $\text{Con}_{\mathcal{K}} \mathbf{A}$.

By a UD we shall mean a sentence of the form

$$(\forall x_0, \dots, x_{n-1}) \left(\bigwedge_{i < k} \sigma_{i0}(\bar{x}) \approx \tau_{i0}(\bar{x}) \rightarrow \bigvee_{i < m} \sigma_{i1}(\bar{x}) \approx \tau_{i1}(\bar{x}) \right),$$

where $\sigma_{i\varepsilon}(\bar{x})$ and $\tau_{i\varepsilon}(\bar{x})$ are terms in the variables x_0, \dots, x_{n-1} and k may be 0, but $m > 0$. Every universal sentence that is valid in a one-element algebra is equivalent to a finite conjunction of UD's, and every UD is such a universal sentence.

Here is the principal theorem of this section.

THEOREM 6.3. *Let \mathcal{W} be a quasivariety of finite signature possessing pseudo-complemented congruence lattices. Let n be a positive integer, and put $\mathcal{V} = H(\mathcal{W})$. Let ϕ be any universal sentence in the language of \mathcal{W} . Then $\mathcal{L} = \mathcal{W} \cap SP(\mathcal{V}_n) \cap SP(\text{Mod}(\phi))$ is finitely axiomatizable relative to \mathcal{W} , provided there exists some quasivariety \mathcal{E} with the weak extension property such that $\mathcal{L} \subseteq \mathcal{E} \subseteq SP(\text{Mod}(\phi))$.*

COROLLARY 6.4. *Every finitely generated quasivariety of finite signature which satisfies the WEP and the PCC is finitely axiomatizable.*

COROLLARY 6.5. *Every finitely generated relatively congruence distributive quasivariety of finite signature is finitely axiomatizable.*

PROOF OF COROLLARY 6.4. We take, as usual \mathcal{W} to be a finitely axiomatizable quasivariety with Willard terms, $\mathcal{W} \supseteq \mathcal{K}$. We can assume that $\mathcal{K} = SP(\mathcal{M})$ where $\mathcal{M} \subseteq \mathcal{W}_n$, $\mathcal{V} = H(\mathcal{W})$. We can also assume that every subalgebra of an algebra in \mathcal{M} belongs to \mathcal{M} and that \mathcal{M} contains a one-element algebra. Then there is a universal sentence ϕ such that the models of ϕ are precisely those algebras isomorphic to an algebra in \mathcal{M} . Thus $\mathcal{K} = \mathcal{W} \cap SP(\mathcal{V}_n) \cap SP(\text{Mod}(\phi))$. Since \mathcal{K} has the weak extension property, then Theorem 6.3 gives that \mathcal{K} is finitely axiomatizable relative to \mathcal{W} , and hence is finitely axiomatizable. ■

PROOF OF COROLLARY 6.5. This corollary is the theorem of D. Pigozzi [30]. It is an immediate consequence of Theorem 6.1, Corollary 6.2, Theorem 2.3 and Corollary 6.4. ■

PROOF OF THEOREM 6.3. By Theorem 5.4 and its proof, $\mathcal{W} \cap SP(\mathcal{V}_n)$ is finitely axiomatizable relative to \mathcal{W} and for all m , we have a formula pcd_m which defines PCD_m over this class.

Let \mathcal{K} be the union of $\text{Mod}(\phi)$ and the class of all one-element models of the signature of \mathcal{W} . There is a universal sentence ϕ' such that $\mathcal{K} = \text{Mod}(\phi')$. Since $SP(\mathcal{K}) = SP(\text{Mod}(\phi))$, we can assume that ϕ has one-element models.

For any UD

$$\psi : (\forall x_0, \dots, x_{p-1}) \left(\bigwedge_{i < k} \sigma_{i0}(\bar{x}) \approx \tau_{i0}(\bar{x}) \rightarrow \bigvee_{i < m} \sigma_{i1}(\bar{x}) \approx \tau_{i1}(\bar{x}) \right),$$

we take $pcd(\psi)$ to be the sentence

$$(\forall \bar{x}, x, y) \left(\bigwedge_{i < k} pcd_2(x, \sigma_{i0}(\bar{x}), y, \tau_{i0}(\bar{x})) \rightarrow pcd_{m+1}(x, \sigma_{01}(\bar{x}), \dots, \sigma_{m-1,1}(\bar{x}), y, \tau_{01}(\bar{x}), \dots, \tau_{m-1,1}(\bar{x})) \right).$$

We can assume that ϕ is the sentence $\bigwedge_{i < N} \phi_i$ where ϕ_i is the sentence

$$(\forall x_0, \dots, x_{p-1}) \left(\bigwedge_{j < K_i} \sigma_{ij0}(\bar{x}) \approx \tau_{ij0}(\bar{x}) \rightarrow \bigvee_{j < M_i} \sigma_{ij1}(\bar{x}) \approx \tau_{ij1}(\bar{x}) \right),$$

We claim that under the assumptions of this theorem, the quasivariety $\mathcal{L} = \mathcal{W} \cap SP(\mathcal{V}_n) \cap SP(\text{Mod}(\phi))$ is axiomatized relative to $\mathcal{W} \cap SP(\mathcal{V}_n)$ by $\bigwedge_{i < N} pcd(\phi_i)$.

To prove this, first let $\mathbf{A} \in \mathcal{L}$ and choose any $i < N$. We want to show that $\mathbf{A} \models pcd(\phi_i)$. So let $\{a, b\} \subseteq A$, $\bar{a} \in A^p$. Write $\eta_{a,b}$ for the pseudo-complement of $\theta_{\mathbf{A}}(a, b)$. Now suppose that

$$\mathbf{A} \models \bigwedge_{j < K_i} pcd_2(a, \sigma_{ij0}(\bar{a}), b, \tau_{ij0}(\bar{a})).$$

Since $\mathbf{A} \in \mathcal{W} \cap SP(\mathcal{V}_n)$, this means that for all $j < K_i$, $(\sigma_{ij0}(\bar{a}), \tau_{ij0}(\bar{a})) \in \eta_{a,b}$. We are trying to prove that $\mathbf{A} \models pcd(\phi_i)$. So to derive a contradiction, suppose that

$$\mathbf{A} \models \neg pcd_{M_i+1}(a, \sigma_{i01}(\bar{a}), \dots, \sigma_{i, M_i-1, 1}(\bar{a}), b, \tau_{i01}(\bar{a}), \dots, \tau_{i, M_i-1, 1}(\bar{a})).$$

This means that

$$\theta_{\mathbf{A}}(a, b) \cap \bigcap_{j < M_i} \theta_{\mathbf{A}}(\sigma_{ij1}(\bar{a}), \tau_{ij1}(\bar{a})) \neq 0_A$$

and it implies that for no $j < M_i$ is $(\sigma_{ij1}(\bar{a}), \tau_{ij1}(\bar{a})) \in \eta_{a,b}$.

The algebra \mathbf{A} has pseudo-complemented congruences, since $\mathbf{A} \in \mathcal{W}$; and \mathbf{A} belongs to the quasivariety \mathcal{E} with the weak extension property. Hence $\mathbf{A}/\eta_{a,b} \in \mathcal{E}$. Choose

$$(e, f) \in \theta_{\mathbf{A}}(a, b) \cap \bigcap_{j < M_i} \theta_{\mathbf{A}}(\sigma_{ij1}(\bar{a}), \tau_{ij1}(\bar{a})), \quad e \neq f.$$

Write x' for $x/\eta_{a,b}$ when $x \in A$. Now (e', f') is a pair of unequal elements of $\mathbf{A}/\eta_{a,b}$ which belong to each principal congruence $\theta_{\mathbf{A}/\eta_{a,b}}(\sigma_{ij1}(\bar{a}'), \tau_{ij1}(\bar{a}'))$, $j < M_i$. Moreover, we have $\mathbf{A}/\eta_{a,b} \models \sigma_{ij0}(\bar{a}') \approx \tau_{ij0}(\bar{a}')$ for all $j < K_i$. But this implies that every homomorphism χ of $\mathbf{A}/\eta_{a,b}$ into an algebra satisfying ϕ_i must have $\chi(e') = \chi(f')$. This contradicts the fact that $\mathbf{A}/\eta_{a,b} \in \mathcal{E} \subseteq SP(\text{Mod}(\phi))$.

Thus we have shown that every algebra in \mathcal{L} satisfies $\bigwedge_{i < N} pcd(\phi_i)$. Conversely, suppose that $\mathbf{A} \in \mathcal{W} \cap SP(\mathcal{V}_n)$ and $\mathbf{A} \models \bigwedge_{i < N} pcd(\phi_i)$. We need to show that $\mathbf{A} \in SP(\text{Mod}(\phi))$. We can assume that \mathbf{A} is countable.

Let a, b be any pair of distinct elements of \mathbf{A} . We seek a homomorphism $\chi : \mathbf{A} \rightarrow \mathbf{C}$ where $\mathbf{C} \models \phi$ and $\chi(a) \neq \chi(b)$. Once again, we form a sequence $(a, b) = (e_0, f_0) \geq (e_1, f_1) \geq \dots \geq (e_r, f_r) \geq \dots$ of pairs of distinct elements of \mathbf{A} so that where $\theta_i = \eta_{e_i, f_i}$ (the pseudo-complement) and $\theta = \bigcup_{i < \omega} \theta_i$, we have that for all $\{x, y\} \subseteq A$, either $(x, y) \in \theta$ or else $(x, y) \geq (e_i, f_i)$ for some i . Let $\mathbf{C} = \mathbf{A}/\theta$ and χ be the quotient map. Thus $\chi(a) \neq \chi(b)$. We claim, of course, that $\mathbf{C} \models \phi$.

Suppose that $\mathbf{C} \models \neg\phi_i$. Let $a_0/\theta, \dots, a_{p-1}/\theta$ witness that $\mathbf{C} \models \neg\phi_i$. This means that for large r we have $(\sigma_{ij0}(\bar{a}), \tau_{ij0}(\bar{a})) \in \theta_r$ for all $j < K_i$ and we also have $(e_r, f_r) \leq (\sigma_{ij1}(\bar{a}), \tau_{ij1}(\bar{a}))$ for all $j < M_i$. But then $e_r, f_r, a_0, \dots, a_{p-1}$ witnesses a failure of $pcd(\phi_i)$ in \mathbf{A} . This contradiction finishes our proof of Theorem 6.3. ■

7. Striving for generality

The two chief new results of this paper have been Theorem 5.4 and Theorem 6.3. Each asserts that $\mathcal{L} = \mathcal{W} \cap \mathcal{W}' \cap SP(\text{Mod}(\phi))$ is finitely axiomatizable relative to \mathcal{W} where $\mathcal{W}, \mathcal{W}'$ are quasivarieties, \mathcal{W} has pseudo-complemented congruence lattices, and ϕ is a universal sentence, subject to the satisfaction of additional hypotheses. Our only reason for assuming that \mathcal{W} is of finite signature, and for putting $\mathcal{W}' = SP(\mathcal{V}_n)$ with $\mathcal{V} = H(\mathcal{W})$, was to ensure that there is a finitely axiomatizable quasivariety extending \mathcal{L} in which the $2m$ -ary relation PCD_m is first-order definable for every positive integer m . We can simply assume that this holds. Then our arguments prove the following theorem, which includes both Theorem 5.4 and Theorem 6.3.

THEOREM 7.1. *Suppose that \mathcal{W} is a quasivariety with pseudo-complemented congruences. Let ϕ be a universal sentence and put $\mathcal{L} = \mathcal{W} \cap SP(\text{Mod}(\phi))$. Assume that there is a finitely axiomatizable quasivariety $\mathcal{K} \supseteq \mathcal{L}$ such that for every positive integer m , the $2m$ -ary relation PCD_m is first-order definable over \mathcal{K} . If either (i) ϕ is positive; or (ii) there is a quasivariety \mathcal{E} with the weak extension property such that $\mathcal{L} \subseteq \mathcal{E} \subseteq SP(\text{Mod}(\phi))$, then \mathcal{L} is finitely axiomatizable relative to \mathcal{W} .*

Problems. For each of these properties, characterize the family of all quasivarieties \mathcal{K} that possess it: $\mathcal{K}\text{-SD}(\wedge)$, EP, WEP.

References

- [1] BAKER, K., ‘Primitive satisfaction and equational problems for lattices and other algebras’, *Transactions Amer. Math. Soc.* 190:125–150, 1974.
- [2] BAKER, K., ‘Finite equational bases for finite algebras in a congruence distributive equational class’, *Advances in Math.* 24:207–243, 1977.
- [3] BAKER, K. and JU WANG, ‘Definable principal subcongruences’, (preprint).
- [4] BAKER, K., G. MCNULTY, and JU WANG, ‘An extension of Willard’s finite basis theorem: congruence meet semidistributive varieties of finite critical depth’, (preprint).
- [5] BELKIN, V. P., ‘Quasi-identities of finite rings and lattices’, *Algebra i Logika* 17:247–259, 1978.
- [6] BESTSENNYI, I. P., ‘On quasi-identities of finite nilpotent algebras’, *Algebra Universalis* 36:60–278, 1996 (no. 2).
- [7] BRYANT, R., ‘The laws of finite pointed groups’, *Bull. London Math. Soc.* 14:119–123, 1982.
- [8] CZELAKOWSKI, J. and W. DZIOBIAK, ‘Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class’, *Algebra Universalis* 27:128–149, 1990 (no. 1).
- [9] DZIOBIAK, W., ‘Finitely generated congruence distributive quasivarieties of algebras’, *Fundamenta Mathematicae* 133:47–57, 1989.
- [10] DZIOBIAK, W., ‘Finite bases for finitely generated relatively congruence distributive quasivarieties’, *Algebra Universalis* 28:303–323, 1991.
- [11] GERHARD, J. A., ‘The lattice of equational classes of idempotent semigroups’, *Journal of Algebra* 15:195–224, 1970.
- [12] HOBBY, D. and R. MCKENZIE, *The structure of finite algebras*, American Math. Soc., Contemporary Mathematics, vol. 76, 1988.
- [13] JEŽEK, J., M. MARÓTI, and R. MCKENZIE, ‘Quasi-equational theories of flat algebras’, to appear in *Czechoslovak Math. J.*

- [14] KEARNES, K. and R. MCKENZIE, 'Commutator theory for relatively modular quasivarieties' *Trans. Amer. Math. Soc.* 331:465–502, 1992.
- [15] KRUSE, R. L., 'Identities satisfied by a finite ring', *Journal of Algebra* 26:298–318, 1973.
- [16] L'VOV, I. V. 'Varieties of associative rings, I, II,' *Algebra i Logika* 12:267–298, 667–668, 735, 1973.
- [17] LAWRENCE J. and R. WILLARD, 'On finitely based groups and nonfinitely based quasivarieties', *Journal of Algebra* 203:9–11, 1998 (no. 1).
- [18] LYNDON, R., 'Identities in two-valued calculi', *Transactions Amer. Math. Soc.* 71:457–465, 1951.
- [19] MARGOLIS S. and M. V. SAPIR, 'Quasi-identities of finite semigroups and symbolic dynamics', *Israel J. Math.* 92:317–331, 1995 (no.1–3).
- [20] MCKENZIE, R., 'A new product of algebras and a type reduction theorem', *Algebra Universalis* 18:29–69, 1984.
- [21] MCKENZIE, R., 'Finite equational bases for congruence modular varieties', *Algebra Universalis* 24:224–250, 1987.
- [22] MCKENZIE, R., 'The residual bound of a finite algebra is not computable', *International Journal of Algebra and Computation* 6:29–48, 1996.
- [23] MCKENZIE, R. 'Tarski's finite basis problem is undecidable', *International Journal of Algebra and Computation* 6:49–104, 1996.
- [24] McNULTY, G. and Ju WANG, 'The class of subdirectly irreducible groups generated by a finite group is finitely axiomatizable', (preprint).
- [25] MURSKIĬ, V. L., 'The existence in three-valued logic of a closed class with finite basis not having a finite complete set of identities', *Dokl. Akad. Nauk. SSR* 163:815–818, 1965; English translation in *Soviet Math. Dokl.* 6:1020–1024, 1965.
- [26] OATES S. and M. B. POWELL, 'Identical relations in finite groups', *Journal of Algebra* 1:11–39, 1964.
- [27] OL'SHANSKIĬ, A. Yu, 'Conditional identities in finite groups', *SIBER. MATH. J.* 15:1000–1003, 1974.
- [28] PERKINS, P., 'Bases for equational theories of semigroups', *Journal of Algebra* 11:293–314, 1969.
- [29] PIGOZZI, D., 'A finite basis theorem for quasivarieties', *Algebra Universalis* 22:1–13, 1986.
- [30] PIGOZZI, D., 'Finite basis theorems for relatively congruence distributive quasivarieties', *Trans. Amer. Math. Soc.* 310:499–533, 1988.
- [31] POLIN, S. V., 'On the identities of finite algebras', *Siberian Math. J.* 17:1356–1366, 1976.

- [32] SAPIR, M. V., ‘Inherently non-finitely based finite semigroups’, *Mat. Sb. (N.S.)* 133 (175):154–166, 270; English translation in *MATH. USSR-SB.* 61:155–166, 1988 (no. 1).
- [33] TUMANOV, V. I., ‘On finite lattices not having an independent basis of quasi-identities’, *Mat. Zametki* 36:625–634, 1984.
- [34] VISHIN, V. V., ‘Identity transformations in a four-valued logic’, *Dokl. Akad. Nauk. SSSR* 150:719–721, 1963; English translation in *Soviet Math. Dokl.* 4:724, 1963.
- [35] WILLARD, R., ‘Tarski’s finite basis problem via $\mathbf{A}(T)$ ’, *Transactions Amer. Math. Soc.* 349:2755–2774, 1997.
- [36] WILLARD, R., ‘A finite basis theorem for residually finite, congruence meet-semidistributive varieties’, *Journal of Symbolic Logic* 65:187–200, 2000.
- [37] WILLARD, R., ‘Extending Baker’s theorem’, *Algebra Universalis* 45:335–344, 2001.

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