

Fuzzy Topology and Łukasiewicz Logics from the Viewpoint of Duality Theory*

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Abstract

This paper explores relationships between many-valued logic and fuzzy topology from the viewpoint of duality theory. We first show a fuzzy topological duality for the algebras of Łukasiewicz n -valued logic with truth constants, which generalizes Stone duality for Boolean algebras to the n -valued case via fuzzy topology. Then, based on this duality, we show a fuzzy topological duality for the algebras of modal Łukasiewicz n -valued logic with truth constants, which generalizes Jónsson-Tarski duality for modal algebras to the n -valued case via fuzzy topology. We emphasize that fuzzy topological spaces naturally arise as spectrums of algebras of many-valued logics.

Keywords: fuzzy topology; Stone duality; Jónsson-Tarski duality; algebraic logic; many-valued logic; modal logic; Kripke semantics; compactness

1 Introduction

This paper aims to explore relationships between many-valued logic and fuzzy topology from the viewpoint of duality theory. In particular, we consider fuzzy topological dualities for the algebras of Łukasiewicz n -valued logic L_n^c with truth constants and for the algebras of modal Łukasiewicz n -valued logic ML_n^c with truth constants.

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Roughly speaking, a many-valued logic is a logical system in which there are more than two truth values (for a general introduction, see [13, 15, 21]). In many-valued logic, a proposition may have a truth value different from 0 (false) and 1 (true). Łukasiewicz many-valued logic is one of the most prominent many-valued logics. Many-valued logics have often been studied from the algebraic point of view (see, e.g., [2, 6, 15]). MV-algebra introduced in [4] provides algebraic semantics for Łukasiewicz infinite-valued logic. MV_n -algebra introduced in [14] provides algebraic semantics for Łukasiewicz n -valued logic introduced in [20] ([14] also gives an axiomatization of Łukasiewicz n -valued logic). L_n^c -algebra in this paper is considered as MV_n -algebra enriched by constants.

Kripke semantics for modal logic is naturally extended to the many-valued case by allowing for more than two truth values at each possible world and so we can define modal many-valued logics by such many-valued Kripke semantics, including modal Łukasiewicz many-valued logics. Modal many-valued logics have already been studied by several authors (see [9, 10, 22, 29]).

As a major branch of fuzzy mathematics, fuzzy topology is based on the concept of fuzzy set introduced in [30, 11], which is defined by considering many-valued membership function. For example, a $[0, 1]$ -valued fuzzy set μ on a set X is defined as a function from X to $[0, 1]$. Then, for $x \in X$ and $r \in [0, 1]$, $\mu(x) = r$ intuitively means that the proposition “ $x \in \mu$ ” has a truth value r . A fuzzy topology on a set is defined as a collection of fuzzy sets on the set which satisfies some conditions (for details, see Section 3). Historically, Chang [5] introduced the concept of $[0, 1]$ -valued fuzzy topology and thereafter Goguen [12] introduced that of lattice-valued fuzzy topology. There have been many studies on fuzzy topology (see, e.g., [19, 25, 27]).

Stone duality for Boolean algebras (see [17, 28]) is one of the most important results in algebraic logic and states that there is a categorical duality between Boolean algebras (i.e., the algebras of classical propositional logic) and Boolean spaces (i.e., zero-dimensional compact Hausdorff spaces). Since both many-valued logic and fuzzy topology can be considered as based on the idea that there are more than two truth values, it is natural to expect that there is a duality between the algebras of many-valued logic and “fuzzy Boolean spaces.” Stone duality for Boolean algebras was extended to Jónsson-Tarski duality (see [1, 3, 16, 26]) between modal algebras and relational spaces (or descriptive general frames), which is another classical theorem in duality theory. Thus, it is also natural to expect that there is a duality between the algebras of modal many-valued logic and “fuzzy relational spaces.”

In this paper, we realize the above expectations in the cases of \mathbf{L}_n^c and \mathbf{ML}_n^c . We first develop a categorical duality between the algebras of \mathbf{L}_n^c and \mathbf{n} -fuzzy Boolean spaces (see Definition 4.5), which is a generalization of Stone duality for Boolean algebras to the \mathbf{n} -valued case via fuzzy topology. This duality is developed based on the following insights:

- The spectrum of an algebra of \mathbf{L}_n^c can be naturally equipped with a certain \mathbf{n} -fuzzy topology (see Definition 4.9).
- The notion of clopen subset of Boolean space in Stone duality for Boolean algebras corresponds to that of continuous function from \mathbf{n} -fuzzy Boolean space to \mathbf{n} ($= \{0, 1/(n-1), 2/(n-1), \dots, 1\}$) equipped with the \mathbf{n} -fuzzy discrete topology in the duality for the algebras of \mathbf{L}_n^c . This means that the zero-dimensionality of \mathbf{n} -fuzzy topological spaces is defined in terms of continuous function into \mathbf{n} (see Definition 4.4).

Moreover, based on the duality for the algebras of \mathbf{L}_n^c , we develop a categorical duality between the algebras of \mathbf{ML}_n^c and \mathbf{n} -fuzzy relational spaces (see Definition 6.3), which is a generalization of Jónsson-Tarski duality for modal algebras to the \mathbf{n} -valued case via fuzzy topology. Note that an \mathbf{n} -fuzzy relational space is also defined in terms of continuous functions into \mathbf{n} (see the items 1 and 2 in the object part of Definition 6.3).

There have been some studies on dualities for algebras of many-valued logics (see, e.g., [2, 7, 18, 23, 24, 8, 29]). However, they are based on the ordinary topology and therefore do not reveal relationships between many-valued logic and fuzzy topology. By the results in this paper, we can notice that fuzzy topological spaces naturally arise as spectrums of algebras of some many-valued logics and that there are categorical dualities connecting fuzzy topology and those many-valued logics which generalize Stone and Jónsson-Tarski dualities via fuzzy topology.

This paper is organized as follows. In Section 2, we define \mathbf{L}_n^c and \mathbf{L}_n^c -algebras, and show basic properties of them. In Section 3, we review basic concepts related to fuzzy topology. In Section 4, we define \mathbf{n} -fuzzy Boolean spaces and show a fuzzy topological duality for \mathbf{L}_n^c -algebras, which is a main theorem in this paper. In Section 5, we define \mathbf{ML}_n^c and \mathbf{ML}_n^c -algebras, and show basic properties of them, including a compactness theorem for \mathbf{ML}_n^c . In Section 6, we define \mathbf{n} -fuzzy relational spaces and show a fuzzy topological duality for \mathbf{ML}_n^c -algebras, which is the other main theorem.

2 \mathbb{L}_n^c -algebras and basic properties

Throughout this paper, n denotes a natural number more than 1.

Definition 2.1. \mathbf{n} denotes $\{0, 1/(n-1), 2/(n-1), \dots, 1\}$. We equip \mathbf{n} with all constants $r \in \mathbf{n}$ and the operations $(\wedge, \vee, *, \wp, \rightarrow, (-)^\perp)$ defined as follows:

$$\begin{aligned} x \wedge y &= \min(x, y); \\ x \vee y &= \max(x, y); \\ x * y &= \max(0, x + y - 1); \\ x \wp y &= \min(1, x + y); \\ x \rightarrow y &= \min(1, 1 - (x - y)); \\ x^\perp &= 1 - x. \end{aligned}$$

We define Łukasiewicz n -valued logic with truth constants, which is denoted by \mathbb{L}_n^c . The connectives of \mathbb{L}_n^c are

$$(\wedge, \vee, *, \wp, \rightarrow, (-)^\perp, 0, 1/(n-1), 2/(n-1), \dots, 1),$$

where $(\wedge, \vee, *, \wp, \rightarrow)$ are binary connectives, $(-)^{\perp}$ is a unary connective, and $(0, 1/(n-1), 2/(n-1), \dots, 1)$ are constants. The formulas of \mathbb{L}_n^c are recursively defined in the usual way. Let \mathbf{PV} denote the set of propositional variables and \mathbf{Form} denote the set of formulas of \mathbb{L}_n^c .

$x \leftrightarrow y$ is the abbreviation of $(x \rightarrow y) \wedge (y \rightarrow x)$. For $m \in \omega$ with $m \neq 0$, $*^m x$ is the abbreviation of $x * \dots * x$ (m -times). For instance, $*^3 x = x * x * x$.

Definition 2.2. A function $v : \mathbf{Form} \rightarrow \mathbf{n}$ is an \mathbf{n} -valuation iff it satisfies:

- $v(\varphi @ \psi) = v(\varphi) @ v(\psi)$ for $@ = \wedge, \vee, *, \wp, \rightarrow$;
- $v(\varphi^\perp) = (v(\varphi))^\perp$;
- $v(r) = r$ for $r \in \mathbf{n}$.

Define $\mathbb{L}_n^c = \{\varphi \in \mathbf{Form} ; v(\varphi) = 1 \text{ for any } \mathbf{n}\text{-valuation } v\}$.

\mathbb{L}_n^c -algebras and homomorphisms are defined as follows.

Definition 2.3. $(A, \wedge, \vee, *, \wp, \rightarrow, (-)^\perp, 0, 1/(n-1), 2/(n-1), \dots, 1)$ is an \mathbb{L}_n^c -algebra iff it satisfies the following set of equations: $\{\varphi = \psi ; \varphi \leftrightarrow \psi \in \mathbb{L}_n^c\}$.

A homomorphism of \mathbb{L}_n^c -algebras is defined as a function which preserves the operations $(\wedge, \vee, *, \wp, \rightarrow, (-)^\perp, 0, 1/(n-1), 2/(n-1), \dots, 1)$.

Throughout this paper, we do not distinguish between formulas of \mathbb{L}_n^c and terms of \mathbb{L}_n^c -algebras.

Definition 2.4. $\varphi \in \mathbf{Form}$ is idempotent iff $\varphi * \varphi \leftrightarrow \varphi \in \mathbb{L}_n^c$.

For an \mathbb{L}_n^c -algebra A , $a \in A$ is idempotent iff $a * a = a$.

$\mathcal{B}(A)$ denotes the set of all idempotent elements of an \mathbb{L}_n^c -algebra A .

Let A be an \mathbb{L}_n^c -algebra. Then, we have the following facts: (i) For $a \in A$, $*^{n-1}a$ is always idempotent. (ii) If $a \in A$ is idempotent, then either $v(a) = 1$ or $v(a) = 0$ holds for any homomorphism $v : A \rightarrow \mathbf{n}$. (iii) If $a, b \in A$ are idempotent, then $a * b = (*^{n-1}a) * (*^{n-1}b) = (*^{n-1}a) \wedge (*^{n-1}b) = a \wedge b$ and $a \wp b = (*^{n-1}a) \wp (*^{n-1}b) = (*^{n-1}a) \vee (*^{n-1}b) = a \vee b$.

It is easy to verify the following:

Proposition 2.5. For an \mathbb{L}_n^c -algebra A , $\mathcal{B}(A)$ forms a Boolean algebra. In particular, $a \vee a^\perp = 1$ for any idempotent element a of A .

In the following, we define a formula $\mathbb{T}_r(x)$ for $r \in \mathbf{n}$, which intuitively means that the truth value of x is exactly r .

Lemma 2.6. Let A be an \mathbb{L}_n^c -algebra and $r \in \mathbf{n}$. There is an idempotent formula $\mathbb{T}_r(x)$ with one variable x such that, for any homomorphism $v : A \rightarrow \mathbf{n}$ and any $a \in A$, the following hold:

- $v(\mathbb{T}_r(a)) = 1$ iff $v(a) = r$;
- $v(\mathbb{T}_r(a)) = 0$ iff $v(a) \neq r$.

Proof. If $r = 0$, then we can set $\mathbb{T}_r(x) = *^{n-1}(x^\perp)$. If $r = 1$, then we can set $\mathbb{T}_r(x) = *^{n-1}x$.

Let $r = k/(n-1)$ for $k \in \{1, \dots, n-2\}$. If k is a divisor of $n-1$, then we can set

$$\mathbb{T}_r(x) = *^{n-1}(x \leftrightarrow (\wp^{\frac{n-1}{k}-1}x)^\perp).$$

For a rational number q , let $[q]$ denote the greatest integer n such that $n \leq q$. If k is not a divisor of $n-1$, then

$$\begin{aligned} v(x) = k/(n-1) \quad \text{iff} \quad v(\wp^{\lceil \frac{n-1}{k} \rceil}x) &= \frac{k}{n-1} \left\lceil \frac{n-1}{k} \right\rceil (< 1) \\ \text{iff} \quad v((\wp^{\lceil \frac{n-1}{k} \rceil}x)^\perp) &= 1 - \frac{k}{n-1} \left\lceil \frac{n-1}{k} \right\rceil. \end{aligned}$$

Since

$$1 - \frac{k}{n-1} \left\lceil \frac{n-1}{k} \right\rceil < \frac{k}{n-1},$$

this lemma follows by induction on k . □

The above lemma is more easily proved by using truth constants $r \in \mathbf{n}$. However, it must be stressed that the above proof works even if we consider Lukasiewicz n -valued logic without truth constants.

Note that any homomorphism preserves the operation $T_r(-)$.

Lemma 2.7. *Let A be an L_n^c -algebra and $a_i \in A$ for a finite set I and $i \in I$. Then, (i) $T_1(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} T_1(a_i)$; (ii) $T_1(\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I} T_1(a_i)$.*

Proof. Since \mathbf{n} is totally ordered, we have (i). (ii) is immediate. \square

By (ii) in the above lemma, $T_1(-)$ is order preserving.

Lemma 2.8. *Let A be an L_n^c -algebra and $r \in \mathbf{n}$. There is an idempotent formula $U_r(x)$ with one variable x such that, for any homomorphism $v : A \rightarrow \mathbf{n}$ and any $a \in A$, the following two conditions hold: (i) $v(U_r(a)) = 1$ iff $v(a) \geq r$; (ii) $v(U_r(a)) = 0$ iff $v(a) \not\geq r$.*

Proof. It suffices to let $U_r(x) = \bigvee \{T_s(x) ; r \leq s\}$ by Lemma 2.6. \square

Note that any homomorphism preserves the operation $U_r(-)$.

Lemma 2.9. *Let A be an L_n^c -algebra and $r \in \mathbf{n}$. There is a formula $S_r(x)$ with one variable x such that, for any homomorphism $v : A \rightarrow \mathbf{n}$ and any $a \in A$, the following two conditions hold: (i) $v(S_r(a)) = r$ iff $v(a) = 1$; (ii) $v(S_r(a)) = 0$ iff $v(a) \neq 1$.*

Proof. Let $S_r(x) = (T_1(x) \rightarrow r) \wedge ((T_1(x))^{\perp} \rightarrow 0)$. \square

Note that any homomorphism preserves the operation $S_r(-)$.

Lemma 2.10. *Let A be an L_n^c -algebra. Let v and u be homomorphisms from A to \mathbf{n} . Then, (i) $v = u$ iff (ii) $v^{-1}(\{1\}) = u^{-1}(\{1\})$.*

Proof. Clearly, (i) implies (ii). We show the converse. Assume that $v^{-1}(\{1\}) = u^{-1}(\{1\})$. Suppose for contradiction that $v(a) \neq u(a)$ for some $a \in A$. Let $r = v(a)$. Then $v(T_r(a)) = 1$ and $u(T_r(a)) = 0$, which contradicts $v^{-1}(\{1\}) = u^{-1}(\{1\})$. \square

For an L_n^c -algebra A and $a, b \in A$, we mean $a \vee b = b$ by $a \leq b$.

Lemma 2.11. *Let A be an L_n^c -algebra. For any $a, b \in A$, the following holds:*

$$\bigwedge_{r \in \mathbf{n}} (T_r(a) \leftrightarrow T_r(b)) \leq a \leftrightarrow b.$$

Proof. This is proved by straightforward computation. \square

For a partially ordered set (M, \leq) , $X \subset M$ is called an upper set iff if $x \in X$ and $x \leq y$ for $y \in M$ then $y \in X$.

Definition 2.12. Let A be an L_n^c -algebra. A non-empty subset F of A is called an **n-filter** of A iff F is an upper set and is closed under $*$. An **n-filter** F of A is called proper iff $F \neq A$.

An **n-filter** of A is closed under \wedge , since $a * b \leq a \wedge b$ for any $a, b \in A$.

Definition 2.13. Let A be an L_n^c -algebra. A proper **n-filter** P of A is prime iff, for any $a, b \in A$, $a \vee b \in P$ implies either $a \in P$ or $b \in P$.

Proposition 2.14. Let A be an L_n^c -algebra and F an **n-filter** of A . For $b \in A$, assume $b \notin F$. Then, there is a prime **n-filter** P of A such that $F \subset P$ and $b \notin P$.

Proof. Let Z be the set of all those **n-filters** G of A such that $F \subset G$ and $b \notin G$. Then $F \in Z$. Clearly, every chain of Z has an upper bound in Z . Thus, by Zorn's lemma, we have a maximal element P in Z . Note that $F \subset P$ and $b \notin P$.

To complete the proof, it suffices to show that P is a prime **n-filter** of A . Assume $x \vee y \in P$. Additionally, suppose for contradiction that $x \notin P$ and $y \notin P$. Then, since P is maximal, there exists $\varphi_x \in A$ such that $\varphi_x \leq b$ and $\varphi_x = (*^{n-1}x) * p_x$ for some $p_x \in P$. Similarly, there exists $\varphi_y \in A$ such that $\varphi_y \leq b$ and $\varphi_y = (*^{n-1}y) * p_y$ for some $p_y \in P$. Now, we have the following:

$$\begin{aligned} b &\geq ((*^{n-1}x) * p_x) \vee ((*^{n-1}y) * p_y) \\ &\geq (*^{n-1}(x * p_x)) \vee (*^{n-1}(y * p_y)) \\ &= *^{n-1}((x * p_x) \vee (y * p_y)) \\ &\geq *^{n-1}((x \vee (y * p_y)) * (p_x \vee (y * p_y))) \\ &\geq *^{n-1}((x \vee y) * p_y * p_x), \end{aligned}$$

where note that $*^{n-1}(x \vee y) = (*^{n-1}x) \vee (*^{n-1}y)$ and $x \vee (y * z) \geq (x \vee y) * (x \vee z)$ for any $x, y, z \in A$. Since $p_x, p_y, x \vee y \in P$, we have $b \in P$, which is a contradiction. Hence P is a prime **n-filter** of A . \square

We do not use $(-)^{\perp}$ or \rightarrow in the above proof and therefore the above proof works even for algebras of “intuitionistic Łukasiewicz n -valued logic.”

Definition 2.15. Let A be an L_n^c -algebra. A subset X of A has finite intersection property (f.i.p.) with respect to $*$ iff, for any $n \in \omega$ with $n \neq 0$, if $a_1, \dots, a_n \in X$ then $a_1 * \dots * a_n \neq 0$.

Corollary 2.16. *Let A be an \mathbb{L}_n^c -algebra and X a subset of A . If X has f.i.p. with respect to $*$, then there is a prime \mathbf{n} -filter P of A with $X \subset P$.*

Proof. By the assumption, we have a proper \mathbf{n} -filter F of A generated by X . By letting $b = 0$ in Proposition 2.14, we have a prime \mathbf{n} -filter P of A with $X \subset P$. \square

Proposition 2.17. *Let A be an \mathbb{L}_n^c -algebra. For a prime \mathbf{n} -filter P of A , define $v_P : A \rightarrow \mathbf{n}$ by $v_P(a) = r \Leftrightarrow T_r(a) \in P$. Then, v_P is a bijection from the set of all prime \mathbf{n} -filters of A to the set of all homomorphisms from A to \mathbf{n} with $v_P^{-1}(\{1\}) = P$.*

Proof. Note that v_P is well-defined as a function. We prove that v_P is a homomorphism. We first show $v_P(a * b) = v_P(a) * v_P(b)$ for $a, b \in A$. Let $r = v_P(a)$ and $s = v_P(b)$. Then $T_r(a) \in P$ and $T_s(b) \in P$. It is easy to see that $T_r(a) \wedge T_s(b) \leq T_{r*s}(a * b)$, which intuitively means that if the truth value of a is r and if the truth value of b is s then the truth value of $a * b$ is $r * s$. Since $T_r(a) \in P$ and $T_s(b) \in P$, we have $T_{r*s}(a * b) \in P$, whence we have $v_P(a * b) = r * s = v_P(a) * v_P(b)$.

Next we show that $v_P(a^\perp) = v_P(a)^\perp$. Let $r = v_P(a)$. It is easy to see that $T_r(a) \leq T_{r^\perp}(a^\perp)$. By $T_r(a) \in P$, we have $T_{r^\perp}(a^\perp) \in P$, whence $v_P(a^\perp) = r^\perp = v_P(a)^\perp$. As is well-known, $(\wedge, \vee, \wp, \rightarrow)$ can be defined by using only $(*, (-)^\perp)$ (see [6]) and so v_P preserves the operations $(\wedge, \vee, \wp, \rightarrow)$. Clearly, v_P preserves any constant $r \in \mathbf{n}$. Thus, v_P is a homomorphism. The remaining part of the proof is straightforward. \square

3 \mathbf{n} -valued fuzzy topology

Let us review basic concepts from fuzzy set theory and fuzzy topology.

3.1 \mathbf{n} -valued fuzzy set theory

An \mathbf{n} -fuzzy set on a set S is defined as a function from S to \mathbf{n} . For \mathbf{n} -fuzzy sets μ, λ on S , define an \mathbf{n} -fuzzy set $\mu @ \lambda$ on S by $(\mu @ \lambda)(x) = \mu(x) @ \lambda(x)$ for $@ = \wedge, \vee, *, \wp, \rightarrow$, and define an \mathbf{n} -fuzzy set μ^\perp on S by $(\mu^\perp)(x) = (\mu(x))^\perp$.

Let X, Y be sets and f a function from X to Y . For an \mathbf{n} -fuzzy set μ on X , define the direct image $f(\mu) : Y \rightarrow \mathbf{n}$ of μ under f by

$$f(\mu)(y) = \bigvee \{\mu(x) ; x \in f^{-1}(\{y\})\} \text{ for } y \in Y.$$

For $f : X \rightarrow Y$ and an \mathbf{n} -fuzzy set λ on Y , define the inverse image $f^{-1}(\lambda) : X \rightarrow \mathbf{n}$ of λ under f by $f^{-1}(\lambda) = \lambda \circ f$. Note that f^{-1} commutes with \bigvee , i.e., $f^{-1}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} f^{-1}(\mu_i)$ for \mathbf{n} -fuzzy sets μ_i on Y .

For a relation R on a set S and an \mathbf{n} -fuzzy set μ on S , define an \mathbf{n} -fuzzy set $R^{-1}[\mu]$ on S , which is called the inverse image of μ under R , by $R^{-1}[\mu](x) = \bigvee \{\mu(y) ; xRy\}$ for $x \in S$. Note that $R^{-1}[\bigvee_{i \in I} \mu_i] = \bigvee_{i \in I} (R^{-1}[\mu_i])$.

3.2 \mathbf{n} -valued fuzzy topology

For sets X and Y , Y^X denotes the set of all functions from X to Y . We do not distinguish between $r \in \mathbf{n}$ and the constant function whose value is always r .

Definition 3.1 ([30, 12, 27]). For a set S and a subset \mathcal{O} of \mathbf{n}^S , (S, \mathcal{O}) is an \mathbf{n} -fuzzy space iff the following hold:

- $r \in \mathcal{O}$ for any $r \in \mathbf{n}$;
- if $\mu_1, \mu_2 \in \mathcal{O}$ then $\mu_1 \wedge \mu_2 \in \mathcal{O}$;
- if $\mu_i \in \mathcal{O}$ for $i \in I$ then $\bigvee_{i \in I} \mu_i \in \mathcal{O}$,

Then, we call \mathcal{O} the \mathbf{n} -fuzzy topology of (S, \mathcal{O}) , and an element of \mathcal{O} an open \mathbf{n} -fuzzy set on (S, \mathcal{O}) . An \mathbf{n} -fuzzy set λ on S is a closed \mathbf{n} -fuzzy set on (S, \mathcal{O}) iff $\lambda = \mu^\perp$ for some open \mathbf{n} -fuzzy set μ on (S, \mathcal{O}) . A clopen \mathbf{n} -fuzzy set on (S, \mathcal{O}) means a closed and open \mathbf{n} -fuzzy set on (S, \mathcal{O}) .

An \mathbf{n} -fuzzy space (S, \mathcal{O}) is often denoted by its underlying set S .

Definition 3.2. For a set S , \mathbf{n}^S is called the discrete \mathbf{n} -fuzzy topology on S . (S, \mathbf{n}^S) is called a discrete \mathbf{n} -fuzzy space.

Definition 3.3. Let S_1 and S_2 be \mathbf{n} -fuzzy spaces. Then, $f : S_1 \rightarrow S_2$ is continuous iff, for any open \mathbf{n} -fuzzy set μ on S_2 , $f^{-1}(\mu)$ (i.e., $\mu \circ f$) is an open \mathbf{n} -fuzzy set on S_1 .

A composition of continuous functions between \mathbf{n} -fuzzy spaces is also continuous (as a function between \mathbf{n} -fuzzy spaces).

Definition 3.4. Let (S, \mathcal{O}) be an \mathbf{n} -fuzzy space. Then, an open basis \mathcal{B} of (S, \mathcal{O}) is a subset of \mathcal{O} such that the following holds: (i) \mathcal{B} is closed under \wedge ; (ii) for any $\mu \in \mathcal{O}$, there are $\mu_i \in \mathcal{B}$ for $i \in I$ with $\mu = \bigvee_{i \in I} \mu_i$.

Definition 3.5. An \mathbf{n} -fuzzy space S is Kolmogorov iff, for any $x, y \in S$ with $x \neq y$, there is an open \mathbf{n} -fuzzy set μ on S with $\mu(x) \neq \mu(y)$.

Definition 3.6. An \mathbf{n} -fuzzy space S is Hausdorff iff, for any $x, y \in S$ with $x \neq y$, there are $r \in \mathbf{n}$ and open \mathbf{n} -fuzzy sets μ, λ on S such that $\mu(x) \geq r$, $\lambda(y) \geq r$ and $\mu \wedge \lambda < r$.

Definition 3.7 ([12]). Let S be an \mathbf{n} -fuzzy space. An \mathbf{n} -fuzzy set λ on S is compact iff, if $\lambda \leq \bigvee_{i \in I} \mu_i$ for open \mathbf{n} -fuzzy sets μ_i on S , then there is a finite subset J of I such that $\lambda \leq \bigvee_{i \in J} \mu_i$.

Let 1 denote the constant function on S whose value is always 1 . Then, S is compact iff, if $1 = \bigvee_{i \in I} \mu_i$ for open \mathbf{n} -fuzzy sets μ_i on S , then there is a finite subset J of I such that $1 = \bigvee_{i \in J} \mu_i$.

We can construct an operation $(-)^*$ which turns an \mathbf{n} -fuzzy space into a topological space (in the classical sense) as follows.

Definition 3.8. Let (S, \mathcal{O}) be an \mathbf{n} -fuzzy space. Define

$$\mathcal{O}^* = \{\mu^{-1}(\{1\}); \mu \in \mathcal{O}\}.$$

Then, S^* denotes a topological space (S, \mathcal{O}^*) (see the below proposition).

Lemma 3.9. *Let (S, \mathcal{O}) be an \mathbf{n} -fuzzy space. Then, S^* forms a topological space.*

Proof. Since $0 \in \mathcal{O}$ and $\emptyset = 0^{-1}(\{1\})$, we have $\emptyset \in \mathcal{O}^*$. Similarly, $S \in \mathcal{O}^*$. Assume $X_i \in \mathcal{O}$ for $i \in I$. Then, $X_i = \mu_i^{-1}(\{1\})$ for some $\mu_i \in \mathcal{O}$. Since \mathbf{n} is totally ordered, $\bigcup_{i \in I} X_i = (\bigvee_{i \in I} \mu_i)^{-1}(\{1\})$. Thus, by $\bigvee_{i \in I} \mu_i \in \mathcal{O}$, we have $\bigcup_{i \in I} X_i \in \mathcal{O}^*$. It is easy to verify that $X, Y \in \mathcal{O}$ implies $X \cap Y \in \mathcal{O}^*$. \square

4 A fuzzy topological duality for L_n^c -algebras

In this section, we show a fuzzy topological duality for L_n^c -algebras, which is a generalization of Stone duality for Boolean algebras via fuzzy topology, where note that L_2^c -algebras coincide with Boolean algebras.

Definition 4.1. $L_n^c\text{-Alg}$ denotes the category whose objects are L_n^c -algebras and whose arrows are homomorphisms of L_n^c -algebras.

Our aim in this section is to show that the category $L_n^c\text{-Alg}$ is dually equivalent to the category FBS_n , which is defined in the following subsection.

4.1 Category \mathbf{FBS}_n

We equip \mathbf{n} with the discrete \mathbf{n} -fuzzy topology.

Definition 4.2. Let S be an \mathbf{n} -fuzzy space. Then, $\text{Cont}(S)$ is defined as the set of all continuous functions from S to \mathbf{n} . We endow $\text{Cont}(S)$ with the operations $(\wedge, \vee, *, \wp, \rightarrow, (-)^\perp, 0, 1/(n-1), 2/(n-1), \dots, 1)$ defined pointwise: For $f, g \in \text{Cont}(S)$, define $(f@g)(x) = f(x)@g(x)$, where $@ = \wedge, \vee, *, \wp, \rightarrow$. For $f \in \text{Cont}(S)$, define $f^\perp(x) = (f(x))^\perp$. Finally, $r \in \mathbf{n}$ is defined as the constant function on S whose value is always r .

We show that the operations of $\text{Cont}(S)$ are well-defined:

Lemma 4.3. *Let S be an \mathbf{n} -fuzzy space. Then, $\text{Cont}(S)$ is closed under the operations $(\wedge, \vee, *, \wp, \rightarrow, (-)^\perp, 0, 1/(n-1), \dots, (n-2)/(n-1), 1)$*

Proof. For any $r \in \mathbf{n}$, a constant function $r : S \rightarrow \mathbf{n}$ is continuous, since any $s \in \mathbf{n}$ is an open \mathbf{n} -fuzzy set on S by Definition 3.1. Then it suffices to show that, if $f, g \in \text{Cont}(S)$, then f^\perp and $f@g$ are continuous for $@ = \wedge, \vee, *, \wp, \rightarrow$. Throughout this proof, let $f, g \in \text{Cont}(S)$ and μ an open \mathbf{n} -fuzzy set on \mathbf{n} , i.e., a function from \mathbf{n} to \mathbf{n} . For $r \in \mathbf{n}$, define $\mu_r : \mathbf{n} \rightarrow \mathbf{n}$ by

$$\mu_r(x) = \begin{cases} \mu(r) & \text{if } x = r \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have $\mu = \bigvee_{r \in \mathbf{n}} \mu_r$.

We show that $(f^\perp)^{-1}(\mu)$ is an open \mathbf{n} -fuzzy set on S . Now, we have

$$(f^\perp)^{-1}(\mu) = (f^\perp)^{-1}\left(\bigvee_{r \in \mathbf{n}} \mu_r\right) = \bigvee_{r \in \mathbf{n}} ((f^\perp)^{-1}(\mu_r)).$$

Thus it suffices to show that $(f^\perp)^{-1}(\mu_r)$ is an open \mathbf{n} -fuzzy set on S for any $r \in \mathbf{n}$. Define $\lambda_r : \mathbf{n} \rightarrow \mathbf{n}$ by

$$\lambda_r(x) = \begin{cases} \mu(r) & \text{if } x = 1 - r \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that $(f^\perp)^{-1}(\mu_r) = f^{-1}(\lambda_r)$. Since f is continuous and since λ_r is an open \mathbf{n} -fuzzy set on \mathbf{n} , $f^{-1}(\lambda_r)$ is an open \mathbf{n} -fuzzy set on S .

Next, we show that $(f * g)^{-1}(\mu)$ is an open \mathbf{n} -fuzzy set on S . By the same argument as in the case of f^\perp , it suffices to show that $(f * g)^{-1}(\mu_r)$ is

an open \mathbf{n} -fuzzy set on S for any $r \in \mathbf{n}$. For $p \in \mathbf{n}$, define $\theta_{r,p} : \mathbf{n} \rightarrow \mathbf{n}$ by

$$\theta_{r,p}(x) = \begin{cases} \mu(r) & \text{if } x = p \\ 0 & \text{otherwise.} \end{cases}$$

For $r \neq 0$, define $\kappa_{r,p} : \mathbf{n} \rightarrow \mathbf{n}$ by

$$\kappa_{r,p}(x) = \begin{cases} \mu(r) & \text{if } x = r - p + 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 0$, define $\kappa_{r,p} : \mathbf{n} \rightarrow \mathbf{n}$ by

$$\kappa_{r,p}(x) = \begin{cases} \mu(r) & \text{if } x \leq r - p + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that

$$(f * g)^{-1}(\mu_r) = \bigvee_{p \in \mathbf{n}} (f^{-1}(\theta_{r,p}) \wedge g^{-1}(\kappa_{r,p})).$$

Since $f, g \in \text{Cont}(S)$, the right-hand side is an open \mathbf{n} -fuzzy set on S .

As is well-known, $(\wedge, \vee, \wp, \rightarrow)$ can be defined by using only $(*, (-)^\perp)$ (see [6]) and so $(f@g)^{-1}(\mu)$ is an open \mathbf{n} -fuzzy set for $@ = \wedge, \vee, \wp, \rightarrow$. \square

Definition 4.4. For an \mathbf{n} -fuzzy space S , S is zero-dimensional iff $\text{Cont}(S)$ forms an open basis of S .

Definition 4.5. For an \mathbf{n} -fuzzy space S , S is an \mathbf{n} -fuzzy Boolean space iff S is zero-dimensional, compact and Kolmogorov.

Definition 4.6. FBS_n is defined as the category of \mathbf{n} -fuzzy Boolean spaces and continuous functions.

Proposition 4.7. *Let S be an \mathbf{n} -fuzzy space. Then, (i) S is an \mathbf{n} -fuzzy Boolean space iff (ii) S is zero-dimensional, compact and Hausdorff.*

Proof. Clearly, (ii) implies (i). We show the converse. Assume that S is an \mathbf{n} -fuzzy Boolean space. It suffices to show that S is Hausdorff. Let $x, y \in S$ with $x \neq y$. Since S is Kolmogorov and since S is zero-dimensional, there is $\mu \in \text{Cont}(S)$ with $\mu(x) \neq \mu(y)$. Let $s = \mu(x)$. Then, $T_s \circ \mu(x) = 1$ and $(T_s \circ \mu)^\perp(y) = 1$. Since $T_s : \mathbf{n} \rightarrow \mathbf{n}$ is continuous, $T_s \circ \mu \in \text{Cont}(S)$ and $(T_s \circ \mu)^\perp \in \text{Cont}(S)$ by Lemma 4.3. Since S is zero-dimensional, $T_s \circ \mu$ and $(T_s \circ \mu)^\perp$ are open \mathbf{n} -fuzzy sets on S . We also have $(T_s \circ \mu) \wedge (T_s \circ \mu)^\perp = 0$. Thus, S is Hausdorff. \square

Next we show that $(-)^*$ turns an \mathbf{n} -fuzzy Boolean space into a Boolean space, i.e., a zero-dimensional compact Hausdorff space.

Proposition 4.8. *Let S be an \mathbf{n} -fuzzy Boolean space. Then, S^* forms a Boolean space.*

Proof. By Lemma 3.9, S^* is a topological space.

First, we show that S^* is zero-dimensional in the classical sense. Let $\mathcal{B}^* = \{\mu^{-1}(\{1\}) ; \mu \in \text{Cont}(S)\}$, where, since S is zero-dimensional and so $\mu \in \text{Cont}(S)$ is an open \mathbf{n} -fuzzy set on S , $\mu^{-1}(\{1\})$ is an open subset of S^* . We claim that \mathcal{B}^* forms an open basis of S^* . It is easily verified that \mathcal{B}^* is closed under \cap . Assume that O is an open subset of S^* , i.e., $O = \mu^{-1}(\{1\})$ for some open \mathbf{n} -fuzzy set μ on S . Since S is zero-dimensional, there are $\mu_i \in \text{Cont}(S)$ with $\mu = \bigvee_{i \in I} \mu_i$. Since \mathbf{n} is totally ordered, $O = \bigcup_{i \in I} \mu_i^{-1}(\{1\})$. It follows from $\mu_i \in \text{Cont}(S)$ that $\mu_i^{-1}(\{1\}) \in \mathcal{B}^*$ for any $i \in I$. This completes the proof of the claim. If $\mu \in \text{Cont}(S)$, then

$$(\mu^{-1}(\{1\}))^c = ((T_1 \circ \mu)^\perp)^{-1}(\{1\}).$$

Since $T_1 : \mathbf{n} \rightarrow \mathbf{n}$ is continuous, $T_1 \circ \mu \in \text{Cont}(S)$, whence, by Lemma 4.3, $(T_1 \circ \mu)^\perp \in \text{Cont}(S)$. Thus the right-hand side is open in S^* and so $\mu^{-1}(\{1\})$ is open in S^* for $\mu \in \text{Cont}(S)$. Hence, S^* is zero-dimensional.

Second, we show that S^* is compact in the classical sense. Assume that $S^* = \bigcup_{i \in I} O_i$ for some open subsets O_i of S^* . Since \mathcal{B}^* forms an open basis of S^* , we may assume that $S^* = \bigcup_{i \in I} \mu_i^{-1}(\{1\})$ for some $\mu_i \in \text{Cont}(S)$. Then, $1 = \bigvee_{i \in I} \mu_i$ where 1 denotes the constant function on S ($= S^*$) whose value is always 1. Since S is zero-dimensional, μ_i is an open \mathbf{n} -fuzzy set on S . Thus, since S is compact, there is a finite subset J of I such that $1 = \bigvee_{j \in J} \mu_j$, whence $S^* = \bigcup_{j \in J} \mu_j^{-1}(\{1\})$. Hence S^* is compact.

Finally, we show that S^* is Hausdorff in the classical sense. Since S^* is zero-dimensional, it suffices to show that S^* is Kolmogorov in the classical sense. Assume $x, y \in S^*$ with $x \neq y$. Since S is Kolmogorov, there is an open \mathbf{n} -fuzzy set μ on S with $\mu(x) \neq \mu(y)$. Since S is zero-dimensional, $\mu = \bigvee_{i \in I} \mu_i$ for some $\mu_i \in \text{Cont}(S)$. There is $i \in I$ with $\mu_i(x) \neq \mu_i(y)$. Let $r = \mu_i(x)$. Then, we have $T_r \circ \mu_i(x) = 1$ and $T_r \circ \mu_i(y) = 0$, whence we have $x \in (T_r \circ \mu_i)^{-1}(\{1\})$ and $y \notin (T_r \circ \mu_i)^{-1}(\{1\})$. Since $T_r : \mathbf{n} \rightarrow \mathbf{n}$ is continuous, it follows from $\mu_i \in \text{Cont}(S)$ that $T_r \circ \mu_i \in \text{Cont}(S)$, whence $T_r \circ \mu_i$ is an open \mathbf{n} -fuzzy set on S and so $(T_r \circ \mu_i)^{-1}(\{1\})$ is an open subset of S^* . Hence S^* is Kolmogorov. \square

4.2 Functors Spec and Cont

We define the spectrum $\text{Spec}(A)$ of an L_n^c -algebra A as follows.

Definition 4.9. For an \mathbf{L}_n^c -algebra A , $\text{Spec}(A)$ is defined as the set of all homomorphisms (of \mathbf{L}_n^c -algebras) from A to \mathbf{n} equipped with the \mathbf{n} -fuzzy topology generated by $\{\langle a \rangle; a \in A\}$, where $\langle a \rangle : \text{Spec}(A) \rightarrow \mathbf{n}$ is defined by

$$\langle a \rangle(v) = v(a).$$

The operations $(\wedge, \vee, *, \wp, \rightarrow, (-)^\perp)$ on $\{\langle a \rangle; a \in A\}$ are defined pointwise as in Definition 4.2.

$\{\langle a \rangle; a \in A\}$ forms an open basis of $\text{Spec}(A)$, since $\langle a \rangle \wedge \langle b \rangle = \langle a \wedge b \rangle$.

Definition 4.10. We define a contravariant functor $\text{Spec} : \mathbf{L}_n^c\text{-Alg} \rightarrow \text{FBS}_n$.

For an object A in $\mathbf{L}_n^c\text{-Alg}$, define $\text{Spec}(A)$ as in Definition 4.9.

For an arrow $f : A_1 \rightarrow A_2$ in $\mathbf{L}_n^c\text{-Alg}$, define $\text{Spec}(f) : \text{Spec}(A_2) \rightarrow \text{Spec}(A_1)$ by $\text{Spec}(f)(v) = v \circ f$ for $v \in \text{Spec}(A_2)$.

The well-definedness of the functor Spec is proved by Proposition 4.15 and Proposition 4.16 below.

Since \mathbf{n} is a totally ordered complete lattice, we have:

Lemma 4.11. Let μ_i be an \mathbf{n} -fuzzy set on a set S for a set I and $i \in I$. Then, (i) $\mathbf{T}_1 \circ \bigvee_{i \in I} \mu_i = \bigvee_{i \in I} (\mathbf{T}_1 \circ \mu_i)$; (ii) $\mathbf{T}_1 \circ \bigwedge_{i \in I} \mu_i = \bigwedge_{i \in I} (\mathbf{T}_1 \circ \mu_i)$.

Lemma 4.12. Let A be an \mathbf{L}_n^c -algebra. Then, $\text{Spec}(A)$ is compact.

Proof. Assume that $1 = \bigvee_{j \in J} \mu_j$ for open \mathbf{n} -fuzzy sets μ_j on $\text{Spec}(A)$, where 1 denotes the constant function defined on $\text{Spec}(A)$ whose value is always 1. Then, since $\{\langle a \rangle; a \in A\}$ is an open basis of $\text{Spec}(A)$, we may assume that $1 = \bigvee_{i \in I} \langle a_i \rangle$ for some $a_i \in A$. It follows from Lemma 4.11 that $1 = \mathbf{T}_1 \circ 1 = \mathbf{T}_1 \circ \bigvee_{i \in I} \langle a_i \rangle = \bigvee_{i \in I} \mathbf{T}_1 \circ \langle a_i \rangle = \bigvee_{i \in I} \langle \mathbf{T}_1(a_i) \rangle$. Thus, we have

$$0 = \left(\bigvee_{i \in I} \langle \mathbf{T}_1(a_i) \rangle \right)^\perp = \bigwedge_{i \in I} \langle (\mathbf{T}_1(a_i))^\perp \rangle.$$

Then, there is no homomorphism $v : A \rightarrow \mathbf{n}$ such that $v((\mathbf{T}_1(a_i))^\perp) = 1$ for any $i \in I$. Therefore, by Proposition 2.17, there is no prime \mathbf{n} -filter of A which contains $\{(\mathbf{T}_1(a_i))^\perp; i \in I\}$. Thus, by Corollary 2.16, $\{(\mathbf{T}_1(a_i))^\perp; i \in I\}$ does not have f.i.p. with respect to $*$ and so there is a finite subset $\{i_1, \dots, i_m\}$ of I such that $(\mathbf{T}_1(a_{i_1}))^\perp * \dots * (\mathbf{T}_1(a_{i_m}))^\perp = 0$, whence $\mathbf{T}_1(a_{i_1}) \wp \dots \wp \mathbf{T}_1(a_{i_m}) = 1$. Since $\mathbf{T}_1(a_{i_k})$ is idempotent for any $k \in \{1, \dots, m\}$, we have $\mathbf{T}_1(a_{i_1}) \vee \dots \vee \mathbf{T}_1(a_{i_m}) = 1$ and, by Lemma 2.7, $\mathbf{T}_1(a_{i_1} \vee \dots \vee a_{i_m}) = 1$. By $\mathbf{T}_1(x) \leq x$, we have $a_{i_1} \vee \dots \vee a_{i_m} = 1$, whence $\langle a_{i_1} \vee \dots \vee a_{i_m} \rangle = 1$. This completes the proof. \square

Lemma 4.13. *Let A be an L_n^c -algebra. Then, $\text{Spec}(A)$ is Kolmogorov.*

Proof. Let $v_1, v_2 \in \text{Spec}(A)$ with $v_1 \neq v_2$. Then there is $a \in A$ such that $v_1(a) \neq v_2(a)$, whence we have $\langle a \rangle(v_1) \neq \langle a \rangle(v_2)$. \square

Lemma 4.14. *Let A be an L_n^c -algebra. Then, $\text{Spec}(A)$ is zero-dimensional.*

Proof. Since $\{\langle a \rangle ; a \in A\}$ forms an open basis of $\text{Spec}(A)$, it suffices to show that

$$\text{Cont} \circ \text{Spec}(A) = \{\langle a \rangle ; a \in A\}.$$

We first show that $\text{Cont} \circ \text{Spec}(A) \supset \{\langle a \rangle ; a \in A\}$, i.e., $\langle a \rangle$ is continuous for any $a \in A$. Let $a \in A$ and μ an \mathbf{n} -fuzzy set on \mathbf{n} . Then, by Lemma 2.9,

$$\langle a \rangle^{-1}(\mu) = \mu \circ \langle a \rangle = \bigvee_{r \in \mathbf{n}} (\mathbb{S}_{\mu(r)} \circ \mathbb{T}_r) \circ \langle a \rangle = \langle \bigvee_{r \in \mathbf{n}} (\mathbb{S}_{\mu(r)}(\mathbb{T}_r(a))) \rangle.$$

Hence $\langle a \rangle$ is continuous.

Next we show $\text{Cont} \circ \text{Spec}(A) \subset \{\langle a \rangle ; a \in A\}$. Let $f \in \text{Cont} \circ \text{Spec}(A)$ and $r \in \mathbf{n}$. Define an \mathbf{n} -fuzzy set λ_r on \mathbf{n} by $\lambda_r(x) = 1$ for $x = r$ and $\lambda_r(x) = 0$ for $x \neq r$. Since f is continuous, $f^{-1}(\lambda_r) = \bigvee_{i \in I} \langle a_i \rangle$ for some $a_i \in A$. Now the following holds:

$$1 = f^{-1}(\lambda_r) \vee (f^{-1}(\lambda_r))^\perp = \left(\bigvee_{i \in I} \langle a_i \rangle \right) \vee (f^{-1}(\lambda_r))^\perp.$$

Here, we have $(f^{-1}(\lambda_r))^\perp = (\lambda_r \circ f)^\perp = \lambda_r^\perp \circ f = f^{-1}(\lambda_r^\perp)$. Since $f^{-1}(\lambda_r^\perp)$ is an open \mathbf{n} -fuzzy set, $(f^{-1}(\lambda_r))^\perp$ is an open \mathbf{n} -fuzzy set on $\text{Spec}(A)$. Since $\text{Spec}(A)$ is compact by Lemma 4.12, there is a finite subset J of I such that $1 = (\bigvee_{j \in J} \langle a_j \rangle) \vee (f^{-1}(\lambda_r))^\perp$. Thus, $f^{-1}(\lambda_r) \leq \bigvee_{j \in J} \langle a_j \rangle$. Since $\bigvee_{j \in J} \langle a_j \rangle \leq \bigvee_{i \in I} \langle a_i \rangle = f^{-1}(\lambda_r)$, we have $f^{-1}(\lambda_r) = \bigvee_{j \in J} \langle a_j \rangle$. Since J is finite, $f^{-1}(\lambda_r) = \bigvee_{j \in J} \langle a_j \rangle = \langle \bigvee_{j \in J} a_j \rangle$. Let $a_r = \bigvee_{j \in J} a_j$. Note that if $v \in f^{-1}(\{r\})$ then $v(a_r) = 1$ and that if $v \notin f^{-1}(\{r\})$ then $v(a_r) = 0$. We claim that $f = \langle \bigvee_{r \in \mathbf{n}} (r \wedge a_r) \rangle$. If $v \in f^{-1}(\{s\})$ for $s \in \mathbf{n}$, then

$$\left\langle \bigvee_{r \in \mathbf{n}} (r \wedge a_r) \right\rangle(v) = v \left(\bigvee_{r \in \mathbf{n}} (r \wedge a_r) \right) = \bigvee_{r \in \mathbf{n}} (r \wedge v(a_r)) = s = f(v).$$

This completes the proof. \square

By the above lemmas, we obtain the following proposition.

Proposition 4.15. *Let A be an object in $L_n^c\text{-Alg}$. Then, $\text{Spec}(A)$ is an object in the category FBS_n .*

Proposition 4.16. *Let A_1 and A_2 be objects in $\mathbf{L}_n^c\text{-Alg}$ and $f : A_1 \rightarrow A_2$ an arrow in $\mathbf{L}_n^c\text{-Alg}$. Then, $\text{Spec}(f)$ is an arrow in \mathbf{FBS}_n .*

Proof. Since the inverse image $(\text{Spec}(f))^{-1}$ commutes with \bigvee , it suffices to show that $(\text{Spec}(f))^{-1}(\langle a \rangle)$ is an open \mathbf{n} -fuzzy set on $\text{Spec}(A_2)$ for any $a \in A_1$. For $v \in \text{Spec}(A_2)$, we have

$$(\text{Spec}(f))^{-1}(\langle a \rangle)(v) = \langle a \rangle \circ \text{Spec}(f)(v) = \langle a \rangle(v \circ f) = v \circ f(a) = \langle f(a) \rangle(v).$$

Hence $(\text{Spec}(f))^{-1}(\langle a \rangle) = \langle f(a) \rangle$, which is an open \mathbf{n} -fuzzy set. \square

Definition 4.17. We define a contravariant functor $\text{Cont} : \mathbf{FBS}_n \rightarrow \mathbf{L}_n^c\text{-Alg}$.

For an object S in \mathbf{FBS}_n , $\text{Cont}(S)$ is defined as in Definition 4.2.

For an arrow $f : S \rightarrow T$ in \mathbf{FBS}_n , $\text{Cont}(f) : \text{Cont}(T) \rightarrow \text{Cont}(S)$ is defined by $\text{Cont}(f)(g) = g \circ f$ for $g \in \text{Cont}(T)$.

Since the operations of $\text{Cont}(S)$ are defined pointwise, $\text{Cont}(S)$ is an \mathbf{L}_n^c -algebra and the following holds, whence Cont is well-defined.

Proposition 4.18. *Let S_1 and S_2 be objects in \mathbf{FBS}_n , and $f : S_1 \rightarrow S_2$ an arrow in \mathbf{FBS}_n . Then, $\text{Cont}(f)$ is an arrow in $\mathbf{L}_n^c\text{-Alg}$.*

Definition 4.19. Let A be an \mathbf{L}_n^c -algebra. Then, $\text{Spec}_2(\mathcal{B}(A))$ is defined as the set of all homomorphisms of Boolean algebras from $\mathcal{B}(A)$ to $\mathbf{2}$ equipped with the (ordinary) topology generated by $\{\langle a \rangle_2 ; a \in \mathcal{B}(A)\}$, where $\langle a \rangle_2 = \{v \in \text{Spec}_2(\mathcal{B}(A)) ; v(a) = 1\}$.

Proposition 4.20. *Let A be an \mathbf{L}_n^c -algebra. Define a function t_1 from $\text{Spec}(A)^*$ to $\text{Spec}_2(\mathcal{B}(A))$ by $t_1(v) = T_1 \circ v$. Then, t_1 is a homeomorphism.*

Proof. By Lemma 2.10, t_1 is injective. We show that t_1 is surjective. Let $v \in \text{Spec}_2(\mathcal{B}(A))$. Define $u \in \text{Spec}(A)$ by $u(a) = r \Leftrightarrow T_r(a) \in v^{-1}(\{1\})$ for $a \in A$, where note $T_r(a) \in \mathcal{B}(A)$. Then, in a similar way to Proposition 2.17, it is verified that u is a homomorphism (i.e., $u \in \text{Spec}(A)$). Moreover, we have $t_1(u) = v$ on $\mathcal{B}(A)$. Thus t_1 is bijective. It is straightforward to verify the remaining part of the proof. Note that, for $\langle a \rangle_{\mathbf{n}} = \{v \in \text{Spec}(A) ; v(a) = 1\}$, $\{\langle a \rangle_{\mathbf{n}} ; a \in A\}$ forms an open basis of $\text{Spec}(A)^*$ and that $t_1(\langle a \rangle_{\mathbf{n}}) = \langle T_1(a) \rangle_2$ for $a \in A$. \square

4.3 A fuzzy topological duality for \mathbf{L}_n^c -algebras

Theorem 4.21. *Let A be an \mathbf{L}_n^c -algebra. Then, there is an isomorphism between A and $\text{Cont} \circ \text{Spec}(A)$ in the category $\mathbf{L}_n^c\text{-Alg}$.*

Proof. Define $\langle - \rangle : A \rightarrow \text{Cont} \circ \text{Spec}(A)$ as in Definition 4.9. In the proof of Lemma 4.14, it has already been proven that $\langle - \rangle$ is well-defined and surjective. Since the operations of $\text{Cont} \circ \text{Spec}(A)$ are defined pointwise, $\langle - \rangle$ is a homomorphism.

Thus it suffices to show that $\langle - \rangle$ is injective. Assume that $\langle a \rangle = \langle b \rangle$ for $a, b \in A$, which means that, for any $v \in \text{Spec}(A)$, we have $v(a) = v(b)$. Thus, for any $v \in \text{Spec}(A)$ and any $r \in \mathbf{n}$, we have $v(\mathbb{T}_r(a)) = v(\mathbb{T}_r(b))$. Thus, it follows from Proposition 2.17 that, for any prime \mathbf{n} -filter P of A and any $r \in \mathbf{n}$, $\mathbb{T}_r(a) \in P$ iff $\mathbb{T}_r(b) \in P$.

We claim that $\mathbb{T}_r(a) = \mathbb{T}_r(b)$ for any $r \in \mathbf{n}$. Suppose for contradiction that $\mathbb{T}_r(a) \neq \mathbb{T}_r(b)$ for some $r \in \mathbf{n}$. We may assume without loss of generality that $\mathbb{T}_r(a) \not\leq \mathbb{T}_r(b)$. Let $F = \{x \in A ; \mathbb{T}_r(a) \leq x\}$. Then, since $\mathbb{T}_r(a)$ is idempotent, F is an \mathbf{n} -filter of A . Clearly, $\mathbb{T}_r(b) \notin F$. Thus, by Lemma 2.14, there is a prime \mathbf{n} -filter P of A such that $F \subset P$ and $\mathbb{T}_r(b) \notin P$. By $F \subset P$, we have $\mathbb{T}_r(a) \in P$, which contradicts $\mathbb{T}_r(b) \notin P$, since we have already shown that $\mathbb{T}_r(a) \in P$ iff $\mathbb{T}_r(b) \in P$. Thus, $\mathbb{T}_r(a) = \mathbb{T}_r(b)$ for any $r \in \mathbf{n}$, whence $\bigwedge_{r \in \mathbf{n}} (\mathbb{T}_r(a) \leftrightarrow \mathbb{T}_r(b)) = 1$. Hence, it follows from Lemma 2.11 that $a = b$, and therefore $\langle - \rangle$ is injective. \square

Theorem 4.22. *Let S be an \mathbf{n} -fuzzy Boolean space. Then, there is an isomorphism between S and $\text{Spec} \circ \text{Cont}(S)$ in the category FBS_n .*

Proof. Define $\Psi : S \rightarrow \text{Spec} \circ \text{Cont}(S)$ by $\Psi(x)(f) = f(x)$ for $x \in S$ and $f \in \text{Cont}(S)$. Since the operations of $\text{Cont}(S)$ are defined pointwise, $\Psi(x)$ is a homomorphism and so Ψ is well-defined.

We show that Ψ is continuous. Let $f \in \text{Cont}(S)$. Then $\Psi^{-1}(\langle f \rangle) = f$ by the following:

$$(\Psi^{-1}(\langle f \rangle))(x) = \langle f \rangle \circ \Psi(x) = \Psi(x)(f) = f(x).$$

Since $f \in \text{Cont}(S)$ and S is zero-dimensional, f is an open \mathbf{n} -fuzzy set and so $\Psi^{-1}(\langle f \rangle)$ is an open \mathbf{n} -fuzzy set on S . Since the inverse image Ψ^{-1} commutes with \bigvee , it follows that Ψ is continuous.

Next we show that Ψ is injective. Let $x, y \in S$ with $x \neq y$. Since S is Kolmogorov and zero-dimensional, there is $f \in \text{Cont}(S)$ with $f(x) \neq f(y)$. Thus, $\Psi(x)(f) = f(x) \neq f(y) = \Psi(y)(f)$, whence Ψ is injective.

Next we show that Ψ is surjective. Let $v \in \text{Spec} \circ \text{Cont}(S)$. Consider $\{f^{-1}(\{1\}) ; v(f) = 1\}$. Define $\mu : \mathbf{n} \rightarrow \mathbf{n}$ by $\mu(1) = 0$ and $\mu(x) = 1$ for $x \neq 1$. Since $f^{-1}(\mu) (= \mu \circ f)$ is an open \mathbf{n} -fuzzy set on S for $f \in \text{Cont}(S)$, $(\mu \circ f)^{-1}(\{1\})$ is an open subset of S^* . Since $(\mu \circ f)^{-1}(\{1\}) = (f^{-1}(\{1\}))^c$, $f^{-1}(\{1\})$ is a closed subset of S^* for $f \in \text{Cont}(S)$.

We claim that $\{f^{-1}(\{1\}); v(f) = 1\}$ has the finite intersection property. Since $f^{-1}(\{1\}) \cap g^{-1}(\{1\}) = (f \wedge g)^{-1}(\{1\})$ for $f, g \in \text{Cont}(S)$, it suffices to show that if $v(f) = 1$ then $f^{-1}(\{1\})$ is not empty. Suppose for contradiction that $v(f) = 1$ and $f^{-1}(\{1\}) = \emptyset$. Since $f^{-1}(\{1\}) = \emptyset$, we have $T_1(f) = 0$. Thus $v(T_1(f)) = 0$ and so $v(f) \neq 1$, which contradicts $v(f) = 1$.

By Proposition 4.8, S^* is compact. Thus, there is $z \in S$ such that $z \in \bigcap \{f^{-1}(\{1\}); v(f) = 1\}$. We claim that $\Psi(z) = v$. By the definition of z , if $v(f) = 1$ then $\Psi(z)(f) = 1$. We show the converse. Suppose for contradiction that $\Psi(z)(f) = 1$ and $v(f) \neq 1$. Then $v(T_1(f)) = T_1(v(f)) = 0$ and so $v((T_1(f))^\perp) = 1$. By the definition of z , $(T_1(f))^\perp(z) = 1$ and so $(T_1(f))(z) = 0$. Thus $f(z) \neq 1$, which contradicts $\Psi(z)(f) = 1$. Hence, for any $f \in \text{Cont}(S)$, $v(f) = 1$ iff $\Psi(z)(f) = 1$. By Lemma 2.10, we have $\Psi(z) = v$. Hence, Ψ is surjective.

Finally we show that Ψ^{-1} is an arrow in the category FBS_n . It suffices to show that, for any open \mathbf{n} -fuzzy set λ on S , $\Psi(\lambda)$ is an open \mathbf{n} -fuzzy set on $\text{Spec} \circ \text{Cont}(S)$. Since S is zero-dimensional, there are $f_i \in \text{Cont}(S)$ with $\lambda = \bigvee_{i \in I} f_i$. For $v \in \text{Spec} \circ \text{Cont}(S)$, the following holds:

$$\Psi(\lambda)(v) = \bigvee \{\lambda(x); x \in \Psi^{-1}(\{v\})\} = \lambda(z) = v(\lambda) = v\left(\bigvee_{i \in I} f_i\right) = \left(\bigvee_{i \in I} \langle f_i \rangle\right)(v),$$

where z is defined as the unique element x such that $\Psi(x) = v$ (for the definition of the direct image of an \mathbf{n} -fuzzy set, see Subsection 3.1). Hence $\Psi(\lambda) = \bigvee_{i \in I} \langle f_i \rangle$ and so $\Psi(\lambda)$ is an open \mathbf{n} -fuzzy set on $\text{Spec} \circ \text{Cont}(S)$. \square

By Theorem 4.21 and Theorem 4.22, we obtain a fuzzy topological duality for \mathbb{L}_n^c -algebras, which is a generalization of Stone duality for Boolean algebras to the n -valued case via fuzzy topology.

Theorem 4.23. *The category $\mathbb{L}_n^c\text{-Alg}$ is dually equivalent to the category FBS_n via the functors Spec and Cont .*

Proof. Let Id_1 denote the identity functor on $\mathbb{L}_n^c\text{-Alg}$ and Id_2 denote the identity functor on FBS_n . Then, we define two natural transformations $\epsilon : \text{Id}_1 \rightarrow \text{Cont} \circ \text{Spec}$ and $\eta : \text{Id}_2 \rightarrow \text{Spec} \circ \text{Cont}$. For an \mathbb{L}_n^c -algebra A , define $\epsilon_A : A \rightarrow \text{Cont} \circ \text{Spec}(A)$ by $\epsilon_A = \langle - \rangle$ (see Theorem 4.21). For an \mathbf{n} -fuzzy Boolean space S , define $\eta_S : S \rightarrow \text{Spec} \circ \text{Cont}(S)$ by $\eta_S = \Psi$ (see Theorem 4.22). It is straightforward to see that η and ϵ are natural transformations. By Theorem 4.21 and Theorem 4.22, η and ϵ are natural isomorphisms. \square

5 ML_n^c -algebras and basic properties

We define modal Łukasiewicz n -valued logic with truth constants ML_n^c by \mathbf{n} -valued Kripke semantics. The connectives of ML_n^c are a unary connective \Box and the connectives of L_n^c . \mathbf{Form}_\Box denotes the set of formulas of ML_n^c .

Definition 5.1. Let (W, R) be a Kripke frame (i.e., R is a relation on a set W). Then, e is a Kripke \mathbf{n} -valuation on (W, R) iff e is a function from $W \times \mathbf{Form}_\Box$ to \mathbf{n} which satisfies: For each $w \in W$ and $\varphi, \psi \in \mathbf{Form}_\Box$,

- $e(w, \Box\varphi) = \bigwedge \{e(w', \varphi) ; wRw'\}$;
- $e(w, \varphi @ \psi) = e(w, \varphi) @ e(w, \psi)$ for $@ = \wedge, \vee, *, \wp, \rightarrow$;
- $e(w, \varphi^\perp) = (e(w, \varphi))^\perp$;
- $e(w, r) = r$ for $r \in \mathbf{n}$.

Then, (W, R, e) is called an \mathbf{n} -valued Kripke model. Define ML_n^c as the set of all those formulas $\varphi \in \mathbf{Form}_\Box$ such that $e(w, \varphi) = 1$ for any \mathbf{n} -valued Kripke model (W, R, e) and any $w \in W$.

By straightforward computation, we have the following lemma. Recall the definition of U_r (Definition 2.8).

Lemma 5.2. Let $\varphi, \psi \in \mathbf{Form}_\Box$ and $r \in \mathbf{n}$. (i) $U_r(\Box\varphi) \leftrightarrow \Box U_r(\varphi) \in \text{ML}_n^c$. (ii) $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi \in \text{ML}_n^c$ and $\Box 1 \leftrightarrow 1 \in \text{ML}_n^c$. (iii) $\Box(\varphi * \psi) \leftrightarrow (\Box\varphi) * (\Box\psi) \in \text{ML}_n^c$ and $\Box(\varphi \wp \psi) \leftrightarrow (\Box\varphi) \wp (\Box\psi) \in \text{ML}_n^c$.

Definition 5.3. For $X \subset \mathbf{Form}_\Box$, X is satisfiable iff there are an \mathbf{n} -valued Kripke model (W, R, e) and $w \in W$ such that $e(w, \varphi) = 1$ for any $\varphi \in X$.

ML_n^c -algebras and homomorphisms are defined as follows.

Definition 5.4. Let A be an L_n^c -algebra. Then, (A, \Box) is an ML_n^c -algebra iff it satisfies the following set of equations: $\{\varphi = \psi ; \varphi \leftrightarrow \psi \in \text{ML}_n^c\}$.

A homomorphism of ML_n^c -algebras is defined as a homomorphism of L_n^c -algebras which additionally preserves the operation \Box .

Throughout this paper, we do not distinguish between formulas of ML_n^c and terms of ML_n^c -algebras.

Definition 5.5. Let A be an ML_n^c -algebra. Define a relation R_\Box on $\text{Spec}(A)$ by

$$vR_\Box u \Leftrightarrow \forall r \in \mathbf{n} \forall x \in A (v(\Box x) \geq r \text{ implies } u(x) \geq r).$$

Define $e : \text{Spec}(A) \times A \rightarrow \mathbf{n}$ by $e(v, x) = v(x)$ for $v \in \text{Spec}(A)$ and $x \in A$. Then, $(\text{Spec}(A), R_\Box, e)$ is called the \mathbf{n} -valued canonical model of A .

Proposition 5.6. *Let A be an ML_n^c -algebra. Then, the \mathbf{n} -valued canonical model $(\text{Spec}(A), R_\square, e)$ of A is an \mathbf{n} -valued Kripke model. In particular, $e(v, \square x) = v(\square x) = \bigwedge \{u(x); vR_\square u\}$ for $x \in A$ and $v \in \text{Spec}(A)$.*

Proof. It suffices to show that e is a Kripke \mathbf{n} -valuation. Since v is a homomorphism of L_n^c -algebras, it remains to show $e(v, \square x) = \bigwedge \{u(x); vR_\square u\}$. To prove this, it is enough to show that, for any $r \in \mathbf{n}$, (i) $v(\square x) \geq r$ iff (ii) $vR_\square u$ implies $u(x) \geq r$. By the definition of R_\square , (i) implies (ii). We show the converse. To prove the contrapositive, assume $v(\square x) \not\geq r$, i.e., $U_r(\square x) \notin v^{-1}(\{1\})$. Let

$$F_0 = \{U_s(x); s \in \mathbf{n} \text{ and } U_s(\square x) \in v^{-1}(\{1\})\}.$$

Let F be the \mathbf{n} -filter of A generated by F_0 . We claim that $U_r(x) \notin F$. Suppose for contradiction that $U_r(x) \in F$. Then, there is $\varphi \in A$ such that $\varphi \leq U_r(x)$ and φ is constructed from $*$ and elements of F_0 . Since $U_s(x)$ is idempotent, $U_{s_1}(x_1) * U_{s_2}(x_2) = U_{s_1}(x_1) \wedge U_{s_2}(x_2)$ and so we may assume that $\varphi = \bigwedge \{U_s(x); U_s(x) \in F_1\}$ for some finite subset F_1 of F_0 . By Lemma 5.2, $\square\varphi = \bigwedge \{U_s(\square x); U_s(x) \in F_1\}$. By the definition of F_0 , $U_s(\square x) \in v^{-1}(\{1\})$ for any $U_s(x) \in F_1$ and so $\square\varphi \in v^{-1}(\{1\})$. Since $\varphi \leq U_r(x)$, we have $\square\varphi \leq \square U_r(x) = U_r(\square x)$. Thus, $U_r(\square x) \in v^{-1}(\{1\})$, which contradicts $U_r(\square x) \notin v^{-1}(\{1\})$. Hence $U_r(x) \notin F$. By Proposition 2.14, there is a prime \mathbf{n} -filter P of A such that $U_r(x) \notin P$ and $F \subset P$. By Proposition 2.17, $v_P \in \text{Spec}(A)$. Since $U_r(x) \notin P$, we have $v_P(x) \not\geq r$. Since $F_0 \subset F \subset P$, we have $vR_\square v_P$. Thus, (ii) does not hold. \square

The following is a compactness theorem for ML_n^c .

Theorem 5.7. *Let $X \subset \mathbf{Form}_\square$. Assume that any finite subset of X is satisfiable. Then, X is satisfiable.*

Proof. Let A be the Lindenbaum algebra of ML_n^c . We may consider $X \subset A$. We show that X has f.i.p. with respect to $*$. If not, then there are $n \in \omega$ with $n \neq 0$ and $x_1, \dots, x_n \in X$ such that $x_1 * \dots * x_n = 0$, which is a contradiction, since $\{x_1, \dots, x_n\}$ is satisfiable by assumption. Thus, by Proposition 2.16, there is a prime \mathbf{n} -filter P of A with $X \subset P$. By Proposition 2.17, v_P is a homomorphism, i.e., $v_P \in \text{Spec}(A)$. Consider the \mathbf{n} -valued canonical model $(\text{Spec}(A), R_\square, e)$ of A . Then, $e(v_P, x) = v_P(x) = 1$ for any $x \in X$ by Proposition 2.17. Thus, X is satisfiable. \square

Proposition 5.8. *Let A be an ML_n^c -algebra. Then, $\mathcal{B}(A)$ forms a modal algebra.*

Proof. If $x \in A$ is idempotent, then $\Box x$ is also idempotent, since $\Box x * \Box x = \Box(x * x) = \Box x$ by Lemma 5.2. Thus, $\mathcal{B}(A)$ is closed under \Box . By Lemma 5.2, $\mathcal{B}(A)$ forms a modal algebra. \square

Definition 5.9. Let A be an ML_n^c -algebra. Define a relation R_{\Box_2} on $\text{Spec}_2(\mathcal{B}(A))$ by $vR_{\Box_2}u \Leftrightarrow \forall x \in \mathcal{B}(A) (v(\Box x) = 1 \text{ implies } u(x) = 1)$.

Proposition 5.10. Let A be an ML_n^c -algebra. For $v, u \in \text{Spec}(A)$, $vR_{\Box}u$ iff $t_1(v)R_{\Box_2}t_1(u)$ (for the definition of t_1 , see Proposition 4.20).

Proof. By $\Box T_1(x) = T_1(\Box x)$, if $vR_{\Box}u$ then $t_1(v)R_{\Box_2}t_1(u)$. We show the converse. Assume $t_1(v)R_{\Box_2}t_1(u)$. In order to show $vR_{\Box}u$, it suffices to prove that, for any $r \in \mathbf{n}$ and any $x \in A$, $v(\Box U_r(x)) = 1$ implies $u(U_r(x)) = 1$, which follows from the assumption, since we have $U_r(x) \in \mathcal{B}(A)$ and $T_1(U_r(x)) = U_r(x)$. \square

6 A fuzzy topological duality for ML_n^c -algebras

In this section, based on the fuzzy topological duality for L_n^c -algebras, we show a fuzzy topological duality for ML_n^c -algebras, which is a generalization of Jónsson-Tarski duality for modal algebras via fuzzy topology, where note that ML_2^c -algebras coincide with modal algebras.

Definition 6.1. $\text{ML}_n^c\text{-Alg}$ denotes the category of ML_n^c -algebras and homomorphisms of ML_n^c -algebras.

Our aim in this section is to show that the category $\text{ML}_n^c\text{-Alg}$ is dually equivalent to the category FRS_n , which is defined in Definition 6.3 below.

For a Kripke frame (S, R) , we can define a modal operator \Box on the “ \mathbf{n} -valued powerset algebra” \mathbf{n}^S of S as follows.

Definition 6.2. Let (S, R) be a Kripke frame and f a function from S to \mathbf{n} . Define $\Box_R f : S \rightarrow \mathbf{n}$ by $(\Box_R f)(x) = \bigwedge \{f(y) ; xRy\}$.

Recall: For a Kripke frame (S, R) and an \mathbf{n} -fuzzy set μ on S , an \mathbf{n} -fuzzy set $R^{-1}[\mu]$ on S is defined by $R^{-1}[\mu](x) = \bigvee \{\mu(y) ; xRy\}$ for $x \in S$.

Definition 6.3. We define the category FRS_n as follows.

An object in FRS_n is a tuple (S, R) such that S is an object in FBS_n and that a relation R on S satisfies the following conditions:

1. if $\forall f \in \text{Cont}(S)((\Box_R f)(x) = 1 \Rightarrow f(y) = 1)$ then xRy ;

2. if $\mu \in \text{Cont}(S)$, then $R^{-1}[\mu] \in \text{Cont}(S)$.

An arrow $f : (S_1, R_1) \rightarrow (S_2, R_2)$ in FRS_n is an arrow $f : S_1 \rightarrow S_2$ in FBS_n which satisfies the following conditions:

1. if xR_1y then $f(x)R_2f(y)$;
2. if $f(x_1)R_2x_2$ then there is $y_1 \in S_1$ such that $x_1R_1y_1$ and $f(y_1) = x_2$.

An object in FRS_n is called an \mathbf{n} -fuzzy relational space.

The item 1 in the object part of Definition 6.3 is an \mathbf{n} -fuzzy version of the tightness condition of descriptive general frames in classical modal logic (for the definition of the tightness condition in classical modal logic, see [3]).

Definition 6.4. We define a contravariant functor $\text{RSpec} : \text{ML}_n^c\text{-Alg} \rightarrow \text{FRS}_n$. For an object A in $\text{ML}_n^c\text{-Alg}$, define $\text{RSpec}(A) = (\text{Spec}(A), R_\square)$. For an arrow $f : A \rightarrow B$ in $\text{ML}_n^c\text{-Alg}$, define $\text{RSpec}(f) : \text{RSpec}(B) \rightarrow \text{RSpec}(A)$ by $\text{RSpec}(f)(v) = v \circ f$ for $v \in \text{Spec}(B)$.

We call $\text{RSpec}(A)$ the relational spectrum of A . The well-definedness of RSpec is shown by Proposition 6.6 and Proposition 6.7 below.

Definition 6.5. Let A be an ML_n^c -algebra. Then, we define $\text{RSpec}_2(\mathcal{B}(A))$ as $(\text{Spec}_2(\mathcal{B}(A)), R_{\square_2})$. Let A_1 and A_2 be ML_n^c -algebras and $f : \mathcal{B}(A_1) \rightarrow \mathcal{B}(A_2)$. Then, we define $\text{RSpec}_2(f) : \text{RSpec}_2(\mathcal{B}(A_2)) \rightarrow \text{RSpec}_2(\mathcal{B}(A_1))$ by $\text{RSpec}_2(f)(v) = v \circ f$ for $v \in \text{RSpec}_2(\mathcal{B}(A_2))$.

Proposition 6.6. For an ML_n^c -algebra A , $\text{RSpec}(A)$ is an object in FRS_n .

Proof. It suffices to show the items 1 and 2 in the object part of Definition 6.3. We first show the item 1 by proving the contrapositive. Assume $(v, u) \notin R_\square$, i.e., there are $r \in \mathbf{n}$ and $x \in A$ such that $v(\square x) \geq r$ and $u(x) \not\geq r$. By Lemma 2.8, $v(\text{U}_r(\square x)) = 1$ and $u(\text{U}_r(x)) = 0$. Then, $\langle \text{U}_r(x) \rangle(u) = 0$. By Proposition 5.6 and Lemma 5.2,

$$(\square_R \langle \text{U}_r(x) \rangle)(v) = \bigwedge \{ \langle \text{U}_r(x) \rangle(v') ; vR_\square v' \} = v(\square \text{U}_r(x)) = v(\text{U}_r \square x) = 1.$$

As is shown in the proof of Lemma 4.14, $\langle \text{U}_r(x) \rangle$ is continuous.

We show the item 2. Since $\text{Cont} \circ \text{Spec}(A) = \{ \langle x \rangle ; x \in A \}$ as is shown in the proof of Lemma 4.14, it suffices to show that, for any $x \in A$, $R_\square^{-1}(\langle x \rangle) \in \text{Cont} \circ \text{Spec}(A)$. Let $\diamond x$ denote $(\square(x^\perp))^\perp$. Since $(R_\square^{-1}(\langle x \rangle))(v) = \bigvee \{ u(x) ; vR_\square u \} = v(\diamond x)$, we have $R_\square^{-1}(\langle x \rangle) = \langle \diamond x \rangle \in \text{Cont} \circ \text{Spec}(A)$. \square

Proposition 6.7. For ML_n^c -algebras A_1 and A_2 , let $f : A_1 \rightarrow A_2$ be a homomorphism of ML_n^c -algebras. Then, $\text{RSpec}(f)$ is an arrow in FRS_n .

Proof. Define $f_* : \mathcal{B}(A_1) \rightarrow \mathcal{B}(A_2)$ by $f_*(x) = f(x)$ for $x \in \mathcal{B}(A_1)$. By Proposition 5.8, f_* is a homomorphism of modal algebras. Consider $\text{RSpec}_2(f_*) : \text{RSpec}_2(\mathcal{B}(A_2)) \rightarrow \text{RSpec}_2(\mathcal{B}(A_1))$. By Jónsson-Tarski duality for modal algebras (see [16, 1]), $\text{RSpec}_2(f_*)$ is an arrow in FRS_2 .

We first show that $\text{RSpec}(f)$ satisfies the item 2 in the arrow part of Definition 6.3. Assume $\text{RSpec}(f)(v_2)R_{\square}u_1$ for $v_2 \in \text{RSpec}(A_2)$ and $u_1 \in \text{RSpec}(A_1)$. By Proposition 5.10, $t_1(\text{RSpec}(f)(v_2))R_{\square_2}t_1(u_1)$. It follows from $t_1(\text{RSpec}(f)(v_2)) = T_1 \circ v_2 \circ f = \text{RSpec}_2(f_*)(t_1(v_2))$ that we have $\text{RSpec}_2(f_*)(t_1(v_2))R_{\square_2}t_1(u_1)$. Since $\text{RSpec}_2(f_*)$ is an arrow in FRS_2 , there is $u_2 \in \text{RSpec}_2(\mathcal{B}(A_2))$ such that $t_1(v_2)R_{\square_2}u_2$ and $\text{RSpec}_2(f_*)(u_2) = t_1(u_1)$. Define $u'_2 \in \text{RSpec}(A_2)$ by $u'_2(x) = r \Leftrightarrow u_2(T_r(x)) = 1$. It is verified in a similar way to Proposition 2.17 that u'_2 is a homomorphism.

We claim that $v_2R_{\square}u'_2$ and $\text{RSpec}(f)(u'_2) = u_1$. Let $x \in A_2$ and $r \in \mathbf{n}$. If $v_2(\square x) \geq r$ then $(t_1(v_2))(\square U_r(x)) = 1$ and, since $t_1(v_2)R_{\square_2}u_2$, we have $u_2(U_r(x)) = 1$, whence $u'_2(x) \geq r$. Thus, $v_2R_{\square}u'_2$. Next we show $\text{RSpec}(f)(u'_2) = u_1$. Let $r = (\text{RSpec}(f)(u'_2))(x)$ for $x \in A_1$. Then, $u_2(T_r(f(x))) = 1$ and so $(\text{RSpec}_2(f_*)(u_2))(T_r(x)) = 1$. It follows from $\text{RSpec}_2(f_*)(u_2) = t_1(u_1)$ that $(t_1(u_1))(T_r(x)) = 1$ and so $u_1(T_r(x)) = 1$, whence $u_1(x) = r = (\text{RSpec}(f)(u'_2))(x)$. Thus $\text{RSpec}(f)$ satisfies the item 2.

It is easier to verify that $\text{RSpec}(f)$ satisfies the item 1 in the arrow part of Definition 6.3. \square

Definition 6.8. A contravariant functor $\text{MCont} : \text{FRS}_n \rightarrow \text{ML}_n^c\text{-Alg}$ is defined as follows. For an object (S, R) in FRS_n , define $\text{MCont}(S, R) = (\text{Cont}(S), \square_R)$. For an arrow $f : (S_1, R_1) \rightarrow (S_2, R_2)$ in FRS_n , define $\text{MCont}(f) : \text{MCont}(S_2, R_2) \rightarrow \text{MCont}(S_1, R_1)$ by $\text{MCont}(f)(g) = g \circ f$ for $g \in \text{Cont}(S_2)$.

The well-definedness of MCont is shown by the following propositions.

Proposition 6.9. For an object (S, R) in FRS_n , $\text{MCont}(S, R)$ is an ML_n^c -algebra.

Proof. We first show that if $f \in \text{Cont}(S)$ then $\square_R f \in \text{Cont}(S)$. Let $f \in \text{Cont}(S)$ and μ an open \mathbf{n} -fuzzy set on \mathbf{n} . Define μ_r as in the proof of Lemma 4.3 and then it suffices to show that $(\square_R f)^{-1}(\mu_r)$ is an open \mathbf{n} -fuzzy set on S for any $r \in \mathbf{n}$. By Lemma 2.8,

$$(\square_R f)^{-1}(\mu_r) = R^{-1}[\mu_r \circ f] \wedge (R^{-1}[(U_r \circ f)^\perp])^\perp.$$

Since both $\mu_r \circ f$ and $(U_r \circ f)^\perp$ are elements of $\text{Cont}(S)$, the right-hand side is an element of $\text{Cont}(S)$ by the definition of R and so is an open \mathbf{n} -fuzzy set on S , since S is zero-dimensional. Thus $\Box_R f \in \text{Cont}(S)$.

Next we show that $\text{MCont}(S, R)$ satisfies $\{\varphi = \psi ; \varphi \leftrightarrow \psi \in \text{ML}_n^c\}$. Consider $\text{Cont}(S)$ as the set of propositional variables. Since $\text{Cont}(S)$ is closed under the operations of $\text{Cont}(S)$, an element of \mathbf{Form}_\square may be seen as an element of $\text{Cont}(S)$. Define $e : S \times \mathbf{Form}_\square \rightarrow \mathbf{n}$ by $e(w, f) = f(w)$ for $w \in S$ and $f \in \text{Cont}(S)$. Then, (S, R, e) is an \mathbf{n} -valued Kripke model by the definition of the operations of $\text{Cont}(S)$. Since $e(w, f) = 1$ for any $w \in S$ iff $f = 1$, it follows from the definition of ML_n^c that $\text{MCont}(S, R)$ satisfies $\{\varphi = \psi ; \varphi \leftrightarrow \psi \in \text{ML}_n^c\}$. \square

Proposition 6.10. *Let $f : (S_1, R_1) \rightarrow (S_2, R_2)$ be an arrow in FRS_n . Then, $\text{MCont}(f)$ is a homomorphism of ML_n^c -algebras.*

Proof. It remains to show that $\text{MCont}(f)(\Box g_2) = \Box(\text{MCont}(f)(g_2))$ for $g_2 \in \text{Cont}(S_2)$. For $x_1 \in S_1$, $(\text{MCont}(f)(\Box g_2))(x_1) = \bigwedge \{g_2(y_2) ; f(x_1)R_2y_2\}$. Let a denote the right-hand side. We also have $(\Box(\text{MCont}(f)(g_2)))(x_1) = \bigwedge \{g_2(f(y_1)) ; x_1R_1y_1\}$. Let b denote the right-hand side. Since $x_1R_1y_1$ implies $f(x_1)R_2f(y_1)$, we have $a \leq b$. By the item 2 in the arrow part of Definition 6.3, we have $a \geq b$. Hence $a = b$. \square

Theorem 6.11. *Let A be an object in $\text{ML}_n^c\text{-Alg}$. Then, A is isomorphic to $\text{MCont} \circ \text{RSpec}(A)$ in the category $\text{ML}_n^c\text{-Alg}$.*

Proof. We claim that $\langle - \rangle : A \rightarrow \text{MCont} \circ \text{RSpec}(A)$ is an isomorphism of ML_n^c -algebras. By Theorem 4.21, it remains to show that $\langle \Box x \rangle = \Box_{R_\square} \langle x \rangle$ for $x \in A$. By Proposition 5.6, we have the following for $v \in \text{Spec}(A)$: $(\Box_{R_\square} \langle x \rangle)(v) = \bigwedge \{u(x) ; vR_\square u\} = v(\Box x) = \langle \Box x \rangle(v)$. \square

Theorem 6.12. *Let (S, R) be an object in FRS_n . Then, (S, R) is isomorphic to $\text{RSpec} \circ \text{MCont}(S, R)$ in the category FRS_n .*

Proof. Define $\Phi : (S, R) \rightarrow \text{RSpec} \circ \text{MCont}(S, R)$ by $\Phi(x)(f) = f(x)$ for $x \in S$ and $f \in \text{Cont}(S)$. We show: For any $x, y \in S$, xRy iff $\Phi(x)R_{\Box_R}\Phi(y)$. Assume xRy . Let $r \in \mathbf{n}$ and $f \in \text{Cont}(S)$ with $\Phi(x)(\Box_R f) \geq r$. Since $\Phi(x)(\Box_R f) = \bigwedge \{f(z) ; xRz\}$, we have $\Phi(y)(f) = f(y) \geq r$. Next we show the converse. To prove the contrapositive, assume $(x, y) \notin R$. By Definition 6.3, there is $f \in \text{Cont}(S)$ such that $(\Box_R f)(x) = 1$ and $f(y) \neq 1$. Then, $\Phi(x)(\Box_R f) = 1$ and $\Phi(y)(f) \neq 1$. Thus, we have $(\Phi(x), \Phi(y)) \notin R_{\Box_R}$.

By Theorem 4.22, it remains to prove that Φ and Φ^{-1} satisfy the item 2 in the arrow part of Definition 6.3, which follows from the above fact that xRy iff $\Phi(x)R_{\Box_R}\Phi(y)$, since Φ is bijective. \square

By Theorem 6.11 and Theorem 6.12, we obtain a fuzzy topological duality for ML_n^c -algebras, which is a generalization of Jónsson-Tarski duality for modal algebras to the n -valued case via fuzzy topology.

Theorem 6.13. *The category $ML_n^c\text{-Alg}$ is dually equivalent to the category FRS_n via the functors $R\text{Spec}(-)$ and $M\text{Cont}(-)$.*

Proof. By arguing as in the proof of Theorem 4.23, this theorem follows immediately from Theorem 6.11 and Theorem 6.12. \square

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