Logical Consequence and the Paradoxes

Edwin Mares[1], Francesco Paoli[2]
[1]School of History, Philosophy, Political Science and International Relations, Victoria University, Wellington;
[2]Department of Pedagogy, Psychology and Philosophy, University of Cagliari.

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Abstract

We group the existing variants of the familiar set-theoretical and truth-theoretical paradoxes into two classes: *connective* paradoxes, which can in principle be ascribed to the presence of a contracting connective of some sort, and *structural* paradoxes, where at most the faulty use of a structural inference rule can possibly be blamed. We impute the former to an equivocation over the meaning of logical constants, and the latter to an equivocation over the notion of consequence. Both equivocation sources are tightly related, and can be cleared up by adopting a particular substructural logic in place of classical logic. We then argue that our perspective can be justified via an informational semantics of contraction-free substructural logics.

Keywords. Paradoxes; structural contraction; V-Curry paradox; informational semantics; substructural logics.

1 Introduction

We were all taught early on, in our basic courses on logic and the methodology of science, that mathematical theories have two sorts of postulates: logical postulates — the 'underlying logic' of the theory — and specific postulates, governing the behaviour of the very notions the theory is about. Likewise, we were all taught that, when faced with an inconsistent mathematical theory, some of these postulates (logical and/or specific) have to go. It is well-known that the immediate reaction to the discovery of inconsistencies in Cantor's set theory was to stick to its underlying logic and weaken its specific postulates, the axioms of comprehension and extensionality. Despite all the differences in their respective approaches, Russell, Zermelo, Bernays, Gödel and other pioneers of contemporary set theory all shared this stance. This choice had an obvious allure: classical logic was well-understood, and had been until very recently the only logic on the market. Giving it up was certainly not a move to be taken too lightly. Sure, there was an evident drawback as well, since the principles of comprehension and extensionality seemed, at least in the eyes of some participants in the discussion, to encode all there was to say about the notion of set. Starting from the late Fifties, Thoralf Skolem (see e.g. [59]), one of the protagonists of the early debate on axiomatic set theory, suggested the opposite way out: if we are ready to weaken the logical basis of set theory and embrace a *nonclassical* logic, we can retain the specific axioms of Cantor's theory, together with all their intuitive appeal, and still be in a position to dodge the paradoxes. In particular, he proposed the adoption of Łukasiewicz's infinite-valued logic.

Since that landmark attempt, suggestions to the effect that some nonclassical logic has to be used as a logical underpinning for naïve set theory, or naïve truth theory — whose unrestricted T-scheme is jeopardised in an analogous way by the Liar paradox[1] — have been made by the score. However, the revisionary camp is in turn divided into two factions. While Zermelo, and other supporters of a classical approach, took naïve comprehension and the paradoxes to be equally *unacceptable*, and were therefore ready to dispose with the former to avoid the latter, some nonclassical logicians, most notably Graham Priest (see e.g. [44]), take all of these claims to be equally *acceptable*, and look for a nontrivial (dialetheic) logic that can vindicate the paradoxes as *theorems* about inconsistent mathematical objects. Other authors, instead (see e.g. [24], [12], [16], [26]), take naïve comprehension and extensionality to be *acceptable* and the paradoxes to be *unacceptable*, and aim at a logic where you can have your cake (i.e. keep Cantor's axioms) and eat it, too (i.e. avoid the paradoxes). We definitely belong in this second stream and will contend in the present paper that this is the most reasonable way to go about the whole issue. Although, as we have just seen, there is nothing new in such a general approach, we will try to make a few specific points of our own:

1. *Thesis One*. The familiar set-theoretical and truth-theoretical paradoxes can most conveniently be grouped into two classes: *connective* paradoxes, which can in principle be ascribed to the presence of a contracting connective of some sort, and *structural* paradoxes, where at most the faulty use of a structural inference rule can possibly be blamed.

2. *Thesis Two*. Classical logic is not a *wrong* logic, but an *ambiguous*[2] logic; set-theoretical and semantical paradoxes are, like paradoxes of implication [42], nothing but paralogisms, or fallacies of equivocation. There are two tightly related, but distinct sources of equivocation in classical logic: an equivocation over the meaning of logical constants, which we take to be responsible for the connective paradoxes, and an equivocation over the meaning of consequence, which we take to be responsible for the structural paradoxes. Classical logic fails to take into account the distinction between

[1] Under some respects, discussing the paradoxes in the context of set theory or of truth theory is just a matter of preference. There are, however, important differences between these two contexts. Here, we will in general refrain from discussing these differences.

[2] Stewart Shapiro suggested in conversation that it would be more appropriate to talk of *polysemy*, rather than ambiguity. Throughout the present discussion, however, we will disregard this distinction.

extensional and *intensional* logical constants, as well as the distinction between *external* and *internal* consequence.

3. *Thesis Three.* To successfully tackle the paradoxes, we need a logic where both distinctions make sense. We suggest as a plausible candidate a fragment of linear logic with no exponentials and no additive constants.

The paper is structured as follows. In Section 2 we present four recent challenges to the plausibility of nonclassical approaches, whose alleged common upshot is that the Russell and Curry paradoxes unexpectedly reappear even in the context of very weak nonclassical logics. We will see that all of these arguments are meant to be relatively 'logic-neutral', in that they assume very little in terms of controversial logical theorems. Some of them, moreover, are presented as structural paradoxes, because no logical constant seems to play any rôle in the proof. In Section 3 we set the stage for a rejoinder, comparing different concepts of logical consequence and selecting one which is broad enough to encompass the relation that is of interest to us. In Section 4 we argue in favour of the substructural logic **LL** (a fragment of linear logic) as a nonclassical logical basis for set theory, and we suggest a unified approach to implicational, semantical and set-theoretical paradoxes. In Section 5 we illustrate our replies to the four arguments of Section 2. Finally, in Section 6 we take up Beall's and Murzi's gauntlet [8], trying to meet their challenge to buttress our perspective by means of a 'new metaphysical account of validity' for which the rejection of some structural rules of derivation makes sense.

2 Resurfacing Paradoxes

It has been recently contended on at least four occasions that even nonclassical set theories, or truth theories, may after all conceal versions of the paradoxes, if we are not careful enough in our assumptions. Let us examine these arguments one by one.

2.1 Restall (2008)

Greg Restall [48] argues that nonclassical logics offer a viable solution to set-theoretical and semantical paradoxes only if we are ready to pay prices that would seem untoward to many, such as giving up intuitively plausible logical principles (like distributivity of conjunction over disjunction and vice versa). Restall, himself an advocate of a nonclassical approach to logical paradoxes, does not purport to show that using a heterodox logic in this context is wrong-headed; he only claims that "nonclassical solutions are not the 'easy way out' of the paradoxes" [48, p. 262].

More precisely, Restall shows that Curry's paradox surreptitiously reappears in naïve truth theory even in the presence of very weak and seemingly uncontentious logical principles:

- a transitive relation of consequence;
- the availability of a weak form of T-scheme ($T\left(\overline{A}\right) \wedge C \vdash A$, to be read as: $T\left(\overline{A}\right)$, together with the set of all required background contraints C, entails that A; and, similarly, $A \wedge C \vdash T\left(\overline{A}\right)$);
- a conjunction $\wedge$ that is: 1) commutative; 2) idempotent; 3) such that $A \vdash B$ and $A \vdash C$ if and only if $A \vdash B \wedge C$; 4) residuated, i.e. such that for some connective $\rightarrow$ we have that $A \wedge B \vdash C$ iff $A \vdash B \rightarrow C$;
- the availability of a diagonalisation technique needed to express a sentence λ which is equivalent to $T\left(\overline{\lambda}\right) \rightarrow A$ (where A is an arbitrary sentence and $\rightarrow$ is the residual of conjunction).

If we are willing to accept this much, Curry's paradox reappears in our chosen logic, as Restall deftly shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \wedge T\left(\overline{\lambda}\right) \vdash \lambda \qquad \lambda \vdash T\left(\overline{\lambda}\right) \rightarrow A}{C \wedge T\left(\overline{\lambda}\right) \vdash T\left(\overline{\lambda}\right) \rightarrow A}}{C \wedge T\left(\overline{\lambda}\right) \wedge T\left(\overline{\lambda}\right) \vdash A}\,(*)}{C \wedge T\left(\overline{\lambda}\right) \vdash A}}{C \vdash T\left(\overline{\lambda}\right) \rightarrow A} \qquad T\left(\overline{\lambda}\right) \rightarrow A \vdash \lambda}{C \vdash \lambda} \qquad C \wedge \lambda \vdash T\left(\overline{\lambda}\right)}{C \vdash T\left(\overline{\lambda}\right)} \qquad \dfrac{\dfrac{\text{from } (*)}{C \wedge T\left(\overline{\lambda}\right) \vdash A}}{T\left(\overline{\lambda}\right) \vdash C \rightarrow A}}{\dfrac{C \vdash C \rightarrow A}{\dfrac{C \wedge C \vdash A}{C \vdash A}}}$$

The nonclassical logician is thus left with surprisingly few choices. Abandoning the unrestricted T-scheme, or comprehension axiom, would be tantamount to throwing in the towel. Giving up the transitivity of entailment would be almost as hard to swallow. The most plausible culprit, then, would seem to be the fact that conjunction is residuated, if not for a small catch: simplifying a bit, *any* logic where conjunction distributes over disjunction admits a residual for conjunction. And, Restall concludes, dropping the distribution principle does not seem too enticing a perspective.

2.2 Restall (2010)

In [50], Restall discusses another way leading to Curry's paradox. Once more, the raw material necessary to build our paradox is amazingly minimal. We only need to have in our favourite logic a conditional $\rightarrow$ that obeys identity and modus ponens, and two propositional constants: a truth constant t, which entails a sentence A iff A is true, and a falsity (or untruth[3]) constant u, which is

[3] Restall carefully distinguishes falsity from untruth because the dialectical targets of his attack include partisans of truth value gaps. We will faithfully stick to his terminology when presenting his argument.

entailed by a sentence A iff A is untrue. Given these ingredients, Restall defines the new conditional connective

$$A \Rightarrow B =_{def.} (A \wedge t) \to (B \vee u)\,.$$

For a start, $\Rightarrow$ can be shown to satisfy modus ponens. Here is Restall's natural deduction proof, in his own notation:

$$\cfrac{\cfrac{\cfrac{A \qquad \bar{t}}{A \wedge t} \qquad (A \wedge t) \to (B \vee u)}{B \vee u}}{B \qquad \underline{u}}$$

The last line of this proof means that we have either B or u, but the latter is impossible, since u is untrue. We can therefore conclude that B. Summing up, if $A \Rightarrow B$ is true, then either A is untrue or B is true. We now suppose, conversely, that either A is untrue or B is true, and go through a case-splitting argument. If the former, then A entails u. Identity gives us $u \to u$. Since A entails u we may strengthen the antecedent to $A \to u$. Since $A \wedge t$ entails A, strengthening again gives $A \wedge t \to u$. Finally, since u entails $B \vee u$, we may weaken the consequent to get $A \Rightarrow B$. Similarly, it is possible to show that if B is true, $A \Rightarrow B$ is equally true. In conclusion, our new conditional $A \Rightarrow B$ is true if and only if either A is untrue or B is true. This means that this new conditional is sufficiently close to material implication to permit a derivation of Curry's paradox, as witnessed by an argument by Meyer, Routley and Dunn [36].

Our prospects, like at the end of the previous subsection, look bleak. To prevent Curry's paradox from resurfacing, for want of better alternatives, we are forced to another unpalatable move, that is to impoverish our logical vocabulary through the rejection of truth and falsity constants.

2.3 Hinnion and Libert (2003); Restall (2012)

Many proponents of nonclassical approaches to the paradoxes put the blame on the presence, in classical logic, of questionable principles concerning negation — for example, the law of excluded middle — or concerning the conditional — for example, the contraction principle $(A \to (A \to B)) \to (A \to B)$. In [49, p. 90], following [29], Restall delivers a third attack to such approaches, contending that the dubious principles cannot be at fault, for it is possible to concoct versions of the paradoxes where no logical constant seems to play any rôle. In our terminology, these versions are the structural paradoxes.

Let us see one of these variants. Restall assumes the principles of comprehension and extensionality in the following forms:

$$\frac{\Gamma, \phi(a) \vdash \Delta}{\Gamma, a \in \{x : \phi(x)\} \vdash \Delta}\,(\in L) \qquad \frac{\Gamma \vdash \phi(a), \Delta}{\Gamma \vdash a \in \{x : \phi(x)\}, \Delta}\,(\in R)$$

$$\frac{\Gamma, x \in a \vdash x \in b, \Delta \qquad \Gamma, x \in b \vdash x \in a, \Delta}{\Gamma \vdash a = b, \Delta}\,(Ext_{\in})$$

These 'sequents' are to be interpreted as claims about assertions and denials: $\Gamma \vdash \Delta$ means that it is incoherent to assert all of the Γ's and deny all of the Δ's, where Γ and Δ are *sets* of statements. This is how Restall justifies the plausibility of his formulation of the comprehension principle, or Axiom (V) (as Frege called it):

> Independent of concerns over conditionality, there is a central core to commitment to axiom (V): $\phi(a)$ and $a \in \{x : \phi(x)\}$ stand and fall together. The assertion of $\phi(a)$ has the same upshot as the assertion of $a \in \{x : \phi(x)\}$; a denial of $\phi(a)$ has the same upshot as a denial of $a \in \{x : \phi(x)\}$. Anyone prepared to assert $\phi(a)$ but to deny $a \in \{x : \phi(x)\}$ rejects condition (V). Similarly, anyone prepared to deny $\phi(a)$ but to assert $a \in \{x : \phi(x)\}$ also rejects condition (V) [49, p. 90].

In addition to this, Restall assumes that his entailment relation is reflexive and monotonic, i.e. that $\Gamma \vdash \Delta$ whenever $\Gamma \cap \Delta \neq \emptyset$, and that it satisfies Cut, i.e. that we can infer $\Gamma \vdash \Delta$ from $\Gamma \vdash \Delta, A$ and $A, \Gamma \vdash \Delta$. These principles are claimed to be more or less straightforward, given the previous reading of sequents in terms of assertion and denial. Identity, moreover, must satisfy a number of principles including

$$\frac{\Gamma, \phi(a) \vdash \Delta}{\Gamma, a = b, \phi(b) \vdash \Delta}\,(= L_l)$$

Now, let $\mathfrak{H} = \{x : \{y : x \in x\} = \{y : A\}\}$, where A is arbitrary. For convenience, we will break the proof of the paradox in three parts. Let δ_1 be the following derivation:

$$\frac{\dfrac{\dfrac{\mathfrak{H} \in \mathfrak{H} \vdash \mathfrak{H} \in \mathfrak{H}}{\mathfrak{H} \in \mathfrak{H} \vdash x \in \{y : \mathfrak{H} \in \mathfrak{H}\}}\,(\in R) \qquad \dfrac{\dfrac{A \vdash A}{x \in \{y : A\} \vdash A}\,(\in L)}{x \in \{y : \mathfrak{H} \in \mathfrak{H}\}, \{y : \mathfrak{H} \in \mathfrak{H}\} = \{y : A\} \vdash A}\,(= L_l)}{\mathfrak{H} \in \mathfrak{H}, \{y : \mathfrak{H} \in \mathfrak{H}\} = \{y : A\} \vdash A}\,(Cut)}{\mathfrak{H} \in \mathfrak{H} \vdash A}\,(\in L)$$

The next piece of the derivation is δ_2 below:

$$\cfrac{\cfrac{\cfrac{A \vdash x \in \{y : \mathfrak{H} \in \mathfrak{H}\}, A}{x \in \{y : A\} \vdash x \in \{y : \mathfrak{H} \in \mathfrak{H}\}, A}\,(\in L) \qquad \cfrac{\begin{matrix}\delta_1 \\ \vdots \\ \mathfrak{H} \in \mathfrak{H} \vdash A\end{matrix}}{x \in \{y : \mathfrak{H} \in \mathfrak{H}\} \vdash x \in \{y : A\}, A}\,(\in L, \in R)}{\vdash \{y : \mathfrak{H} \in \mathfrak{H}\} = \{y : A\}, A}\,(Ext_{\in})}{\vdash \mathfrak{H} \in \mathfrak{H}, A}\,(\in R)$$

Combining δ_1 and δ_2,

$$\cfrac{\begin{matrix}\delta_1 \\ \vdots \\ \mathfrak{H} \in \mathfrak{H} \vdash A\end{matrix} \qquad \begin{matrix}\delta_2 \\ \vdots \\ \vdash A, \mathfrak{H} \in \mathfrak{H}\end{matrix}}{\vdash A}\,(Cut)$$

Restall comments:

> At some stage the derivation of A is to break down, but where? Orthodoxy tells us that the rules to reject (at least where $\in$ expresses class membership) are $(\in L)$ or $(\in R)$, and the underlying assumption that every predicate determines a set: to reject Law (V). For defenders of Law (V), however, some other move must be rejected. For defenders of Law (V) concerning classes, the pickings seem extremely thin: either defend Law (V) despite rejecting $(\in L)$ or $(\in R)$ – in the face of criticism that to reject $(\in L)$ or $(\in R)$ is to reject what we meant by Law (V) in the first place – or reject $(Ext_{\in})$ in the face that this was what we meant by *extensionality* in the first place – or finally, find fault in $(= L_l)$, (Cut) or (Id). What option can the defender of Law (V) take? [...] Evading this paradox will, at least, help clarify what is at stake in taking a non-classical position on classes in defence of Law (V) (p. 93).

We parenthetically observe that Humberstone [27] is unpersuaded by Restall's claim that there is actually no logical connective lurking behind this argument. Let us define $\widehat{\phi} = \{v : \phi\}$ (v not free in ϕ), and $\phi \leftrightarrow \psi$ as $\widehat{\phi} = \widehat{\psi}$. Then from the above postulates we derive 1) $\phi \leftrightarrow \psi, \phi \vdash \psi$; 2) $\phi \leftrightarrow \psi, \psi \vdash \phi$; 3) $\phi, \psi \vdash \phi \leftrightarrow \psi$; 4) $\vdash \phi, \psi, \phi \leftrightarrow \psi$. The connective $\leftrightarrow$ is, therefore, sufficiently close to the classical biconditional to expose the Hinnion-Libert paradox as a variant of the *biconditional Curry paradox*, discussed e.g. in [51] or [28]. If so, it is a connective paradox rather than a structural paradox. We will argue below, however, that it has features in common with both classes of paradoxes and that it is an ideal case study to discuss the relations between such classes.

2.4 Beall and Murzi (201+)

Beall and Murzi [8] join Restall in his diagnosis that being paradox-free has nothing to do with a deviant treatment of logical constants in nonclassical logics.

In particular, although it has often been said that Curry-style paradoxes depend on the presence, in one's logic of choice, of some form of *contracting implication* (see e.g. [15], [7]), it is possible to give a structural version of the argument which does not presuppose any contracting connective at all[4]. In particular, Beall and Murzi focus on the semantic, rather than on the set-theoretical, Curry paradox. Once again, the ingredients of their so-called *V-Curry paradox* are minimal. We only have to assume that:

- In full analogy with the disquotational T-scheme $A \longleftrightarrow T\left(\overline{A}\right)$, one has a *disquotational validity* scheme V-S saying that the binary validity predicate applies to an ordered pair of sentence names $\overline{A}, \overline{B}$ just in case I have a valid argument from A to B. In symbols: $Val\left(\overline{A}, \overline{B}\right)$ iff $A \vdash B$.
- One has a detachable biconditional $\longleftrightarrow$ and some means of achieving self-reference (like diagonalisation, quotation, etc.) that allow, for some π, to express the sentence $\pi \longleftrightarrow Val\,(\overline{\pi}, A)$, where A is arbitrary.

Here is the paradox, in natural deduction formulation.

1.		$\pi \longleftrightarrow Val\,(\overline{\pi}, A)$	
2.	(2)	π	Ass.
3.	(2)	$Val\,(\overline{\pi}, A)$	1, 2, MP
4.	(2)	A	2, 3, V-S
5.		$Val\,(\overline{\pi}, A)$	2-4, V-S
6.		π	1, 5, MP
7.		A	5, 6, V-S

On the face of it, it looks like this version of the antinomy is hard to defuse for *any* logic: how can the mild assumptions listed above be reasonably denied? Beall and Murzi, however, point at a feature of this derivation that is easily overlooked — the natural deduction proof we just illustrated *employs contraction at a structural level*. If we conscientiously keep track of the assumptions we used, for instance by piling them in a sequent to the left of the conclusion we are establishing at each step, we notice that π gets used twice in the subderivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi \vdash \pi \quad \vdash \pi \to Val(\overline{\pi}, \overline{A})}{\pi \vdash Val(\overline{\pi}, \overline{A})} \quad \pi \vdash \pi}{\pi, \pi \vdash A}}{\pi \vdash A}}{\vdash Val(\overline{\pi}, \overline{A})} \quad \vdash Val(\overline{\pi}, \overline{A}) \to \pi}{\vdash \pi} \quad \vdash Val(\overline{\pi}, \overline{A})}{\vdash A}$$

Thus the whole proof only goes through if we have at our disposal the rule of Structural Contraction

$$\text{If } \Gamma, A, A \vdash B \text{ then } \Gamma, A \vdash B.$$

[4] Andrea Cantini observed in conversation that the original version of Curry's paradox in [13] is closer to the structural version by Beall and Murzi than to the connective paradox usually encountered in the literature.

As Beall and Murzi put it:

> Instead of either treating truth and validity differently or 'going unified' along broadly Tarskian lines, one may extend the rcf lesson in the obvious fashion: just as [the standard Curry paradox] teaches us that our connectives don't contract, so too V-Curry teaches us that validity fails to contract. In other words, not only is contracting behaviour for our connectives (in particular, conditionals) to be rejected, but contraction at the structural level, namely, Structural Contraction, [...] is to be rejected.

After discarding a few avenues of reply, Beall and Murzi consider (although they do not endorse) the possibility of dropping Structural Contraction, a rule that is indeed rejected in some substructural logics [41]. Some objections that would seem to put us off making such a move are duly countered by the authors; however, they suspend their judgment as to whether a structurally contraction-free logic can be independently motivated other than as a means to block the V-Curry paradox.

3 Concepts of Consequence

So far, we have talked about logical consequence and entailment quite informally. We repeatedly used the symbol '$\vdash$' to denote entailment relations in different logics, but we have been anything but precise about the properties of these relations. For one, we have not even settled on picking a *domain* for such relations (do they hold between a set of formulas and a single formula, or otherwise?) Actually, several different concepts of consequence coexist in the literature. The purpose of the present section is comparing them to one another and selecting one that is especially convenient for our aims.

3.1 Consequence *à la* Tarski vs Consequence *à la* Scott

It is well-known that the first contemporary[5] logician who defined logical consequence at an abstract level was Alfred Tarski [60]. According to Tarski, a *consequence relation* over a propositional language $\mathcal{L}$ is a relation $\vdash \subseteq \wp(Fm(\mathcal{L})) \times Fm(\mathcal{L})$ obeying the following conditions for all $A \in Fm(\mathcal{L})$ and for all $\Gamma, \Delta \subseteq Fm(\mathcal{L})$:

1. $\Gamma \vdash A$ if $A \in \Gamma$ *(Reflexivity)*;
2. If $\Gamma \vdash A$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash A$ *(Monotonicity)*;
3. If $\Delta \vdash A$ and $\Gamma \vdash B$ for every $B \in \Delta$, then $\Gamma \vdash A$ *(Cut)*.

[5] The qualification is needed, since an abstract logical consequence concept of sorts can be found already in Bolzano's *Wissenschaftslehre* (1837).

Thus, a consequence relation holds between a set of formulas and a single formula of a given language. Building on Tarski's definition, it has become customary in abstract algebraic logic to define a (propositional) *logic* over $\mathcal{L}$ as an ordered pair $(\mathbf{Fm}(\mathcal{L}), \vdash)$, where $\mathbf{Fm}(\mathcal{L})$ is the absolutely free algebra of formulas of $\mathcal{L}$ and $\vdash$ is a *substitution-invariant* consequence relation over $\mathcal{L}$, where this label means that if $\Gamma \vdash A$ and σ is a substitution on $\mathbf{Fm}(\mathcal{L})$, then $\{\sigma(C) : C \in \Gamma\} \vdash \sigma(A)$. A logic and its associated consequence relation, so defined, may or may not be *finitary*: they are such if whenever $\Gamma \vdash A$, there is always some *finite* $\Delta \subseteq \Gamma$ s.t. $\Delta \vdash A$. Consequence relations that are defined syntactically, for example as derivability relations of a Hilbert-style axiomatic system, are finitary because proofs are finite objects; observe, however, that Tarski's definition does not dictate any constraint to that effect — logics can be defined semantically, via the equational consequence relation of a class of algebras, or by other means. This concept of propositional logic is different from the one you will often encounter in other branches of logic, especially philosophical logic, where a logic is sometimes identified with a set of *formulas* (the 'theorems' or 'tautologies' of the logic in question). The present, more discriminating concept is better suited for our context, because there exist distinct logics (in an acceptation similar to the present one) which share the same set of theorems (where A is a theorem of $(\mathbf{Fm}(\mathcal{L}), \vdash)$ in case $\emptyset \vdash A$) but, as we will see, suggest different recipes for dealing with the paradoxes.

Tarski's concept seemed too restrictive to many. In particular, his definition was questioned on the ground of its alleged failure to encompass some features of logical consequence that require due consideration:

- *Why just one conclusion?* Some logicians judged the restriction to a single conclusion arbitrary, for, on the one hand, it renders the concept patently asymmetrical, and on the other hand there seem to be some natural uses and motivations for multiple conclusion logic ([58], [55], [5]).

- *Why just formulas?* When engaged in derivations, proofs, or arguments, we sometimes manipulate more than plain formulas: we may work with whole sequents, with equations (e.g. when deriving in the equational consequence relations associated to classes of algebras), with labelled formulas containing additional information on the formulas themselves ([19], [10]).

- *Why just sets?* Sometimes, e.g. when we want to keep track of how many times we resort to a given assumption, or to monitor the order in which the premisses were used, sets are far from optimal. We need more sensitive ways of aggregating premisses, like multisets or sequences ([19], [5]).

- *Why these three postulates?* The conditions of Reflexivity, Monotonicity, and Cut have been variously challenged. Monotonicity, in particular, has been deemed unfortunate in inferential situations where relevance or belief revision play a rôle ([19], [5]).

The introduction of a multiple-conclusion generalisation of Tarski's definition, where entailments with multiple succedents are allowed in analogy with the sequents of Gentzen's calculus **LK** for classical logic, is generally credited[6] to Dana Scott (see for example his [55]). A Scott *consequence relation* over a propositional language $\mathcal{L}$ is a relation $\vdash \subseteq \wp(Fm(\mathcal{L})) \times \wp(Fm(\mathcal{L}))$ obeying the following conditions for all $A \in Fm(\mathcal{L})$ and for all $\Gamma, \Delta \subseteq Fm(\mathcal{L})$:

1. $\Gamma \vdash \Delta$ if $\Gamma \cap \Delta \neq \emptyset$ *(Reflexivity)*;
2. If $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \Delta$ *(Monotonicity)*;
3. If $\Gamma, A \vdash \Delta$ and $\Gamma \vdash A, \Delta$, then $\Gamma \vdash \Delta$ *(Cut)*.

3.2 Further generalisations

While Scott modified Tarski's definition to make room for multiple-conclusion entailments, Gabbay [19] and Blok and Jónsson [10] suggested to lift the restrictions to formulas as objects to be manipulated in the entailments themselves. Blok and Jónsson, in particular, go absolutely general and allow *any set* X to take the place of $Fm(\mathcal{L})$ in Tarski's definition. Their principal motivation, as hinted above, is to account for equational consequence relations of quasivarieties of algebras and for consequence relations of sequent systems along with more traditional Tarskian consequence relations on formulas.

Remark 1 *Following this approach, Blok and Jónsson give a definition of* equivalence *of consequence relations that is now a benchmark in algebraic logic. Let $\vdash_1, \vdash_2$ be consequence relations over the sets A_1, A_2, respectively. We say that $\vdash_1$ and $\vdash_2$ are equivalent*[7] *iff there exist a mapping $\tau : A_1 \to \wp(A_2)$ and a mapping $\rho : A_2 \to \wp(A_1)$ such that the following conditions hold for every $X \cup \{a\} \subseteq A_1$ and for every $Y \cup \{b\} \subseteq A_2$:*

S1 $X \vdash_1 a$ *iff* $\tau(X) \vdash_2 \tau(a)$;

S2 $Y \vdash_2 b$ *iff* $\rho(Y) \vdash_1 \rho(b)$;

S3 $a \dashv\vdash_1 \rho(\tau(a))$;

S4 $b \dashv\vdash_2 \tau(\rho(b))$.

When $\vdash_1$ is a Tarskian consequence relation on formulas, and $\vdash_2$ is the equational consequence relation of a quasivariety, we have as a special case the relation of algebraisability *of a deductive system [11]. This definition of equivalence does justice to the intuition that, for example, the Hilbert calculus for classical logic and Gentzen's* **LK** *are just different presentations of the 'same' logic. It still does not cover, however, obvious cases of equivalence such as we have when the same logic is expressed in different languages. Gyuris [25] suggests a notion of* deductive equivalence *that is meant to be the logical analogue of the algebraic notion of term equivalence between varieties. Galatos and Gil-Férez [20] have recently attempted a common abstraction of these two concepts.*

[6] We will follow this attribution, although it would be historically fair to acknowledge priority to Dov Gabbay [18].

[7] The original definition has been somewhat simplified for the sake of conciseness.

Finally, both Gabbay [19] and Avron (see e.g. his [5], [6]) argue in favour of a multiple-conclusion notion of consequence but insist that it is appropriate to consider more finely structured collection of objects, rather than sets, as arguments of the relation: multisets, sequences or even non-associative structured lists. Moreover, they call into question Tarski's postulate of Monotonicity. All these changes are advocated to include notions of consequence that evade the Tarskian account: flexible, inductive and abductive reasoning, logics of data and resources, some temporal logics. Finally, while Avron does not question Tarski's restriction to formulas, Gabbay contends that we should also allow more complex objects, like labelled formulas, as elements of his structured lists. The next Table summarises the extent to which the previously discussed concepts go beyond Tarski's original definition.

	One concl.	*Only form.*	*Only sets*	*Ref., Mon., Cut*
Tarski	x	x	x	x
Scott		x	x	x
Blok-Jónsson	x		x	x
Gabbay				
Avron		x		

In what follows, we will never need to deviate from Tarski's orthodoxy as much as is permitted by the most revolutionary proposal among those listed in the preceding Table, namely Gabbay's one. Indeed, Avron's concept, hereafter reproduced in full, will suffice for our purposes and will be adopted throughout the paper. According to Avron, a *multiset consequence relation* over a propositional language $\mathcal{L}$ is a binary relation $\vdash$ between finite[8] multisets of formulas of $\mathcal{L}$ obeying the following conditions for all $A \in Fm(\mathcal{L})$ and for all multisets $\Gamma, \Delta, \Pi, \Sigma$ of members of $Fm(\mathcal{L})$:

1. $A \vdash A$ *(Reflexivity)*;
2. If $\Gamma, A \vdash \Delta$ and $\Pi \vdash A, \Sigma$, then $\Gamma, \Pi \vdash \Delta, \Sigma$ *(Cut)*.

Observe that this more general formulation of Cut, which is equivalent to the standard one for Tarskian consequence relations, is made necessary by the concurrence of two factors: i) the replacement of sets by multisets, whereby it is no longer possible to 'contract' several occurrences of a same formula into a single one; ii) the absence of Monotonicity and the restrictions placed on Reflexivity, whereby we can no longer freely add formula occurrences to the left or to the right of our turnstile.

[8] The restriction to finite multisets in Avron's definition rules out non-finitary Tarskian consequence relations as instances of the concept. In the context of the present discussion, this circumstance need not concern us.

3.3 Sequent Calculi: Internal and External Consequence

To anticipate the content of the next Section, we are going to argue in favour of the relevant, substructural logic **LL** as a framework for a unified approach to the paradoxes of material and strict implication, as well as to set-theoretical and semantical paradoxes. In this logic, we can draw a crucial distinction between an internal and an external notion of consequence. This terminology obviously calls for an explanation, which it is the purpose of the present subsection to offer.

We will be cursory on substructural logics, since this area of nonclassical logics is by now very well-known ([47]; [41]). We only recall that the term originates within the context of Gentzen's sequent calculi, to denote what is obtained from the calculi for classical or intuitionistic logic by deleting, or appropriately restricting, one or more of the structural rules of exchange, weakening and contraction. Although the phrase has come to acquire a more comprehensive meaning over time [21], sequent calculi still represent a privileged tool for much work done in the field, and are home to the distinction alluded to in the subsection title and in our beginning paragraph, to which we now turn.

Is there any consequence relation (in our sense) we can naturally extract out of a sequent calculus? There is an obvious candidate for an affirmative answer. Sequents, the formal objects manipulated in sequent calculi derivations, informally represent entailments: for example, in the multiple-conclusion sequent calculus **LK** for classical logic, we are often invited to interpet intuitively a sequent $\Gamma \Rightarrow \Delta$ as 'the disjunction of the formulas in Δ follows from the conjunction of the formulas in Γ'. Therefore, we might simply associate with a sequent calculus **S** of our choice (over the language $\mathcal{L}$) the following relation: if Γ, Δ are finite multisets of $\mathcal{L}$-formulas, $\Gamma \vdash^{I}_{\mathbf{S}} \Delta$ holds whenever $\Gamma \Rightarrow \Delta$ is a provable sequent of **S**. This relation, sometimes termed the *internal consequence relation* of **S** [6], is easily seen to be a multiset consequence relation for any calculus that has all instances of $A \Rightarrow A$ among its provable sequents and where Cut is at least an admissible rule.

However, such relation is seldom considered. This is surprising only up to a point. If you are immersed in the Tarskian paradigm, you are likely to feel a distaste for internal consequence relations. In your eyes, they will look awkward under too many respects: they are multiple-conclusion, they call into play multisets rather than sets, they are made finitary by *fiat*, they are not monotonic unless the calculus has full weakening rules. Fortunately, there is something you might want to look at in their place. For Γ a finite multiset of $\mathcal{L}$-formulas and A a *single* $\mathcal{L}$-formula, let $\Gamma \vdash^{E}_{\mathbf{S}} A$ hold whenever $\Rightarrow A$ is provable in the calculus obtained from **S** by adding as initial sequents all the sequents $\Rightarrow B$, for B in Γ, as well as Cut as a primitive rule. This relation, sometimes called the *external consequence relation* of **S** [6], is less likely to cause misgivings to the Tarskian loyalist: just replace 'finite multiset' by '(possibly infinite) set' and you are fully in the Tarskian framework back again. As it stands, however, all instances[9] of an external consequence relation can be seen — by identifying

[9] As Humberstone [27] rightly observes when discussing Scott consequence relations (or

$\Gamma \vdash^{E}_{\mathbf{S}} A$ with $\Gamma \vdash^{E}_{\mathbf{S}} \{A\}$ — as instances of a multiset consequence relation that automatically satisfies Monotonicity and contraction even though **S** has no weakening or contraction rules. Not that the difference matters so much if you work in **LK**: the two relations can be shown to coincide when Δ is a singleton. There are sequent calculi for substructural logics, however, where the internal and the external relation genuinely differ even in such a case. For example, if **S** has no weakening rules, then $A, B \vdash^{E}_{\mathbf{S}} A$ while it is not the case that $A, B \vdash^{I}_{\mathbf{S}} A$.

4 The Paradoxes as Fallacies of Equivocation

In this paper we argue that nonclassical solutions, and in particular solutions that make recourse to substructural logics, are *really* the easy way out of the set-theoretical and semantical paradoxes. The only 'cost' of the perspective we defend, which after all can be seen as an additional bonus in that it yields as a windfall a clearer logical analysis of natural language, amounts to acknowledging that classical logic is *ambiguous*: each primitive binary connective of classical logic is ambiguous between an extensional and an intensional connective, and each consequence statement of classical logic is ambiguous between an external and an internal statement. Although this idea, or at least its former half, is not new at all ([4], [45]), most authors working on the philosophical foundations of substructural logics do not seem to have taken it at face value; nor do they make it the centrepiece of their responses to the challenge issued by paradoxes. Contrary to the idea, quite widespread in nonclassical milieus, that you have to 'give up' some classical inferential principles to preserve naïve comprehension and extensionality (or the unrestricted T-scheme) and keep the paradoxes from the door, we maintain that the paradoxes, in a sense, can be solved for free. There is no need to give up any inferential principle of classical logic — only to recognise that bad things can happen when principles holding of different connectives are used, in the course of a derivation, as holding of the same ambiguous connective, or when structural rules holding of different entailment concepts are used, in the course of a derivation, as holding of the same ambiguous notion of consequence. In another paper [42], one of the present authors claimed that such a view is best defended against the backdrop of a proof-conditional approach to the meaning of logical constants, although we will argue in the final section of this article that it may go along just as well with an informational account of the matter ([33], [1]).

generalised consequence relations, as he calls them): "No consequence relation is a generalized consequence relation — henceforth, mostly abbreviated to 'gcr'. Thus the sense in which gcr's generalize the notion of a consequence relation is not that the latter are special cases of gcr's, but that individual $\vdash$-statements for $\vdash$ a consequence relation also count as $\vdash$-statements for $\vdash$ a gcr". Here, the situation is analogous.

4.1 Internal vs External Consequence: A Philosophical Interpretation

In his early writings (e.g. [30, p. 352]), C.I. Lewis makes an enlightening distinction between two senses of 'follows from' *not in ordinary language, but in mathematics*. This is not to be confused with his later distinction between ordinary and logistic derivability which we find in his *Symbolic Logic* [31, p. 253 ff.], nor do we claim, of course, that the dichotomy we are going to defend here is historically faithful to Lewis's intentions; we merely think that it could serve as a nice introduction to what we are going to say.

It makes perfect sense to say, for example, that a given mathematical theorem T 'follows from' the axioms of the theory it belongs to. Are we thereby claiming that all the axioms of the theory have been *used* to establish it, or even that they are relevant in content to it? No, of course: only that whenever we grant the axioms themselves, we are committed to accept T. This is a sense of 'follows from' for which weakening and contraction seem beyond doubt. But it does not appear to be the same meaning that we have in mind when claiming that T 'follows from', say, a lemma L. For us to appropriately affirm that, it seems necessary that L be at least *used* in the proof of T. Yet, there is more to it. Suppose you have a proof of T that requires two applications of the axiom of choice and that we challenge you to find a proof where the same axiom is applied only *once*. You may or may not succeed — if you don't, perhaps you will start wondering whether Structural Contraction is adequate for this particular sense of 'follows from'.

We like the Lewis discussion because it takes place in the context of axiomatic systems for mathematical theories, and not as an attempt to faithfully represent the inferential practice in everyday reasoning. In real life contexts, it is all the more reasonable to maintain that we draw inferences on the basis of the information available to us: our assumptions collectively make up some body of relevant information needed to infer the sentences we are asserting. If so, it is not implausible to assume that sentences of our formal language stand for *information tokens* (not types) and that premisses must be aggregated in a multiset (or even in some more finely structured collection, although we will not pursue this suggestion in the present paper), not in a set. In general, when we say that some conclusion A follows from the premisses in Γ, we can mean either of two different things (at least):

- given the rules of the logic at issue, we can extract the information that A from the combined information provided by the sentences in Γ;
- Γ yields grounds for asserting A; i.e. whenever we accept Γ we are committed to accepting A.

The former meaning calls for some clarification. *Information extraction*, as used here, is a different notion than information preservation. The notion of information preservation, discussed at length e.g. in [2], presupposes a view of what information is and what it is to be preserved in a context. We doubt whether there are any such notions that are not heavily theory-laden and depend essentially on the interpretation of particular logics. The notion of information extraction, too, can depend on the logic concerned (as we shall illustrate presently), but the concept of information involved is neutral between them. The interpretation of the logic, if there is one, should provide justification for the rules of the system and tell us what 'extraction' means exactly in the case at hand, just as the interpretation of a particular modal logic should make legitimate the rules governing 'necessity' and tell us what that operator means. So, for example, if we have Brady's logic **MC** (and apply his interpretation of it), then the first sense of 'B follows from A' means that the meaning of A contains the meaning of B and, as such, we can extract B from A [12]. On Restall's channel theoretic interpertation of the logic **RW**, on the other hand, $A \to B$ means that there is, in a particular context, a channel from the information that A to the information that B. This interpretation of implication gives us a way of talking about the flow of information in a particular situation, given specific connections between it and other situations [46]. The consequence relation generalises this and tells us that $A \vdash B$ if and only if, regardless of the specific connections between situations, we can always get the information that B from the information that A. Thus, the turnstile represents the notion of a universal channel, whereas implication is used to represent local channels. Thus, $A \vdash B$ can be taken to mean that we can, regardless of situation, extract the information that B from the information that A.

The next issue that needs to be clarified is to what extent this distinction overlaps with the dichotomy between external and internal consequence discussed above. Indeed, we think there is a great deal of overlap. External consequence, in fact, has to do with the *preservation of the warrant to assert*[10]. It is the kind of notion we have in mind when we affirm, for example, that the (right) introduction rules of connectives in a sequent calculus provide us with the meaning of those connectives, by specifying when we are in a position to infer, say, a conjunctive sentence from its conjuncts. Here, we look at the sequent rule and we are interested in the way its conclusion 'follows from' its premisses in the sense encoded by the fraction line that separates them; we do not look at the individual sequents in the rule, because we are not interested in the way their succedents 'follow from' their antecedents in the sense encoded by the sequent symbol $\Rightarrow$ that separates them. For this *vertical* notion of inference, weakening and contraction make perfect sense. But for the other *horizontal*, informational reading this is less clear. It is fine to say that we extract the information that B by applying the information that $A \to B$ to the information that A; however, although it is just as fine to claim that whenever I accept A, B I am committed to A, it is far more dubious that I extracted the information that A by *applying* the information that A to the information that B.

If weakening and contraction hold for external but not for internal conse-

[10] To simplify our picture, we focussed on a single-conclusion version of external consequence. If we had gone multiple-conclusion, it would have been more appropriate to state that external consequence has to do with the preservation of the warrant to assert, as well as with the converse preservation of the warrant to deny (cp. [49]).

quence, is there any property of internal consequence that external consequence lacks? An interesting example is *the deduction theorem*, intuitively understood as the statement that $A \to B$ follows from Γ just in case B follows from Γ, A. While built into the very notion of internal consequence, it is questionable for the external one. Returning to the Lewis example, suppose A is an axiom of our theory that is not employed in deriving B, and that Γ comprises the remaining axioms of the theory: then B follows from Γ, A in the external sense, but it would be awkward to say that $A \to B$ follows —even in that sense — from Γ, if $\to$ has to be a decent relevant conditional of any kind. On a more formal note: the deduction theorem for the internal notion of consequence is a direct outcome of the right introduction rule for implication, in any substructural logic. Most substructural logics, on the other hand, fail to prove the deduction theorem for the external notion of consequence and for the relevant, intensional conditional[11].

Distinctions analogous to the above-sketched one have been drawn from time to time in the literature. Prawitz, for example [43, pp. 29-30], claims that "we assert sentences hypothetically, i.e. on some given assumptions", and goes on to define inferring as "the act of asserting the conclusion on the grounds of the premisses and on the appropriate hypotheses (depending on what hypotheses the premisses are asserted on)". According to Prawitz, we ought to discriminate sharply between the *grounds* we have for asserting a given sentence and the *assumptions* on which we assert it. An approximately similar distinction also appears in the writings of Dana Scott, when he discusses 'horizontal' and 'vertical' inference [55, p. 802].

We will argue below that, in order to solve the paradoxes, this cut-off cannot be disregarded. Of course, this is compatible with the choice of several different logics which, unlike classical logic, keep this distinction in force. The logic we advocate here is **LL**, roughly, linear logic with no exponentials and no additive constants; partly because it is one of the strongest logics with this property, and partly because one of us argued elsewhere that it is an appropriate environment to deal with paradoxes of implication [42]. Since, ideally, we do not want to have a separate logic for each kind of philosophical riddle, having a unified framework for implicational, semantical and set-theoretical paradoxes looks to us like a considerable advantage of our theory.

4.2 Intensional vs Extensional Connectives

In his 1934 dissertation, where he introduced **LK**, Gerhard Gentzen gave the following introduction rules for conjunction (where Γ, Δ are finite, possibly empty

[11] It is essential to remark that it is this form of the deduction theorem that we are talking about. Most external relations of substructural logics have some local, or even global form of the deduction theorem in the sense of abstract algebraic logic: $\Gamma, A \vdash B$ iff $\Gamma \vdash A^n \to B$ for some n, or $\Gamma, A \vdash B$ iff $\Gamma \vdash A \sqcap t \to B$.

multisets of formulae of a standard propositional language[12]):

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}(\wedge \mathrm{L}) \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}(\wedge \mathrm{R})$$

Later, it turned out that the following pair of rules allow to formalise classical conjunction just as well:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}(\wedge \mathrm{L'}) \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma, B}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, A \wedge B}(\wedge \mathrm{R'})$$

In fact, using the structural rules of weakening and contraction, we can derive (∧L') and (∧R') in Gentzen's original formulation of **LK**, and conversely, it is possible to derive (∧L) and (∧R) in the calculus which has the alternative rules as primitive.

A similar situation holds for disjunction, where we have the alternative pairs of rules:

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}(\vee \mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}(\vee \mathrm{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Pi \Rightarrow \Sigma}{A \vee B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}(\vee \mathrm{L'}) \quad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}(\vee \mathrm{R'})$$

It is important to observe that the equivalence proofs for the previous pairs of logical rules *rest essentially on the presence of weakening and contraction inferences*. If, as previously advocated, we ban from **LK** the rules of weakening and contraction, it matters whether we choose to introduce, say, conjunction by means of the pair (∧L)-(∧R) or of the pair (∧L')-(∧R') — and similarly for disjunction. Classical conjunction (∧) and classical disjunction (∨) would both split. We would end up with:

- an extensional conjunction, or *meet* (⊓), defined by the rules (∧L)-(∧R) (henceforth rechristened (⊓L)-(⊓R));
- an intensional conjunction, or *fusion* (⊗), defined by the rules (∧L')-(∧R') (henceforth rechristened (⊗L)-(⊗R);
- an extensional disjunction, or *join* (⊔), defined by the rules (∨L)-(∨R) (henceforth rechristened (⊔L)-(⊔R);
- an intensional disjunction, or *fission* (⊕), defined by the rules (∨L')-(∨R') (henceforth rechristened (⊕L)-(⊕R).

If we choose natural deduction calculi instead of sequent calculi as our privileged framework, we can attain the same distinction. Confining ourselves to disjunction, the pair (⊕L)-(⊕R) is replaced there by conditional proof as an

[12] Gentzen had *sequences*, not multisets, of formulas. This forced him to assume structural rules of exchange which our choice renders unnecessary.

introduction rule, and disjunctive syllogism as an elimination rule, while the rôle played by (⊔L)-(⊔R) is taken up by addition as an introduction rule, and proof by cases as an elimination rule. A fission $A \oplus B$, therefore, can be asserted whenever each disjunct follows from the negation of the other, while a join $A \sqcup B$ can be asserted whenever we can assert at least one of its disjuncts:

$$\dfrac{\begin{matrix}[\neg A]\\ \vdots \\ B\end{matrix}}{A \oplus B}\,(\oplus I) \qquad \dfrac{\neg A \quad A \oplus B}{B}(\oplus E) \qquad \dfrac{A}{A \sqcup B}\;\dfrac{B}{A \sqcup B}(\sqcup I) \qquad \dfrac{A \sqcup B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}(\sqcup E)$$

Neither constant has, individually, all the logical properties of classical disjunction $A \vee B$; but each classical theorem involving disjunction holds of at least one of these two connectives. Differently from what most relevant logicians claim, we do not maintain that there are *invalid* inferential principles in classical propositional logic; if properly disambiguated, i.e. given the right interpretation of the logical constants contained therein, all laws of classical logic can be salvaged in **LL**.

Failing to disambiguate, however, can yield paralogisms. In [42], one of us argued that Prior's 'Tonk' argument and Lewis's independent proof of the *ex absurdo quodlibet* have a surprisingly common structure: if recast in sequent calculi terms, the former involves a disharmonious connective ('Tonk') which obeys (⊓L) as a left introduction rule and (⊔R) as a right introduction rule, while the latter involves another disharmonious connective (classical disjunction) which obeys (⊕L) as a left introduction rule and (⊔R) as a right introduction rule. Here, 'harmonious' means that in the cut elimination process, the non-atomic cut on the compound formula at issue can be replaced by cuts on its immediate subformulae. In the case of classical disjunction, this moving cuts upwards succeeds only if we have at our disposal the rule of weakening, which as we have seen is questionable on its own right. Or, if we see it the other way around, the unpalatability of the *ex absurdo quodlibet* often lamented by relevant logicians can be taken as a *reductio* of the soundness of weakening rules.

4.3 Application to the Paradoxes

The situation with set-theoretical and semantical paradoxes is, in our opinion, quite similar. Nonclassical solutions to these paradoxes typically proceed along the following lines: although naïve truth theory and naïve set theory, with their strong intuitive appeal, are retained in their entirety, they are superimposed on a nonclassical (usually subclassical) logical basis. In some cases, e.g. in the dialetheist approach, paradoxes remain provable but the theory itself avoids trivialisation in virtue of its paraconsistent consequence relation; in this perspective, paradoxes are viewed as *theorems* about inconsistent objects (or concepts). In other approaches, switching to a weaker nonclassical basis simply prevents paradoxes from arising. What we want to defend is just a variant of the latter perspective: classical naïve truth theory and classical naïve set theory encode sound pre-theoretical intuitions in an ambiguous formal theory, and therefore allows for gross fallacies of equivocation — the paradoxes themselves.

For the time being, let us confine ourselves to the connective paradoxes, and let us take Russell's paradox as an example. At some stage in its proof, one infers that $R \in R$ (where R denotes Russell's set) on the assumption that $R \notin R$, and similarly, that $R \notin R$ on the assumption that $R \in R$. This is not yet the contradiction we are after — we need to prove that $R \in R$ and that $R \notin R$ *unconditionally.* However, in classical logic we also have $R \in R \vee R \notin R$. Therefore, using the classical disjunction elimination, i.e., proof by cases, we get:

$$\dfrac{R \in R \vee R \notin R \quad \begin{matrix}[R \in R] \\ \vdots \\ R \in R\end{matrix} \quad \begin{matrix}[R \notin R] \\ \vdots \\ R \in R\end{matrix}}{R \in R}$$

Similarly, we prove that $R \notin R$. And now we have our contradiction.

What is wrong with this argument is that it conflates fission and extensional disjunction. The premiss $R \in R \vee R \notin R$ is only assertable as a fission, for otherwise we should be in a position to assert either $R \in R$ or $R \notin R$ unconditionally, which is patently not the case. But a fission does not have the proof-by-cases elimination rule. It has the disjunctive syllogism elimination rule. So, the attempt to extract a contradiction from the supposedly offending formula $R \in R \vee R \notin R$ is blocked, because it needs to be read intensionally in order to defend its plausibility and extensionally in order to derive a contradiction from it. Thus, Russell's paradox is a clear-cut case of *quaternio terminorum.* And that's all there is to it. Other variants of Russell's paradox, as well as Curry's paradox, the Liar and their numerous kins are blocked similarly.

5 Rejoinder to the Arguments of § 2

5.1 Restall (2008) and (2010)

Recall that Restall's argument in [48] presupposed several requirements about conjunction, here denoted through the ambiguous symbol $\wedge$: in particular, it had to be idempotent, such that $A \vdash B$ and $A \vdash C$ if and only if $A \vdash B \wedge C$, and residuated, i.e. for some connective $\rightarrow$ we had that $A \wedge B \vdash C$ iff $A \vdash B \rightarrow C$. The reader may by now have guessed our tack on the issue. These properties are conflicting requirements that cannot coexist in the same connective. More precisely, **LL** dictates that meet is idempotent and conditionally adjunctive but not residuated, and vice versa for fusion. In neither case we have all we need to let Restall's paradoxical argument go through. Yet, Restall goes on to point out a number of cases where we are *forced* to accept these properties at the same time and for the same connective. In particular, in any logic where conjunction distributes over disjunction the former connective is necessarily residuated, and we cannot escape the paradox. Since the distribution law can be motivated quite naturally from an analysis of natural language, Restall suggests that this undercuts any attempt to block the paradox by giving up residuation of conjunction.

How does **LL** fare in the light of this criticism? At first sight, not very well. It is well known that linear logic lacks a distribution principle of meet over join (and conversely). Among relevant logicians, this failure has been interpreted for a long time as meaning that the same logic fails to be distributive *tout court*. The reasons of this situation are manifold, and they have to do with some peculiar features of Anderson's and Belnap's preferred logical system **R**, as well as with their notational habit of using the same symbol for extensional relevant conjunction and for classical conjunction. However, **LL** does not have to give up distribution, for it contains a distribution principle of *fusion* over join. Does this mean that there are no unpleasant renunciations to be made? One could argue that rejecting even a single form of distribution is too heavy a price. Restall, for example, challenges us to provide a natural language counterexample to extensional distribution. Once we accept the ambiguity of conjunction and disjunction, it is not clear how we should understand the difficulty of finding counterexamples to distribution. Should we say that the lack of counterexamples shows that **LL** is not a good representation of natural language inference? Or should we rather say that natural language distributions should be understood in terms of **LL** valid forms of distribution? We take the latter approach and at the same time reject purely extensional distribution.[13]

The argument in [50], on the other hand, seems to jeopardise **LL** more seriously than does this previous one. After all, this logic has an intensional conditional $\rightarrow$ satisfying identity and modus ponens, as well as two intensional truth and falsity constants t and u which do precisely the job they are supposed to do (under a suitable interpretation of such): namely, t entails A whenever A is valid and A entails u whenever $\neg A$ is valid[14]. Therefore, Restall's argument has its bite. However, for it to be conclusive, one needs to show that modus ponens holds for the new conditional $\Rightarrow$.

If this proof is carried out in **LL**, something goes wrong, for $A \Rightarrow B$ fails to satisfy modus ponens.[15] Indeed, we have nothing to query until the penultimate step, where $B \vee u$ is derived from $A, A \Rightarrow B$. But then an elimination rule is used which looks very classical, and fails to hold in **LL**. As we have seen, in **LL**

[13] Here is one case in which we find distribution dubious. If we are provided the information that "Paul is a logician and Paul is Belgian" (where "and" can be formalised as you please) we can infer neither that "Paul is a logician and Paul is Flemish" nor that "Paul is Walloon". Therefore, we are in no position to infer the *extensional* disjunction of the above sentences.

[14] Sure, in the above definition there are two connectives in need of a disambiguation, but the only plausible interpretation here is to read them as *extensional* connectives, for $(A \otimes t) \rightarrow (B \oplus u)$ is **LL**-equivalent to $A \rightarrow B$.

[15] For a countermodel, let $\mathbf{Z}$ be the residuated lattice of the integers, where implication is interpreted as the converse of subtraction. Let A, B be atomic formulas, and let $v(A) = +1, v(B) = -1$, and $v(t) = v(u) = 0$. $\langle \mathbf{Z}, v \rangle$ is a model for **LL** with the nonnegative integers as designated values. Yet, $v(A), v(A \Rightarrow B) \geq 0$ whereas $v(B) < 0$.

joins are eliminated via the proof-by-cases rule:

$$\dfrac{A \sqcup B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}$$

Switching to our disambiguated notation, the only plausible application of this rule we can see in the above proof is the following:

$$\dfrac{B \sqcup u \quad \dfrac{[B]}{B} \quad \dfrac{[u]}{B}}{B}$$

But here a further fallacy of equivocation is lurking. How do you prove an arbitrary B from the assumption u? Classically, of course, the unique available falsity constant implies any old sentence, and many substructural logics (although not **LL**) have extensional falsity constants which validate this *ex absurdo quodlibet* rule. But even in these more expressive logics it is not possible to replace intensional constants with extensional ones in the definition of $A \Rightarrow B$, on pain of having $A \Rightarrow B$ and $A \rightarrow B$ collapse onto each other. Therefore, we have to stick to our intensional constants t and u, and for the latter the *ex absurdo quodlibet* is an invalid rule. Once more, the way to Curry's paradox seems to be blocked.

5.2 Hinnion and Libert (2003); Restall (2012)

The tactic in this proof, as we have seen, is to introduce a consequence relation that satisfies weakening and contraction. On Restall's interpretation, $\Gamma \vdash \Delta$ means that we are not permitted to accept all the members of Γ at the same time as denying all the members of Δ. This, we agree, is a coherent notion of consequence. On it, Γ and Δ are taken to be sets, and the comma on both sides of the turnstile is to be understood extensionally. On our view, this is a form of external consequence. It allows us to formulate certain norms of assertion and denial. What it does not do is to formalise a substantive notion of information extraction.

In this context, we need to ask which of the rules used is appropriate for this external notion of consequence. We assume, with the object of Restall, Hinnion and Libert's attack, that a strong form of Frege's law V is correct. We can replace $\varphi(y)$ with $y \in \{x : \varphi(x)\}$ in any context, and vice versa. As we said at the outset of this paper, we do not want to block the paradoxes by restricting comprehension. So, we must reject some other principle or principles used in their argument.

First, let us consider $Ext_\in$:

$$\dfrac{\Gamma, x \in a \vdash x \in b, \Delta \qquad \Gamma, x \in b \vdash x \in a, \Delta}{\Gamma \vdash a = b, \Delta} \; (Ext_\in)$$

The idea here is that if we can deduce $x \in b$ from $x \in a$ and the converse, then the sets a and b are identical. This is a version of the axiom of extensionality. But it does not accord with the substructural view of identity. In order to infer that $a = b$, we need to have known that $\forall x(x \in a \leftrightarrow x \in b)$. The biconditional here is the central biconditional of the substructural logic, which residuates with fusion, not with meet. So, if we interpret the comma in $Ext_{\in}$ as intensional (and Γ and Δ as multisets), we can accept it. But we must reject it if the turnstile is to be interpreted as an external form of consequence and, correspondingly, the commas are understood as extensional.

Now let's look at $= L_l$:

$$\frac{\Gamma, \phi(a) \vdash \Delta}{\Gamma, a = b, \phi(b) \vdash \Delta}\,(= L_l)$$

If the turnstile is taken to be internal consequence, then we can derive the following. Where a does not occur in ϕ,

$$\cfrac{\cfrac{\phi \vdash \phi}{a = b, \phi \vdash \phi}{=}L_l}{a = b \vdash \phi \to \phi}{\to}R$$

From a relevance point of view, the conclusion is unacceptable. The standard (and in our opinion correct) way of blocking this inference is to restrict $= L_l$ to extensional contexts (see [32]).[16] Therefore, whereas $Ext_{\in}$ must be read as rule governing internal consequence, $= L_l$, weakening and contraction can only be read as laws of external consequence. We cannot combine both rules in the way required by the Hinnion, Libert, and Restall argument. And so the argument is blocked.

5.3 Beall and Murzi (201+)

This typically structural paradox underscores the ambiguity of the classical concept of consequence. On the internal reading, as we have seen, Structural Contraction is utterly suspect[17], and in fact, it fails in the internal consequence relation of **LL**, which therefore avoids the difficulty. On the other hand, one could suggest to switch to an external reading, where such a rule is again available. Does that mean that we are back in trouble? No, for external consequence lacks anything like the deduction theorem, and therefore the disquotational validity schema — in particular its Val-In part, namely if $A \vdash B$ then $Val\left(\overline{A}, \overline{B}\right)$ — becomes problematic. This, of course, if $Val\left(\overline{A}, \overline{B}\right)$ is understood (correctly, we think) as a predicate appropriate for *internal* validity. Graham Priest

[16] There are other reasons for rejecting $= L_l$, interpreted intensionally. For one, it trivializes Dunn's lovely theory of relevant predication [14]. If we add this rule, every case of predication becomes relevant.

[17] Structural Contraction has been blamed for leading to what is essentially a version of a V-Curry paradox in [56], in the more general context of a deflationist account of logical consequence. An early reference to the rôle of Structural Contraction in the debate on truth, paradox and logical consequence can be found in a paper of (Stewart) Shapiro from 2003 [57].

suggested in conversation that we could regain the paradox if we had a corresponding predicate for external validity, for then an external reading of the turnstile would support all the ingredients needed for the argument to succeed. We think, however, that our theory should contain no such predicate, exactly as it contains no implication that residuates extensional conjunction — and Val, for us, *is* a form of implication.

Although obviously sympathetic to such an approach, however, Beall and Murzi resist endorsing it explicitly. We recap hereafter the three main objections they raise either in that same paper, or in a sequel written by Murzi alone ([38]):

1. The substructural view of consequence is clearly at odds with the standard account of consequence and validity in terms of truth preservation. This is the main reason behind, for example, Field's reluctance to even envisage this way out of the paradoxes [16, pp. 10-11]. Beall and Murzi concede that the rejection of Structural Contraction goes along much better with certain proof-theoretical conceptions of validity, but — they seem to imply — any bid for an independent justification of these views would have to face serious difficulties of its own. Consequently, they conclude by urging enemies of Structural Contraction to provide "a new metaphysical account of validity—one for which the rejection of Structural Contraction is perfectly natural", supposedly different from an account along proof-conditional lines.

2. Structural Contraction is heavily used in mathematics, not only as a means to get paradoxes, but also as a tool for deriving sound theorems, including e.g. Cantor's theorem. This suggests that it may not be so wise to renounce *all* the applications of Structural Contraction. If so, how do you tell the grain from the chaff? Is there a principled criterion according to which you can retain the 'good' uses of structural contraction while abandoning the 'bad' ones? To stretch this line of thought just a bit: we have shown what we *can't* do using **LL** (we can't prove certain versions of the paradoxes), but we haven't shown what we *can* do with it. Can we reconstruct significant fragments of naïve set theory with our logic as a basis? Or is it altogether to weak to do anything interesting? After all, one could argue in favour of an *empty* consequence relation on the grounds that it's safely paradox-free — however, this is of course mad since it would amount to throwing out the baby (set theory) with the bathwater (the paradoxes).

3. Suppose we have a substructural theory **S** of naïve truth and validity. An argument using Löb's Theorem can be given to the effect that

   Validity and derivability-in-**S** must peel apart: either some arguments that are derivable in **S** are not valid, or some valid arguments cannot be derived in **S**. Lest **S** is unsound, validity must outrun derivability-in-**S** [38, p. xx].

   According to Murzi, this result need not be unacceptable to the propounder of a substructural account. One might say that for any formal theory **T** satisfying the Hilbert-Bernays derivability conditions and rich enough to contain disquotational principles for validity, there are valid inferences which cannot be formalised in **T**. Once we embrace this option, on the other hand, most of the appeal of the suggested revisionary approach is gone. Have these views not been propounded in order to obtain a closer fit of our theoretical notions of truth, validity or consequence to the informal ones, or even to avoid the distinction between language and metalanguage? Biting the bullet, here, means giving up one of the main motivations for revising classical logic in the first place.

The next Section is entirely devoted to a reply to Objection 1. Although we do not share the devaluative view of proof-conditional theories of truth, meaning and consequence that Beall and Murzi imply to be commonplace in the scholarly community (almost as if showing that a rejection of Structural Contraction is only consistent with a proof-conditional account of the matter amounted to a *reductio ad absurdum* of the plausibility of this move), we accept the challenge of providing a *different*, information-based, account of validity for which dropping Structural Contraction in the only natural thing to do. In the remainder of this Section, we briefly and jointly address the other two objections.

It can reasonably be argued that partisans of substructural and other nonclassical logics are playing a home match when advancing their solutions to the paradoxes: they simply have to point at whatever classical inferential principle they may find problematic and show that it is essential in deriving the faulty conclusion at issue. But then a more difficult away match remains to be played — are the logical principles that survive their lopping sufficient to retrieve the sound parts of, say, naïve set theory? Several authors (e.g. [26]), indeed, have undertaken such a *pars construens*. They set up nonclassical axiomatic systems for naïve set theory, proved them to be consistent, and then tried to reconstruct as much set theory as possible within them. Would not a similar strategy look promising also in the case of **LL**?

We do not think so. A proof system (axiomatic or otherwise) for set theory would presuppose, at least, a proof system for first order logic — and in substructural logics *that's exactly what is beyond the state of the art*. The current approaches to first order substructural logics superimpose to propositional logics which contain intensional and extensional connectives a first-order upper layer *that only contains extensional quantifiers*, simply because there are several conflicting intuitions about what intensional quantifiers look like or what rules they should obey. For the monadic fragment, there have been some attempts to develop formal theories of intensional quantifiers in the literature on substructural logics ([37], [9]; [17]; see also [39]). The fascinating proposal in the direction of a contraction-free theory of naïve truth recently advanced by Elia Zardini [62], where intensional quantifiers of sorts play an important rôle, stands in need of a closer assessment. However, until some more light is shed on what we consider one of the most important philosophical, as well as technical, problems about substructural logics, we have to confine ourselves to blocking those paradoxes arising at the level of propositional connexion, while being unable to follow up with a positive proposal as to the form our alternative set theory, or formal truth theory, should assume. These remarks put Objection 2 on hold, so to speak. But for the same reason, Objection 3 loses some of its bite. The substructural naïve truth theory **S** envisioned by Murzi is entirely hypothetical, but the fact that he needs Löb's Theorem to carry out his argument leads to suppose that, whatever its logical basis, it is (in some sense of this word) at least as strong or expressive as Robinson's arithmetic. What treatment of the quantifiers are we to expect from this formal system? Again, using only extensional quantifiers may be enough to prove the result Murzi is after, but does not do justice to our approach.

6 Informational Semantics

There are two elements to Beall and Murzi's challenge. First, there is the problem of providing a philosophical interpretation of the model theoretic semantics for contraction-free relevant logics. Second, there is the difficulty of showing that in the framework of this semantics we can have a reasonable treatment of their operator Val that does not lead to Curry's paradox.

We deal with the first of these problems by presenting a modified version of the theory of situated inference from [33]. The theory in that book was created to provide a philosophical reading of the logic **R** and needs some modification in order to understand **LL**. Our interpretation fits in a fairly straightforward way to the model theory of Ono [40], Goldblatt [23], Wansing [61] and Girard [22]. An interpretation of the model theory of Allwein and Dunn [3] is possible along the same lines, but it would be somewhat more complicated.

On the current version theory of situated inference, the indices of the model theory are situations – parts (or potential parts) of worlds. The notion of a situation that we use is an abstraction from the notion of a concrete situation. A concrete situation is a concrete part of the universe. Concrete situations contain, or fail to contain, information. The situation that contains all and only the information available in Edwin's study on 23 March 2012, for example, does not contain the information about whether or not it is raining in Cagliari on that day (unless of course, his computer is on the internet, but we will discount that). But it does contain the information that Edwin's desk is brown, that it is not red, and so on. So too with the abstract situations that we use in the semantics. They contain certain positive or negative information and are silent on other matters.

On the current version of situation semantics, we take situations to be very abstract characterizations of parts of worlds. One particularly important aspect of the view is that the same situation can be instantiated more than once in the same world. For example, we might have an abstract situation that merely tells us that there is a dog barking. Clearly, there can be more than one dog barking in the world, and hence we say that this situation is instantiated more than once in the actual world.

The doctrine of situated inference says that an implication $A \to B$ is to be understood as licensing an inference from there being a situation in one's world which contains the information that A to there also being a situation instantiated in that world that contains B. Our understanding of this license in the current context is as a weaker notion than that of [33]. The implication $A \rightarrow B$ tells us that A warrants the acceptance of B and this is how we should understand the licensing of the inference. By 'warrant' here we do not mean the mere extensional relationship that is represented by the external notion of inference, but rather an intensional relation: if $A \rightarrow B$ in a situation s, then the norms of evidence and the other information available tell us that there is a relevant evidential connection between A and B. The very abstract notion of a situation that is in use here allows us to show that contraction is not supported by our interpretation. The instantiation of a situation twice may warrant conclusions that a single instantiation does not warrant. To take a well-worn example from [41], if the situation that says that a person saw X steal the jewels is instantiated twice it may be sufficient to warrant the assertion that X did steal the jewels, whereas a single eyewitness may be insufficient.

The theory of situated inference that we apply here is different in at least two ways from that of [33]. First, the information available in a situation is to be understood in terms of informational tokens. This is a theme that we have stressed throughout this paper. Second, the model theory for substructural logics has distinguished situations, at which we verify the theorems of the logic. These are variously called "normal situations", "logical situations", or "base situations" in the literature. One of the important characteristics of base situations is that they satisfy a model theoretic form of the deduction theorem and its converse, that is, for all situations s in a model, if $s \models A$ then $s \models B$ if and only if for all base situations b, $b \models A \rightarrow B$. Our understanding of base situations is, in particular, constructed to provide an explanation of this property. We do so by borrowing a notion from intuitionism and from the philosophical view known as response dependence. Base situations are idealizations that contain an agent who has an unlimited capacity to reflect on the nature of the complete model. If every situation that contains the information that A also contains the information that B, then this epistemologically perfect agent will accept that it is a principle that A warrants the acceptance of B, i.e., that $A \rightarrow B$.

We can represent situated inferences in the framework of a natural deduction system. Instead of adding numbers to indicate dependencies on assumptions (as in the proof in section 2.4), an assumption is always of the form $s_k \models \varphi$, where s_k is a new situation to the proof – it is the assumption that a particular formula holds at an arbitrary situation. Implication introduction, then, can be represented schematically as

$$\begin{array}{|l} s_k \models A \\ \vdots \\ s_\alpha \models B \end{array}$$
$$s_{a-k} \models A \rightarrow B$$

α is a multiset of numbers that contains at least one occurrence of k. $\alpha - k$ is the result of removing exactly one occurrence of k from α. s_α is a situation that we infer to exist at our world from the information in the situations the numbers of which occur in α (and used the number of times they occur in α). In short, the natural deduction system for situated inference repeats the natural deduction system for the logic, with explicit reference to situations.

Here is the Beall-Murzi argument reinterpreted using our theory of situated inference. The notation $\mathcal{M} \models A$ means that A is satisfied by all the base situations in $\mathcal{M}$. The subproof assumes that there is an arbitrary situation which contains the information that π.

1.	$\mathcal{M} \models \pi \leftrightarrow Val(\overline{\pi}, \overline{A})$	
2.	$\quad \mid\ s_2 \models \pi$	Ass
3.	$\quad \mid\ s_2 \models Val(\overline{\pi}, \overline{A})$	$1, 2$
4.	$\quad \mid\ s_2 \models A$	$2, 3$
5.	$\mathcal{M} \models Val(\overline{\pi}, \overline{A})$	$2-4,\ Val\ I$
6.	$\mathcal{M} \models \pi$	$1, 5$
7.	$\mathcal{M} \models A$	$5, 6,\ Val\ E$

The rule $Val\ I$ is 'Val introduction' and $Val\ E$ is 'Val elimination'. The key step here is the move from lines 2,3 to 4. We argued above that this move fails to be internally valid, while the move from 2-4 to 5 fails to be externally valid. However, even if we grant step 4 in this form, we have that the formula

$$Val(\overline{\pi}, \overline{A}) \sqcap \pi \rightarrow A$$

must also be valid (by the model theoretic version of the deduction theorem). This formula is recognizable to anyone working in the field as a variant of *pseudo-modus ponens*, a very close relative of contraction and a formula that we reject as a theorem.

Beall and Murzi, in effect, justify the application of pseudo-modus ponens by the definition of Val, in this case, that $Val(\overline{\pi}, \overline{A})$ if and only if $\pi \vdash A$. But the usual way of understanding such definitions in the model theory for relevant and substructural logics is that this biconditional only holds at base situation – in our theory, the ideal situations that contain perfectly reflective agents. This is not enough to justify the use of detachment at arbitrary situations, such as s_2 in the inference presented above. From the point of view of situated inference, if one is in a situation which contains the information that $Val(\overline{\pi}, \overline{A})$ and the information that π, then she has adequate warrant to assert that A. But for us all that means is that she has warrant to infer that there is some situation that contains the information that A.

If Val is interpreted in terms of reasonable warrant, then we could just stop here. Situations are not closed under reasonable warrant, and so pseudo-modus ponens is invalid. But what is supposed to be new about Beall and Murzi's use of $Val(\overline{\pi}, \overline{A})$ in the paradox is that, in our terminology, this is supposed to *mean* that every situation that contains π also contains A. So, in knowing that one's own situation contains π and $Val(\overline{\pi}, \overline{A})$ should be sufficient to know that it also contains A. This, however, does not follow. Situations are to be understood informationally. If we have the information in s that for every situation if it has the property P then it has the property Q, we need to know that s is in its own domain of quantification.[18] To demand this is to say that every situation must contain the information that it is itself a situation. This demand seems too strong.

For those readers who are not familar with or suspicious of the theory of situated inference, consider a closely related example, but one based on classical normal modal logic rather than on substructural logic. In the Gödel-Löb logic, **GL**, necessity is to be understood to mean provablility. So, we should have $\Box A$ iff $\vdash A$. It is true that this holds in the logic (i.e. that $\Box A$ is a theorem iff $\vdash A$) and that it can hold in a particular model (i.e. there is a model $\mathcal{M}$ such that $\mathcal{M} \models A$ iff $\vdash A$).[19] But in models for **GL**, there typically are worlds that satisfy necessitives that are not theorems of the logic. So, although there is a sense in which the biconditional $\Box A$ iff $\vdash A$ holds, and the box is to be understood as provability, this biconditional does not holds at all worlds. The case of the Beall-Murzi biconditional ($Val(\overline{A}, \overline{B})$ iff $A \vdash B$) in the model theory for substructural logics is very similar. If one takes the possible worlds semantics for GL to be a reasonable interpretation of that logic, then he or she should be with us in rejecting that the Beall-Murzi biconditional should hold for all situations.

7 Conclusion

If we are ready to grant that classical logic equivocates over both the meaning of logical constants and the meaning of consequence, we have a powerful argument to the effect that ambiguity has to be blamed for the paradoxes. In particular, we tried to provide a unified informational account of both meanings that does not justify our jointly upholding Structural Contraction and disquotational validity (or other seemingly paradox-conducive principles), except for *different* notions of consequence. We have seen that such a move is in line with a stream of recent papers, all of them, for various reasons, suspicious of Structural Contraction. We have not mentioned so far a more cautious approach, which rejects contraction only in a restricted manner, and we do it now. Peter Schröder-Heister ([52], [53], [54]), for example, suggested in a series of articles that in any antinomy we only need to contract over occurrences of the paradoxical formulas, whereby it would suffice to block contraction in such cases to avoid the paradoxes and to achieve at the same time the classical recapture needed to regain back the sound parts of our mathematical theories. This may well be the way to go in a prospective *pars construens*. However, to declare a principle invalid is to affirm that there is at least one counterexample to it, and this is perfectly consistent with the story we told above. For the time being, we are content with indicating

[18] The semantics for the universal quantifier of [34] has constant domains. But this was a simplified version of the "true semantics", which has variable domains. Surely not every situation contains information about every possible (or impossible) object.

[19] One such model is the canonical model. Readers might worry about the appeal to the canonical model, since its accessibility relation is not converse well-founded (and so the canonical model is not a **GL** model as usually defined). But it is a model of **GL**, even though **GL** is not complete over the frame of the canonical model.

that the prospects for the 'new account of validity' urged by Beall and Murzi as a prerequisite for a rejection of Structural Contraction are not as hopeless as they might seem to be.

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