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Abstract

The process of the mathematization of physical situations through differential calculus requires an understanding of the justification for and the meaning of the differential in the context of physics. In this work, four different conceptions about the differential in physics are identified and assessed according to their utility for the mathematization process. We also present an empirical study to probe the conceptions about the differential that are used by students in physics, as well students' perceptions of how they are expected to use differential calculus in physics. The results support the claim that students have a quasi-exclusive conception of the differential as an infinitesimal increment and that they perceive that their teachers only expect them to use differential calculus in an algorithmic way, without a sound understanding of what are they doing and why. These results are related to the lack of attention paid by conventional physics teaching to the mathematization process. Finally, some proposals for action are put forward.

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Obstacles to Mathematization in Physics: The Case of the Differential

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Abstract The process of the mathematization of physical situations through differential calculus requires an understanding of the justification for and the meaning of the differential in the context of physics. In this work, four different conceptions about the differential in physics are identified and assessed according to their utility for the mathematization process. We also present an empirical study to probe the conceptions about the differential that are used by students in physics, as well students' perceptions of how they are expected to use differential calculus in physics. The results support the claim that students have a quasi-exclusive conception of the differential as an infinitesimal increment and that they perceive that their teachers only expect them to use differential calculus in an algorithmic way, without a sound understanding of what are they doing and why. These results are related to the lack of attention paid by conventional physics teaching to the mathematization process. Finally, some proposals for action are put forward.

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1 Problem Statement: Why Differentials? Why is it Important to Analyse How Differentials are Taught and Learned in Physics Classes?

Differential calculus (DC) provides quantitative research methods for studying the process of change and the dependency of one variable on others (Aleksandrov et al. 1956). Its invention in the seventeenth century represented a major boost for many branches of science, and it has been considered as 'the most powerful theoretical tool ever constructed by men throughout history' (Rossi 1997, p. 199) and 'the main quantitative tool for the research of scientific problems for the last three centuries (…) without which physics and modern technology would not exist' (Kleiner 2001, p. 138). The calculus is one of the great triumphs of modern civilization (Dray and Manogue 2010), it lies at the foundation of our scientific world view and it is important for an understanding of who we are as a society (Bressoud 1992).

Because of its relevance in the progress of scientific knowledge, DC is first used in Spain in the teaching of maths and physics in the final years of high school (in the techno-scientific branch, for 17- to 18-year-old students) and becomes an essential part of university education. Typically, DC in physics is not a straightforward application of the ideas already learned in maths. A reinterpretation of these ideas in the physics context is required (Meredith and Marrongelle 2008; Uhden et al. 2012). Maths used in physics, and pure maths, have distinct objectives because the aim of physics is the description and understanding of a physical system, and not the resolution of equations and the expressing of the AQ3 most abstract possible relationships (Redish 2006). This latter author proposes a model to describe the bare bones of how we use maths in physics (see Fig. 1).

Once the physical analysis of a real-world problem or situation has been made, step #1 is a mathematization process or mathematical modelling that consists of expressing the ideas of the initial analysis through mathematical concepts and relationships. Uhden et al. (2012) distinguish different levels of mathematization and propose a greater intertwining of physics and mathematics models.

This 'translation' from the physical context to the abstract level of the mathematical one requires an understanding of the mathematical framework to be used, that is to say, the mathematical concepts and their relationships (White and Mitchelmore 1996). Usually, conventional teaching at all levels pays no attention to the first step and focuses on step #2: how to operate with the initial mathematical expressions to get a numerical or algebraic result. This imbalance has clear consequences for students: on the one hand, it leads them to turn physics problem-solving into maths problem-solving, and on the other hand, it affects their beliefs on how they are expected to use mathematics in physics (Redish 2006).

When using DC in physics situations, the basic concept that appears in the mathematization process is the one of the differential calculus, referring to independent variables

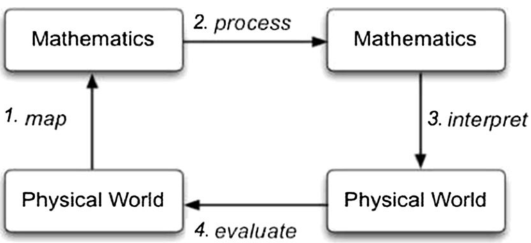

Fig. 1 A model for the use of maths in science (Redish 2006)

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and as part of differential expressions. For instance, $F\cdot \mathrm{d}x$ is used to introduce the concept of work; $\mathrm{d}q = \rho\cdot \mathrm{d}x$ for calculating the charge of a unidimensional charged rod; $\mathrm{d}N = -\lambda\cdot N\cdot \mathrm{d}t$ to study nuclear decay; $\mathrm{d}I = R^2\cdot \mathrm{d}M$ to calculate the moment of inertia; and $\mathrm{d}T = -\alpha\cdot(T-T_a)\cdot \mathrm{d}t$ to obtain the functional expression for the cooling of an object over time. If we want students to learn how to mathematize real-world problems when DC is required, a conceptual understanding of this kind of expression and the situations in which these expressions are necessary is essential.

But differential is a polysemic concept. Its meaning and role are different in maths and physics—in maths, the differential is concerned with substantiating and formalizing calculus beyond the physical context, and in physics, it is centred on the productive use of a set of concepts and reasoning regardless of rigour (Artigue and Viennot 1987; Dunn and AQ4 Barbanel 2000). This polysemic character of the differential is also found in education. It is not only its role and meaning that are different in the teaching of maths and physics, but even possible to acknowledge different conceptions within these areas (Alibert et al. 1989; Artigue and Viennot 1987; Dray and Manogue 2010).

Our aim in this paper is to identify the conception or conceptions of the differential that are used by students in physics and assess whether these conceptions are the most appropriate to allow them to mathematize or, on the contrary, whether they are only useful to allow students to handle in a mathematical way expressions that have already been given mathematically.

This search for conceptual transparency must be pursued from the very introduction of DC in physics teaching (as it is, in our experience, in physics courses in upper high school and in the first courses of college physics). Precisely because of their introductory nature, when the physics situations that are being studied are not very complex, reasoning, and a clear justification of what we do when we face real-world physics problems, should be distinctive characteristics of these physics courses. Otherwise, it might happen that the teaching never addresses the necessary requirements for mathematizing physics situations and, in this way, we might encourage mechanical behaviour in students and incomplete or incoherent conceptions about the use of DC in physics, with consequent feelings of insecurity when students try to mathematize physics problems.

This is the reason why we have studied how students in their final year of high school, after the conventional introduction of differential calculus in physics, conceive and justify differential expressions, what perceptions they have of what they are expected to be able to do with DC in physics and, also, the extent to which these ideas and perceptions remain unchanged among university students.

This work is organized in two parts. In the first part, we will summarize some findings AQ5 of other studies on the usual approach to teaching and learning differential calculus [2]. Then, we will introduce [3, 4] some different conceptions of the differential from the point of view of both physics and mathematics (our students are affected by both). And in the last section of the first part, we will assess the possible usefulness and shortcomings of these conceptions [5] for helping in the mathematization process.

In the second part, we will describe the experimental design [6], and we will analyse the data [7] on students' ideas about the use of the differential, and the justification for that use, and their perceptions of how they are supposed to use DC in physics. In the last section [8], the main conclusions and implications for teaching that are derived from the results are summarized.

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2 Some Prior Work About the Teaching and Students' Use of Differential Calculus

In this section, we will try to give a brief summary of some contributions of the literature on how DC is taught and learned in mathematics and how it is used in physics.

Research in mathematics education has highlighted the poor conceptual understanding that students, and also teachers, have of DC.[1] That poor understanding affects not only the main ideas of calculus, like derivative or integral, but also other related concepts such as variable, function, limit and infinite. Some ideas have been identified as acting as barriers to good learning, such as conceiving the tangent as a straight line that touches the curve at only one point (Ferrini-Mundy and Lauten 1994; Speiser and Walter 1994), considering the surface to be generated by the accretion of lines or the volume by the accretion of surfaces (Schneider 1991; Turégano 1998), rejecting the existence of so-called mental objects as opposed to empirical ones (Schneider 1992; Speiser and Walter 1994), or performing known operations on misunderstood symbols (White and Mitchelmore 1996).

Most of the above-mentioned authors point to the merely algorithmic approach used in teaching [that is, the exclusively pragmatic perspective of the mathematics (Uhden et al. 2012)] as the basis of these shortcomings. After all, a large number of maths teachers and of maths teachers' trainers explicitly assume an instrumentalist view of maths (Moreno 2001; Mura 1993, 1995; Pereira de Ataíde and Greca 2013). The research of Nagy et al. (1991) highlighted this algorithmic tendency in calculus teaching: an analysis of sessions focused on studying calculus, including exams, showed a clear predominance of the *techniques* category over other categories relating to the meaning of concepts, when and why they should be used, etc. This operational view is absorbed by the students, who end up believing that doing maths is restricted to performing specific operations with meaningless symbols (Habre and Abboud 2006; Porter and Masingila 2000), and this does not necessarily lead to greater procedural confidence (Engelbrecht et al. 2005).

If we add to this algorithmic approach the usual tendency to give mathematics only a technical role in physics, emphasizing mathematical manipulations, it seems logical that when students use differential calculus in physics, they know how to perform the calculations, but they have difficulty connecting the physical world with mathematics. As several works have shown, although students know how to calculate integrals in a specific physics problem, they have difficulties related to the conceptual understanding. For instance, at all levels when approaching physical situations, students have difficulties in understanding the integral as a limit and interpreting the result as an exact value, in deciding when it is necessary to use the integral concept, in establishing the integration limits and, especially, in writing down the correct differential expression that represents a concrete physical situation or giving meaning to the product $f(x)\cdot \mathrm{d}x$ when constructing an integral.[2]

The above survey highlights the shortcomings in teaching and the difficulties for students in performing the mathematization process. The relevant role of the differential in this process means that it is necessary to address the understanding of its role and meaning

[1] Artigue et al. (1989), Berry and Nyman (2003), Ferrini-Mundy and Gaudard (1992), Ferrini-Mundy and Graham (1991), Labraña (2001), Mahir (2009), Nagle et al. (2013), Orton (1983a, b), Porter and Masingila (2000), Schneider (1991, 1992), Tall (1985, 1992), Thompson (1994), among many others.

[2] See, for example, Meredith and Marrongelle (2008), Hu and Rebello (2013), Sealey (2014), Von Korff and Rebello (2014), and Wilcox et al. (2013).

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from the mathematical perspective, as well as to identify and assess the different conceptions about the differential as used in physics.

3 The Concept of the Differential in Teaching Mathematics

In this section, we will briefly characterize some different, and more frequent, roles and meanings that the differential has in teaching maths.

1. *The differential as a merely formal instrument with no meaning in itself*. DC teaching has remained loyal to nineteenth century mathematics, represented by the work of Cauchy who, by means of an accurate definition of limit, banished from calculus the ambiguity and lack of rigour that can both be attributed to Leibnitz's differential (Martínez Torregrosa et al. 2006). Cauchy defined the differential as an expression involving the derivative: $\mathrm{d}f = f'(x)\cdot \mathrm{d}x$, with an arbitrary (big or small) increment dx of the variable, and it thus became a simple formal instrument, necessary for the abbreviation of certain proofs. The differential was then detached from the ambiguity of the infinitely small quantities, but it was devoid of all physical meaning: it was just the result of multiplying the derivative by the increment of the independent variable. As Freudenthal (1973, p. 550) says: 'Useless differentials can readily be dismissed. If dy and dx occur only in the combination dy/dx, or under the integral sign after the integrand, the question as to what dx and dy mean individually is as meaningful as to ask what the "l", "o", "g" in "log" mean'.
2. *The differential as a linear approximation* (*but never used in practice*). Modern calculus texts, if they introduce the differential and assign it some meaning, usually do this as a linear approximation of the increment: $\Delta y \approx f'\mathrm{d}x = \mathrm{d}y$. However, after that, in practice, the differential only appears as part of algorithmic developments and plays a similar role to that in conception #1. This conception has been criticized because, although it usefully refers to the idea of linear approximation, the identification with the differential is unnecessary (Dray and Manogue 2010). These authors suspect that this unnecessary identification is done to avoid any risk of identifying differentials with infinitesimals.

In recent decades, two conceptions of the differential have been proposed that give it back its central role in the structure of DC: the Fréchet differential and the infinitesimal differential.

3. *The differential as a tangential linear estimate*. The Fréchet differential, whose original definition in 1911 had its origin in the analysis of functions of infinite variables, is used in some textbooks to introduce the analysis of functions of one variable (Del Castillo 1980; Hallez 1989, p. 67). For this particular case, Fréchet would define differentiability and the differential in this way:

   > A function $f(x)$ admits a differential, in my sense, in point x_0 if there is a homogeneous and linear function of the increment, let it be $A\cdot\Delta x$, that does not differ from the increment of the function Δf, that starts from the value $f(x_0)$, in more than an infinitely small value in relation with Δx. The differential is then, by definition, $\mathrm{d}f = A\cdot\Delta x$.
   > (…) This definition is expressed by the formula: $\Delta f = \mathrm{d}f + \varepsilon\cdot\Delta x$, where ε goes to zero when Δx goes to zero. It reminds us of the old definition, as the principal part,

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and it presents all its advantages, but it overcomes the objections of lack of rigour that quite correctly had been put forward to it (Artigue 1989, p. 34).

This then means that df is an approximation of Δf and that it is linear respect to Δx. However, it cannot be any linear approximation but it is the one that meets an additional condition: (Δf–df) must be infinitely small in relation to Δx. This does not mean that Δf or df is infinitely small. The condition imposed on the differential can also be expressed by saying that (Δf–df) goes to zero faster than Δx. This last condition is equivalent to either of the two following ones:

- The Riemann sum of the approximations (df) must coincide with the increment: $\lim_{N\to\infty} \mathrm{d}f_i = \lim_{N\to\infty} \Delta f_i = \Delta f$, that is, $\int \mathrm{d}f = \Delta f$. This step from the approximation (df) to the exactness (Δf) is only possible if the approximation fulfils Fréchet's condition. In fact, for the Riemann sum $\int(\mathrm{d}f - \Delta f)$ to be zero, the limit of $N \cdot (\mathrm{d}f - \Delta f)$ when N goes to infinity must be zero, that is, the limit of $(\mathrm{d}f - \Delta f)/\Delta x$ must be zero when Δx goes to zero, but this is a precise condition required by Fréchet for the differential.
- The differential quotient df/dx equals the derivative f'. We will see that if the approximation df fulfils Fréchet's condition, then its slope will coincide with the derivative. In effect, as df is a linear function of the increment, its slope df/Δx has a constant value and therefore the Fréchet condition can be expressed as: $\lim_{\Delta x\to 0} \frac{(\mathrm{d}f - \Delta f)}{\Delta x} = 0 \to \frac{\mathrm{d}f}{\mathrm{d}x} - \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} = 0 \to \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}$. The second member of this last equality is the derivative of the function, if it exists, and since for a single independent variable $\Delta x = \mathrm{d}x$, then: $f' = \mathrm{d}f/\mathrm{d}x$.

The combination of these two forms of expressing the Fréchet condition leads directly to the fundamental theorem: $\int \square \cdot \mathrm{d}x$ equals ΔF if and only if: $\square \cdot \mathrm{d}x = \mathrm{d}F$, that is, $\square = \mathrm{d}F/\mathrm{d}x = F'$. We have addressed these ideas in more detail in other works (Martínez Torregrosa et al. 2002, 2006).

4. *The differential as an infinitesimal*. This last conception brings back the original idea of Leibniz: the differential as an infinitesimal, but this time in a rigorous way based on the non-standard analysis introduced by Robinson in the 1960s. Robinson built up a large set of numbers that includes real numbers, infinitesimals and infinite numbers. The infinitesimals are nonzero numbers that are lower than any real number. There are a few introductory calculus texts based on Robinson's ideas; among these, the text of Keisler (2000) is worth quoting. Keisler introduces the main DC ideas (derivative, differential, integral and the fundamental theorem) by means of this new set of numbers and without the concept of limit. Next, we briefly present the conception of the differential as an infinitesimal, but we recommend the study of Keisler's text.

If Δx is an infinitesimal, then Δf is also an infinitesimal and the derivative f' is defined, if it exists, as the nearest real number to the quotient $\Delta f/\Delta x$ (which is called the standard part of this quotient) (Keisler 2000, pp. 55–57). The differential is defined as the product of a real number times an infinitesimal: $\mathrm{d}f = f' \cdot \Delta x$, which is also an infinitesimal. Here again, for the independent variable, $\Delta x = \mathrm{d}x$, and thus, we can write: $\mathrm{d}f = f' \cdot \mathrm{d}x$. The so-called theorem of the increment demonstrates that if Δx is an infinitesimal, then there exists another infinitesimal ε that satisfies: $\Delta f = \mathrm{d}f + \varepsilon \cdot \Delta x$ (Keisler 2000, pp. 55–57).

We highlight some characteristics of this conception to avoid any possible inadequate interpretations. The differential is an infinitesimal, but not an infinitesimal increment. df is an approximation of the increment Δf; however, it is not just any approximation but it is the

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one that meets an additional condition related to the quotient (Δf−df)/Δx. For the infinitesimal differential, the condition is expressed by saying that this quotient must be an infinitesimal, while in the case of the Fréchet differential, it is said that the limit of this quotient goes to zero when Δx goes to zero.

The fundamental difference between the two is that for the infinitesimal differential, the emphasis is on its value (which is lower than that of any real number), while the Fréchet differential puts the emphasis on the linear nature of this estimation of Δf, highlighting what the French mathematician Dieudonné (1960, p. 145) considered to be *the fundamental idea of calculus*: the approximation of any function by linear functions (Artigue and Viennot 1987; Martínez Torregrosa et al. 2006).

Sofronas et al. (2011) have shown that the conventional teaching of calculus does not incorporate the idea of approximation. Similarly, the analysis of French calculus texts by Alibert et al. (1989, p. 10) concludes that, although the idea of approximation associated with the differential in mathematics was introduced in the 1960s, there were no changes in practice, and it continued to be considered as a merely formal instrument, useful for developing operations.

4 The Differential Concept in Physics Teaching

In this section, we will briefly describe some different conceptions that are used or could be used in physics teaching. This description is based on the results of an analysis of high school physics texts (in Spain) and university physics texts for the first courses of science and engineering degrees (López-Gay 2002), on our experience as physics teachers in high schools and for the first courses of engineering degrees, and on the study presented above from the mathematical perspective. From now on, we will continue to use the mathematical notation (f, x…) when referring to general physics equations and magnitudes, and we will only change this notation for specific physics situations.

1. *The meaningless differential*. A few (fortunately!) physics textbooks use differentials without assigning them any meaning, whether explicit or implicit. It seems that they consider differentials to be part of an intermediate routine that leads to derivatives and integrals, in a similar way to conceptions #1 and #2 of the previous section. The following fragment comes from an upper high school standard physics textbook: 'According to the equation $F = q \cdot v \cdot B \cdot \sin\alpha$, the force exerted by the magnetic field on the charge dq is: dF = d$q \cdot v \cdot B \cdot \sin\alpha$ (…) that is the force that an *element* of electric current will suffer inside a magnetic field'.
2. *The differential as an infinitesimal increment*. According to this conception, df is equal to an infinitesimal Δf, produced by an also infinitesimal Δx. The terms infinitesimal, very small, tiny or extremely small are used to express the same idea. Indeed, explicit definitions of this conception are sometimes expressed: 'df is the limit of Δf when Δx goes to zero'. But if this is the case, if $f(x)$ is continuous, then df will always be zero.

This intuitive conception is similar to the original idea of Leibniz, an ambiguous idea removed from calculus in the nineteenth century. As we have already seen, infinitesimals were reinserted into mathematics, free of any suspicion, in the 1960s, although their use in DC teaching continues to be marginal. Moreover, from the mathematical perspective, the infinitesimal differential is not equal to Δf. Therefore, modern calculus legitimates the use of infinitesimals but not the interpretation of df as an infinitesimal Δf.

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3. *The differential as an infinitesimal approximation*. This conception considers that d*f* is an infinitesimal that is very close to the value of Δf when Δx is infinitesimal. Here again, the terms very small or extremely small are used to express the same idea. In this conception, in the expression $\mathrm{d}f = \Box \cdot \mathrm{d}x$, dx is an infinitesimal Δx, so small that $\Box$ practically remains unchanged, so that d*f* is practically identical to the extremely small Δf.

This conception acknowledges the idea of approximation as well as the linear dependence of d*f* on Δx (although in other words: '$\Box$ *practically* remains unchanged'), but, at the same time, it uses infinitesimals or very small quantities in order to assert that (d*f*–Δ*f*) is negligible in practice; that is, d*f* can be replaced by Δf without any error. This is a very similar conception to the mathematical one of the differential as an infinitesimal (Sect. 3, #4), but it fails to state the additional condition that must be fulfilled by the linear approximation.

4. *The differential as linear estimate*. In this conception, d*f* is an approximation of Δf, an approximation that consists in supposing that Δf changes uniformly with Δx, without any reference to whether Δx, Δf, d*f* or (Δ*f*–d*f*) have big or small values. According to this idea, in the expression $\mathrm{d}f = \Box \cdot \mathrm{d}x$, d*x* (independent variable) is a Δx and d*f* is an approximation of the corresponding Δf that is calculated by assuming that $\Box$ is kept constant along that Δx.

This conception highlights the idea of linear approximation by clearly stating the difference between d*f* and Δf, and it does not emphasize the idea that they can be interchanged. It is similar to the Fréchet conception (Sect. 3, #3) but, in this case, there is no explicit expression of the additional condition that must be fulfilled by this linear estimate.

The last three conceptions have in common that they consider d*f* as a change, as Δf or as an approximation of Δf. Some studies indicate that in certain physical situations, the interpretation is different: d*f* can be considered as an amount and not as a change (Von Korff and Rebello 2014). Therefore, when they write $\mathrm{d}\phi = B \cdot \mathrm{d}S$, they interpret $\mathrm{d}\phi$ as the amount of flux passing through an amount of surface. However, this distinction is not necessary because $\mathrm{d}\phi$ can be considered as a change in the amount of flux passing through a surface, when this surface changes by an amount d*S*. A different issue is that many physical examples are developed in terms of accumulation of quantities rather than accumulation of changes. We do not argue in this work about whether it is adequate to proceed in this way, but, if this were the case, the change from the interpretation as change to that as amount can be done without difficulties. Therefore, from now on, we will interpret the differential as a change.

5 Critical Assessment of the Different Conceptions of the Differential in Physics Teaching

Our concern for the differential is caused by its important role in the mathematization process of physics problems and situations requiring the use of DC for their solution, that is, in the first step of Redish's schema about the use of mathematics in physics. Such a process, from our standpoint, requires an understanding of the role and meaning of the differential in the physical context. This means that students should be able to answer the following questions in *concrete* physics situations: Why is it necessary to use differentials? What is the meaning of the differential in physics? Why do we write down exactly that

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differential expression and not another one? Next, we will appraise the different conceptions of the differential according to the answers to those questions.

5.1 Why is it Necessary to Use Differentials?

- From conception 4#1, the meaningless differential, there is no answer to this question; in fact, 'differential' is used when terms like 'elemental' or 'element' appear as cues in the text to introduce the differential.
- From conception 4#2, the differential as an infinitesimal increment, we need differentials because the magnitudes are very small. Differentials are in fact written just to indicate that we are in the realm of the *very small*; hence, we write the expression $\mathrm{d}v = a \cdot \mathrm{d}t$ because we are referring to a very tiny Δt.
- From conception 4#4, the differential as a linear estimate of the increment, the reason we must use the differential is that the change in a magnitude is non-uniform with changes in the other variable, which prevents us from calculating the Δf produced by a Δx. Differentials are, in fact, written to indicate a non-real change, the change that would occur if the change were uniform along the Δx. Thus, the expression $\mathrm{d}v = a \cdot \mathrm{d}t$ is written because the speed does not change uniformly with time, that is, because the acceleration is not constant during that time interval.
- From conception 4#3, the differential as an infinitesimal approximation, differentials are needed for a combination of the reasons given for conceptions 4#2 and 4#4.

5.2 What is the Meaning of the Differential in Physics?

- For conception 4#1, this is a nonsensical question since the differential is considered to be an instrument without any meaning. When certain cues like *elemental displacement* appear, differentials must be used.
- For conception 4#2, if $\mathrm{d}v = a \cdot \mathrm{d}t$, $\mathrm{d}v$ is the very small change in the velocity produced in the very small time interval $\mathrm{d}t$. This conception assumes that is impossible to assign numerical values to $\mathrm{d}v$ and $\mathrm{d}t$ because they can always be even smaller; thus, it is difficult to interpret the numerical value of the acceleration at any *instant*, for instance, $a(t = 7) = 2$ (SI units).
- For conception 4#3, if $\mathrm{d}v = a \cdot \mathrm{d}t$, $\mathrm{d}v$ is an approximation of the change in the velocity in a very small time interval $\mathrm{d}t$, so small that the acceleration can be considered constant in that interval; hence, $\mathrm{d}v$ is practically equal to the small Δv in that small Δt. The same difficulties as in 4#2 appear when assigning numerical values to $\mathrm{d}v$ and $\mathrm{d}t$, or interpreting the numerical value of the acceleration at an instant.
- For conception 4#4, if $\mathrm{d}v = a \cdot \mathrm{d}t$, $\mathrm{d}v$ is an estimate of the velocity change that would occur in any time interval $\mathrm{d}t$; this estimate is made by assuming that the acceleration is constant throughout that time interval. The numerical value of the acceleration in an instant, for instance, $a(t = 7) = 2$ (SI units) means that, from $t = 7$ s, if the acceleration is kept constant, the speed would change by 2 m/s every second. Similarly, if $\mathrm{d}t = 5$ s, then $\mathrm{d}v = 10$ m/s, and this would be the change in speed from t = 7 to t = 12 if the acceleration was kept constant.

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5.3 Why Exactly that Differential Expression and not Another One?

- For conception 4#1, there is no answer: any expression could be written, if it is called elemental and we use the differential symbol.
- For conception 4#2, which identifies the differential as an infinitesimal increment, when we are addressing a physical situation in which we start from an already defined expression for finite increments ($\Delta x = v \cdot \Delta t$, $\Delta m = \rho \cdot dV \ldots$), the differential expression will be exactly the same because it only indicates that those same changes are now very small ($dx = v \cdot dt$, $dm = \rho \cdot dV \ldots$). The trouble with this conception arises in physical situations in which an expression for finite increments has not been previously defined. Let us, for instance, look at the disintegration process of a radioactive sample. In this case, $dN = -\lambda \cdot N \cdot dt$ is written with confidence, as if it were possible to know what happens in very small time intervals but not in 'normal' time intervals. In fact, starting from that expression, via integration, we arrive at: $\Delta N = N \cdot (1 - e^{-\lambda \cdot \Delta t})$, and, therefore, in order to match this conception, the differential expression should be written as $dN = N \cdot (1 - e^{-\lambda \cdot dt})$. Starting from a particular premise, a conclusion that contradicts that premise is obtained.
- For conception 4#4, in physical situations of the first kind (we have already defined expressions for the uniform cases: $\Delta x = v \cdot \Delta t$, $\Delta m = \rho \cdot dV \ldots$), the corresponding differential expression is similar, because the differential approximation consists precisely in assuming that there is linear behaviour even though we know that actually there is not; hence, $dx = v \cdot dt$, $dm = \rho \cdot dV \ldots$ In relation to the second kind of physical situation mentioned above, we must search for an expression that represents uniform behaviour although we know that actually the behaviour is not uniform. From this perspective, the following differential expressions could represent the corresponding linear estimates for the radioactive disintegration process: $dN = -\lambda \cdot N \cdot dt$, $dN = -\lambda \cdot N^2 \cdot dt$, $dN = -\lambda \cdot dt/N \ldots$ Any linear function of dt could be a good mathematical candidate. However, we know that only one of them complies with the Fréchet condition, $dN/dt = N'$, but as we do not know $N(t)$, any of them may remain valid. As in many other physical situations, we must conceive each of the linear estimates as a plausible hypothesis, work mathematically on it ('via integration') and find a functional expression for ΔN or of $N(t)$, the validity of which must be empirically probed in the physical situation at hand or by its coherence with other findings in the same field. In such cases, we must say that differential expressions (initially) have a hypothetical nature.
- For conception 4#3, the answer to this question is similar to that for conception 4#2 or for conception 4#4, depending on whether the emphasis is on the infinitesimal value or on the character of the differential as a linear approximation. Our experience is that the infinitesimal value is emphasized. This conception is often based on the intuition, which is apparently correct, that the sum of many thousands of very tiny approximations ends up giving an accurate result because the error in each approximation is practically zero. In fact, when university physics students and high school physics teachers are asked to analyse a mathematical development that begins with a 'reasonable' differential expression but produces a physically or geometrically erroneous result, almost none of them regards the differential expression (called by us the 'differential hypothesis') as doubtful. They usually check the mathematical operations again and again from beginning to the end, without finding the 'mistake'.

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Some contradictory situations like this, in the context of the calculus of surfaces and the volumes of regular geometric objects, have been addressed by certain authors (Artigue et al. 1989, pp. 31–38; Schneider 1991). When the infinitesimal character is emphasized instead of the linear estimation one, it is difficult to justify why some differential expressions lead to a correct result and others do not.

5.4 Which is the more Suitable Concept of the Differential for Teaching Introductory Physics Courses?

The answer depends on the aim of the teaching. If we wanted to focus only on the mathematical operations, starting from expressions that are already known, then conception 4#1 could be sufficient. We are committed to helping students to learn the modelling process, so we discard that conception.

Conception 4#2 is intuitive and, in our experience, is the most frequently used one in physics teaching. However, we have identified some shortcomings in this conception:

- It does not allow one to identify the situations in which it is necessary to use the differential, since it does not justify the need take infinitesimals when approaching a concrete problem.
- It does not allow one to assign numerical values to explain the meaning of the differential, a didactic exercise that, in general, promotes understanding; neither does it provide an easy explanation of the numerical values of derivatives as quotients since, again, it is necessary to refer to the quotient of two quantities that do not admit numerical values.
- It makes it impossible to acknowledge the hypothetical nature of the differential expression in some physical situations, because there is no criterion for deciding between one expression and another.
- It fails in its internal logic: starting with an expression for an infinitesimal Δf, which is considered to be correct, it permits one, via integration, to obtain another different expression that is also valid for an infinitesimal Δf.

For these reasons (besides its incorrectness from a mathematical viewpoint, including the viewpoint of the non-standard calculus), conception 4#2 is not the most suitable one for the mathematization of physical situations.

Conception 4#3 preserves the intuitive character and the reference to infinitesimals that is so frequent in physics teaching and, besides, avoids some of the shortcomings mentioned before:

- It clearly identifies the situations that require that a differential expression be written: these are when there is non-uniform or nonlinear behaviour.
- It highlights the character of the differential as an approximation and imposes a condition that must be fulfilled to justify the transition from approximation to exactitude via *integral* that the slope of the approximation ($\mathrm{d}f/\mathrm{d}x$) should coincide with the derivative (f'), that is to say, with the quotient of two infinitesimals.
- It permits one to acknowledge the hypothetical nature of the differential expressions in some physical situations that have already been described.
- Contradictions disappear.

However, some drawbacks that come from the identification of differentials with infinitesimals remain. One of them is that this conception does not permit numerical values to be assigned and, in this way, an explanation of the meaning of differential and the

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interpretation of concrete values of the derivative to be facilitated. The other shortcoming is that it is extremely difficult to understand the meaning in physics of any differential expression in the realm of the infinitely small, and this becomes a major drawback in selecting a hypothetical differential expression when required. Moreover, this drawback sometimes goes unnoticed as a result of the intuitive, but erroneous, idea that any approximation will end up giving the exact result if sufficiently small Δx are taken.

And lastly, conception 4#4 maintains the advantages of conception 4#3 while avoiding the objections noted above. In particular, the differential expression makes sense for any Δx whether large or small, which allows the physical meaning of the approximation to be interpreted more clearly and hypothetical differential expressions to be written through physical analysis, without taking extremely small values for which it seems that, in the end, 'anything goes'.

In accordance with these assessments, conceptions of the differential that highlight the idea of approximation, whether as an infinitesimal approximation or as a linear estimate, are the most suitable for facilitating the mathematization of physical situations through DC. Between the two, we find some important additional advantages in the conception of the differential as a linear estimate of the increment. However, we are aware that these appraisals depend on our aim and, therefore, that the assessment and selection can be different from another perspective and for other purposes (Ostebee and Zorn 1997).

6 Objectives and Experimental Design

Our experimental study seeks to gather data on how final-year high school students justify the use of the differential in physics, assign meaning to differential expressions and perceive how they are supposed to use DC in physics, and on the extent to which these aspects evolve in university students. In this way, we expect to identify students' conceptions and the persistence of these conceptions, for a better understanding of the difficulties students can have in mathematizing physical situations requiring DC.

We have designed a set of instruments that includes both qualitative and quantitative tools:

- *Four written questions (3 closed and 1 open).* These are designed for getting data on students' conceptions of the justification and meaning of differential expressions and on their perceptions of how they are expected to use DC in physics. They are measured using a Likert-type scale. When the results are presented, the content of the questions will be explained.
- *Three problems to be solved by students*, on physics topics with which the students were familiar. In two of these problems, paragraphs were included to remind students of the necessary conceptual grounds in physics for solving the problems. The wording of each problem explicitly requested students to write explanatory comments, especially each time they used differential calculus in solving the problem.
- *An individual semi-structured interview on one written solved problem* from which the explanations had been removed. We divided the solution with horizontal lines into a total of seven sections that were gradually revealed to the student during the interview. For each section, the student was asked the corresponding question that appears in the Appendix. The data obtained from questions 4–6 are not used in this work. The objective of this audio-recorded interview was to obtain complementary qualitative

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information to illustrate and support the interpretation of the quantitative results obtained from the students' written responses to the questions and problems.

The considerable time period for our work has provided us with large sample sizes. In total, the sample was made up of 488 students belonging to six high schools and four Spanish universities and fulfilled a diversity of criteria with regard to both the origin of the students and that of the teachers of the physics or maths courses. The respondents were divided into three subgroups: 190 final-year physics high school students (in the techno-scientific branch, aged 17–18 years), 153 university freshmen and 145 second- to fourth-year university students. Over half the university students included in the sample were physics undergraduates and the rest were studying for other scientific or engineering degrees. All the students in the sample were taking a physics subject and, for the youngest, a maths subject that included differential calculus topics.

Because our interest was in the teaching of physics in high school (when DC is introduced for the first time), seven bright students in the final year of high school were selected by their own teachers to be interviewed. We also interviewed four recent graduates in physics or technical studies who are training to become high school teachers. We chose these four recent graduates, who are now our students in a Master's programme on Teaching Physics and Chemistry at secondary level, because we believe them to be representative of university students.

In order to decide whether there were significant differences between the results obtained by the different sample subgroups, the Student's t test was used with a significance level lower than 5 %.

7 Results and Discussion

The results, instruments and interpretations will be grouped around the three objectives mentioned above. The interpretations will be accompanied by verbatim fragments from the interviews. A more detailed and thorough analysis can be found in López-Gay (2002).

7.1 Results on When and Why it is Necessary to Use Differentials

These results come from data obtained from two closed questions and from the analysis of the problems solved by the students (each of the three problems was different, and adapted, for each subgroup).

The first question aims to discover what students consider to be the best reason to justify the change from increments to differentials. Each of the response items is related to one of the conceptions of the differential as used in physics [4]. We used item (d) to distinguish between students who justify the use of the differential by the existence of a dependency relationship (Meredith and Marrongelle 2008) and those who refer to the non-uniform behaviour during Δt (see Table 1).

Fifty-eight percentage of the secondary school students justify the step from increments to differentials because these are infinitely small values (conceptions 4#2 and 4#3), and this percentage increases significantly for the university degree students, with 82 % of students on these higher courses choosing this option. Overall, 69 % of students chose this option. By contrast, the option that justifies the step from increments to differentials because of the non-uniform behaviour (conceptions 4#3 and 4#4) was chosen by only 11 % of the high school students and by only 3 % of the university students. Finally, the statement related to

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Table 1 Wording and results obtained from the first closed question about *justification**

In a text on kinematics, one reaches the following expression: $\Delta v = a{\cdot}\Delta t$, which is then written as follows: $dv = a{\cdot}dt$ Tick (✔) for which of the following reasons you think justifies more accurately the need for taking this step	Last year high school (N = 149) % (SD)	1st U. (N = 92) % (SD)	≥2nd U. (N = 90) % (SD)
a. Because we want to finish with a derivative or an integral	20.1 (3.3)	15.2 (3.8)	15.2 (3.5)
b. Because infinitely small times are being considered	57.7 (4.1)	73.9 (4.6)	82.2 (4.1)
c. Because acceleration depends on time	10.7 (2.5)	2.2 (1.5)	3.3 (1.9)
d. Because speed depends on time	17.4 (3.1)	10.9 (3.3)	6.7 (2.6)
e. I don't know	4.7 (1.7)	4.3 (2.1)	2.2 (1.6)

* Although we asked students to choose only one statement, some of them chose more than one. This is the reason for the sum of percentages of each column being higher than 100 %

conception 4#1 ("Because we want to finish with a derivative or an integral") was chosen by 20 % of the high school students and a slightly lower percentage of university students.

The second closed question was used to obtain information on the students not through what they say but through what they do: we wanted to know their criteria for using DC in a physics situation. This would give us information on the implicit justification for using calculus. The wording of the question and the results obtained are presented in Table 2.

Only 15 % of the high school students correctly identify that it is necessary to use DC when nonlinear relationships appear (conceptions 4#3 and 4#4); this percentage significantly decreases in university students, falling to 5 % of the students in higher years of university courses. Overall, 89 % of the students do not know the characteristics of a situation in which DC is needed (as would be expected if they were thinking with conceptions 4#1 and 4#2), and this percentage is higher as the education level increases.

At high school, 5 % of students consider it necessary to use DC in all situations, and this percentage increases significantly during university education, reaching 30 % of students in the higher years of university courses. On the other hand, 59 % of high school students and a higher percentage of university students consider that it is necessary to use DC in all cases in which position depends on time.

The analysis of the problems solved by students provides information on what students say and do when they use differentials. The results related to justification and meaning are presented in Table 3.

Table 2 Wording and results obtained from the second closed question about *justification*

We know the position equation of four different moving objects. We want to calculate the instantaneous speed of each object. Tick (✔) for the cases in which the use of differential calculus (derivatives, differentials, integrals …) is necessary	Last year high school ($N = 117$) % (SD)	1st U. ($N = 89$) % (SD)	≥2nd U. ($N = 43$) % (SD)
a. $x = 12$	5.1 (2.0)	11.2 (3.4)	30.2 (7.1)
b. $x = 8 + 3t^2$	87.2 (3.1)	91.0 (3.0)	97.7 (2.3)
c. $x = 6t - 2$	72.6 (4.1)	80.9 (4.2)	93.0 (3.9)
d. $x = 5 \cos 3t$	87.2 (3.1)	91.0 (3.0)	100 (–)
Tick all the options (*a*, *b*, *c*, *d*)	3.4 (1.7)	10.1 (3.2)	30.2 (7.1)
Tick cases where $x = f(t)$ (*b*, *c*, *d*)	59.0 (4.6)	68.5 (5.0)	62.8 (7.5)
Tick only nonlinear cases (*b*, *d*)	14.5 (3.3)	9.0 (3.1)	4.7 (3.3)

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Table 3 Results on the justification and explicit meaning of the differential when solving physics problems

When they solve problems in which they are specifically asked to include explanatory comments each time, they use differential calculus	Last year high school ($N = 57$) % (SD)	1st U. ($N = 95$) % (SD)	≥2nd U. ($N = 105$) % (SD)
They use differentials	17.5 (5.1)	50.5 (5.2)	44.8 (4.9)
Of them			
Try to justify why they use differentials	10.0 (10.0)	4.2 (2.9)	10.7 (4.6)
Write any meaning of the differential	0 (–)	6.4 (3.6)	21.2 (6.0)

The low percentage of students who use DC to solve the problems is striking. For high school students, this could be due to two main difficulties: mathematizing and using DC. However, it is even harder to explain why nearly half of the university students of all courses do not use DC when it is necessary to solve the problem, despite knowing the physics of the problem and being given a reminder of the main physics ideas. Moreover, only one out of ten students who use DC to solve the problem tries to justify its use, despite being clearly asked for an explanation for the use of DC. Given the low number of students who try to justify the use of DC, we do not consider it useful to distinguish between the different kinds of justifications.

The results of Tables 1, 2, and 3 show that, when using the differential or DC to solve a physics situation, most of the students cannot identify which characteristics of the situation at hand lead them to use it. Moreover, when they are given certain statements, their justifications refer to infinitesimal values and not to the idea of linear approximation. We interpret these results as showing the predominance—more marked in higher education levels—of the conception of the differential as an infinitesimal increment, as seen in both their explicit statements and the absence of criteria for when to use the differential.

The following extracts from interviews illustrate the difficulties that students have in justifying the step from increments to differentials:

Extract 1. Pedro, a bright upper high school pupil

P: *The top one [$\Delta m/\Delta V$] is the definition of density and the bottom one [dm/dV] is the same but for very, very small pieces… It would be the same but maybe here mass could be written depending on the volume…*

In: And couldn't you do the same thing using the top expression?

P: *Yes, I suppose, but I don't really know…*

Extract 2. Maria, teacher in training

M: *He has turned the increments into differentials, and I don't know why he does that. What I don't understand is why he doesn't directly replace an increment. He has gone from increments to differential…*

In: Why? Couldn't he have done the same with increments?

M: *I don't know why he goes to differentials*

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7.2 Results on the Meaning of Differential Expressions

The open question is aimed at obtaining information on the meaning that students give to the differential in a context of a familiar physics situation that is briefly described. The complete wording of the question is shown in Table 6.

We analysed the responses to this question in order to see which conception of the differential appears from the responses of each student. In the light of the responses given, we have added a new unexpected category: the differential is an increment, without any reference to its value (Table 4).

We did not consider category *b* as a new conception, but as an incomplete view of what category c represents. Taking into account this criterion, 61 % of the students (58 % of high school and university students in higher years and 70 % of freshmen) expressed their conception of the differential as a small or very small infinitesimal increment. Only *one* out of 122 students expressed the conception of the differential as an infinitesimal approximation (4#3).

Category *a* includes those students who do not know any meaning of 'differential', as well as those who think that differential has no meaning. Hence, we cannot distinguish how many students have the conception of the differential as a formal instrument.

When they are directly asked, between 58 and 72 % of students are able to express a conception of the differential (Table 4). In contrast, the results obtained when analysing the problems solved by students (Table 3) show how little sense students make when explaining the meaning of differentials when they use them, despite being asked for explanations. This could be interpreted as an effect of the algorithmic approach of calculus.

Our experience as teachers leads us to assume that the concept of the differential that prevails in physics teaching is that of a small or very small infinitesimal increment. Why then do 'only' 61 % of the students express this openly? We believe that it is likely that the algorithmic approach of calculus induces inconsistency and a lack of confidence among students, which prevent them from clearly expressing any assertion about the only, but nebulous, conception they have.

Over the course of the interviews, a lack of meaning appeared in certain cases (extract 3) or an identification of the differential and an increment, *with no additional conditions* (extract 4). However, the most frequent answer was that it was a matter of *very small* increments, although the arguments converged on operational ideas (extract 5).

Table 4 Results of the analysis of responses to the open question about the meaning of dN

	Last year high school ($N = 52$) % (SD)	1st U ($N = 37$) % (SD)	≥2nd U. ($N = 33$) % (SD)
a. They do not write any meaning	42.3 (6.9)	27.1 (7.4)	42.4 (8.7)
b. It is ΔN (without more accuracy)	34.6 (6.7)	29.7 (7.6)	36.4 (8.5)
c. It is a very small ΔN (without more accuracy)	23.1 (5.9)	40.5 (8.2)	21.2 (7.2)
d. It is a small ΔN that, so small that any other magnitude is thought to be constant	0 (–)	2.7 (2.7)	0 (–)
e. It is a linear estimation of ΔN	0 (–)	0 (–)	0 (–)

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Extract 3. Isa, a bright high school pupil

I: *dm is as if the mass is changed with regard to… I'm getting mixed up!*

In: Do you know any meaning of that expression?

I: *I know that by doing the integral you remove the "d", but I can't think of one at the moment*

Extract 4. Julia, teacher in training

J: *I have always taken an increment to mean the difference between the final and the initial mass, and a differential when you do not place any limits between what is varying the mass*

In: But, does it vary?

J: *Of course, so it has to have limits*

In: What difference is there then between increments and differentials?

J: *There is no difference*

Extract 5. Juan, a bright high school pupil

J: *I understand differential to mean when you want to study the parts as "tiny" as you want*

In: So what is *dm*?

J: *Well "tiny" little pieces of the…* (unable to finish sentence)

In: Of what?

J: *Of the mass…*

J: (…) *Every time we use differentials, my teacher says: "in order to study this curve we are going to take the straight lines as small as we please…"*

J: (…) *dV, dh are volume increments, height increments… he seems to take it like that*

In: And is that what you think?

J: *I don't understand it… In truth, I know how to calculate integrals, but I haven't actually understood the differentials that occur, I see them in writing but I don't know what they are… and, why am I going to bother to ask, since they are going to tell me: "these are the little pieces…"*

7.3 Results on Students' Perceptions of How they are Supposed to Use Differential Calculus in Physics

Different works on teaching and using differential calculus [2] have revealed the predominance of the algorithmic approach, so the regular use of calculus in physics does not take into account the mathematization process. It focuses only on the manipulation of mathematical expressions. This situation is clearly reflected in the results obtained from the analysis of the problems solved by students (Table 3): most of the students are unable to mathematize a familiar physics situation using DC and, if they are, they cannot express either the reasons that have led them to use DC or the meaning of the differential expression they write.

We believe that this situation will affect students' perception of how they are supposed to use DC in physics. To obtain information on this perception, we wrote two complementary statements: one refers to what students perceive from their teachers and the other refers to their own actions when using calculus. We asked students to express their degree of agreement with each of these two statements according to a Likert scale from 1 to 5 (1 Totally agree—2 Agree—3 Neutral—4 Disagree—5 Totally disagree).

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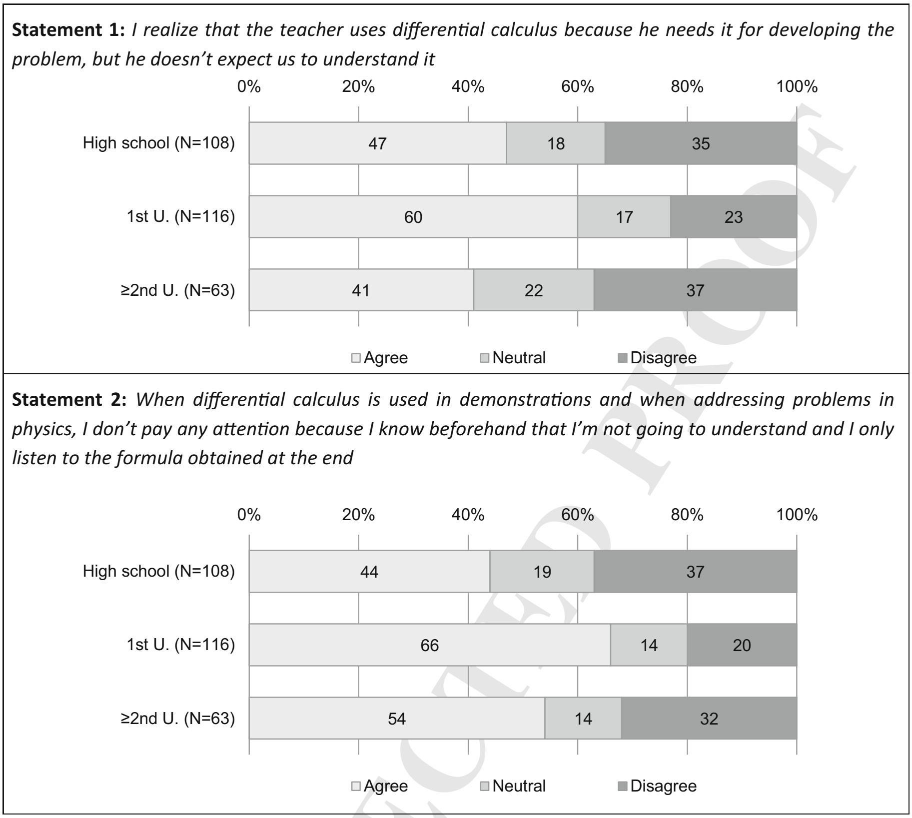
Table 5 Wording and results for the statements related to students' perception of how they are supposed to use differential calculus in physics



The wording of the two statements and the results obtained by grouping the responses given by students into three categories (1–2 Agree, 3 Neutral, and 4–5 Disagree) are shown in Table 5.

According to Table 5, 47 % of secondary students perceive that their teacher does not expect them to understand the use of DC and, indeed, 44 % of them refuse to understand it. This perception is even more widespread among freshmen: 60 % of them perceive that their teacher does not expect them to understand the use of DC, and 66 % of them refuse to understand it. Although this growth does not reach as far as university students in higher years, it is a significant fact that 41 % of them still perceive that the teacher does not expect them to understand the use of DC, and 54 % refuse to understand it (Table 6).

We can conclude that the percentage of students who perceive that what is expected of them is to use DC without understanding it and only to use it mechanically is always greater than that of students who do not share that perception, independently of the statement and the subgroup of students.

This situation, particularly at university level, would be unsustainable if it were not for an unspoken agreement between teachers and students: although differential calculus is

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Table 6 Open question on the meaning of the differential

A radioactive substance is one whose nuclei are being transformed into other nuclei or particles, that is, they are disintegrating. If we call N the number of nuclei of a radioactive substance at an instant t, this number will decrease in an interval of time due to the number of disintegrations that have taken place
The law of radioactive disintegration refers to the number of disintegrated nuclei in a certain radioactive substance over an interval of time, and its initial mathematical expression is the following: $\mathrm{d}N = -\lambda \cdot N \cdot \mathrm{d}t$, with λ being a characteristic constant of each radioactive element
In your own words, and as clearly as possible, explain the physical meaning of dN that is deduced from the above mathematical expression

used, the teacher does not expect his or her students to understand it, and nor do the students aspire to understand it; what matters is that the students know how to apply the rules. The following extracts from interviews illustrate this lack of understanding among students and a generalized mechanical attitude.

Extract 6. Julia, teacher in training

In: Did you learn when your physics teachers or textbooks used differential calculus?

J: *I learnt how to perform calculus, but not what it actually was, I've never learnt that*

Extract 7. Sergio, a bright upper high school pupil

S: *Seeing all that on the board is a shock, at first sight it's horrendous; of course it puts you off.*

In: Why do you think that is?

S: *We are all for being practical, and seeing so many operations well, it frightens you a little, but it doesn't really because finally what's important is the result, from a practical point of view. The same thing happens to me, but then I get home and I manage to do it*

Extract 8. Juan, a bright upper high school pupil

J: *The truth is, when there are some integrals—for example, some that have got a "little zero" in the middle that I don't know what they're about—and I can see them, but I don't study them because I can waste too much time trying to understand them*

In: And the others?

J: *If I "get" them quickly, yes*

In: But didn't you say you don't know their meaning?

J: *Yes, but I know how to do them*

8 Summary and Implications

In this study, we have shown four conceptions of the differential as used in physics: as a merely formal instrument, as an infinitesimal increment, as an infinitesimal approximation and as a linear estimate of the increment. We have assessed each of these according to how useful they are in helping in the mathematization process of physics situations using differential calculus, a process that generally leads to the use of differentials. The conclusion, according to the explicit assessment criteria that we have established [5], is that the last two conceptions, especially that of a linear estimate of the increment, are the more suitable. However, global analysis of different results obtained using different instruments supports the claim that the main conception of students in physics contexts, especially university students, is the one that identifies the differential with an infinitesimal increment; this constitutes an obstacle to students' ability to mathematize. We have come to this conclusion from the students' direct answers when they are asked about the meaning of the

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differential in a physics context (61 %, see Table 4), from the reasons they give to justify the use of differentials (69 %, see Table 1), as well as from their difficulties in identifying those cases in which it is necessary to use differential calculus, difficulties that are associated with this conception of the differential (89 %, see Table 2).

Furthermore, the data seem to indicate that students perceive that all that is expected of them is a mechanical use of differential calculus in physics. The percentage of students who hold this perception is always greater than that of students who do not hold it, no matter what their academic level or the instrument used to obtain data.

Although we have not proved that there is a cause and effect relationship between the prevailing conception of the differential in students and their perception of what is expected of them, we think that the two results are linked. In effect, as we have explained, the idea of the differential as an infinitesimal increment is inappropriate for the mathematization process of physical situations by calculus. The fact that this is the prevailing conception in students can only be explained by concluding that the use of calculus is focused on an operational process (isolate a variable, replace it, solve derivatives and integrals …) on a mathematical expression that has already been written, in order to get a result, neglecting the process that begins with a physical analysis of the situation at hand and leads to that starting expression. This claim is consistent with the results of different studies about the algorithmic approach of the teaching and use of calculus. In this context, it seems reasonable for students to perceive that they may only use calculus in a mechanical way.

In our opinion, if the aim is to teach students the mathematization process when they use calculus in physics, and to change their perceptions, the conception of the differential that is usually used should be changed. To students who are familiar with the idea of the infinitesimal increment, it may be easier to promote this change to the conception of the differential as an infinitesimal approximation. In the case of students who not only are going to start learning about differential calculus but will also be using it to do physics, we think that, even though the idea of an infinitesimal approximation could be valid, the idea of a linear estimation of the increment is better because it allows one to see a clearer relationship between physics analysis and the written differential expressions.

Anyway, it is necessary to avoid approaching physics problems and theoretical developments as if the initial steps of mathematization were self-evident or a mechanical response to cues like 'infinitesimal' or 'elemental'. For our part, we are working on the design and implementation of physics teaching sequences for upper high schools that systematically incorporate the conception of the differential as a linear estimation, to help students to use differential calculus with understanding and good sense.

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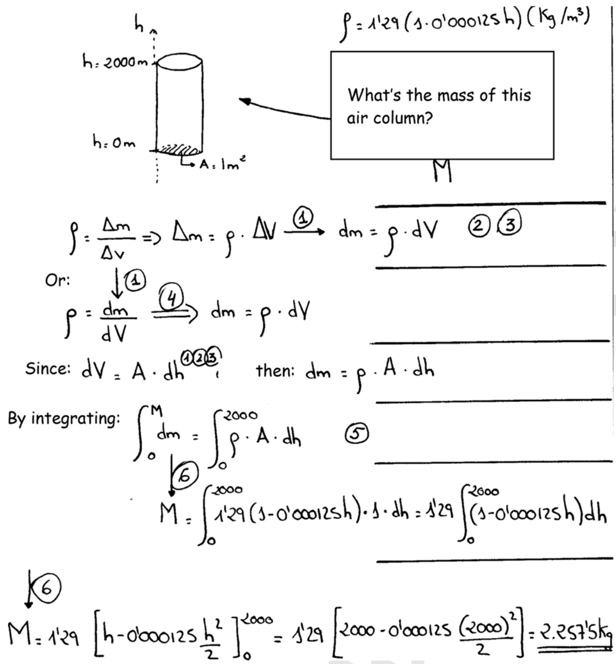
Appendix: Document and Guidelines for the Semi-Structured Interview

PROBLEM STATEMENT: We know that the density of air (ρ) decreases with height (h) according to the following equation: $\rho = 1.29{\cdot}(1\text{–}0.000125{\cdot}h)$. That equation is written for the International System, that is, if h is written in metres, density is obtained in kg/m^3. The value $h = 0$ represents sea level. ***What would be the mass of a cylindrical column of air measuring 1 m^2 at the base that rises 2000 m above sea level?***



1. *Why do you take that step? (Be it the step from increment to differential or from incremental quotient to differential quotient)*
2. *What is the meaning of that expression? (We insist on searching for an explanation that goes beyond the use of key words or literal reading)*
3. *What could be the value of dm, dV…? (If they answer with a numerical value, we check on its meaning and functional nature)*
4. *How must that expression be read? Is it correct to isolate the differential? (We are referring to the expression of the derivative, and want to know if they consider it as a true differentials quotient)*
5. *What is the meaning of those integrals? (They may adhere to the idea of the anti-derivative, or go further and identify Riemann sums)*
6. *Why is the result of that integral precisely that? (We will inquire to see if they are capable of justifying why the integral of a differential is a macroscopic increment, or why infinite sums are necessarily calculated using anti-derivatives)*
7. *Do you understand properly when your teacher or the textbook use differential calculus in physics lessons?*
8. *Where have you best learnt the use and meaning of differential calculus, in physics or maths lessons?*
9. *In general, do you think that the use of differential calculus makes students like physics more or less? Why do you think so? And do you think that is the case for you too?*

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