

# SUPERATOMIC BOOLEAN ALGEBRAS CONSTRUCTED FROM STRONGLY UNBOUNDED FUNCTIONS

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ABSTRACT. Using Koszmider's strongly unbounded functions, we show the following consistency result:

Suppose that  $\kappa, \lambda$  are infinite cardinals such that  $\kappa^{+++} \leq \lambda$ ,  $\kappa^{<\kappa} = \kappa$  and  $2^\kappa = \kappa^+$ , and  $\eta$  is an ordinal with  $\kappa^+ \leq \eta < \kappa^{++}$  and  $\text{cf}(\eta) = \kappa^+$ . Then, in some cardinal-preserving generic extension there is a superatomic Boolean algebra  $\mathbb{B}$  such that  $\text{ht}(\mathbb{B}) = \eta + 1$ ,  $\text{wd}_\alpha(\mathbb{B}) = \kappa$  for every  $\alpha < \eta$  and  $\text{wd}_\eta(\mathbb{B}) = \lambda$  (i.e. there is a locally compact scattered space with cardinal sequence  $\langle \kappa \rangle_\eta \frown \langle \lambda \rangle$ ).

Especially,  $\langle \omega \rangle_{\omega_1} \frown \langle \omega_3 \rangle$  and  $\langle \omega_1 \rangle_{\omega_2} \frown \langle \omega_4 \rangle$  can be cardinal sequences of superatomic Boolean algebras.

## 1. INTRODUCTION

A Boolean algebra  $\mathcal{B}$  is *superatomic* iff every homomorphic image of  $\mathcal{B}$  is atomic. Under Stone duality, homomorphic images of a Boolean algebra  $\mathcal{A}$  correspond to closed subspaces of its Stone space  $S(\mathcal{A})$ , and atoms of  $\mathcal{A}$  correspond to isolated points of  $S(\mathcal{A})$ . Thus  $\mathcal{B}$  is superatomic iff its dual space  $S(\mathcal{B})$  is *scattered*, i.e. every non-empty (closed) subspace has some isolated point.

For every Boolean algebra  $\mathcal{A}$ , let  $\mathcal{I}(\mathcal{A})$  be the ideal generated by the atoms of  $\mathcal{A}$ . Define, by induction on  $\alpha$ , the  $\alpha^{\text{th}}$  *Cantor-Bendixson ideal*  $\mathcal{J}_\alpha(\mathcal{A})$ , and the  $\alpha^{\text{th}}$  *Cantor-Bendixson derivative*  $\mathcal{A}^{(\alpha)}$  of  $\mathcal{A}$  as follows. If  $\mathcal{J}_\alpha(\mathcal{A})$  has been defined, put  $\mathcal{A}^{(\alpha)} = \mathcal{A}/\mathcal{J}_\alpha(\mathcal{A})$  and let  $\pi_\alpha : \mathcal{A} \rightarrow \mathcal{A}^{(\alpha)}$  be the canonical map. Define  $\mathcal{J}_0(\mathcal{A}) = \{0_{\mathcal{A}}\}$ ,  $\mathcal{J}_{\alpha+1}(\mathcal{A}) = \pi_\alpha^{-1}[\mathcal{I}(\mathcal{A}^{(\alpha)})]$ , and for  $\alpha$  a limit  $\mathcal{J}_\alpha(\mathcal{A}) = \bigcup\{\mathcal{J}_{\alpha'}(\mathcal{A}) : \alpha' < \alpha\}$ . It is easy to see that the sequence of the ideals  $\mathcal{J}_\alpha(\mathcal{A})$  is increasing. And it is a well-known fact that a non-trivial Boolean algebra  $\mathcal{A}$  is superatomic iff there is an ordinal  $\alpha$  such that  $\mathcal{A} = \mathcal{J}_\alpha(\mathcal{A})$  (see [4, Proposition 17.8]).

Assume that  $\mathcal{B}$  is a superatomic Boolean algebra. The *height* of  $\mathcal{B}$ ,  $\text{ht}(\mathcal{B})$ , is the least ordinal  $\delta$  such that  $\mathcal{B} = \mathcal{J}_\delta(\mathcal{B})$ . This ordinal  $\delta$  is always a successor ordinal. Then, we define the *reduced height* of  $\mathcal{B}$ ,  $\text{ht}^-(\mathcal{B})$ , as the least ordinal  $\delta$  such that  $\mathcal{B} = \mathcal{J}_{\delta+1}(\mathcal{B})$ . It is well-known that if  $\text{ht}^-(\mathcal{B}) = \delta$ , then  $\mathcal{J}_{\delta+1}(\mathcal{B}) \setminus \mathcal{J}_\delta(\mathcal{B})$  is a finite set. For each  $\alpha < \text{ht}^-(\mathcal{B})$  let  $\text{wd}_\alpha(\mathcal{B}) = |\mathcal{J}_{\alpha+1}(\mathcal{B}) \setminus \mathcal{J}_\alpha(\mathcal{B})|$ , the

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number of atoms in  $\mathcal{B}/\mathcal{J}_\alpha(\mathcal{B})$ . The *cardinal sequence* of  $\mathcal{B}$ ,  $CS(\mathcal{B})$ , is the sequence  $\langle wd_\alpha(\mathcal{B}) : \alpha < ht^-(\mathcal{B}) \rangle$ .

Let us turn now our attention from Boolean algebras to topological spaces for a moment. Given a scattered space  $X$ , define, by induction on  $\alpha$ , the  $\alpha^{th}$  *Cantor-Bendixson derivative*  $X^\alpha$  of  $X$  as follows:  $X^0 = X$ ,  $X^\alpha = \bigcap_{\beta < \alpha} X^\beta$  for  $\alpha$  a limit, and  $X^{\alpha+1} = X^\alpha \setminus I(X^\alpha)$ , where  $I(Y)$  denotes the set of isolated points of a space  $Y$ . The set  $I_\alpha(X) = X^\alpha \setminus X^{\alpha+1}$  is the  $\alpha^{th}$  *Cantor-Bendixson level* of  $X$ . The *reduced height* of  $X$ ,  $ht^-(X)$ , is the least ordinal  $\delta$  such that  $X^\delta$  is finite (and so  $X^{\delta+1} = \emptyset$ ). For  $\alpha < ht^-(X)$  let  $wd_\alpha(X) = |I_\alpha(X)|$ . The *cardinal sequence* of  $X$ ,  $CS(X)$ , is defined as  $\langle wd_\alpha(X) : \alpha < ht^-(X) \rangle$ .

It is well-known that if  $\mathcal{B}$  is a superatomic Boolean algebra, then the dual space of  $\mathcal{B}^{(\alpha)}$  is  $(S(\mathcal{B}))^{(\alpha)}$  (see [4, Construction 17.7]). So  $ht^-(\mathcal{B}) = ht^-(S(\mathcal{B}))$ , and  $wd_\alpha(\mathcal{B}) = wd_\alpha(S(\mathcal{B}))$  for each  $\alpha < ht^-(\mathcal{B})$ , that is,  $\mathcal{B}$  and  $S(\mathcal{B})$  have the same cardinal sequences.

In this paper we consider the following problem: given a sequence  $\mathbf{s}$  of infinite cardinals, construct a superatomic Boolean algebra having  $\mathbf{s}$  as its cardinal sequence.

For basic facts and results on superatomic Boolean algebras and cardinal sequences we refer the reader to [4] and [8]. We shall use the notation  $\langle \kappa \rangle_\alpha$  to denote the constant  $\kappa$ -valued sequence of length  $\alpha$ . Let us denote the concatenation of two sequences  $f$  and  $g$  by  $f \frown g$ . If  $\eta$  is an ordinal we denote by  $\mathcal{C}(\eta)$  the family of all cardinal sequences of superatomic Boolean algebras whose reduced height is  $\eta$ .

**Definition 1.** If  $\kappa, \lambda$  are infinite cardinals and  $\eta$  is an ordinal, we say that a superatomic Boolean algebra  $\mathcal{B}$  is a  $(\kappa, \eta, \lambda)$ -*Boolean algebra* iff  $CS(\mathcal{B}) = \langle \kappa \rangle_\eta \frown \langle \lambda \rangle$ , i.e. if  $ht(\mathcal{B}) = \eta + 1$ ,  $wd_\alpha(\mathcal{B}) = \kappa$  for each  $\alpha < \eta$  and  $wd_\eta(\mathcal{B}) = \lambda$ .

An  $(\omega, \omega_1, \omega_2)$ -Boolean algebra is called a *very thin-thick Boolean algebra*. And, for an infinite cardinal  $\kappa$ , a  $(\kappa, \kappa^+, \kappa^{++})$ -Boolean algebra is called a  $\kappa$ -*very thin-thick Boolean algebra*.

By using the combinatorial notion of the *new  $\Delta$  property (NDP)* of a function, it was proved by Roitman that the existence of an  $(\omega, \omega_1, \omega_2)$ -Boolean algebra is consistent with ZFC (see [7] and [8]). It is worth to mention that [7] was the first paper in which such a special function was used to guarantee the chain condition of a certain poset. Roitman's result was generalized in [3], where for every infinite regular cardinal  $\kappa$ , it was proved that the existence of a  $(\kappa, \kappa^+, \kappa^{++})$ -Boolean algebra is consistent with ZFC. Then, our aim here is to prove the following stronger result.

**Theorem 2.** *Assume that  $\kappa, \lambda$  are infinite cardinals such that  $\kappa^{+++} \leq \lambda$ ,  $\kappa^{<\kappa} = \kappa$  and  $2^\kappa = \kappa^+$ . Then for each ordinal  $\eta$  with  $\kappa^+ \leq \eta < \kappa^{++}$  and  $cf(\eta) = \kappa^+$ , in some cardinal-preserving generic extension there is a  $(\kappa, \eta, \lambda)$ -Boolean algebra, i.e.  $\langle \kappa \rangle_\eta \frown \langle \lambda \rangle \in \mathcal{C}(\eta + 1)$ .*

**Corollary 3.** *The existence of an  $(\omega, \omega_1, \omega_3)$ -Boolean algebra is consistent with ZFC. An  $(\omega_1, \omega_2, \omega_4)$ -Boolean algebra may also exist.*

In order to prove Theorem 2, we shall use the main result of [5].

**Definition 4.** Assume that  $\kappa, \lambda$  are infinite cardinals such that  $\kappa$  is regular and  $\kappa < \lambda$ . We say that a function  $F : [\lambda]^2 \longrightarrow \kappa^+$  is a  $\kappa^+$ -strongly unbounded function on  $\lambda$  iff for every ordinal  $\delta < \kappa^+$ , every cardinal  $\nu < \kappa$  and every family  $A \subseteq [\lambda]^\nu$  of pairwise disjoint sets with  $|A| = \kappa^+$ , there are different  $a, b \in A$  such that  $F\{\alpha, \beta\} > \delta$  for every  $\alpha \in a$  and  $\beta \in b$ .

The following result was proved in [5].

**Koszmider's Theorem .** *If  $\kappa, \lambda$  are infinite cardinals such that  $\kappa^{+++} \leq \lambda$ ,  $\kappa^{<\kappa} = \kappa$  and  $2^\kappa = \kappa^+$ , then there is a  $\kappa$ -closed and cardinal-preserving partial order that forces the existence of a  $\kappa^+$ -strongly unbounded function on  $\lambda$ .*

So, in order to prove Theorem 2 it is enough to show the following result.

**Theorem 5.** *Assume that  $\kappa, \lambda$  are infinite cardinals with  $\kappa^{+++} \leq \lambda$  and  $\kappa^{<\kappa} = \kappa$ , and  $\eta$  is an ordinal with  $\kappa^+ \leq \eta < \kappa^{++}$  and  $\text{cf}(\eta) = \kappa^+$ . Assume that there is a  $\kappa^+$ -strongly unbounded function on  $\lambda$ . Then, there is a cardinal-preserving partial order that forces the existence of a  $(\kappa, \eta, \lambda)$ -Boolean algebra.*

In [3], [6], [7] and in many other papers, the authors proved the existence of certain superatomic Boolean algebras in such a way that instead of constructing the algebras directly, they actually produced certain “graded posets” which guaranteed the existence of the wanted superatomic Boolean algebras. From these constructions, Bagaria, [1], extracted the following notion and proved the Lemma 7 below which was implicitly used in many earlier papers.

**Definition 6** ([1]). Given a sequence  $\mathfrak{s} = \langle \kappa_\alpha : \alpha < \delta \rangle$  of infinite cardinals, we say that a poset  $\langle T, \prec \rangle$  is an  $\mathfrak{s}$ -poset iff the following conditions are satisfied:

- (1)  $T = \bigcup \{T_\alpha : \alpha < \delta\}$  where  $T_\alpha = \{\alpha\} \times \kappa_\alpha$  for each  $\alpha < \delta$ .
- (2) For each  $s \in T_\alpha$  and  $t \in T_\beta$ , if  $s \prec t$  then  $\alpha < \beta$ .
- (3) For every  $\{s, t\} \in [T]^2$  there is a finite subset  $i\{s, t\}$  of  $T$  such that for each  $u \in T$ :

$$(u \preceq s \wedge u \preceq t) \text{ iff } u \preceq v \text{ for some } v \in i\{s, t\}.$$

- (4) For  $\alpha < \beta < \delta$ , if  $t \in T_\beta$  then the set  $\{s \in T_\alpha : s \prec t\}$  is infinite.

**Lemma 7** ([1, Lemma 1]). *If there is an  $\mathfrak{s}$ -poset then there is a superatomic Boolean algebra with cardinal sequence  $\mathfrak{s}$ .*

Actually, if  $\mathcal{T} = \langle T, \prec \rangle$  is an  $\mathfrak{s}$ -poset, we write  $U_{\mathcal{T}}(x) = \{y \in T : y \preceq x\}$  for  $x \in T$ , and we denote by  $X_{\mathcal{T}}$  the topological space on  $T$  whose subbase is the family

- (1)  $\{U_{\mathcal{T}}(x), T \setminus U_{\mathcal{T}}(x) : x \in T\},$

then  $X_{\mathcal{T}}$  is a locally compact, Hausdorff, scattered space whose cardinal sequence is  $\mathfrak{s}$ , and so the clopen algebra of the one-point compactification of  $X_{\mathcal{T}}$  is the required superatomic Boolean algebra with cardinal sequence  $\mathfrak{s}$ .

So, to prove Theorem 5 it will be enough to show that  $\langle \kappa \rangle_{\eta} \frown \langle \lambda \rangle$ -posets may exist for  $\kappa, \eta$  and  $\lambda$  as above.

The organization of this paper is as follows. In Section 2, we shall prove Theorem 5 for the special case in which  $\kappa = \omega$  and  $\lambda \geq \omega_3$ , generalizing in this way the result proved by Roitman in [7]. In Section 3, we shall define the combinatorial notions that make the proof of Theorem 5 work. And in Section 4, we shall present the proof of Theorem 5.

## 2. GENERALIZATION OF ROITMAN'S THEOREM

In this section, our aim is to prove the following result.

**Theorem 8.** *Let  $\lambda$  be a cardinal with  $\lambda \geq \omega_3$ . Assume that there is an  $\omega_1$ -strongly unbounded function on  $\lambda$ . Then, in some cardinal-preserving generic extension for each ordinal  $\eta$  with  $\omega_1 \leq \eta < \omega_2$  and  $\text{cf}(\eta) = \omega_1$  there is an  $(\omega, \eta, \lambda)$ -Boolean algebra.*

The theorem above is a bit stronger than Theorem 5 for  $\kappa = \omega$ , because the generic extension does not depend on  $\eta$ . However, as we will see, its proof is much simpler than the proof of the general case.

By Lemma 7, it is enough to construct a c.c.c. poset  $\mathcal{P}$  such that in  $V^{\mathcal{P}}$  for each  $\eta < \omega_2$  with  $\text{cf}(\eta) = \omega_1$  there is an  $\langle \omega \rangle_{\eta} \frown \langle \lambda \rangle$ -poset.

For  $\eta = \omega_1$  it is straightforward to obtain a suitable  $\mathcal{P}$ : all we need is to plug Kosmider's strongly unbounded function into the original argument of Roitman. For  $\omega_1 < \eta < \omega_2$  this simple approach does not work, but we can use the "stepping-up" method of Er-rhaimini and Veličkovic from [2]. Using this method, it will be enough to construct a single  $\langle \omega \rangle_{\omega_1} \frown \langle \lambda \rangle$ -poset (with some extra properties) to obtain  $\langle \omega \rangle_{\eta} \frown \langle \lambda \rangle$ -posets for each  $\eta < \omega_2$  with  $\text{cf}(\eta) = \omega_1$ .

To start with, we adapt the notion of a skeleton introduced in [2] to the cardinal sequences we are considering.

**Definition 9.** Assume that  $\mathcal{T} = \langle T, \prec \rangle$  is an  $\mathfrak{s}$ -poset such that  $\mathfrak{s}$  is a cardinal sequence of the form  $\langle \kappa \rangle_{\mu} \frown \langle \lambda \rangle$  where  $\kappa, \lambda$  are infinite cardinals with  $\kappa < \lambda$  and  $\mu$  is a non-zero ordinal. Let  $i$  be the infimum function associated with  $\mathcal{T}$ . Then:

- (a) For  $\gamma < \mu$  we say that  $T_{\gamma}$ , the  $\gamma^{\text{th}}$ -level of  $\mathcal{T}$ , is a *bone level* iff the following holds:
  - (1)  $i\{s, t\} = \emptyset$  for every  $s, t \in T_{\gamma}$  with  $s \neq t$ .
  - (2) If  $x \in T_{\gamma+1}$  and  $y \prec x$  then there is a  $z \in T_{\gamma}$  with  $y \preceq z \prec x$ .
- (b) We say that  $\mathcal{T}$  is a  $\mu$ -*skeleton* iff  $T_{\gamma}$  is a bone level of  $\mathcal{T}$  for each  $\gamma < \mu$ .

The next statement can be proved by a straightforward modification of the proof of [2, Theorem 2.8].

**Theorem 10.** *Let  $\kappa, \lambda$  be infinite cardinals. If there is a  $\langle \kappa \rangle_{\kappa^+} \widehat{\langle \lambda \rangle}$ -poset which is a  $\kappa^+$ -skeleton, then for each  $\eta < \kappa^{++}$  with  $cf(\eta) = \kappa^+$  there is a  $\langle \kappa \rangle_{\eta} \widehat{\langle \lambda \rangle}$ -poset.*

So, to get Theorem 8 it is enough to prove the following result.

**Theorem 11.** *Let  $\lambda$  be a cardinal with  $\lambda \geq \omega_3$ . Assume that there is an  $\omega_1$ -strongly unbounded function on  $\lambda$ . Then, in some c.c.c. generic extension there is an  $\langle \omega \rangle_{\omega_1} \widehat{\langle \lambda \rangle}$ -poset which is an  $\omega_1$ -skeleton.*

Let  $F : [\lambda]^2 \longrightarrow \omega_1$  be an  $\omega_1$ -strongly unbounded function on  $\lambda$ . In order to prove Theorem 11, we shall define a c.c.c. forcing notion  $\mathcal{P} = \langle P, \leq \rangle$  that adjoins an  $\mathfrak{s}$ -poset  $\mathcal{T} = \langle T, \preceq \rangle$  which is an  $\omega_1$ -skeleton, where  $\mathfrak{s}$  is the cardinal sequence  $\langle \omega \rangle_{\omega_1} \widehat{\langle \lambda \rangle}$ .

First, we define the underlying set of the required  $\mathfrak{s}$ -poset.

**Definition 12.**

- (a) We put  $T = \bigcup \{T_\alpha : \alpha \leq \omega_1\}$  where  $T_\alpha = \{\alpha\} \times \omega$  for  $\alpha < \omega_1$  and  $T_{\omega_1} = \{\omega_1\} \times \lambda$ .
- (b) If  $s = (\alpha, \nu) \in T$ , we write  $\pi(s) = \alpha$  and  $\xi(s) = \nu$ .

**Definition 13.** We define the poset  $\mathcal{P} = \langle P, \leq \rangle$  as follows.

- (a) We say that  $p = \langle X, \preceq, i \rangle \in P$  iff the following conditions hold:
  - (P1)  $X$  is a finite subset of  $T$ .
  - (P2)  $\preceq$  is a partial order on  $X$  such that  $s \prec t$  implies  $\pi(s) < \pi(t)$ .
  - (P3)  $i : [X]^2 \longrightarrow [X]^{<\omega}$  is an infimum function, that is, a function such that for every  $\{s, t\} \in [X]^2$  we have:
$$\forall x \in X ([x \preceq s \wedge x \preceq t] \text{ iff } x \preceq v \text{ for some } v \in i\{s, t\}).$$
  - (P4) If  $s, t \in X \cap T_{\omega_1}$  and  $v \in i\{s, t\}$ , then  $\pi(v) \in F\{\xi(s), \xi(t)\}$ .
  - (P5) If  $s, t \in X$  with  $\pi(s) = \pi(t) < \omega_1$ , then  $i\{s, t\} = \emptyset$ .
  - (P6) If  $s, t \in X$ ,  $s \prec t$  and  $\pi(t) = \alpha + 1$ , then there is a  $u \in X$  such that  $s \preceq u \prec t$  and  $\pi(u) = \alpha$ .
- (b) If  $\langle X, \preceq, i \rangle, \langle X', \preceq', i' \rangle \in P$  we put  $\langle X', \preceq', i' \rangle \leq \langle X, \preceq, i \rangle$  iff  $X \subseteq X'$ ,  $\preceq = \preceq' \cap (X \times X)$  and  $i \subseteq i'$ .

We will need condition (P4) in order to show that  $\mathcal{P}$  is c.c.c.

**Lemma 14.** *Assume that  $p = \langle X, \preceq, i \rangle \in P$ ,  $t \in X$ ,  $\alpha < \pi(t)$  and  $n < \omega$ . Then, there is a  $p' = \langle X', \preceq', i' \rangle \in P$  with  $p' \leq p$  and there is an  $s \in X' \setminus X$  with  $\pi(s) = \alpha$  and  $\xi(s) > n$  such that, for every  $x \in X$ ,  $s \preceq' x$  iff  $t \preceq x$ .*

*Proof.* Let  $L = \{\alpha\} \cup \{\xi : \alpha < \xi < \pi(t) \wedge \exists j < \omega \xi + j = \pi(t)\}$ . Let  $\alpha = \alpha_0, \dots, \alpha_\ell$  be the increasing enumeration of  $L$ . Since  $X$  is finite, we can pick an  $s_j \in T_{\alpha_j} \setminus X$  with  $\xi(s_j) > n$  for  $j \leq \ell$ . Let  $X' = X \cup \{s_j : j \leq \ell\}$  and let

$$\preceq' = \preceq \cup \{(s_j, y) : j \leq \ell, t \preceq y\} \cup \{(s_j, s_k) : j < k \leq \ell\}.$$

Now, we put  $i'\{x, y\} = i\{x, y\}$  if  $x, y \in X$ ,  $i'\{s_j, y\} = \{s_j\}$  if  $t \preceq y$ ,  $i'\{s_j, s_k\} = s_{\min(j,k)}$ , and  $i'\{s_j, y\} = \emptyset$  otherwise. Clearly,  $\langle X', \preceq', i' \rangle$  is as required.  $\square$

**Lemma 15.** *If  $\mathcal{P}$  preserves cardinals, then  $\mathcal{P}$  adjoins an  $\langle \omega \rangle_{\omega_1} \frown \langle \lambda \rangle$ -poset which is an  $\omega_1$ -skeleton.*

*Proof.* Let  $\mathcal{G}$  be a  $\mathcal{P}$ -generic filter. We put  $p = \langle X_p, \preceq_p, i_p \rangle$  for  $p \in \mathcal{G}$ . By Lemma 10 and standard density arguments, we have

$$(2) \quad T = \bigcup \{X_p : p \in \mathcal{G}\},$$

and taking

$$(3) \quad \preceq = \bigcup \{\preceq_p : p \in \mathcal{G}\},$$

the poset  $\langle T, \preceq \rangle$  is an  $\langle \omega \rangle_{\omega_1} \frown \langle \lambda \rangle$ -poset. Especially, Lemma 14 ensures that  $\langle T, \preceq \rangle$  satisfies (4) in Definition 6. Properties (P5) and (P6) guarantee that  $\langle T, \preceq \rangle$  is an  $\omega_1$ -skeleton.  $\square$

Now, we prove the key lemma for showing that  $\mathcal{P}$  adjoins the required poset.

**Lemma 16.**  *$\mathcal{P}$  is c.c.c.*

*Proof.* Assume that  $R = \langle r_\nu : \nu < \omega_1 \rangle \subseteq P$  with  $r_\nu \neq r_\mu$  for  $\nu < \mu < \omega_1$ . For  $\nu < \omega_1$ , write  $r_\nu = \langle X_\nu, \preceq_\nu, i_\nu \rangle$  and put  $L_\nu = \pi[X_\nu]$ . By the  $\Delta$ -System Lemma, we may suppose that the set  $\{X_\nu : \nu < \omega_1\}$  forms a  $\Delta$ -system with root  $X^*$ . By thinning out  $R$  again if necessary, we may assume that  $\{L_\nu : \nu < \omega_1\}$  forms a  $\Delta$ -system with root  $L^*$  in such a way that  $X_\nu \cap T_\alpha = X_\mu \cap T_\alpha$  for every  $\alpha \in L^* \setminus \{\omega_1\}$  and  $\nu < \mu < \omega_1$ . Without loss of generality, we may assume that  $\omega_1 \in L^*$ . Since  $\beta \setminus \alpha$  is a countable set for  $\alpha, \beta \in L^*$  with  $\alpha < \beta < \omega_1$ , we may suppose that  $L^* \setminus \{\omega_1\}$  is an initial segment of  $L_\nu$  for every  $\nu < \omega_1$ . Of course, this may require a further thinning out of  $R$ . Now, we put  $Z_\nu = X_\nu \cap T_{\omega_1}$  for  $\nu < \omega_1$ . Without loss of generality, we may assume that the domains of the forcing conditions of  $R$  have the same size and that there is a natural number  $n > 0$  with  $|Z_\nu \setminus X^*| = |Z_\mu \setminus X^*| = n$  for  $\nu < \mu < \omega_1$ . We consider in  $T_{\omega_1}$  the well-order induced by  $\lambda$ . Then, by thinning out  $R$  again if necessary, we may assume that for every  $\{\nu, \mu\} \in [\omega_1]^2$  there is an order-preserving bijection  $h = h_{\nu, \mu} : L_\nu \rightarrow L_\mu$  with  $h \upharpoonright L^* = \text{id}$  that lifts to an isomorphism of  $X_\nu$  with  $X_\mu$  satisfying the following:

- (A) For every  $\alpha \in L_\nu \setminus \{\omega_1\}$ ,  $h(\alpha, \xi) = (h(\alpha), \xi)$ .
- (B)  $h$  is the identity on  $X^*$ .
- (C) For every  $i < n$ , if  $x$  is the  $i^{\text{th}}$ -element in  $Z_\nu \setminus X^*$  and  $y$  is the  $i^{\text{th}}$ -element in  $Z_\mu \setminus X^*$ , then  $h(x) = y$ .
- (D) For every  $x, y \in X_\nu$ ,  $x \preceq_\nu y$  iff  $h(x) \preceq_\mu h(y)$ .
- (E) For every  $\{x, y\} \in [X_\nu]^2$ ,  $h[i_\nu\{x, y\}] = i_\mu\{h(x), h(y)\}$ .

Now, we deduce from condition (P4) and the fact that  $R$  is uncountable that if  $\{x, y\} \in [X^*]^2$  then  $i_\nu\{x, y\} \subseteq X^*$  for every  $\nu < \omega_1$ . So if  $\{x, y\} \in [X^*]^2$ , then  $i_\nu\{x, y\} = i_\mu\{x, y\}$  for  $\nu < \mu < \omega_1$ .

Let  $\delta = \max(L^* \setminus \{\omega_1\})$ . Since  $F$  is an  $\omega_1$ -strongly unbounded function on  $\lambda$ , there are ordinals  $\nu, \mu$  with  $\nu < \mu < \omega_1$  such that if we put  $a = \{\xi \in \lambda : (\omega_1, \xi) \in Z_\nu \setminus X^*\}$  and  $a' = \{\xi \in \lambda : (\omega_1, \xi) \in Z_\mu \setminus X^*\}$ , then  $F\{\xi, \xi'\} > \delta$  for every  $\xi \in a$  and every  $\xi' \in a'$ . Our purpose is to prove that  $r_\nu$  and  $r_\mu$  are compatible in  $\mathcal{P}$ . We put  $p = r_\nu$  and  $q = r_\mu$ . And we write  $p = \langle X_p, \preceq_p, i_p \rangle$  and  $q = \langle X_q, \preceq_q, i_q \rangle$ . Then, we define the extension  $r = \langle X_r, \preceq_r, i_r \rangle$  of  $p$  and  $q$  as follows. We put  $X_r = X_p \cup X_q$ . We define  $\preceq_r = \preceq_p \cup \preceq_q$ . Note that  $\preceq_r$  is a partial order on  $X_r$ , because  $L^* \setminus \{\omega_1\}$  is an initial segment of  $\pi[X_p]$  and  $\pi[X_q]$ . Now, we define the infimum function  $i_r$ . Assume that  $\{x, y\} \in [X_r]^2$ . We put  $i_r\{x, y\} = i_p\{x, y\}$  if  $x, y \in X_p$ , and  $i_r\{x, y\} = i_q\{x, y\}$  if  $x, y \in X_q$ . Suppose that  $x \in X_p \setminus X_q$  and  $y \in X_q \setminus X_p$ . Note that  $x, y$  are not comparable in  $\langle X_r, \preceq_r \rangle$  and there is no  $u \in (X_p \cup X_q) \setminus X^*$  such that  $u \preceq_r x, y$ . Then, we define  $i_r\{x, y\} = \{u \in X^* : u \prec_r x, y\}$ . It is easy to check that  $r \in P$ , and so  $r \leq p, q$ .  $\square$

After finishing the proof of Theorem 5 for  $\kappa = \omega$ , try to prove it for  $\kappa = \omega_1$ . So, assume that  $2^\omega = \omega_1$ ,  $\omega_4 \leq \lambda$ , and there is an  $\omega_2$ -strongly unbounded function on  $\lambda$ . We want to find  $\langle \omega_1 \rangle_\eta \frown \langle \lambda \rangle$ -posets for each ordinal  $\eta < \omega_3$  with  $\text{cf}(\eta) = \omega_2$  in some cardinal-preserving generic extension. Since the “stepping-up” method of Er-rhaimini and Veličkovic worked for  $\kappa = \omega$ , it is natural to try to apply Theorem 10 for the case  $\kappa = \omega_1$ . That is, we can try to find a cardinal-preserving generic extension that contains an  $\langle \omega_1 \rangle_{\omega_2} \frown \langle \lambda \rangle$ -poset which is an  $\omega_2$ -skeleton. For this, first we should consider the forcing construction given in [3, Section 4] to add an  $\langle \omega_1 \rangle_{\omega_2} \frown \langle \omega_3 \rangle$ -poset, and then try to extend this construction to add the required  $\omega_2$ -skeleton. However, the construction from [3] is  $\sigma$ -complete and requires that CH holds in the ground model. Then, the following results show that the forcing construction of an  $\langle \omega_1 \rangle_{\omega_2} \frown \langle \lambda \rangle$ -poset which is an  $\omega_2$ -skeleton is quite hopeless, at least by using the standard forcing from [3].

If  $X$  is the topological space associated with a skeleton and  $x \in X$ , we denote by  $t(x, X)$  the tightness of  $x$  in  $X$ . Also, if  $A$  is a subset of points of  $X$  we denote by  $A'$  the set of all points  $x \in X$  such that  $x$  is an accumulation point of  $A$ .

**Proposition 17.** *Assume that  $\mathcal{T} = \langle T, \prec \rangle$  is a  $\mu$ -skeleton,  $\alpha < \mu$  and  $x \in I_{\alpha+1}(X_{\mathcal{T}})$ . Then,  $t(x, X_{\mathcal{T}}) = \omega$ .*

*Proof.* Assume that  $A \subseteq T$  and  $x \in A'$ . We can assume that  $a \prec x$  for each  $a \in A$ .

Let

$$(4) \quad U = \{u \in I_\alpha(X_{\mathcal{T}}) : u \prec x \wedge \exists a_u \in A \ a_u \preceq u\}.$$

Since  $y \prec x$  iff  $y \preceq u$  for some  $u \prec x$  with  $u \in I_\alpha(X_{\mathcal{T}})$ , the set  $U$  is infinite.

Pick  $V \in [U]^\omega$ , and put  $B = \{a_v : v \in V\}$ . We claim that  $x \in B'$ . Indeed, if  $y \prec x$  then there is a  $u \in I_\alpha(X_{\mathcal{T}})$  such that  $y \preceq u \prec x$ . So  $|\{b \in B : b \preceq y\}| \leq 1$ . Hence  $y \notin B'$ . However,  $B$  has an accumulation point because  $B \subseteq U_{\mathcal{T}}(x)$  and  $U_{\mathcal{T}}(x)$  is compact in  $X_{\mathcal{T}}$ . So,  $B$  should converge to  $x$ .  $\square$

**Corollary 18.** *If  $\mathcal{T}$  is a  $\mu$ -skeleton, then  $\mu \leq |I_0(X_{\mathcal{T}})|^\omega$ . Especially, under CH an  $\langle \omega_1 \rangle_{\omega_2} \frown \langle \lambda \rangle$ -poset can not be an  $\omega_2$ -skeleton.*

Thus, we are unable to use Theorem 10 to prove Theorem 5 even for  $\kappa = \omega_1$ . Instead of this stepping-up method, in the next two sections we will construct  $\langle \omega_1 \rangle_{\eta} \frown \langle \lambda \rangle$ -posets directly using the method of orbits from [6]. This method was used to construct by forcing  $\langle \omega_1 \rangle_{\eta}$ -posets for  $\omega_2 \leq \eta < \omega_3$ . It is not difficult to get an  $\langle \omega_1 \rangle_{\omega_2}$ -poset by means of countable ‘‘approximations’’ of the required poset. However, for  $\omega_2 \leq \eta < \omega_3$  we need the notion of orbit and a much more involved forcing to obtain  $\langle \omega_1 \rangle_{\eta}$ -posets (see [6]).

### 3. COMBINATORIAL NOTIONS

In this section, we define the combinatorial notions that will be used in the proof of Theorem 5.

If  $\alpha, \beta$  are ordinals with  $\alpha \leq \beta$  let

$$(5) \quad [\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}.$$

**Definition 19.**

(a) We say that  $I$  is an *ordinal interval* iff there are ordinals  $\alpha$  and  $\beta$  with  $\alpha \leq \beta$  and  $I = [\alpha, \beta)$ . Then, we write  $I^- = \alpha$  and  $I^+ = \beta$ .

(b) Assume that  $I = [\alpha, \beta)$  is an ordinal interval. If  $\beta$  is a limit ordinal, let  $E(I) = \{\varepsilon_\nu^I : \nu < \text{cf}(\beta)\}$  be a cofinal closed subset of  $I$  having order type  $\text{cf}(\beta)$  with  $\alpha = \varepsilon_0^I$ , and then put

$$(6) \quad \mathcal{E}(I) = \{[\varepsilon_\nu^I, \varepsilon_{\nu+1}^I) : \nu < \text{cf}(\beta)\}.$$

If  $\beta = \beta' + 1$  is a successor ordinal, put  $E(I) = \{\alpha, \beta'\}$  and

$$(7) \quad \mathcal{E}(I) = \{[\alpha, \beta'), \{\beta'\}\}.$$

(c) If  $\kappa$  is an infinite cardinal and  $\eta$  is an ordinal with  $\kappa^+ \leq \eta < \kappa^{++}$  and  $\text{cf}(\eta) = \kappa^+$ , we define  $\mathbb{I}_\eta = \bigcup \{\mathcal{I}_n : n < \omega\}$  where:

$$(8) \quad \mathcal{I}_0 = \{[0, \eta)\} \text{ and } \mathcal{I}_{n+1} = \bigcup \{\mathcal{E}(I) : I \in \mathcal{I}_n\}.$$

Note that  $\mathbb{I}_\eta$  is a cofinal tree of intervals in the sense defined in [6]. So, the following conditions are satisfied:

- (i) For every  $I, J \in \mathbb{I}_\eta$ ,  $I \subseteq J$  or  $J \subseteq I$  or  $I \cap J = \emptyset$ .
- (ii) If  $I, J$  are different elements of  $\mathbb{I}_\eta$  with  $I \subseteq J$  and  $J^+$  is a limit, then  $I^+ < J^+$ .
- (iii)  $\mathcal{I}_n$  partitions  $[0, \eta)$  for each  $n < \omega$ .
- (iv)  $\mathcal{I}_{n+1}$  refines  $\mathcal{I}_n$  for each  $n < \omega$ .
- (v) For every  $\alpha < \eta$  there is an  $I \in \mathbb{I}_\eta$  such that  $I^- = \alpha$ .

**Definition 20.**

(a) For each  $\alpha < \eta$  and  $n < \omega$  we define  $I(\alpha, n)$  as the unique interval  $I \in \mathcal{I}_n$  such that  $\alpha \in I$ .



(b) For each  $\alpha < \eta$  we define  $n(\alpha)$  as the least natural number  $n$  such that there is an interval  $I \in \mathcal{I}_n$  with  $I^- = \alpha$ .

Note that if  $n(\alpha) = k$ , then for every  $m \geq k$  we have  $I(\alpha, m)^- = \alpha$ .

The following notion will be essential in our forcing construction.

**Definition 21.** Assume that  $\alpha < \eta$ . If  $m < n(\alpha)$ , we put  $o_m(\alpha) = E(I(\alpha, m)) \cap \alpha$ . Then, we define the *orbit* of  $\alpha$  (with respect to  $\mathbb{I}_\eta$ ) as

$$(9) \quad o(\alpha) = \bigcup \{o_m(\alpha) : m < n(\alpha)\}.$$

For basic facts on orbits and trees of intervals, we refer the reader to [6, Section 1]. In particular, we have  $|o(\alpha)| \leq \kappa$  for every  $\alpha < \eta$ .

We write  $E([0, \eta)) = \{\varepsilon_\nu : \nu < \kappa^+\}$ .

**Claim 22.**  $o(\varepsilon_\nu) = \{\varepsilon_\zeta : \zeta < \nu\}$  for  $\nu < \kappa^+$ .

*Proof.* Clearly  $I(\varepsilon_\nu, 0) = [0, \eta)$  and  $I(\varepsilon_\nu, 1) = [\varepsilon_\nu, \varepsilon_{\nu+1})$ . So  $n(\varepsilon_\nu) = 1$ . Thus  $o(\varepsilon_\nu) = o_0(\varepsilon_\nu) = E(I(\varepsilon_\nu, 0)) \cap \varepsilon_\nu = E([0, \eta)) \cap \varepsilon_\nu = \{\varepsilon_\zeta : \zeta < \nu\}$ .  $\square$

For  $\alpha < \beta < \eta$  let

$$(10) \quad j(\alpha, \beta) = \max\{j : I(\alpha, j) = I(\beta, j)\},$$

and put

$$(11) \quad J(\alpha, \beta) = I(\alpha, j(\alpha, \beta) + 1).$$

For  $\alpha < \eta$  let

$$(12) \quad J(\alpha, \eta) = I(\alpha, 1).$$

**Claim 23.** If  $\varepsilon_\zeta \leq \alpha < \varepsilon_{\zeta+1} \leq \beta \leq \eta$ , then  $J(\alpha, \beta) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$ .

*Proof.* For  $\beta = \eta$ ,  $J(\alpha, \beta) = I(\alpha, 1) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$ .

Now assume that  $\beta < \eta$ . Since  $I(\alpha, 0) = I(\beta, 0) = [0, \eta)$ , but  $I(\alpha, 1) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$  and  $I(\beta, 1) = [\varepsilon_\xi, \varepsilon_{\xi+1})$  for some  $\varepsilon_\xi$  with  $\varepsilon_{\zeta+1} \leq \varepsilon_\xi$ , we have  $j(\alpha, \beta) = 0$  and so  $J(\alpha, \beta) = [\varepsilon_\zeta, \varepsilon_{\zeta+1})$ .  $\square$

#### 4. PROOF OF THE MAIN THEOREM

In order to prove Theorem 5, suppose that  $\kappa, \lambda$  are infinite cardinals with  $\kappa^{+++} \leq \lambda$  and  $\kappa^{<\kappa} = \kappa$ ,  $\eta$  is an ordinal with  $\kappa^+ \leq \eta < \kappa^{++}$  and  $\text{cf}(\eta) = \kappa^+$ , and there is a  $\kappa^+$ -strongly unbounded function on  $\lambda$ . We will use a refinement of the arguments given in [6] and [3, Section 4].

First, we define the underlying set of our construction.

**Definition 24.**

- (a) We put  $T = \bigcup\{T_\alpha : \alpha \leq \eta\}$  where  $T_\alpha = \{\alpha\} \times \kappa$  for every  $\alpha < \eta$  and  $T_\eta = \{\eta\} \times \lambda$ .  
 (b) We write  $T_{<\eta} = T \setminus T_\eta$ .

**Definition 25.**

- (a) We put  $\mathbb{I} = \mathbb{I}_\eta$ .  
 (b) We define  $E = E([0, \eta)) = \{\varepsilon_\nu : \nu < \kappa^+\}$ .

Since there is a  $\kappa^+$ -strongly unbounded function on  $\lambda$  and  $\text{cf}(\eta) = \kappa^+$  there is a function  $F : [\lambda]^2 \longrightarrow E$  such that the following condition holds:

- ( $\star$ ) For every ordinal  $\gamma < \eta$  and every family  $A \subseteq [\lambda]^{<\kappa}$  of pairwise disjoint sets with  $|A| = \kappa^+$ , there are different  $a, b \in A$  such that  $F\{\alpha, \beta\} > \gamma$  for every  $\alpha \in a$  and  $\beta \in b$ .

The following notion will be used in our forcing construction.

**Definition 26.** Let  $\Lambda \in \mathbb{I}$  and  $\{s, t\} \in [T]^2$  with  $\pi(s) < \pi(t)$ . We say that  $\Lambda$  *isolates  $s$  from  $t$*  iff  $\Lambda^- < \pi(s) < \Lambda^+$  and  $\Lambda^+ \leq \pi(t)$ .

**Definition 27.** We define the poset  $\mathcal{P} = \langle P, \leq \rangle$  as follows.

- (a) We say that  $p = \langle X, \preceq, i \rangle \in P$  iff the following conditions hold:  
 (P1)  $X \in [T]^{<\kappa}$ .  
 (P2)  $\preceq$  is a partial order on  $X$  such that  $s \prec t$  implies  $\pi(s) < \pi(t)$ .  
 (P3)  $i : [X]^2 \longrightarrow X \cup \{\text{undef}\}$  is an infimum function, that is, a function such that for every  $\{s, t\} \in [X]^2$  we have:  

$$\forall x \in X ([x \preceq s \wedge x \preceq t] \text{ iff } x \preceq i\{s, t\}).$$
  
 (P4) If  $s, t \in X$  are compatible but not comparable in  $\langle X, \preceq \rangle$ ,  $v = i\{s, t\}$  and  $\pi(s) = \alpha_1$ ,  $\pi(t) = \alpha_2$  and  $\pi(v) = \beta$ , we have:  
 (a) If  $\alpha_1, \alpha_2 < \eta$ , then  $\beta \in o(\alpha_1) \cap o(\alpha_2)$ .  
 (b) If  $\alpha_1 < \eta$  and  $\alpha_2 = \eta$ , then  $\beta \in o(\alpha_1) \cap E$ .  
 (c) If  $\alpha_1 = \eta$  and  $\alpha_2 < \eta$ , then  $\beta \in o(\alpha_2) \cap E$ .  
 (d) If  $\alpha_1 = \alpha_2 = \eta$ , then  $\beta \in F\{\xi(s), \xi(t)\} \cap E$ .  
 (P5) If  $s, t \in X$  with  $s \preceq t$  and  $\Lambda = J(\pi(s), \pi(t))$  isolates  $s$  from  $t$ , then there is a  $u \in X$  such that  $s \preceq u \preceq t$  and  $\pi(u) = \Lambda^+$ .  
 (b) If  $\langle X, \preceq, i \rangle, \langle X', \preceq', i' \rangle \in P$ , we put  $\langle X', \preceq', i' \rangle \leq \langle X, \preceq, i \rangle$  iff  $X \subseteq X'$ ,  $\preceq = \preceq' \cap (X \times X)$  and  $i \subseteq i'$ .

**Lemma 28.** *Assume that  $p = \langle X, \preceq, i \rangle \in P$ ,  $t \in X$ ,  $\alpha < \pi(t)$  and  $\nu < \kappa$ . Then, there is a  $p' = \langle X', \preceq', i' \rangle \in P$  with  $p' \leq p$  and there is an  $s \in X' \setminus X$  with  $\pi(s) = \alpha$  and  $\xi(s) > \nu$  such that, for every  $x \in X$ ,  $s \preceq' x$  iff  $t \preceq' x$ .*

*Proof.* Since  $|X| < \kappa$ , we can take an  $s \in T_\alpha \setminus X$  with  $\xi(s) > \nu$ . Let  $\{I_0, \dots, I_n\}$  be the list of all the intervals in  $\mathbb{I}$  that isolate  $s$  from  $t$  in such a way that

$I_0^+ > I_1^+ > \dots > I_n^+$ . Put  $\gamma_i = I_i^+$  for  $i \leq n$ . We take points  $c_i \in T \setminus X$  with  $\pi(c_i) = \gamma_i$  for  $i \leq n$ . Let  $X' = X \cup \{s\} \cup \{c_i : i \leq n\}$  and let

$$\prec' = \prec \cup \{\langle s, c_i \rangle : i \leq n\} \cup \{\langle s, y \rangle : t \preceq y\} \cup \{\langle c_j, c_i \rangle : i < j\} \cup \{\langle c_i, y \rangle : i \leq n, t \preceq y\}.$$

Note that, for  $z \in X'$  and  $y \in \{s\} \cup \{c_i : i \leq n\}$ , either  $z$  and  $y$  are comparable or they are incompatible with respect to  $\preceq'$ . So, the definition of  $i'$  is clear.

Finally, observe that  $p'$  satisfies (P5) because if  $x \prec' y$  with  $x \in \{s\} \cup \{c_i : i \leq n\}$ ,  $y \in X'$  and  $J(\pi(x), \pi(y))$  isolates  $x$  from  $y$  then either  $J(\pi(x), \pi(y)) = I_k$  for some  $0 \leq k \leq n$  or  $J(\pi(x), \pi(y)) = J(\pi(t), \pi(y))$ . But if  $J(\pi(x), \pi(y)) = I_k$ , then  $c_k$  witnesses (P5) for  $x$  and  $y$ ; and if  $J(\pi(x), \pi(y)) = J(\pi(t), \pi(y))$ , we are done by condition (P5) for  $p$ .  $\square$

**Definition 29.** For  $p \in P$  we write  $p = \langle X_p, \preceq_p, i_p \rangle$ ,  $Y_p = X_p \cap T_{<\eta}$  and  $Z_p = X_p \cap T_\eta$ .

**Lemma 30.** *If  $\mathcal{P}$  preserves cardinals, then forcing with  $\mathcal{P}$  adjoins a  $(\kappa, \eta, \lambda)$ -Boolean algebra.*

*Proof.* Let  $\mathcal{G}$  be a  $\mathcal{P}$ -generic filter. Then

$$(13) \quad T = \bigcup \{X_p : p \in \mathcal{G}\},$$

and taking

$$(14) \quad \preceq = \bigcup \{\preceq_p : p \in \mathcal{G}\}$$

the poset  $\langle T, \preceq \rangle$  is a  $\langle \kappa \rangle_\eta \frown \langle \lambda \rangle$ -poset. Especially, Lemma 28 guarantees that  $\langle T, \prec \rangle$  satisfies (4) from Definition 6. So, by Lemma 7, in  $V[\mathcal{G}]$  there is a  $(\kappa, \eta, \lambda)$ -Boolean algebra.  $\square$

To complete our proof we should check that forcing with  $P$  preserves cardinals. It is straightforward that  $\mathcal{P}$  is  $\kappa$ -closed. The burden of our proof is to verify the following statement, which completes the proof of Theorem 5.

**Lemma 31.**  *$\mathcal{P}$  has the  $\kappa^+$ -chain condition.*

We need to consider the partial order introduced in [6].

**Definition 32.** We define the subposet  $\mathcal{P}_\eta = \langle P_\eta, \leq_\eta \rangle$  of  $\mathcal{P}$  as follows. We put

$$(15) \quad P_\eta = \{p \in P : X_p \subseteq \eta \times \kappa\},$$

and we let  $\leq_\eta = \leq \upharpoonright P_\eta$ .

The poset  $\mathcal{P}_\eta$  was defined in [6, Definition 2.1], and it was proved that  $\mathcal{P}_\eta$  satisfies the  $\kappa^+$ -chain condition. In [6, Lemmas 2.5 and 2.6] it was shown that every set  $R \in [P_\eta]^{\kappa^+}$  has a linked subset of size  $\kappa^+$ . Actually, a stronger statement was proved, and we will use that statement to prove Lemma 31. However, before doing so, we need some preparation.

**Definition 33.** Suppose that  $g : A \longrightarrow B$  is a bijection, where  $A, B \in [T]^{<\kappa}$ . We say that  $g$  is *adequate* iff the following conditions hold:

- (1)  $g[A \cap T_{<\eta}] = B \cap T_{<\eta}$  and  $g[A \cap T_\eta] = B \cap T_\eta$ .
- (2) For every  $s, t \in A$ ,  $\pi(s) < \pi(t)$  iff  $\pi(g(s)) < \pi(g(t))$ .
- (3) For every  $s = \langle \alpha, \nu \rangle \in A \cap T_{<\eta}$ ,  $g(\alpha, \nu) = (\beta, \zeta)$  implies  $\nu = \zeta$ .
- (4) For every  $s, t \in A \cap T_\eta$ ,  $\xi(s) < \xi(t)$  iff  $\xi(g(s)) < \xi(g(t))$ .

For  $A, B \subseteq T_{<\eta}$ , this definition is just [6, Definition 2.2].

**Definition 34.** A set  $Z \subseteq P$  is *separated* iff the following conditions are satisfied:

- (1)  $\{X_p : p \in Z\}$  forms a  $\Delta$ -system with root  $X$ .
- (2) For each  $\alpha < \eta$ , either  $X_p \cap T_\alpha = X \cap T_\alpha$  for every  $p \in Z$ , or there is at most one  $p \in Z$  such that  $X_p \cap T_\alpha \neq \emptyset$ .
- (3) For every  $p, q \in Z$  there is an adequate bijection  $h_{p,q} : X_p \longrightarrow X_q$  which satisfies the following:
  - (a) For any  $s \in X$ ,  $h_{p,q}(s) = s$ .
  - (b) If  $s, t \in X_p$ , then  $s \prec_p t$  iff  $h_{p,q}(s) \prec_q h_{p,q}(t)$ .
  - (c) If  $s, t \in X_p$ , then  $h_{p,q}(i_p\{s, t\}) = i_q\{h_{p,q}(s), h_{p,q}(t)\}$ .

For  $Z \subseteq P_\eta$ , this definition is just [6, Definition 2.3].

**Lemma 35.** Assume that  $Z \in [P]^{\kappa^+}$  is separated and  $X$  is the root of the  $\Delta$ -system  $\{X_p : p \in Z\}$ . If  $s, t$  are compatible but not comparable in  $p \in Z$  and  $s \in X \cap T_{<\eta}$ , then  $i_p\{s, t\} \in X$ .

*Proof.* Assume that  $s, t$  are compatible but not comparable in  $p \in Z$  and  $s \in X \cap T_{<\eta}$ . Assume that  $i_p\{s, t\} \notin X$ . Then since

$$(16) \quad \{i_q\{s, h_{p,q}(t)\} : q \in Z\} = \{h_{p,q}(i_p\{s, t\}) : q \in Z\},$$

the elements of  $\{i_q\{s, h_{p,q}(t)\} : q \in Z\}$  are all different. But this is impossible, because  $\pi(i_q\{s, h_{p,q}(t)\}) \in o(s)$  for all  $q \in Z$  and  $|o(s)| \leq \kappa$ .  $\square$

In [6, Lemmas 2.5 and 2.6], as we explain in the Appendix of this paper, actually the following statement was proved.

**Proposition 36.** For each subset  $R \in [P_\eta]^{\kappa^+}$  there is a separated subset  $Z \in [R]^{\kappa^+}$  and an ordinal  $\gamma < \eta$  such that every  $p, q \in Z$  have a common extension  $r \in P_\eta$  such that the following holds:

- (R1)  $\sup \pi[X_r \setminus (X_p \cup X_q)] < \gamma$ .
- (R2) (a)  $y \prec_r s$  iff  $y \prec_r h_{p,q}(s)$  for each  $s \in X_p$  and  $y \in X_r \setminus (X_p \cup X_q)$ ,  
(b)  $s \prec_r y$  iff  $h_{p,q}(s) \prec_r y$  for each  $s \in X_p$  and  $y \in X_r \setminus (X_p \cup X_q)$ ,  
(c) if  $s \prec_r y$  for  $s \in X_p \cup X_q$  and  $y \in X_r \setminus (X_p \cup X_q)$ , then there is a  $w \in X_p \cap X_q$  with  $s \preceq_r w \prec_r y$ ,

$$(d) \text{ for } s \in X_p \setminus X_q \text{ and } t \in X_q \setminus X_p,$$

$$(17) \quad \begin{aligned} s \prec_r t &\text{ iff } \exists u \in X_p \cap X_q \text{ such that } s \prec_p u \prec_q t, \\ t \prec_r s &\text{ iff } \exists u \in X_p \cap X_q \text{ such that } t \prec_q u \prec_p s. \end{aligned}$$

After this preparation, we are ready to prove Lemma 31.

*Proof of Lemma 31.* We will argue in the following way. Assume that  $R = \langle r_\nu : \nu < \kappa^+ \rangle \subseteq P$ , where  $r_\nu = \langle X_\nu, \preceq_\nu, i_\nu \rangle$ . For each  $\nu < \kappa^+$  we will “push down”  $r_\nu$  into  $P_\eta$ , more precisely, we will construct an isomorphic copy  $r'_\nu \in P_\eta$  of  $r_\nu$ . Using Proposition 36 we can find a separated subfamily  $\{r'_\nu : \nu \in K\}$  of size  $\kappa^+$  and an ordinal  $\gamma < \eta$  such that for each  $\nu, \mu \in K$  with  $\nu \neq \mu$  there is a condition  $r'_{\nu, \mu} \in P_\eta$  such that  $r'_{\nu, \mu} \leq_\eta r'_\nu, r'_\mu$  and (R1)–(R2) hold, especially

$$(18) \quad \sup \pi[X'_{\nu, \mu} \setminus (X'_\nu \cup X'_\mu)] < \gamma.$$

Let  $X$  be the root of  $\{X_\nu : \nu < \kappa^+\}$ ,  $Y = X \setminus T_\eta$  and  $\gamma_0 = \max(\gamma, \sup \pi[Y])$ . Since  $F$  is  $\kappa^+$ -strongly unbounded, there are  $\nu, \mu \in K$  with  $\nu < \mu$  such that

$$(19) \quad \forall s \in (X_\nu \setminus X_\mu) \cap T_\eta \quad \forall t \in (X_\mu \setminus X_\nu) \cap T_\eta \quad F\{\xi(s), \xi(t)\} > \gamma_0.$$

Then we will be able to “pull back”  $r' = r'_{\nu, \mu}$  into  $P$  to get a condition  $r = r_{\nu, \mu}$  which is a common extension of  $r_\nu$  and  $r_\mu$ . Let us remark that  $r$  will not be an isomorphic copy of  $r'$ , rather  $r$  will be a “homomorphic image“ of  $r'$ .

Now we carry out our plan.

Since  $\kappa^{<\kappa} = \kappa$ , by thinning out our sequence we can assume that  $R$  itself is a separated set. So  $\{X_r : r \in R\}$  forms a  $\Delta$ -system with kernel  $\bar{X}$ . We write  $\bar{Y} = \bar{X} \cap T_{<\eta}$  and  $\bar{Z} = \bar{X} \cap T_\eta$ .

Recall that  $E = E([0, \eta]) = \{\varepsilon_\zeta : \zeta < \kappa^+\}$  is a closed unbounded subset of  $\eta$ .

Fix  $\nu < \kappa^+$ . Write  $Y_\nu = X_\nu \cap T_{<\eta}$  and  $Z_\nu = X_\nu \cap T_\eta$ . Pick a limit ordinal  $\zeta(\nu) < \kappa^+$  such that:

- (i)  $\sup(\pi[Y_\nu]) < \varepsilon_{\zeta(\nu)}$ ,
- (ii)  $\zeta(\mu) < \zeta(\nu)$  for  $\mu < \nu$ .

Let  $\theta = \text{tp}(\xi[Z_\nu])$  and  $\alpha = \varepsilon_{\zeta(\nu)}$ . We put  $Z'_\nu = \{\langle \alpha, \xi \rangle : \xi < \theta\}$ . Clearly,  $Z'_\nu \subseteq T_{\varepsilon_{\zeta(\nu)}}$  and  $\text{tp}(\xi[Z'_\nu]) = \text{tp}(\xi[Z_\nu])$ . We consider in  $Z'_\nu$  and  $Z_\nu$  the well-orderings induced by  $\kappa$  and  $\lambda$  respectively. Put  $X'_\nu = Y_\nu \cup Z'_\nu$ , and let  $g_\nu : X'_\nu \rightarrow X_\nu$  be the natural bijection, i.e.  $g_\nu \upharpoonright Y_\nu = \text{id}$  and  $g_\nu(s) = t$  if for some  $\xi < \text{tp}(\xi[Z_\nu])$   $s$  is the  $\xi$ -element in  $Z'_\nu$  and  $t$  is the  $\xi$ -element in  $Z_\nu$ .

Let  $\bar{Z}'_\nu = g_\nu^{-1} \bar{Z}$ . We define the condition  $r'_\nu = \langle X'_\nu, \preceq'_\nu, i'_\nu \rangle \in P_\eta$  as follows: for  $s, t \in X'_\nu$  with  $s \neq t$  we put

$$(20) \quad s \prec'_\nu t \text{ iff } g_\nu(s) \prec_\nu g_\nu(t),$$

and

$$(21) \quad i'_\nu\{s, t\} = i_\nu\{g_\nu(s), g_\nu(t)\}.$$

**Claim 37.**  $r'_\nu \in P_\eta$ .

*Proof.* (P1), (P2) and (P3) are clear because  $g_\nu$  is an isomorphism between  $r'_\nu = \langle X'_\nu, \preceq'_\nu, i'_\nu \rangle$  and  $r_\nu = \langle X_\nu, \preceq_\nu, i_\nu \rangle$ , moreover  $\pi(s) < \pi(t)$  iff  $\pi(g_\nu(s)) < \pi(g_\nu(t))$ . (P4) Since  $X'_\nu \subseteq T_{<\eta}$  we should check just (a). So assume that  $s', t' \in X'_\nu$  are compatible but not comparable in  $\langle X'_\nu, \preceq'_\nu \rangle$  and  $v' = i'_\nu\{s', t'\}$ . Put  $s = g_\nu(s')$ ,  $t = g_\nu(t')$ . Since  $g_\nu \upharpoonright Y_\nu = id$ , we can assume that  $\{s', t'\} \notin [Y_\nu]^2$ , e.g.  $s' \in Z'_\nu$  and so  $s \in Z_\nu$ .

First observe that  $v' \in Y_\nu$ , so  $v' = g_\nu(v')$ .

If  $t' \in Y_\nu$ , then  $t' = g_\nu(t')$ , and  $v' = i_\nu\{s, t'\}$ . By applying (P4)(c) in  $r_\nu$  for  $s$  and  $t'$  we obtain

$$(22) \quad \pi(v') \in E \cap o(\pi(t')) \subseteq E \cap \varepsilon_{\zeta(\nu)} \cap o(\pi(t')) = o(\pi(s')) \cap o(\pi(t'))$$

because  $o(\pi(s')) = E \cap \varepsilon_{\zeta(\nu)}$  by Claim 22.

If  $t' \in Z'_\nu$ , then  $t = g_\nu(t') \in Z_\nu \subseteq T_\eta$ . Since  $v' = i'_\nu\{s', t'\} = i_\nu\{s, t\}$ , applying (P4)(d) in  $r_\nu$  for  $s$  and  $t$  we obtain

$$\pi(v') \in F\{\xi(s), \xi(t)\} \cap E \cap \varepsilon_{\zeta(\nu)} \subseteq E \cap \varepsilon_{\zeta(\nu)} = o(\pi(s')) \cap o(\pi(t'))$$

because  $o(\pi(s')) = o(\pi(t')) = E \cap \varepsilon_{\zeta(\nu)}$  by Claim 22.

(P5) Assume that  $s', t' \in X'_\nu$ ,  $s' \prec'_\nu t'$  and  $\Lambda = J(\pi(s'), \pi(t'))$  isolates  $s'$  from  $t'$ . Then  $s' \in Y_\nu$ , so  $g_\nu(s') = s'$ . Since  $g_\nu \upharpoonright Y_\nu = id$ , we can assume that  $\{s', t'\} \notin [Y_\nu]^2$ , i.e.  $t' \in Z'_\nu$ .

Write  $t = g_\nu(t')$ . Since  $\pi(t') = \varepsilon_{\zeta(\nu)} \in E$ , by Claim 23,  $J(\pi(s'), \pi(t')) = J(\pi(s'), \pi(t)) = [\varepsilon_\zeta, \varepsilon_{\zeta+1}] = I(\pi(s'), 1)$ , where  $\varepsilon_\zeta \leq \pi(s') < \varepsilon_{\zeta+1}$ . Applying (P5) in  $r_\nu$  for  $s'$  and  $t$ , we obtain a  $v \in Y_\nu$  such that  $\pi(v) = \Lambda^+$  and  $s' \prec_\nu v \prec_\nu t$ . Then  $g_\nu(v) = v$ , so  $s' \prec'_\nu v \prec'_\nu t'$ , which was to be proved.  $\square$

Now applying Proposition 36 to the family  $\{r'_\nu : \nu < \kappa^+\}$ , there are  $K \in [\kappa^+]^{\kappa^+}$  and  $\gamma < \eta$  such that  $\{r'_\nu : \nu \in K\}$  is separated and for every  $\nu, \mu \in K$  with  $\nu \neq \mu$  there is a common extension  $r' \in P_\eta$  of  $r'_\nu$  and  $r'_\mu$  such that (R1)-(R2) hold. Let  $\gamma_0 = \max(\gamma, \sup \pi[\bar{Y}])$ . Recall that  $\bar{Y}$  is the root of the  $\Delta$ -system  $\{Y_\nu : \nu \in \kappa^+\}$ . For  $\nu < \mu < \kappa^+$  we denote by  $h'_{\nu, \mu}$  the adequate bijection  $h_{r'_\nu, r'_\mu}$ .

Since  $F$  satisfies  $(\star)$ , there are  $\nu, \mu \in K$  with  $\nu \neq \mu$  such that for each  $s \in (Z_\nu \setminus Z_\mu)$  and  $t \in (Z_\mu \setminus Z_\nu)$  we have

$$(23) \quad F\{\xi(s), \xi(t)\} > \gamma_0.$$

We show that the conditions  $r_\nu$  and  $r_\mu$  have a common extension  $r = \langle X, \preceq, i \rangle \in P$ .

Consider a condition  $r' = \langle X', \preceq', i' \rangle$  which is a common extension of  $r'_\nu$  and  $r'_\mu$  and satisfies (R1)-(R2). We define the condition  $r = \langle X, \preceq, i \rangle$  as follows. Let

$$(24) \quad X = (X' \setminus (Z'_\nu \cup Z'_\mu)) \cup (Z_\nu \cup Z_\mu).$$

Write  $U = X' \setminus (Z'_\nu \cup Z'_\mu) = X \setminus (Z_\nu \cup Z_\mu)$  and  $V = X' \setminus (X'_\nu \cup X'_\mu)$ . Clearly,  $V \subseteq U$ . We define the function  $h : X' \rightarrow X$  as follows:

$$(25) \quad h = g_\nu \cup g_\mu \cup (id \upharpoonright U).$$

Then  $h$  is well-defined,  $h$  is onto,  $h \upharpoonright X' \setminus (\bar{Z}'_\nu \cup \bar{Z}'_\mu)$  is injective, and  $h[\bar{Z}'_\nu] = h[\bar{Z}'_\mu] = \bar{Z}$ .

Now, if  $s, t \in X$  we put

$$(26) \quad s \prec t \text{ iff there is a } t' \in X' \text{ with } h(t') = t \text{ and } s \prec' t'.$$

Finally, we define the meet function  $i$  on  $[X]^2$  as follows:

$$(27) \quad i\{s, t\} = \max_{\prec'} \{i'\{s', t'\} : h(s') = s \text{ and } h(t') = t\}.$$

We will prove in the following claim that the definition of the function  $i$  is meaningful. Then, the proof of Lemma 31 will be complete as soon as we verify that  $r \in P$  and  $r \leq r_\nu, r_\mu$ .

**Claim 38.**  $i$  is well-defined by (27), moreover  $i \supseteq i_\nu \cup i_\mu$ .

*Proof.* We need to verify that the maximum in (27) does exist when we define  $i\{s, t\}$ . So, suppose that  $\{s, t\} \in [X]^2$ .

If  $\{s, t\} \in [X \setminus \bar{Z}]^2$  then there is exactly one pair  $(s', t')$  such that  $h(s') = s$  and  $h(t') = t$ , and hence there is no problem in (27). So if  $\{s, t\} \in [X_\nu]^2$  then  $i\{s, t\} = i'\{s', t'\} = i_\nu\{s, t\}$  by the construction of  $r'_\nu$ . If  $\{s, t\} \in [X_\mu]^2$  proceeding similarly we obtain  $i\{s, t\} = i'\{s', t'\} = i_\mu\{s, t\}$ .

So we can assume that e.g.  $s \in \bar{Z}$ . Then  $h^{-1}(s) = \{s', s''\}$  for some  $s' \in \bar{Z}'_\nu$  and  $s'' \in \bar{Z}'_\mu$ .

First assume that  $t \notin \bar{Z}$ , so there is exactly one  $t' \in X'$  with  $h(t') = t$ . We distinguish the following cases.

*Case 1.*  $t \in V$ .

Note that since  $t \in V$ ,  $t = t'$ . We show that  $i'\{s', t\} = i'\{s'', t\}$ .

Let  $v = i'\{s', t\}$ . Assume that  $v \in X'_\nu \cup X'_\mu$ . Then, by (R2)(c),  $v \prec' t$  and  $t \in V$  imply that there is a  $w \in \bar{Y} = X'_\nu \cap X'_\mu$  such that  $v \preceq' w \prec' t$ . Thus  $v = i'\{s', w\}$  and  $i'\{s', w\} = i'_\nu\{s', w\} = i_\nu\{s, w\} \in \bar{Y}$  by Lemma 35 for  $w \in \bar{Y}$ . Clearly,  $v \prec' t, s''$ . Hence  $v \preceq' i'\{s'', t\}$ .

Now assume that  $v \in V$ . Then  $v \prec' s'$  implies  $v \prec' h'_{\nu, \mu}(s') = s''$  by (R2)(a). So  $v \prec' t, s''$ , thus  $i'\{s', t\} \preceq' i'\{s'', t\}$ .

So, in both cases  $i'\{s', t\} \preceq' i'\{s'', t\}$ . But  $s'$  and  $s''$  are symmetrical, hence  $i'\{s'', t\} \preceq' i'\{s', t\}$ , and so we are done.

*Case 2.*  $t \in X_\nu \setminus \bar{Z}$ .

We show that in this case  $i'\{s'', t\} \preceq' i'\{s', t\}$ .

Let  $v = i'\{s'', t\}$ . If  $v \in V$ , then  $v \prec' s''$  and  $h'_{\nu, \mu}(s') = s''$  imply  $v \prec' s'$  by (R2)(a). Thus  $v \preceq' s', t'$ , and so  $v \preceq' i'\{s', t\}$ .

Now assume that  $v \in X'_\nu \cup X'_\mu$ . Note that if  $v \in \bar{Y} = X'_\nu \cap X'_\mu$ , then  $v \prec' s'$ , and so  $v \prec' i'\{s', t\}$ . We show that  $v \in \bar{Y}$ . For this, assume that  $v \in (X'_\nu \cup X'_\mu) \setminus \bar{Y}$ . Without loss of generality, we may suppose that  $v \in X'_\nu \setminus X'_\mu$ . Then, by (R2)(d),

there is a  $w \in \bar{Y}$  such that  $v \prec' w \prec' s''$ . Thus  $v = i'\{w, t'\} = i'_\nu\{w, t'\} \in \bar{Y}$  by Lemma 35.

Moreover,  $\{s, t\} \in [X_\nu]^2$  and  $i\{s, t\} = i'\{s', t'\} = i_\nu\{s, t\}$  because  $g_\nu(s') = h(s') = s$  and  $g_\nu(t') = h(t') = t$ .

*Case 3.*  $t \in X_\mu \setminus \bar{Z}$ .

Proceeding as in Case 2, we can show that  $i'\{s', t'\} \preceq' i'\{s'', t''\} = i_\mu\{s, t\}$ .

Finally, assume that  $t \in \bar{Z}$ . Then  $h^{-1}(t) = \{t', t''\}$  for some  $t' \in \bar{Z}'_\nu$  and  $t'' \in \bar{Z}'_\mu$ .

Note that by Cases (2) and (3),

$$i'\{s'', t''\} \preceq' i'\{s', t'\} \text{ and } i'\{s', t''\} \preceq' i'\{s'', t''\}.$$

Since  $i'\{s', t'\} = i_\nu\{s, t\} = i_\mu\{s, t\} = i'\{s'', t''\}$  by the construction of  $r'_\nu$  and  $r'_\mu$ , we have

$$(28) \quad i'\{s', t'\} = i'\{s'', t''\} = \max_{\prec'}(i'\{s', t'\}, i'\{s'', t''\}, i'\{s', t''\}, i'\{s'', t'\}).$$

Moreover, in this case  $\{s, t\} \in [X_\nu]^2 \cap [X_\mu]^2$  and we have just proved that  $i\{s, t\} = i_\nu\{s, t\} = i_\mu\{s, t\}$ .  $\square$

By Claim 38 above,  $r$  is well-defined. Since  $i \supseteq i_\nu \cup i_\mu$ , it is easy to check that if  $r \in P$  then  $r \leq r_\nu, r_\mu$ . So, the following claim completes the verification of the chain condition.

**Claim 39.**  $r \in P$ .

*Proof.* (P1) and (P2) are clear.

(P3) Assume that  $\{s, t\} \in [X]^2$ . Without loss of generality, we may assume that  $s, t$  are compatible but not comparable in  $\langle X, \preceq \rangle$ . Note that by (26), (27) and condition (P3) for  $r'$ , we have  $i\{s, t\} \prec s, t$ . So, we have to show that if  $v \prec s, t$  then  $v \preceq i\{s, t\}$ .

Assume that  $v \prec s, t$ . Then,  $v \in U$  and there are  $s', t' \in X'$  such that  $h(s') = s$ ,  $h(t') = t$  and  $v \prec' s', t'$ . By (P3) for  $r'$ ,  $v \preceq' i'\{s', t'\}$ . Now as  $v, i'\{s', t'\}, i\{s, t\} \in U$  and  $h \upharpoonright U = id$ , we infer from (27) that  $v \preceq' i'\{s', t'\} \preceq' i\{s, t\}$  and hence  $v \preceq i\{s, t\}$ .

(P4) Assume that  $s, t \in X$  are compatible but not comparable in  $\langle X, \preceq \rangle$ . Let  $v = i\{s, t\}$ .

(a) In this case  $\pi(s), \pi(t) < \eta$ . Then  $s, t \in X \setminus (Z_\nu \cup Z_\mu) = U$ , so  $h(s) = s$  and  $h(t) = t$ . Thus  $i\{s, t\} = i'\{s, t\}$ . Hence, it follows from condition (P4)(a) for  $r'$  that  $\pi(i\{s, t\}) \in o(s) \cap o(t)$ .

(b) In this case  $\pi(s) < \eta$  and  $\pi(t) = \eta$ . Then  $s \in X \setminus (Z_\nu \cup Z_\mu) = U$  and  $t \in Z_\nu \cup Z_\mu$ .

By (27) and Claim 38, there is a  $t^* \in Z'_\nu \cup Z'_\mu$  such that  $h(t^*) = t$  and  $i\{s, t\} = i'\{s, t^*\}$ .



Now, applying (P4)(a) for  $r'$ , we infer that  $\pi(v) \in o(s) \cap o(t^*)$ . Since  $\pi(t^*) \in E$ , we have  $o(t^*) \subseteq E$  by Claim 22. Then we deduce that  $\pi(v) \in o(s) \cap E$ , which was to be proved.

(c) The same as (b).

(d) In this case  $\pi(s) = \pi(t) = \eta$ . If  $\{s, t\} \in [Z_\nu]^2$  then  $i\{s, t\} = i_\nu\{s, t\}$ , and by (P4)(d) for  $r_\nu$ , we deduce that  $\pi(i\{s, t\}) \in F\{\xi(s), \xi(t)\} \cap E$ . A parallel argument works if  $s, t \in Z_\mu$ .

So we can assume that  $s \in Z_\nu \setminus Z_\mu$  and  $t \in Z_\mu \setminus Z_\nu$ . Note that there are a unique  $s' \in Z'_\nu$  with  $h(s') = s$  and a unique  $t' \in Z'_\mu$  with  $h(t') = t$ . Then,  $v = i\{s, t\} = i'\{s', t'\} \in U$ . Hence either  $v \in V$ , or  $v \in X_\nu \cup X_\mu$  and in this case there is a  $w \in X_\nu \cap X_\mu$  with  $v \preceq' w$  by (R2)(d).

In both cases  $\pi(v) \leq \gamma_0$ . Note that, applying (P4)(a) in  $r'$  for  $s', t'$  and  $v = i'\{s', t'\}$ , we obtain  $\pi(v) \in o(s') \cap o(t')$ . Since  $\pi(s'), \pi(t') \in E$  we have  $o(s') \cup o(t') \subseteq E$  by Claim 22. Thus  $\pi(v) \in E$ . And since  $\pi(v) \leq \gamma_0$ , we have  $\pi(v) \in F\{\xi(s), \xi(t)\} \cap E$ , which was to be proved.

(P5) Assume that  $s, t \in X$ ,  $s \prec t$  and  $\Lambda = J(\pi(s), \pi(t))$  isolates  $s$  from  $t$ . Then  $s \notin T_\eta$ , so  $h(s) = s$ .

If  $t \notin T_\eta$  then  $h(t) = t$ , so we are done because  $r'$  satisfies (P5).

Assume that  $t \in T_\eta$ . As  $s \prec t$ , there is a  $t' \in T_{\varepsilon_{\zeta(\nu)}} \cup T_{\varepsilon_{\zeta(\mu)}}$  such that  $h(t') = t$  and  $s \prec' t'$ . Since  $\pi(t') \in E$ , by Claim 23 we have  $J(\pi(s), \pi(t')) = I(\pi(s), 1) = J(\pi(s), \pi(t))$ . Applying (P5) in  $r'$  for  $s$  and  $t'$ , we obtain a  $v \in X'$  such that  $s \prec' v \preceq' t'$  and  $\pi(v) = \Lambda^+$ . But as  $\zeta(\nu), \zeta(\mu)$  are limit ordinals, we have  $v \prec' t'$ , and hence  $v \in X' \setminus (Z'_\nu \cup Z'_\mu) = U$ . Then  $h(v) = v$ , so  $s \prec v \prec t$ , which was to be proved.  $\square$

Hence we have proved that  $\mathcal{P}$  satisfies the  $\kappa^+$ -chain condition, which completes the proof of Theorem 5.  $\square$

## 5. APPENDIX

We explain in detail how Proposition 36 was proved in [6].

**Definition 40.** Assume that  $Z \subseteq P_\eta$  is a separated set and  $\bar{X}$  is the root of  $\{X_p : p \in Z\}$ .

(a) For every  $n \in \omega$  and every  $I \in \mathcal{I}_n$  with  $\text{cf}(I^+) = \kappa^+$ , we define  $\xi(I) =$  the least ordinal  $\gamma$  such that  $\varepsilon_\gamma^I \supseteq \pi[\bar{X}] \cap I$  and we put  $\gamma(I) = \varepsilon_{\xi(I)+\kappa}^I$ .

(b) For every  $\alpha < \eta$ , if there is an  $n < \omega$  and an interval  $I \in \mathcal{I}_n$  with  $\text{cf}(I^+) = \kappa^+$  such that  $\alpha \in I$  and  $\gamma(I) \leq \alpha$ , we consider the least natural number  $k$  with this property and write  $I(\alpha) = I(\alpha, k)$ . Otherwise, we write  $I(\alpha) = \{\alpha\}$ .

(c) We say that  $Z$  is *pairwise equivalent* iff for every  $p, q \in Z$  and every  $s \in X_p$ ,  $I(\pi(s)) = I(\pi(h_{p,q}(s)))$ .

In [6], the following two lemmas were proved:

**Lemma 41** ([6, Lemma 2.5]). *Every set in  $[P_\eta]^{\kappa^+}$  has a pairwise equivalent subset of size  $\kappa^+$ .*

**Lemma 42** ([6, Lemma 2.6]). *A pairwise equivalent set  $Z \subseteq P_\eta$  of size  $\kappa^+$  is linked.*

To get Proposition 36 we explain that the proof of [6, Lemma 2.6] actually gives the following statement:

*If  $Z \subseteq P_\eta$  is a pairwise equivalent set of size  $\kappa^+$ , then there is an ordinal  $\gamma < \eta$  such that every  $p, q \in Z$  have a common extension  $r \in P_\eta$  satisfying (R1)–(R2).*

As above, we denote by  $\bar{X}$  the root of  $\{X_p : p \in Z\}$ . Assume that  $p, q \in Z$  with  $p \neq q$ . First observe that the ordering  $\prec_r$  is defined in [6, Definition 2.4]. For this, adequate bijections  $g_1 : X_r \setminus (X_p \cup X_q) \rightarrow X_p \setminus \bar{X}$  and  $g_2 : X_r \setminus (X_p \cup X_q) \rightarrow X_q \setminus \bar{X}$  are considered in such a way that  $g_2 = h_{p,q} \circ g_1$ . Then since  $g_2 = h_{p,q} \circ g_1$ , [6, Definition 2.4](b) and (c) imply (R2)(a) and [6, Definition 2.4](d) and (f) imply (R2)(b). Also, (R2)(c) follows directly from [6, Definition 2.4](d) and (f), and (R2)(d) is just [6, Definition 2.4](e) and (g). So, we have verified (R2).

To check (R1), i.e. to get the right  $\gamma$  we need a bit more work. Let

$$(29) \quad \mathcal{J} = \{I(\pi(s)) : s \in X_p\}$$

where  $p \in Z$ . Since  $Z$  is pairwise equivalent,  $\mathcal{J}$  does not depend on the choice of  $p \in Z$ . For every  $I \in \mathbb{I}_\eta$  with  $\text{cf}(I^+) = \kappa^+$  we can choose a set  $D(I) \in [E(I) \cap \gamma(I)]^\kappa$  unbounded in  $\gamma(I)$ . We claim that

$$(30) \quad \gamma = \sup(\bigcup\{D(I) : I \in \mathcal{J}\}) + 1$$

works.

First observe that  $\gamma < \eta$ , because  $\text{cf}(\eta) = \kappa^+$ ,  $|\mathcal{J}| < \kappa$  and  $|D(I)| = \kappa$  for any  $I \in \mathcal{J}$ .

Now assume that  $p, q \in Z$  with  $p \neq q$ . Write  $L_p = \pi[X_p]$ ,  $L_q = \pi[X_q]$  and  $\bar{L} = \pi[\bar{X}]$ . Let  $\{\alpha_\xi : \xi < \delta\}$  and  $\{\alpha'_\xi : \xi < \delta\}$  be the strictly increasing enumerations of  $L_p \setminus \bar{L}$  and  $L_q \setminus \bar{L}$  respectively. In the proof of [6, Lemma 2.6], for each  $\xi < \delta$  an element  $\beta_\xi \in D(I(\alpha_\xi)) = D(I(\alpha'_\xi))$  was chosen, and then a condition  $r \leq_\eta p, q$  was constructed in such a way that  $X_r = X_p \cup X_q \cup Y$  where  $Y \cap (X_p \cup X_q) = \emptyset$  and  $\pi[Y] = \{\beta_\xi : \xi < \delta\}$ . Then since  $\{\beta_\xi : \xi < \delta\} \subseteq \bigcup\{D(I) : I \in \mathcal{J}\}$ , we infer that

$$(31) \quad \sup \pi[X_r \setminus (X_p \cup X_q)] = \sup \pi[Y] < \gamma,$$

which was to be proved.

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## REFERENCES

- [1] J. Bagaria, Locally-generic Boolean algebras and cardinal sequences. *Algebra Universalis* **47** (2002), 283–302.
- [2] K. Er-rhaimini and B. Veličkovic, PCF structure of height less than  $\omega_3$ . To appear in *J. Symbolic Logic*.
- [3] P. Koepke and J.C. Martínez, Superatomic Boolean algebras constructed from morasses. *J. Symbolic Logic* **60** (1995), 940–951.
- [4] S. Koppelberg, General theory of Boolean algebras. Handbook of Boolean algebras 1, edited by J.D. Monk and R. Bonnet ( North-Holland, Amsterdam, 1989).
- [5] P. Koszmider, Universal matrices and strongly unbounded functions. *Math. Res. Lett.* **9** (2002), 549–566.
- [6] J. C. Martínez, A forcing construction of thin-tall Boolean algebras. *Fund. Math.* **159** (1999), 99–113.
- [7] J. Roitman, A very thin thick superatomic Boolean algebra. *Algebra Universalis* **21** (1985), 137–142. (1985).
- [8] J. Roitman, Superatomic Boolean algebras. In: Handbook of Boolean algebras, Vol. 3, edited by J.D. Monk and R. Bonnet, pp. 719–740 ( North-Holland, Amsterdam, 1989).

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