

VINCENZO MARRA  
LUCA SPADA

# The dual adjunction between MV-algebras and Tychonoff spaces

**Abstract.** We offer a proof of the duality theorem for finitely presented MV-algebras and rational polyhedra, a folklore and yet fundamental result. Our approach develops first a general dual adjunction between MV-algebras and subspaces of Tychonoff cubes, endowed with the transformations that are definable in the language of MV-algebras. We then show that this dual adjunction restricts to a duality between semisimple MV-algebras and closed subspaces of Tychonoff cubes. The duality theorem for finitely presented objects is obtained by a further specialisation. Our treatment is aimed at showing exactly which parts of the basic theory of MV-algebras are needed in order to establish these results, with an eye towards future generalisations.

*Keywords:* Lukasiewicz logic, MV-algebras, adjunction, categorical equivalence, duality, Tychonoff cube, compact Hausdorff spaces, Hölder's theorem, Chang's completeness theorem, Wójcicki's theorem, rational polyhedra, piecewise linear maps,  $\mathbb{Z}$ -maps.

## 1. Introduction.

In an address delivered before the Mathematical Society of Japan on the 21<sup>st</sup> of May, 1956, Marshall Stone offered a piece of advice to the working mathematician [22, p. 498]:

It is very often the case that some contact with another field [of mathematics] is necessary before fruitful directions of development [in connection with a specific problem] can be chosen. This is something which the young mathematician needs to keep in mind.

While just over fifty years of age by then, Stone felt that he had already “rounded out his period of youthful energy and creativity” [22, p. 493]. Be that as it may, there is no doubt that he knew what he was talking about, when it came to lay bridges across different fields. Back in the late thirties, his two landmark papers [20, 21] had annihilated the apparent distance between topology and algebra. Stone discovered that the set of prime ideals of a Boolean algebra carries a natural topology, one in which the open sets correspond to arbitrary ideals. Spaces arising in this manner are known

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today as *Stone spaces*. The *clopen* sets — those sets which are both closed and open in the topology — correspond to principal ideals, and hence to elements of the algebra. Thus, the original algebra can be recovered from its space of prime ideals; the bridge is in fact a two-way road. In the Introduction to his book on Stone spaces, Johnstone writes [12, p. XV]:

Now this was a really bold idea. Although the practitioners of abstract general topology [...] had by the early thirties developed considerable expertise in the construction of spaces with particular properties, the motivation of the subject was still geometrical [...] and (as far as I know) nobody had previously had the idea of applying these techniques to the study of spaces constructed from purely algebraic data.

Stone’s “bold idea” germinated, and eventually reached well beyond its Boolean cradle. This is not the place to trace the growth of each shoot; [12] is a useful starting point for the interested reader. Suffice it to say that, while Stone’s motivations were rooted in functional analysis — as indeed his background was — his work is of major importance in the realm of logic: Boolean algebras, of course, are the equivalent algebraic semantics of classical propositional logic; Stone spaces are precisely the spaces of models of theories in classical propositional logic. Generalisations of *Stone duality* — the reformulation of Stone’s results from the thirties in the efficient language of category theory — may therefore be motivated by generalisations of classical propositional logic. A prominent instance of this line of development is provided by the work of the late Leo Esakia. Beginning with [10], Esakia extended Stone’s results to a duality between Heyting algebras (the equivalent algebraic semantics of intuitionistic propositional logic) and a class of partially ordered spaces satisfying certain conditions. Today, such spaces are aptly called *Esakia spaces*, and the ensuing theory is known as *Esakia duality*. It should be emphasised in this connection that the topological morphisms featuring in Esakia duality are not simply continuous, order-preserving maps; the duality here is not with a *full* subcategory of ordered topological spaces. From several similar instances, it appears that lack of fullness is a price one often has to pay in generalising Stone duality: non-trivial conditions on morphisms are frequently required. We shall see another case in point in the present paper. Our initial motivation, once again, comes from non-classical logic.

*Lukasiewicz logic* is a many-valued propositional system going back to the 1920’s; cf. the early survey [14, §3], and its annotated English translation in [23, pp. 38–59]. Completeness of an axiomatisation with respect to the many-valued semantics was established by syntactic means in [18]. Chang

[5] first considered the equivalent algebraic semantics of Łukasiewicz logic, and called the ensuing structures *MV-algebras*. Shortly thereafter, in his ground-breaking paper [6], he obtained an algebraic proof of the completeness theorem. The standard reference for the elementary theory of MV-algebras is [7], whereas [17] is a treatment at the frontier of current research. Let us recall that an MV-algebra is an algebraic structure  $(M, \oplus, \neg, 0)$ , where  $0 \in M$  is a constant,  $\neg$  is a unary operation satisfying  $\neg\neg x = x$ ,  $\oplus$  is a unary operation making  $(M, \oplus, 0)$  a commutative monoid, the element 1 defined as  $\neg 0$  satisfies  $x \oplus 1 = 1$ , and the law

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \quad (*)$$

holds. Any MV-algebra has an underlying structure of distributive lattice bounded below by 0 and above by 1. Joins are defined as  $x \vee y = \neg(\neg x \oplus y) \oplus y$ . Thus, the characteristic law (\*) states that  $x \vee y = y \vee x$ . Meets are defined by the de Morgan condition  $x \wedge y = \neg(\neg x \vee \neg y)$ . To recover the algebraic counterpart of Łukasiewicz implication from the MV-algebraic signature, set  $x \rightarrow y = \neg x \oplus y$ . Conversely, the logical counterpart of the monoidal operation  $\oplus$  is definable in Łukasiewicz logic as  $\alpha \oplus \beta = \neg\alpha \rightarrow \beta$ . The algebraic constants 0 and  $1 = \neg 0$  respectively correspond to an arbitrary but fixed contradiction and tautology of the logic. Boolean algebras are precisely those MV-algebras that are idempotent, meaning that  $x \oplus x = x$  holds, or equivalently, that satisfy the *tertium non datur* law  $x \vee \neg x = 1$ .

The interval (of truth values)  $[0, 1] \subseteq \mathbb{R}$  can be made into an MV-algebra, often called the *standard* MV-algebra. It has 0 as neutral element,  $x \oplus y = \min\{x + y, 1\}$ , and  $\neg x = 1 - x$ . The underlying lattice order of this MV-algebra coincides with the natural order that  $[0, 1]$  inherits from the real numbers.

We are concerned here with the category  $\mathbf{MV}_{\mathfrak{p}}$  of *presented MV-algebras*, i.e. the category whose objects are MV-algebras of the form  $\mathcal{F}_{\mu}/\theta$ , where  $\mu$  is a cardinal,  $\mathcal{F}_{\mu}$  is the MV-algebra freely generated by the set  $\{X_{\alpha} \mid \alpha < \mu, \alpha \text{ an ordinal}\}$ , and  $\theta$  is a congruence on  $\mathcal{F}_{\mu}$ ; morphisms are homomorphisms of MV-algebras. It is an exercise to show that  $\mathbf{MV}_{\mathfrak{p}}$  is equivalent to the category of all MV-algebras.

*Notation.* Throughout,  $\mu$  and  $\nu$  invariably denote cardinal numbers, whereas  $\alpha$  and  $\beta$  invariably denote ordinal numbers. Although elements of  $\mathcal{F}_{\mu}$  are equivalence classes of terms in the language of MV-algebras, we often use single terms as representatives for their equivalence classes. If  $s$  is a term, the notation  $s((X_{\alpha})_{\alpha < \mu})$  means that the (finitely many) variables occurring in  $s$  are among those in the tuple  $(X_{\alpha})_{\alpha < \mu}$ . If  $s((X_{\alpha})_{\alpha < \mu}) \in \mathcal{F}_{\mu}$  and  $\{t_{\alpha}\}_{\alpha < \mu} \subseteq$

$\mathcal{F}_\nu$ , we denote by  $s([X_\alpha \setminus t_\alpha]_{\alpha < \mu})$  the term obtained from  $s$  by uniformly replacing each variable  $X_\alpha$  with the term  $t_\alpha$ . Obviously,  $s([X_\alpha \setminus t_\alpha]_{\alpha < \mu}) \in \mathcal{F}_\nu$ . Further, if  $p \in [0, 1]^\mu$ , then  $s(p)$  denotes the evaluation of the term  $s$  in the MV-algebra  $[0, 1]$  under the assignment  $X_\alpha \mapsto \pi_\alpha(p)$ , where  $\pi_\alpha: [0, 1]^\kappa \rightarrow [0, 1]$  is the projection onto the  $\alpha^{\text{th}}$  coordinate, for each ordinal  $\alpha < \kappa$ .

Since  $[0, 1]$  is an MV-algebra, each subset  $R \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$  determines a subset  $\mathbb{V}(R)$  of the Cartesian product  $[0, 1]^\mu$ , namely, the common solution set over  $[0, 1]^\mu$  of the equations  $s \approx t$ ,  $s, t \in R$ . Conversely, each subset  $S \subseteq [0, 1]^\mu$  determines a subset  $\mathbb{I}(S) \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$ , namely, the set of all pairs of terms  $(s, t)$  such that the evaluations of  $s$  and  $t$  at each element of  $S$  agree. In Section 2 we show that this correspondence yields a dual adjunction between  $\mathbf{MV}_p$  and the category  $\mathbf{T}_{\text{def } \mathbb{Z}}$  whose objects are arbitrary subsets of  $[0, 1]^\mu$  (as  $\mu$  ranges over all cardinals) and whose morphisms are *definable maps*, i.e. those (contravariant) transformations induced by the MV-algebraic homomorphisms, in the following precise sense.

**DEFINITION 1.1.** Given  $S \subseteq [0, 1]^\mu$  and  $T \subseteq [0, 1]^\nu$ , a function  $\lambda: S \rightarrow T$  is *definable* if there exists a  $\nu$ -tuple of terms  $(l_\beta)_{\beta < \nu}$ , with  $l_\beta \in \mathcal{F}_\mu$ , such that

$$\lambda((p_\alpha)_{\alpha < \mu}) = (l_\beta((p_\alpha)_{\alpha < \mu}))_{\beta < \nu}$$

for every  $(p_\alpha)_{\alpha < \mu} \in S$ . We call any such  $\nu$ -tuple a family of *defining terms* for  $\lambda$ . In the special case that  $\nu = 1$ , the  $\nu$ -tuple may be regarded as a single term  $l \in \mathcal{F}_\mu$ , called a *defining term* for  $\lambda$ .

The basic adjunction of Section 2 is formally analogous to the one between affine algebraic varieties over an algebraically closed field, and their structure rings; the  $\mathbb{I}$ – $\mathbb{V}$  notation we adopt here conforms to that model. The proof of the basic adjunction is universal-algebraic: little is needed beyond the observation that  $[0, 1]$  comes with its own MV-algebraic structure. The situation changes when the analogue of the Zariski topology on affine space enters the picture. If we let the definable subsets  $S \subseteq [0, 1]^\mu$  (i.e. those of the form  $\mathbb{V}(R)$ , for some  $R \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$ ) be the closed sets of a topology, the result is precisely the Tychonoff (product) topology on  $[0, 1]^\mu$ , where  $[0, 1]$  is endowed with its Euclidean topology. This is the content of Lemma 3.6; we call it co-Nullstellensatz in that it is the category-theoretic dual of Hilbert’s Nullstellensatz in algebraic geometry. The analogue of the Nullstellensatz itself is Lemma 3.10: the definable subsets  $R \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$  (i.e. those of the form  $\mathbb{I}(S)$ , for some  $S \subseteq [0, 1]^\mu$ ) are precisely the congruences on  $\mathcal{F}_\mu$  such that the quotient algebra  $\mathcal{F}_\mu/R$  is semisimple. (Here, as usual, semisimple algebras are subdirect products of simple algebras.) These results quickly lead to a characterisation of the unit and co-unit of the basic adjunction (Theorem 3.1),

to the effect (Corollary 3.2) that semisimple MV-algebras are dually equivalent to the full subcategory of  $\mathbf{T}_{\text{def } \mathbb{Z}}$  whose objects are closed subspaces of Tychonoff cubes. We then turn to  $\mathbf{MV}_{\text{fp}}$ , the full subcategory of  $\mathbf{MV}_{\text{p}}$  whose objects are finitely presented MV-algebras. A presented MV-algebra  $\mathcal{F}_{\mu}/\theta$  is *finitely presented* if  $\mu$  is a non-negative integer, and  $\theta$  is a finitely generated congruence. Recall [4, II.5.6] that a congruence  $\theta$  on an algebra  $A$  is *finitely generated* if it is the intersection of all congruences on  $A$  containing a finite set  $F \subseteq A \times A$ . Equivalently [4, II.5.7],  $\theta$  is a compact element in the algebraic lattice of congruences on  $A$ . The crucial fact is that finitely presented MV-algebras are semisimple, a result known as *Wójcicki's Theorem* [7, 3.6.9] in the MV-algebraic literature. Since finitely generated free MV-algebras are trivially finitely presentable, and simple MV-algebras are isomorphic to subalgebras of  $[0, 1]$  by the MV-algebraic analogue of Hölder's Theorem for Archimedean totally ordered groups (Lemma 3.8 below), we see that Wójcicki's Theorem entails *Chang's Completeness Theorem* [7, 2.5.3]: the variety of MV-algebras is generated by  $[0, 1]$ . In fact, all proofs we know of Wójcicki's Theorem use the completeness theorem. In our version, the latter is incorporated in Lemma 4.3. Using Wójcicki's Theorem, we obtain a duality between  $\mathbf{MV}_{\text{fp}}$  and the full subcategory of  $\mathbf{T}_{\text{def } \mathbb{Z}}$  whose objects are *finitely definable* subspaces of Tychonoff cubes, namely, sets of the form  $\mathbb{V}(R)$ ,  $R$  finite. Finally, we improve this duality by characterising in geometrical terms the abstract category of finitely definable sets. This yields the geometric duality between finitely presented MV-algebras and the category of *rational polyhedra* with  $\mathbb{Z}$ -maps as morphisms. (Please see Section 4 for definitions.) The result that affords this characterisation is *McNaughton's Theorem* [7, 9.1.5], which allows us to identify definable maps on rational polyhedra with piecewise linear maps having integer coefficients; see Lemma 4.9 for an exact statement. The duality theorem for finitely presented MV-algebras is stated as Corollary 4.12.

The duality theorem for finitely presented MV-algebras is a known result that is best described as folklore;<sup>1</sup> the approach presented here is original with this paper. We have taken pains to tell apart as far as possible results which rest on general considerations, and results with a genuinely MV-algebraic content. Naturally enough, the further we proceed from the abstract (the basic adjunction) to the concrete (the geometric duality for finitely presented objects), the more specific information about MV-algebras

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<sup>1</sup>A recent paper of ours [16] includes a proof of the duality theorem that is instrumental to the problem tackled there. That proof is optimised for brevity, and the present treatment has hardly any overlap with it.

is needed. Future work may explore the applicability of our present approach to general varieties of algebras, subject to appropriate conditions.

## 2. The basic adjunction.

Our aim in this section is to construct a pair of adjoint functors

$$\mathcal{J}: \mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}} \longrightarrow \text{MV}_{\mathbf{p}}, \quad \mathcal{V}: \text{MV}_{\mathbf{p}} \longrightarrow \mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}} .$$

**The functor  $\mathcal{J}$ : Objects.** Given  $S \subseteq [0, 1]^\mu$ , let us define a relation  $\mathbb{I}(S)$  on  $\mathcal{F}_\mu$  by stipulating that, for arbitrary terms  $s, t \in \mathcal{F}_\mu$ ,

$$(s, t) \in \mathbb{I}(S) \text{ if and only if } [0, 1] \models s(p) \approx t(p)$$

for every  $p \in S \subseteq [0, 1]^\mu$ . We call  $\mathbb{I}(S)$  the *vanishing congruence*<sup>2</sup> of  $S$ . When  $S = \{p\}$  is a singleton, we write  $\mathbb{I}(p)$  in place of  $\mathbb{I}(\{p\})$ . Of course, the defining condition for  $\mathbb{I}(S)$  is equivalent to

$$s((p_\alpha)_{\alpha < \mu}) = t((p_\alpha)_{\alpha < \mu}) ,$$

for any  $(p_\alpha)_{\alpha < \mu} \in S$ , where ‘=’ is equality between real numbers.

*Remark 2.1.* For any  $S \subseteq [0, 1]^\mu$ ,  $\mathbb{I}(S)$  is a congruence on  $\mathcal{F}_\mu$ . Indeed,  $\mathbb{I}(S)$  clearly is an equivalence relation, so we only need check that it is compatible with the operations. But that is obvious, too, because if  $s_1, t_1 \in \mathcal{F}_\mu$  and  $s_1(p) = t_1(p)$  for any  $p \in S$ , then  $\neg s_1(p) = 1 - s_1(p) = 1 - t_1(p) = \neg t_1(p)$ , and if, additionally,  $s_2, t_2 \in \mathcal{F}_\mu$  and  $s_2(p) = t_2(p)$ , then  $(s_1 \oplus s_2)(p) = \min \{s_1(p) + s_2(p), 1\} = \min \{t_1(p) + t_2(p), 1\} = (t_1 \oplus t_2)(p)$ .

In view of the preceding remark, for any subset  $S \subseteq [0, 1]^\mu$  we define

$$\mathcal{J}(S) = \mathcal{F}_\mu / \mathbb{I}(S) .$$

**The functor  $\mathcal{J}$ : Arrows.** Given  $S \subseteq [0, 1]^\mu$  and  $T \subseteq [0, 1]^\nu$ , let  $\lambda: S \rightarrow T$  be a definable map, and let  $(l_\beta)_{\beta < \nu}$  be a  $\nu$ -tuple of defining terms for  $\lambda$ . Then there is an induced function

$$\mathcal{J}(\lambda): \mathcal{J}(T) \rightarrow \mathcal{J}(S)$$

which acts on each  $s \in \mathcal{F}_\nu$  by substitution as follows:

$$\frac{s((X_\alpha)_{\beta < \nu})}{\mathbb{I}(T)} \in \mathcal{J}(T) \xrightarrow{\mathcal{J}(\lambda)} \frac{s([X_\beta \setminus l_\beta]_{\beta < \nu})}{\mathbb{I}(S)} \in \mathcal{J}(S) .$$

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<sup>2</sup>The terminology is due to the fact that congruences on MV-algebras are represented by ideals [7, 1.2].

*Remark 2.2.* 1. There can be several distinct defining terms for a definable function  $\lambda: S \rightarrow [0, 1]$ . However, let  $l \in \mathcal{F}_\mu$  be a defining term for  $\lambda$ . Let further  $l'$  be any element of  $\mathcal{F}_\mu$ . Then  $(l, l') \in \mathbb{I}(S)$  if, and only if,  $l'$  is a defining term for  $\lambda$ . Indeed, by definition we have  $(l, l') \in \mathbb{I}(S)$  if, and only if,  $l(p) = l'(p)$  holds for each  $p \in S$ . On the other hand,  $l'$  is a defining term for  $\lambda$  if, and only if,  $\lambda(p) = l'(p)$  holds for each  $p \in S$ . The stated equivalence then follows from the assumption that  $l$  defines  $\lambda$ , i.e.  $\lambda(p) = l(p)$  for each  $p \in S$ .

2. It is clear that the definition of  $\mathcal{S}(\lambda)$  above does not depend on the choice of the representing term  $s$ , for if  $s'$  is another term such that  $(s, s') \in \mathbb{I}(T)$ , then  $s([X_\beta \setminus l_\beta]_{\beta < \nu})$  is congruent to  $s'([X_\beta \setminus l_\beta]_{\beta < \nu})$  modulo  $\mathbb{I}(S)$ , because substitutions commute with congruences. Further, the definition of  $\mathcal{S}(\lambda)$  does not depend on the choice of the family of defining terms  $(l_\beta)_{\beta < \nu}$  either. Indeed, suppose  $(l'_\beta)_{\beta < \nu}$  is another  $\nu$ -tuple of defining terms for  $\lambda$ , and let  $p \in S$ . For each  $\beta < \nu$  we have  $(l_\beta, l'_\beta) \in \mathbb{I}(S)$  by 1 in this remark, so that  $(s((l_\beta)_{\beta < \nu}), s((l'_\beta)_{\beta < \nu})) \in \mathbb{I}(S)$  because congruences are compatible with operations. Thus we see that  $\mathcal{S}$  is well-defined.

LEMMA 2.3. *Let  $\lambda: S \rightarrow T$  be a definable map between subsets  $S \subseteq [0, 1]^\mu$  and  $T \subseteq [0, 1]^\nu$ . Then  $\mathcal{S}(\lambda): \mathcal{S}(T) \rightarrow \mathcal{S}(S)$  is a homomorphism of MV-algebras.*

PROOF. First of all note that, since each  $l_\beta \in \mathcal{F}_\mu$ , the equivalence class  $s([X_\beta \setminus l_\beta]) / \mathbb{I}(S)$  indeed belongs to  $\mathcal{S}(S)$ , as  $s([X_\beta \setminus l_\beta]) \in \mathcal{F}_\mu$ . Let now  $s((X_\beta)_{\beta < \nu}) / \mathbb{I}(T)$  and  $t((X_\beta)_{\beta < \nu}) / \mathbb{I}(T)$  be elements of  $\mathcal{S}(T)$ . The following computation shows that  $\mathcal{S}(\lambda)$  preserves  $\oplus$ .

$$\begin{aligned} \mathcal{S}(\lambda) \left( \frac{s}{\mathbb{I}(T)} \right) \oplus \mathcal{S}(\lambda) \left( \frac{t}{\mathbb{I}(T)} \right) &= \\ &= \frac{s([X_\beta \setminus l_\beta])}{\mathbb{I}(S)} \oplus \frac{t([X_\beta \setminus l_\beta])}{\mathbb{I}(S)} \\ &= \frac{s([X_\beta \setminus l_\beta]) \oplus t([X_\beta \setminus l_\beta])}{\mathbb{I}(S)} \\ &= \frac{(s \oplus t)([X_\beta \setminus l_\beta])}{\mathbb{I}(S)} \\ &= \mathcal{S}(\lambda) \left( \frac{s \oplus t}{\mathbb{I}(T)} \right). \end{aligned}$$

The arguments for  $\neg$  and  $0$  are similar. ■

It is immediately seen that  $\mathcal{S}$  preserves identity arrows. Next lemma shows that  $\mathcal{S}$  is in fact a functor.

LEMMA 2.4. *Let  $\lambda_1: S_1 \rightarrow S_2$  and  $\lambda_2: S_2 \rightarrow S_3$  be definable maps, where each  $S_i$  is a subset of  $[0, 1]^{\mu_i}$ , for some cardinal  $\mu_i$ ,  $i = 1, 2, 3$ . Then  $\mathcal{I}(\lambda_2 \circ \lambda_1) = \mathcal{I}(\lambda_1) \circ \mathcal{I}(\lambda_2)$ .*

PROOF. Let  $(l_\alpha)_{\alpha < \mu_2}$  and  $(m_\beta)_{\beta < \mu_3}$  be families of defining terms for  $\lambda_1$  and  $\lambda_2$ , respectively, and let  $s$  be an element of  $\mathcal{F}_{\mu_3}$ . Then:

$$\begin{aligned} (\mathcal{I}(\lambda_1) \circ \mathcal{I}(\lambda_2)) \left( \frac{s}{\mathbb{I}(S_3)} \right) &= \\ &= \mathcal{I}(\lambda_1) \left( \frac{s([X_\beta \setminus m_\beta]_{\beta < \mu_3})}{\mathbb{I}(S_2)} \right) \\ &= \frac{s([X_\beta \setminus (m_\beta[Y_\alpha \setminus l_\alpha]_{\alpha < \mu_2})]_{\beta < \mu_3})}{\mathbb{I}(S_1)} \\ &= (\mathcal{I}(\lambda_2 \circ \lambda_1)) \left( \frac{s}{\mathbb{I}(S_3)} \right) \end{aligned}$$

The last equality holds because of the fact, proved by direct inspection of the definitions, that if  $(l_\alpha)_{\alpha < \mu_2}$  and  $(m_\beta)_{\beta < \mu_3}$  are families of defining terms for  $\lambda_1$  and  $\lambda_2$  respectively, then  $(m_\beta[Y_\alpha \setminus l_\alpha]_{\alpha < \mu_2})_{\beta < \mu_3}$  is a family of defining terms for  $\lambda_2 \circ \lambda_1$ .  $\blacksquare$

**The functor  $\mathcal{V}$ : Objects.** Given  $R = \{(s_i, t_i) \mid i \in I\} \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$ , for  $I$  an index set, the *vanishing locus*<sup>3</sup> of  $R$  is

$$\mathbb{V}(R) = \{p \in [0, 1]^\mu \mid [0, 1] \models s_i(p) \approx t_i(p) \text{ for each } i \in I\}.$$

Once again, note that  $[0, 1] \models s_i(p) \approx t_i(p)$  means  $s_i(p) = t_i(p)$ . When  $R = \{(s, t)\}$  is a singleton, it will be convenient to write  $\mathbb{V}(s, t)$  as a shorthand for  $\mathbb{V}(\{(s, t)\})$ . By the very definition of  $\mathbb{V}$ , for any congruence  $\theta$  on  $\mathcal{F}_\mu$  we have  $\mathbb{V}(\theta) \subseteq [0, 1]^\mu$ . We therefore set

$$\mathcal{V}(\mathcal{F}_\mu / \theta) = \mathbb{V}(\theta).$$

**The functor  $\mathcal{V}$ : Arrows.** Let  $h: \mathcal{F}_\mu / \theta_1 \rightarrow \mathcal{F}_\nu / \theta_2$  be a homomorphism of MV-algebras. For each  $\alpha < \mu$ , let  $\pi_\alpha$  be the projection term on the  $\alpha^{\text{th}}$  coordinate, and let  $\pi_\alpha / \theta_1$  denote the equivalence class of  $\pi_\alpha$  modulo  $\theta_1$ . Fix, for each  $\alpha$ , an arbitrary  $f_\alpha \in h(\pi_\alpha / \theta_1)$ . For any  $(p_\beta)_{\beta < \nu} \in \mathbb{V}(\theta_2)$ , set

$$\mathcal{V}(h)((p_\beta)_{\beta < \nu}) = (f_\alpha((p_\beta)_{\beta < \nu}))_{\alpha < \mu}.$$

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<sup>3</sup>Cf. Footnote 2.



To see that  $\mathcal{V}(h)$  is well-defined, fix  $\alpha < \mu$ . By definition, if  $p$  is a point of  $\mathbb{V}(\theta_2)$ , and if  $g \in \mathcal{F}_\nu$  is such that  $(f_\alpha, g) \in \theta_2$ , then  $f_\alpha(p) = g(p)$ . Therefore, the definition of  $\mathcal{V}(h)$  does not depend on the choices of the  $f_\alpha$ 's.

LEMMA 2.5. *Given a homomorphism  $h: \mathcal{F}_\mu/\theta_1 \rightarrow \mathcal{F}_\nu/\theta_2$ , the function  $\mathcal{V}(h)$  is a definable map from  $\mathbb{V}(\theta_2)$  to  $\mathbb{V}(\theta_1)$ .*

PROOF. Since each  $f_\alpha$  is an element of  $\mathcal{F}_\nu$ , the function  $\mathcal{V}(h)$  is definable. We now show that the range of  $\mathcal{V}(h)$  is contained in  $\mathbb{V}(\theta_1)$ . To this end, note that if  $(s, t) \in \theta_1$  then

$$s((h(\pi_\alpha/\theta_1))_{\alpha < \mu}) = h(s((\pi_\alpha/\theta_1)_{\alpha < \mu})) = \quad (1)$$

$$h(s((\pi_\alpha/\theta_1)_{\alpha < \mu})) = h(t((\pi_\alpha/\theta_1)_{\alpha < \mu})) = \quad (2)$$

$$h(t((\pi_\alpha/\theta_1)_{\alpha < \mu})) = t((h(\pi_\alpha/\theta_1))_{\alpha < \mu}) . \quad (3)$$

Here, (1) and (3) hold because  $h$  is a homomorphism and thus commutes with terms, and (2) holds because  $(s, t) \in \theta_1$ . For each  $\alpha < \mu$ , choose an  $f_\alpha \in h(\pi_\alpha/\theta_1)$ . We claim that the element  $s((f_\alpha)_{\alpha < \mu})$  of  $\mathcal{F}_\nu$  belongs to  $s((h(\pi_\alpha/\theta_1))_{\alpha < \mu})$ , a congruence class induced on  $\mathcal{F}_\nu$  by  $\theta_2$ . Indeed, we have  $s((h(\pi_\alpha/\theta_1))_{\alpha < \mu}) = s((f_\alpha/\theta_2)_{\alpha < \mu})$  by our choice of the  $f_\alpha$ 's, and  $s((f_\alpha/\theta_2)_{\alpha < \mu}) = s((f_\alpha)_{\alpha < \mu})/\theta_2$  because terms commute with congruences. By the same token,  $t((f_\alpha)_{\alpha < \mu}) \in t((h(\pi_\alpha/\theta_1))_{\alpha < \mu})$ . By (1–3), then,

$$\left( s((f_\alpha)_{\alpha < \mu}), t((f_\alpha)_{\alpha < \mu}) \right) \in \theta_2 . \quad (4)$$

Now let  $(p_\beta)_{\beta < \nu} \in \mathbb{V}(\theta_2)$ . Then

$$\begin{aligned} s(\mathcal{V}(h)((p_\beta)_{\beta < \nu})) &= \\ s\left( (f_\alpha((p_\beta)_{\beta < \nu}))_{\alpha < \mu} \right) &= t\left( (f_\alpha((p_\beta)_{\beta < \nu}))_{\alpha < \mu} \right) \\ &= t(\mathcal{V}(h)((p_\beta)_{\beta < \nu})) , \end{aligned} \quad (5)$$

where (5) holds because of (4) together with the definition of vanishing locus. This shows that the range of  $\mathcal{V}(h)$  indeed is contained in  $\mathbb{V}(\theta_1)$ .  $\blacksquare$

As in the case of  $\mathcal{S}$ , it is readily seen that  $\mathcal{V}$  preserves identity arrows; the following lemma shows that  $\mathcal{V}$  preserves compositions.

LEMMA 2.6. *Let  $h: \mathcal{F}_\mu/\theta_1 \rightarrow \mathcal{F}_\nu/\theta_2$  and  $i: \mathcal{F}_\nu/\theta_2 \rightarrow \mathcal{F}_\xi/\theta_3$  be homomorphisms of MV-algebras. Then*

$$\mathcal{V}(i \circ h) = \mathcal{V}(h) \circ \mathcal{V}(i) .$$

PROOF. Fix, for each  $\alpha$ , an arbitrary  $f_\alpha \in h(\pi_\alpha/\theta_1)$  and, for each  $\beta$ , an arbitrary  $g_\beta \in i(\pi_\beta/\theta_2)$ . Then:

$$(\mathcal{V}(h) \circ \mathcal{V}(i)) = \mathcal{V}(h) \left( (g_\beta)_{\beta < \nu} \right) = \left( f_\alpha \left( (g_\beta)_{\beta < \nu} \right) \right)_{\alpha < \mu} = \mathcal{V}(i \circ h),$$

where the last equality holds because, for each  $\alpha < \mu$ ,

$$\left( f_\alpha \left( (g_\beta)_{\beta < \nu} \right) \right) \in (i \circ h) \left( \frac{\pi_\alpha}{\theta_1} \right).$$

Indeed

$$(i \circ h) \left( \frac{\pi_\alpha}{\theta_1} \right) = i \left( \frac{f_\alpha \left( (g_\beta)_{\beta < \nu} \right)}{\theta_2} \right) = \frac{f_\alpha \left( (i(\pi_\beta))_{\beta < \nu} \right)}{\theta_3} = \frac{f_\alpha \left( (g_\beta)_{\beta < \nu} \right)}{\theta_3}.$$

■

**The basic adjunction.** We shall use some easy facts about Galois connections. For background and further references the reader can consult the survey<sup>4</sup> [9]. Let  $P$  and  $Q$  be posets (partially ordered by  $\leq$ ). A pair of functions  $f: P \rightarrow Q$ ,  $g: Q \rightarrow P$  induce a *Galois connection* between  $P$  and  $Q$  if:

For every  $p \in P$  and  $q \in Q$  we have  $p \leq g(q)$  if, and only if,  $q \leq f(p)$ .

The operators  $\mathbb{V}$  and  $\mathbb{I}$  are functions between partially ordered sets, namely, the powersets of  $\mathcal{F}_\mu \times \mathcal{F}_\mu$  and  $[0, 1]^\mu$ .

LEMMA 2.7 (Basic Galois connection). *For each  $S \subseteq [0, 1]^\mu$  and  $R \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$ ,*

$$R \subseteq \mathbb{I}(S) \text{ if, and only if, } S \subseteq \mathbb{V}(R).$$

*In words, the functions  $\mathbb{V}$  and  $\mathbb{I}$  form a Galois connection. In particular, the following properties are entailed.*

1. For any  $S_1, S_2 \subseteq [0, 1]^\mu$ ,
  - a)  $S_1 \subseteq \mathbb{V}(\mathbb{I}(S_1))$ ,
  - b)  $S_1 \subseteq S_2$  implies  $\mathbb{I}(S_2) \subseteq \mathbb{I}(S_1)$ ,
  - c)  $\mathbb{I}(\mathbb{V}(\mathbb{I}(S_1))) = \mathbb{I}(S_1)$ , and

---

<sup>4</sup>Let us point out that [9] treats covariant, or isotone, Galois connections, whereas here it is more expedient to use the contravariant, or antitone, notion.

d)  $\mathbb{I}$  reverses arbitrary unions:  $\mathbb{I}(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} \mathbb{I}(S_i)$ , where  $S_i$  is a subset of  $[0, 1]^\mu$ , and  $I$  is an arbitrary index set.

2. For any  $R_1, R_2 \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$ ,

a)  $R_1 \subseteq \mathbb{I}(\mathbb{V}(R_1))$ ,

b)  $R_1 \subseteq R_2$  implies  $\mathbb{V}(R_2) \subseteq \mathbb{V}(R_1)$ ,

c)  $\mathbb{V}(\mathbb{I}(\mathbb{V}(R_1))) = \mathbb{V}(R_1)$ , and

d)  $\mathbb{V}$  reverses arbitrary unions:  $\mathbb{V}(\bigcup_{i \in I} R_i) = \bigcap_{i \in I} \mathbb{V}(R_i)$ , where  $R_i$  is a subset of  $\mathcal{F}_\mu \times \mathcal{F}_\mu$ , and  $I$  is an arbitrary index set.

PROOF. Assume  $R \subseteq \mathbb{I}(S)$  and let  $p \in S$ . If  $(s, t) \in R$  then  $s(p) = t(p)$  by the definition of  $\mathbb{I}$ , so that  $p \in \mathbb{V}(R)$ . Conversely, assume  $S \subseteq \mathbb{V}(R)$  and suppose  $(s, t) \in R$ . If  $p \in S$  then  $s(p) = t(p)$  by the definition of  $\mathbb{V}$ , so that  $(s, t) \subseteq \mathbb{I}(S)$ . The remaining assertions are standard facts about Galois connections; see [9] for further references.  $\blacksquare$

*Remark 2.8.* A function  $C: 2^A \rightarrow 2^A$ , where  $2^A$  is the powerset of a set  $A$ , is a *closure operator* on  $A$  [4, I.5.1] if it is *extensive* ( $X \subseteq C(X)$  for each  $X \in 2^A$ ), *isotone* ( $X \subseteq Y$  implies  $C(X) \subseteq C(Y)$  for each  $X, Y \in 2^A$ ), and *idempotent* ( $C(C(X)) = C(X)$  for each  $X \in 2^A$ ). The preceding lemma shows that the composition  $\mathbb{V} \circ \mathbb{I}$  is a closure operator on  $[0, 1]^\mu$ , and the composition  $\mathbb{I} \circ \mathbb{V}$  is a closure operator on  $\mathcal{F}_\mu \times \mathcal{F}_\mu$ .

Throughout the paper, we write  $1_O$  to denote the identity arrow on the object  $O$  of a category  $\mathcal{C}$ , and  $1_{\mathcal{C}}$  to denote the identity functor on  $\mathcal{C}$ . Further, we write composition as juxtaposition whenever convenient, e.g. we write  $\mathcal{V} \mathcal{I}$  in place of  $\mathcal{V} \circ \mathcal{I}$ .

**THEOREM 2.9** (The basic adjunction between MV-algebras and spaces). *The functor  $\mathcal{V}: \text{MV}_{\mathbf{p}} \rightarrow \mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}}$  is left adjoint to the functor  $\mathcal{I}: \mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}} \rightarrow \text{MV}_{\mathbf{p}}$ . In symbols,  $\mathcal{V} \dashv \mathcal{I}$ .*

PROOF. Let us start by exhibiting a co-unit, i.e. a natural transformation  $\varepsilon: \mathcal{V} \mathcal{I} \rightarrow 1_{\mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}}}$ . That is, for any arrow  $\lambda: S \rightarrow T$  in  $\mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}}$  we need to

exhibit components  $\varepsilon_S$  and  $\varepsilon_T$  such that the following diagram commutes.

$$\begin{array}{ccc}
 \mathcal{V} \mathcal{I} (S) & \xrightarrow{\mathcal{V} \mathcal{I} (\lambda)} & \mathcal{V} \mathcal{I} (T) \\
 \varepsilon_S \downarrow & & \downarrow \varepsilon_T \\
 S & \xrightarrow{\lambda} & T
 \end{array}$$

We have  $\mathcal{V} \mathcal{I} (S) = \mathcal{V}(\mathcal{F}_\mu / \mathbb{I}(S)) = \mathbb{V}(\mathbb{I}(S))$ . By 1a) in Lemma 2.7 there is an inclusion arrow  $S \hookrightarrow \mathbb{V}(\mathbb{I}(S))$ ; so for  $\varepsilon_S$  we take its dual arrow in  $\mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}}$ . Let also  $\varepsilon_T$  be defined analogously. If  $\lambda$  is defined by the family of terms  $(l_\alpha)_{\alpha < \mu}$ , direct application of the definitions gives that  $\mathcal{V} \mathcal{I} (\lambda) = (l_\alpha)_{\alpha < \mu}$ . Therefore,

$$\varepsilon_T \circ \mathcal{V} \mathcal{I} (\lambda) = \varepsilon_T \left( (l_\alpha)_{\alpha < \mu} \right) = (l_\alpha)_{\alpha < \mu} = \lambda = (\varepsilon_S \circ \lambda).$$

Now let us construct a natural transformation  $\eta: \mathbb{1}_{\text{MV}_p} \rightarrow \mathcal{I} \mathcal{V}$ , the unit of the adjointness. In other words, for any homomorphism  $h: \mathcal{F}_\mu / \theta_1 \rightarrow \mathcal{F}_\nu / \theta_2$ , we need to exhibit components  $\eta_{\frac{\mathcal{F}_\mu}{\theta_1}}$  and  $\eta_{\frac{\mathcal{F}_\nu}{\theta_2}}$  such that the following diagram commutes.

$$\begin{array}{ccc}
 \mathcal{I} \mathcal{V} \left( \frac{\mathcal{F}_\mu}{\theta_1} \right) & \xrightarrow{\mathcal{I} \mathcal{V} (h)} & \mathcal{I} \mathcal{V} \left( \frac{\mathcal{F}_\nu}{\theta_2} \right) \\
 \eta_{\frac{\mathcal{F}_\mu}{\theta_1}} \uparrow & & \uparrow \eta_{\frac{\mathcal{F}_\nu}{\theta_2}} \\
 \frac{\mathcal{F}_\mu}{\theta_1} & \xrightarrow{h} & \frac{\mathcal{F}_\nu}{\theta_2}
 \end{array}$$

Note that  $\mathcal{I} \mathcal{V}(\mathcal{F}_\mu / \theta_1) = \mathcal{I}(\mathbb{V}(\theta_1)) = \mathcal{F}_\mu / \mathbb{I}(\mathbb{V}(\theta_1))$ . Since, by 1c) in Lemma 2.7,  $\theta_1 \subseteq \mathbb{I}(\mathbb{V}(\theta_1))$ , there is a canonical homomorphism from  $\mathcal{F}_\mu / \theta_1$  to  $\mathcal{I} \mathcal{V}(\mathcal{F}_\mu / \theta_1)$ , which sends a generic element  $s / \theta_1$  of  $\mathcal{F}_\mu / \theta_1$  into  $s / \mathbb{I}(\mathbb{V}(\theta_1))$ . Let  $\eta_{\mathcal{F}_\mu / \theta_1}$  be this arrow. Similarly, let  $\eta_{\mathcal{F}_\nu / \theta_2}$  be the arrow which sends a generic element  $t / \theta_2$  of  $\mathcal{F}_\nu / \theta_2$  to  $t / \mathbb{I}(\mathbb{V}(\theta_2))$ . Next, notice that  $\mathcal{I} \mathcal{V}(h) = \mathcal{I}((f_\alpha)_{\alpha < \mu})$  for an arbitrary  $f_\alpha \in h(\pi_\alpha / \theta_1)$ . So  $\mathcal{I}((f_\alpha)_{\alpha < \mu})$  is the

function that sends a generic equivalence class  $s((X_\alpha)_{\alpha < \mu})/\mathbb{I}(\mathbb{V}(\theta_1))$  into  $s([X_\alpha \setminus f_\alpha]_{\alpha < \mu})/\mathbb{I}(\mathbb{V}(\theta_2))$ . Hence:

$$\begin{aligned}
 \left( \eta_{\frac{\mathcal{F}_\nu}{\theta_2}} \circ h \right) \left( \frac{s((X_\alpha)_{\alpha < \mu})}{\theta_1} \right) &= \\
 &= \eta_{\frac{\mathcal{F}_\nu}{\theta_2}} \left( \frac{h(s((X_\alpha)_{\alpha < \mu}))}{\theta_2} \right) = \left( \frac{h(s((X_\alpha)_{\alpha < \mu}))}{\mathbb{I}(\mathbb{V}(\theta_2))} \right) \\
 &= \frac{s([X_\alpha \setminus f_\alpha]_{\alpha < \mu})}{\mathbb{I}(\mathbb{V}(\theta_2))} = (\mathcal{I} \mathcal{V}(h)) \left( \frac{s((X_\alpha)_{\alpha < \mu})}{\mathbb{I}(\mathbb{V}(\theta_1))} \right) \\
 &= \left( \mathcal{I} \mathcal{V}(h) \circ \eta_{\frac{\mathcal{F}_\nu}{\theta_1}} \right) \left( \frac{s((X_\alpha)_{\alpha < \mu})}{\theta_1} \right) .
 \end{aligned}$$

Next, we need to show that for any  $A \in \text{MV}_p$  the diagram below commutes.

$$\begin{array}{ccccc}
 \mathcal{V}(A) & \xrightarrow{\mathcal{V}(\eta_A)} & \mathcal{V} \mathcal{I} \mathcal{V}(A) & \xrightarrow{\varepsilon_{\mathcal{V}(A)}} & \mathcal{V}(A) \\
 & \searrow & & \nearrow & \\
 & & \mathbb{1}_{\mathcal{V}(A)} & & 
 \end{array}$$

If  $A = \mathcal{F}_\mu/\theta$ , then  $\mathcal{V}(\mathcal{F}_\mu/\theta) = \mathbb{V}(\theta)$  and  $\mathcal{V} \mathcal{I} \mathcal{V}(\mathcal{F}_\mu/\theta) = \mathcal{V} \mathcal{I}(\mathbb{V}(\theta)) = \mathcal{V}(\mathcal{F}_\mu/\mathbb{I}(\mathbb{V}(\theta))) = \mathbb{V}(\mathbb{I}(\mathbb{V}(\theta)))$ . So, by 2c) in Lemma 2.7,  $\mathcal{V}(\mathcal{F}_\mu/\theta) = \mathcal{V} \mathcal{I} \mathcal{V}(\mathcal{F}_\mu/\theta)$ . By definition,  $\eta_A$  sends the projections  $(\pi_\alpha/\theta)_{\alpha < \mu}$  in  $A$  onto the projections  $(\pi_\alpha/\mathbb{I}(\mathbb{V}(\theta)))_{\alpha < \mu}$  in  $\mathcal{I} \mathcal{V}(A)$ . Hence

$$\mathcal{V}(\eta_A)((p_\alpha)_{\alpha < \mu}) = \left( (\pi_\alpha((p_\alpha)_{\alpha < \mu})) \right)_{\alpha < \mu} = (p_\alpha)_{\alpha < \mu} .$$

On the other hand,  $\varepsilon_{\mathcal{V}(A)}$  is the dual arrow of the embedding of  $\mathcal{V}(A)$  into  $\mathcal{V} \mathcal{I} \mathcal{V}(A)$ ; but  $\mathcal{V}(A) = \mathcal{V} \mathcal{I} \mathcal{V}(A)$ , so that  $\varepsilon_{\mathcal{V}(A)}$  is also the identity, as was to be shown.

Finally, for any  $K \in \mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}}$ , the diagram below commutes.

$$\begin{array}{ccccc}
 \mathcal{I}(K) & \xrightarrow{\mathcal{I}(\varepsilon_K)} & \mathcal{I} \mathcal{V} \mathcal{I}(K) & \xrightarrow{\eta_{\mathcal{I}(K)}} & \mathcal{I}(K) \\
 & \searrow & & \nearrow & \\
 & & \mathbb{1}_{\mathcal{I}(K)} & & 
 \end{array}$$

Indeed, the map  $\varepsilon_K$  is the dual of the embedding of  $K$  into  $\mathcal{V} \mathcal{I}(K)$ , so that the projection maps are defining terms for  $\varepsilon_K$ . Thus if  $s((X_\alpha)_{\alpha < \mu})/\mathbb{I}(K) \in$

$\mathcal{I}(K)$  then

$$\begin{aligned} \mathcal{I}(\varepsilon_K)(s(X_\alpha)_{\alpha < \mu}) / \mathbb{I}(K) &= s((X_\alpha \setminus X_\alpha)_{\alpha < \mu}) / \mathbb{I}(\mathbb{V}(\mathbb{I}(K))) \\ &= s((X_\alpha)_{\alpha < \mu}) / \mathbb{I}(K) . \end{aligned}$$

On the other hand, the arrow  $\eta_{\mathcal{I}(K)}$  is the dual of the inclusion of  $\mathcal{I}(K)$  in  $\mathcal{I} \mathcal{V} \mathcal{I}(K)$ . But  $\mathcal{I}(K) = \mathbb{I}(\theta)$  and  $\mathcal{I} \mathcal{V} \mathcal{I}(K) = \mathbb{I}(\mathbb{V}(\mathbb{I}(\theta)))$ , which are equal by 2a) in Lemma 2.7. Hence,  $\eta_{\mathcal{I}(K)}$  is the identity, and the theorem is proved.  $\blacksquare$

### 3. Semisimple algebras.

A congruence on an MV-algebra  $A$  is *maximal* if it is proper (i.e.  $\neq A \times A$ ), and it is not properly contained in any proper congruence. An MV-algebra is *simple* if it has no proper congruences but the identity congruence, and it is *semisimple*<sup>5</sup> if it is a subdirect product of simple MV-algebras. Equivalently, an MV-algebra  $A$  is semisimple if its *radical congruence*  $\text{Rad}(A)$  — the intersection of all maximal congruences on  $A$  — is the identity relation [7, p. 72]. All of these are instances of universal-algebraic notions, see [4, p. 18, II.8.8 and VI.12.1].

In all matters topological we follow Engelking's treatise [8]. From now on, each Cartesian product  $[0, 1]^\mu$  will be endowed with its *Tychonoff*, or *product*, topology, where on  $[0, 1]$  we assume the usual Euclidean topology. Recall [8, 2.3] that the product topology is the coarsest topology on  $[0, 1]^\mu$  that makes all projection functions  $\pi_\alpha: [0, 1]^\mu \rightarrow [0, 1]$  continuous. As a final piece of notation, let us write  $\overline{S}$ , for  $S \subseteq [0, 1]^\mu$ , to denote the closure of  $S$  in  $[0, 1]^\mu$ , i.e. the intersection of all closed subsets of  $[0, 1]^\mu$  containing  $S$ .

Our aim in this section is to establish the following explicit description of the co-unit and unit of the adjunction in Theorem 2.9.

**THEOREM 3.1** (Co-unit & Unit as Closure & Radical).

1. The co-unit  $\varepsilon: \mathcal{V} \mathcal{I} \rightarrow \mathbb{1}_{\mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}}}$  of Theorem 2.9 acts as the closure operator associated to the Tychonoff topology of  $[0, 1]^\mu$ . That is, for any  $S \subseteq [0, 1]^\mu$ , the component  $\varepsilon_S: \mathcal{V} \mathcal{I}(S) \rightarrow S$  is the dual in  $\mathbb{T}_{\text{def } \mathbb{Z}}^{\text{op}}$  of the inclusion arrow  $S \hookrightarrow \overline{S}$  in  $\mathbb{T}_{\text{def } \mathbb{Z}}$  that embeds  $S$  in its closure. Hence,  $\varepsilon_S$  is an isomorphism if, and only, if  $S$  is closed.

---

<sup>5</sup>Note that in [7, p. 70] the trivial, one-element MV-algebra does not count as a simple algebra, and consequently neither does it count as a semisimple algebra.

2. The unit  $\eta: \mathbb{1}_{\mathbf{MV}_p} \rightarrow \mathcal{I}\mathcal{V}$  of Theorem 2.9 acts by modding out radicals. That is, for any congruence  $\theta$  on  $\mathcal{F}_\mu$ , the component  $\eta_{\mathcal{F}_\mu/\theta}: \mathcal{F}_\mu/\theta \rightarrow \mathcal{I}\mathcal{V}(\mathcal{F}_\mu/\theta)$  is the natural quotient map  $\mathcal{F}_\mu/\theta \twoheadrightarrow (\mathcal{F}_\mu/\theta)/\text{Rad}(\mathcal{F}_\mu/\theta)$  induced by the congruence  $\text{Rad}(\mathcal{F}_\mu/\theta)$ . Hence,  $\eta_{\mathcal{F}_\mu/\theta}$  is an isomorphism if, and only if,  $\mathcal{F}_\mu/\theta$  is semisimple.

Let  $\mathbf{MV}_p^{\text{ss}}$  be the full subcategory of  $\mathbf{MV}_p$  whose objects are (presented) semisimple MV-algebras. Further, let  $\mathbf{K}_{\text{def } \mathbb{Z}}^{\text{op}}$  be the full subcategory of  $\mathbf{T}_{\text{def } \mathbb{Z}}^{\text{op}}$  whose objects are closed subsets of Tychonoff cubes.<sup>6</sup> Theorem 3.1 entails at once a duality:

**COROLLARY 3.2** (Duality theorem for semisimple MV-algebras). *The adjunction  $\mathcal{V} \dashv \mathcal{I}$  in Theorem 2.9 restricts to an equivalence of categories between  $\mathbf{MV}_p^{\text{ss}}$  and  $\mathbf{K}_{\text{def } \mathbb{Z}}^{\text{op}}$ .*

We now turn to the proof of Theorem 3.1.

**The co-Nullstellensatz.** We begin summarising our terminology about separation axioms; see [8, 1.5].

A topological space  $X$  is  $T_1$  if its points are closed:  $\{x\}$  is closed for every  $x \in X$ . Further,  $X$  is *Hausdorff* (or  $T_2$ ) if any two distinct points are contained in disjoint open sets, and it is *regular* (or  $T_3$ ) if it is  $T_1$ , and points and closed sets can be separated by open sets: given  $x \in X$  and  $Y \subseteq X$ , if  $x \notin Y$  and  $Y$  is closed, then there are disjoint open sets  $O_1$  and  $O_2$  with  $x \in O_1$  and  $Y \subseteq O_2$ . Moreover,  $X$  is *completely regular*, or *Tychonoff* (or  $T_{3\frac{1}{2}}$ ) if it is  $T_1$ , and points and closed sets can be separated by continuous  $[0, 1]$ -valued functions: if  $x \in X$ ,  $Y \subseteq X$  is closed, and  $x \notin Y$ , there is a continuous function  $f: X \rightarrow [0, 1]$  such that  $Y \subseteq f^{-1}(0)$ , and<sup>7</sup>  $f(x) > 0$ . We now have the implications  $T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$ , none of which can be reversed. It is a standard fact that  $[0, 1]^\mu$  is a Tychonoff space. We are going to prove in Lemma 3.5 below the stronger result that points and closed sets in  $[0, 1]^\mu$  can be separated by *definable* functions.

<sup>6</sup>Each such space is compact and Hausdorff [8, 3.1.2, 2.1.6]; conversely, each compact Hausdorff space can be embedded in some Tychonoff cube [8, 2.3.23]. It should be carefully noted, however, that the notion of definable map between compact Hausdorff spaces  $H$  and  $K$  only makes sense if  $H$  and  $K$  come endowed with a *specific* embedding into  $[0, 1]^\mu$  and  $[0, 1]^\nu$ , respectively. In other words, an object of  $\mathbf{K}_{\text{def } \mathbb{Z}}^{\text{op}}$  cannot be conceived of as an abstract compact Hausdorff space  $K$ , but is rather a continuous embedding  $K \hookrightarrow [0, 1]^\mu$ .

<sup>7</sup>As in [8], it is common to require  $f(x) = 1$  in the definition of complete regularity; the difference is promptly seen to be immaterial.

*Remark 3.3.* Since the operations of the standard MV-algebra  $[0, 1]$  are continuous with respect to the Euclidean topology, it is clear that *any definable map*  $\lambda: S \rightarrow T$  is continuous, where  $S$  and  $T$  are endowed with the subspace topology they inherit from the Tychonoff topology of  $[0, 1]^\mu$  and  $[0, 1]^\nu$ , respectively. Thus, if we regard each object of  $\mathbb{T}_{\text{def } \mathbb{Z}}$  as a topological space with the subspace topology from  $[0, 1]^\mu$ ,  $\mathbb{T}_{\text{def } \mathbb{Z}}$  is a subcategory of the category of Tychonoff spaces and continuous maps.

We shall use a result from [1]. For terms  $s$  and  $t$ , set  $s \odot t = \neg(\neg s \oplus \neg t)$ . Let us write  $nt$  as a shorthand for  $t \oplus \dots \oplus t$  ( $n$  times), and  $t^n$  as a shorthand for  $t \odot \dots \odot t$  ( $n$  times). The term  $s$  is a *basic literal* (in the variable  $X_\alpha$ ) if:

- $s = X_\alpha$ , or
- there is an integer  $n > 0$  such that either  $s = nt$  or  $s = t^n$  for a basic literal  $t$  in the variable  $X_\alpha$ .

Given integers  $n_1 \geq 1$ , and  $n_2, \dots, n_u > 1$ , we write  $(n_1, n_2, \dots, n_u)X_\alpha$  to denote the basic literal  $(\dots((n_i \dots((n_1 X_\alpha)^{n_2} \dots))^{n_{i+1}}) \dots)$ .

LEMMA 3.4. *For each open interval  $(a, b) \subseteq [0, 1]$  and each  $p \in (a, b)$ , there are basic literals  $L = (a_1, \dots, a_u)X_1$  and  $R = (b_1, \dots, b_w)X_1$  such that the function  $\lambda: [0, 1] \rightarrow [0, 1]$  defined by the term  $L \wedge \neg R$  satisfies  $\lambda(p) > 0$  and  $([0, 1] \setminus (a, b)) \subseteq \lambda^{-1}(0)$ .*

PROOF. By [1, Corollary 2.8] there is  $L$  as in the statement such that, if we write  $\lambda_L$  for the function defined by  $L$ ,  $\lambda_L^{-1}(0)$  is an interval  $[0, a']$  with  $a \leq a' < p$ , and  $\lambda_L$  is monotone increasing. By the same result, there is  $R$  as in the statement such that  $\lambda_R^{-1}(1)$  is an interval  $[b', 1]$  with  $p < b' < b$ , and  $\lambda_R$  is monotone increasing. (In the terminology of [1],  $\lambda_L$  *stems* from  $a'$ , and  $\lambda_R$  *culminates* at  $b'$ .) Then  $L \wedge \neg R$  obviously has the desired properties. ■

LEMMA 3.5 (Complete regularity by definable functions). *For any point  $p \in [0, 1]^\mu$  and any closed set  $K \subseteq [0, 1]^\mu$  with  $p \notin K$ , there is a definable function  $\lambda: [0, 1]^\mu \rightarrow [0, 1]$  that takes value 0 over  $K$ , and value  $> 0$  at  $p$ .*

PROOF. Since the space  $[0, 1]^\mu$  is regular [8, 2.3.11], there is an open set  $O$  containing  $p$  such that  $O \cap K = \emptyset$ . Then  $O$  contains a basic open set containing  $p$ , for any fixed base for the topology. By [8, 2.3.1], the family of all sets of the form  $\prod_{\alpha < \mu} W_\alpha$  such that  $W_\alpha \subseteq [0, 1]$  is an open interval, and  $W_\alpha \neq [0, 1]$  for finitely many indices only, is a base for the Tychonoff topology on  $[0, 1]^\mu$ . Therefore, if we write  $\pi_\alpha: [0, 1]^\mu \rightarrow [0, 1]$  as usual for the projection map, there are finitely many ordinals  $\alpha_1, \dots, \alpha_n < \mu$  and open intervals  $(a_1, b_1), \dots, (a_n, b_n) \subseteq [0, 1]$  such that, setting  $U = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}((a_i, b_i))$ ,



we have  $p \in U \subseteq O$ . By Lemma 3.4, let  $\lambda_i: [0, 1]^\mu \rightarrow [0, 1]$  be a definable function whose defining term is built from  $X_{\alpha_i}$  only, that vanishes off  $\pi_{\alpha_i}^{-1}((a_i, b_i))$ , and that is non-zero at  $p$ , for each  $i = 1, \dots, n$ . Then the function  $\lambda: [0, 1]^\mu \rightarrow [0, 1]$  that is the pointwise minimum of  $\lambda_1, \dots, \lambda_n$  is trivially definable (using  $\wedge$ ). Observe that, by construction,  $\lambda$  vanishes off each  $\pi_{\alpha_i}^{-1}((a_i, b_i))$ ,  $i = 1, \dots, n$  — hence it vanishes off  $U$ . Since  $K$  misses  $U$  entirely,  $\lambda$  vanishes on  $K$ . Since, moreover,  $\lambda(p) > 0$  by construction, the lemma is proved. ■

LEMMA 3.6 (Co-Nullstellensatz for MV-algebras).

1. For any  $S \subseteq [0, 1]^\mu$ ,  $\mathbb{V}(\mathbb{I}(S)) = \overline{S}$ .
2. The set  $S \subseteq [0, 1]^\mu$  is closed if, and only if,  $\mathbb{V}(\mathbb{I}(S)) = S$ .

PROOF. 1. If  $X$  is any space, and  $Y$  is Hausdorff, then for any two continuous functions  $f, g: X \rightarrow Y$  the solution set of the equation  $f = g$  is a closed subset of  $X$ , [8, 1.5.4]. Now  $[0, 1]$  is Hausdorff; definable functions are continuous (Remark 3.3); and  $\mathbb{V}(R) = \bigcap_{(s,t) \in R} \mathbb{V}(s, t)$  holds by definition. We conclude that  $\mathbb{V}(R)$  is closed for any subset  $R$  of  $\mathcal{F}_\mu \times \mathcal{F}_\mu$ . Thus the inclusion  $\overline{S} \subseteq \mathbb{V}(\mathbb{I}(S))$  follows from the fact that  $S \subseteq \mathbb{V}(\mathbb{I}(S))$  always holds by 1a) in Lemma 2.7. For the converse inclusion, suppose  $p \in (\mathbb{V}(\mathbb{I}(S)) \setminus \overline{S})$ . Then  $\{p\} \cap \overline{S} = \emptyset$ . By Lemma 3.5, there is a definable function  $\lambda: [0, 1]^\mu \rightarrow [0, 1]$  such that  $\lambda$  vanishes on  $\overline{S}$ , and satisfies  $\lambda(p) > 0$ . If  $s$  is a defining term for  $\lambda$  then the pair  $(s, 0)$  belongs to  $\mathbb{I}(\overline{S})$ , and therefore to  $\mathbb{I}(S)$ . But then, since  $s(p) > 0$ , we have  $p \notin \mathbb{V}(\mathbb{I}(S))$  — a contradiction.

2. Immediate consequence of the previous item in this lemma. ■

*Remark 3.7.* Recall Remark 2.8 and the notation therein. The closure operator  $C$  is *topological* [4, p. 21] if  $C(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} C(X_i)$ , for each *finite* index set  $I$ , and  $X_i \in 2^A$ . The co-Nullstellensatz implies in particular that  $\mathbb{V} \circ \mathbb{I}$  is a topological closure operator, because topological closure is.

**The Nullstellensatz.** The following crucial fact goes back to [11], the founding paper of the theory of ordered groups.<sup>8</sup>

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<sup>8</sup>Lemma 3.8 is [7, 3.5.1] (although it is not called Hölder's Theorem there, and the uniqueness assertion is not stressed). The argument, however, rests on results that amount to our Lemma 3.9 below, in which we will need to apply Hölder's Theorem. We therefore sketch of a proof that only requires the construction of Chang's enveloping group for totally ordered MV-algebras, and Hölder's Theorem for ordered groups.

LEMMA 3.8 (Hölder's Theorem for MV-algebras). *Let  $C$  be a non-trivial, simple MV-algebra. Then there is a unique injective homomorphism  $C \rightarrow [0, 1]$ .*

PROOF. If  $C$  has two incomparable elements  $x, y$ , and  $\sigma: C \hookrightarrow \prod_{i \in I} C_i$  is a subdirect embedding of  $C$ , with each  $C_i$  a totally ordered MV-algebra by [7, 1.3.3], then we must have  $\sigma(x) > \sigma(y)$  for some  $i \in I$  and  $\sigma(y) > \sigma(x)$  for some  $i \neq j \in I$ , so that  $C$  is not subdirectly irreducible. This contradicts the simplicity of  $C$  [4, II.8.4]. Hence  $C$  is totally ordered. By [6, Lemma 5], there is a totally ordered Abelian group<sup>9</sup>  $G$  with a strong order unit  $u$  such that  $C$  is isomorphic to the MV-algebra  $[0, u] = \{g \in G \mid 0 \leq g \leq u\}$ , where the operations are defined as  $g \oplus h = \min\{g + h, 1\}$  and  $\neg g = u - g$ , for each  $g, h \in G$ . This is *Chang's enveloping group* of  $C$ . It is unique to within unit-preserving, order-preserving isomorphisms of ordered groups with a strong unit, by [6, Lemma 6]. Since  $C$  is simple, it is not hard to show that  $G$  is Archimedean (cf. Chang's remarks in [6, bottom of p. 78]), meaning that for every  $g, h \in G$ , if  $pg \leq h$  holds for every integer  $p \geq 0$ , then  $g \leq 0$ . Now Hölder's Theorem for Archimedean totally ordered groups [2, 2.6.3] yields an injective order-preserving homomorphism of groups  $e: G \hookrightarrow \mathbb{R}$ ; direct inspection of the cited proof shows that the latter injection is unique to within the choice of  $e(u)$ , which can be an arbitrary positive real. Let us set  $e(u) = 1$ . Then the restriction of  $e$  to  $[0, u]$  is an MV-algebraic embedding into the standard algebra  $[0, 1] \subseteq \mathbb{R}$ . This embedding is unique because of the uniqueness property of Chang's enveloping group. ■

LEMMA 3.9 (Point=Maximal congruence). *For any set  $S \subseteq [0, 1]^\mu$ , and for any congruence  $\theta$  on  $\mathcal{F}_\mu$ , the following hold.*

1. *If  $\theta$  is a maximal congruence then  $\mathbb{V}(\theta)$  is a singleton.*
2. *If  $S$  is a singleton then  $\mathbb{I}(S)$  is a maximal congruence.*

PROOF. 1. Set  $C = \mathcal{F}_\mu / \theta$ , and let  $q: \mathcal{F}_\mu \rightarrow C$  be the natural quotient map. Since  $\theta$  is maximal,  $C$  is simple by [7, 1.2.10] (more generally, by [4, II.8.9]). By Lemma 3.8, there is an injective homomorphism  $h: C \rightarrow [0, 1]$ . Since  $h$  is injective, it has trivial kernel; since the kernel of  $q$  is  $\theta$  by assumption, we conclude that the composition  $e = h \circ q$  also has kernel  $\theta$ , i.e.  $e(s) = e(t)$  for each  $(s, t) \in \theta$ . Since  $e$  commutes with terms, we have  $s((e(X_\alpha))_{\alpha < \mu}) = t((e(X_\alpha))_{\alpha < \mu})$ . Letting  $p = (e(X_\alpha)_{\alpha < \mu}) \in [0, 1]^\mu$ , this means that  $p \in \mathbb{V}(\theta)$ , i.e.  $\mathbb{V}(\theta)$  is non-empty. Suppose now that  $p \neq q \in \mathbb{V}(\theta)$ , and consider

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<sup>9</sup>For all undefined notions on ordered groups, see [2].

$\mathbb{I}(q)$ . If  $(s, t) \in \theta$  then  $s(q) = t(q)$  because  $q \in \mathbb{V}(\theta)$ , so that  $(s, t) \in \mathbb{I}(q)$ . Hence  $\theta \subseteq \mathbb{I}(q)$ . Since  $[0, 1]^\mu$  is Hausdorff, points are closed. By Lemma 3.5, therefore, there is  $s \in \mathcal{F}_\mu$  such that  $s(q) = 0$  and  $s(p) > 0$ . This shows that  $(s, 0) \in \mathbb{I}(q)$  but  $(s, 0) \notin \theta$ , because  $p \in \mathbb{V}(\theta)$ . Hence  $\theta \subset \mathbb{I}(q)$ . But note that  $\mathbb{I}(q)$  is proper — the top element 1 of  $\mathcal{F}_\mu$  does not vanish at  $q$ , whereas its bottom element 0 does, whence  $(1, 0) \notin \mathbb{I}(q)$ . Hence  $\theta \subset \mathbb{I}(q)$  contradicts the maximality of  $\theta$ , and therefore  $\mathbb{V}(\theta) = \{p\}$ .

2. We first show that if  $\theta$  is any maximal congruence on  $\mathcal{F}_\mu$ , then there is  $q \in [0, 1]^\mu$  such that  $\theta = \mathbb{I}(q)$ . By 2a) in Lemma 2.7,  $\theta \subseteq \mathbb{I}(\mathbb{V}(\theta))$ . By 1 in this lemma,  $\mathbb{V}(\theta) = \{q\}$  for some  $q \in [0, 1]^\mu$ , so  $\theta \subseteq \mathbb{I}(q)$ . Since  $\theta$  is maximal,  $\mathbb{I}(q)$  is either coincident with  $\theta$ , or else with  $\mathcal{F}_\mu \times \mathcal{F}_\mu$ . To see that the latter is not the case, observe that  $\mathbb{I}(q)$  is proper because  $(1, 0) \notin \mathbb{I}(q)$ . Hence  $\theta = \mathbb{I}(q)$ . Now say  $S = \{p\}$ ,  $p \in [0, 1]^\mu$ . Then  $\mathbb{I}(p)$  is proper, again because  $(1, 0) \notin \mathbb{I}(p)$ . Let  $\theta$  be a congruence on  $\mathcal{F}_\mu$  with  $\mathbb{I}(p) \subseteq \theta$ . By [7, 1.2.12 and 1.2.14], we may safely assume that  $\theta$  is maximal. By what we have just shown, there is  $q \in [0, 1]^\mu$  such that  $\theta = \mathbb{I}(q)$ . We show that  $q = p$ . For suppose not. Then, since  $[0, 1]^\mu$  is Hausdorff, so that its points are closed, by Lemma 3.5 there is  $s \in \mathcal{F}_\mu$  such that  $s(p) = 0$ , and  $s(q) > 0$ . Therefore,  $(s, 0) \in \mathbb{I}(p)$ , but  $(s, 0) \notin \mathbb{I}(q)$ . This contradicts the inclusion  $\mathbb{I}(p) \subseteq \mathbb{I}(q)$ , and thus shows that  $q = p$ . ■

For an MV-algebra  $A$ , let us write  $\text{MaxSpec}(A)$  to denote the collection of maximal congruences on  $A$ , the *maximal spectrum* of  $A$ . Further, given  $S \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$ , let us write  $\langle S \rangle$  for the congruence on  $\mathcal{F}_\mu$  generated by  $S$ , i.e. the intersection of all congruences containing  $S$ . When, in particular,  $S = \{(s, t)\}$  consists of a single pair, we write  $\langle (s, t) \rangle$  in place of  $\langle \{(s, t)\} \rangle$ .

LEMMA 3.10 (Nullstellensatz for MV-algebras).

1. For any  $S \subseteq \mathcal{F}_\mu \times \mathcal{F}_\mu$ ,  $\mathbb{I}(\mathbb{V}(S)) = \bigcap \{\theta' \in \text{MaxSpec}(\mathcal{F}_\mu) \mid S \subseteq \theta'\}$   
 $= \text{Rad}(\mathcal{F}_\mu / \langle S \rangle)$ .
2. The algebra  $\mathcal{F}_\mu / \theta$  is semisimple if, and only if,  $\mathbb{I}(\mathbb{V}(\theta)) = \theta$ .

PROOF. 1. Notice that we have

$$\mathbb{I}(\mathbb{V}(S)) = \mathbb{I}\left(\bigcup_{p \in \mathbb{V}(S)} \{p\}\right) = \bigcap_{p \in \mathbb{V}(S)} \mathbb{I}(p),$$

where the last equality is given by 1d) of Lemma 2.7. It thus suffices to show that  $\{\mathbb{I}(p) \mid p \in \mathbb{V}(S)\}$  is the set of all maximal congruences on  $\mathcal{F}_\mu$  containing  $S$ . Indeed, each  $\mathbb{I}(p)$  is a maximal congruence by 2 in Lemma

3.9, and  $\mathbb{I}(p)$  extends  $S$  by 1b) in Lemma 2.7. Vice versa, if  $\theta$  is a maximal congruence, then by 1) in Lemma 3.9 we have  $\mathbb{V}(\theta) = \{q\}$  for some  $q \in [0, 1]^\mu$ . Next, we have  $\theta \subseteq \mathbb{I}(q)$  by 2a) in Lemma 2.7, and therefore  $\theta = \mathbb{I}(q)$  by the maximality of  $\theta$  and  $\mathbb{I}(q)$ . Finally, if  $\theta$  extends  $S$ , then  $\mathbb{V}(\theta) \subseteq \mathbb{V}(S)$  by 2b) in Lemma 2.7, so that  $q \in \mathbb{V}(S)$ .

The last equality in 1) follows from the easily proved universal-algebraic fact that the sublattice of those congruences on  $\mathcal{F}_\mu$  that extend  $\theta$  is isomorphic to the lattice of congruences on  $\mathcal{F}_\mu/\theta$ , [4, II.6.20].

2. Suppose  $\mathbb{I}(\mathbb{V}(\theta)) = \theta$ . Then  $\theta$  is an intersection of maximal congruences, because  $\mathbb{I}(\mathbb{V}(\theta))$  is by the previous item in this lemma. Conversely, suppose  $\theta$  is such an intersection. By 2a) in Lemma 2.7,  $\theta \subseteq \mathbb{I}(\mathbb{V}(\theta))$  always holds. Since, by the previous item in this lemma,  $\mathbb{I}(\mathbb{V}(\theta))$  is the intersection of *all* maximal congruences extending  $\theta$ , we have  $\mathbb{I}(\mathbb{V}(\theta)) \subseteq \theta$ , and the proof is complete. ■

*Remark 3.11.* Recall Remarks 2.8 & 3.7, and the notation therein. The closure operator  $C$  is *algebraic* [4, p. 22] if  $C(X) = \bigcup_{Y \subseteq X} C(Y)$ , where  $X \in 2^A$ , and the union is restricted to *finite* subsets  $Y$ . The closure operator  $\mathbb{I} \circ \mathbb{V}$  is *not* algebraic. Let  $R = \{(s, t) \in \mathcal{F}_1 \times \mathcal{F}_1 \mid s(p) = t(p) \text{ for all } p \in [0, 1] \text{ in an open neighbourhood of } 0\}$ . Then  $\langle R \rangle = R$ . We have  $(X_0, 0) \notin R$ , but  $(X_0, 0) \in \mathbb{I}(\mathbb{V}(R))$ , because  $\mathbb{V}(R) = \{0\}$ . Using these observations, one shows that  $\mathbb{I}(\mathbb{V}(R))$  is not the union of the  $\mathbb{I} \circ \mathbb{V}$ -closure of its finite subsets. The quotient  $\mathcal{F}_1 / \mathbb{I}(\mathbb{V}(R))$  is the two-element Boolean algebra, which is trivially semisimple; the quotient  $\mathcal{F}_1 / R$  is often called *Chang's algebra*, and it is totally ordered, but not semisimple. The former algebra is obtained from the latter by modding out its radical congruence.

**End of Proof of Theorem 3.1.** The theorem is now an immediate consequence of Theorem 2.9, the definitions of  $\mathcal{V}$  and  $\mathcal{S}$  in terms of  $\mathbb{V}$  and  $\mathbb{I}$ , respectively, and Lemmas 3.6 and 3.10. ■

#### 4. Finitely presented algebras.

For the rest of this paper, we let  $m$  be a non-negative integer. The aim of this section is show that the adjointness given by the pair  $\mathcal{S}, \mathcal{V}$  restricts to an equivalence for finitely presented algebras. To this end we introduce the full subcategory of  $\mathbb{T}_{\text{def } \mathbb{Z}}$  which is the  $\mathcal{V}$ -image of finitely presented algebras.

**DEFINITION 4.1.** A subset  $S \subseteq [0, 1]^\mu$  is called *finitely definable* if there is a finite index set  $I$ , along with a set of pairs  $R = \{(s_i, t_i) \in \mathcal{F}_\mu \times \mathcal{F}_\mu \mid i \in I\}$ ,

such that  $S = \mathbb{V}(R)$ . The full subcategory of  $\mathbb{T}_{\text{def } \mathbb{Z}}$  whose objects are finitely definable subsets of  $[0, 1]^m$ , as  $m$  ranges over all non-negative integers, is denoted  $\mathbb{D}_{\text{def } \mathbb{Z}}$ .

*Remark 4.2.* It is an exercise [7, (1.8–1.9)] to show that, given any  $R = \{(s_i, t_i) \in \mathcal{F}_\mu \times \mathcal{F}_\mu \mid i \in I\}$ ,  $I$  finite, there is a term  $s \in \mathcal{F}_\mu$  such that  $\mathbb{V}(R) = \mathbb{V}(s, 0)$ . In particular, finitely generated and principal (=singly generated) congruences coincide.

We will see in Lemma 4.4 that finitely definable sets coincide with the vanishing loci of compact congruences. The proof requires non-trivial results from the theory of MV-algebras, beginning with the next lemma.

**LEMMA 4.3.** *Let  $s, t, u, v$  be elements of  $\mathcal{F}_m$ , then  $(u, v) \in \langle\langle s, t \rangle\rangle$  if, and only if,  $\mathbb{V}(s, t) \subseteq \mathbb{V}(u, v)$ .*

**PROOF.** This is [7, 3.4.8]. The proof involves a geometric argument, Chang's Completeness Theorem, and the easily proved fact that definable functions are piecewise linear maps (the easy implication in Lemma 4.9 below). ■

**LEMMA 4.4.** *Let  $s, t \in \mathcal{F}_m$  then  $\mathbb{V}(s, t) = \mathbb{V}(\langle\langle s, t \rangle\rangle)$ .*

**PROOF.** Since  $\{(s, t)\} \subseteq \langle\langle s, t \rangle\rangle$ , by 2b) in Lemma 2.7 we have  $\mathbb{V}(\langle\langle s, t \rangle\rangle) \subseteq \mathbb{V}(s, t)$ . For the other direction, notice that

$$\mathbb{V}(\langle\langle s, t \rangle\rangle) = \mathbb{V}\left(\bigcup_{(u,v) \in \langle\langle s, t \rangle\rangle} \{(u, v)\}\right) = \bigcap_{(u,v) \in \langle\langle s, t \rangle\rangle} \mathbb{V}(u, v),$$

by the definition of  $\mathbb{V}$ . By Lemma 4.3, whenever  $(u, v) \in \langle\langle s, t \rangle\rangle$  then  $\mathbb{V}(s, t) \subseteq \mathbb{V}(u, v)$ , so we have  $\mathbb{V}(s, t) \subseteq \mathbb{V}(\langle\langle s, t \rangle\rangle)$ , as was to be shown. ■

Using Lemmas 4.3 and 4.4, we obtain:

**LEMMA 4.5 (Wójcicki's Theorem).** *If  $\theta$  is a finitely generated congruence on  $\mathcal{F}_m$ , there exists a set  $D \subseteq [0, 1]^m$  such that  $\theta = \mathbb{I}(D)$ . It follows that every finitely presented MV-algebra is semisimple.*

**PROOF.** The last assertion (the usual statement of Wójcicki's Theorem) is a consequence of the first: if a congruence  $\theta$  on  $\mathcal{F}_\mu$  can be written as  $\mathbb{I}(D)$ , for some  $D \subseteq [0, 1]^\mu$ , then  $\mathbb{I}(\mathbb{V}(\theta)) = \mathbb{I}(\mathbb{V}(\mathbb{I}(D))) = \mathbb{I}(D) = \theta$ , by 1c) in Lemma 2.7; by 2) in Lemma 3.10,  $\mathcal{F}_\mu/\theta$  is semisimple.

By Remark 4.2, we can assume  $\theta = \langle\langle s, t \rangle\rangle$ , for  $s, t \in \mathcal{F}_m$ . Set  $D = \mathbb{V}(s, t)$ . We have  $\langle\langle s, t \rangle\rangle \subseteq \mathbb{I}(\mathbb{V}(\langle\langle s, t \rangle\rangle)) = \mathbb{I}(\mathbb{V}(s, t))$ , where the inclusion

holds because  $\mathbb{I} \circ \mathbb{V}$  is a closure operator (Remark 2.8), and the equality holds by Lemma 4.4. So we have  $\theta = \langle\langle s, t \rangle\rangle \subseteq \mathbb{I}(\mathbb{V}(s, t)) = \mathbb{I}(D)$ . For the other inclusion, let  $(u, v) \in \mathbb{I}(D)$ . By definition, this means that for any  $d \in D$  the equality  $u(d) = v(d)$  holds, whence  $D = \mathbb{V}(s, t) \subseteq \mathbb{V}(u, v)$ . But by Lemma 4.3 this implies  $(u, v) \in \langle\langle s, t \rangle\rangle$ , hence  $\mathbb{I}(D) \subseteq \langle\langle s, t \rangle\rangle = \theta$ , and the lemma is proved. ■

LEMMA 4.6 (Finitely definable set=Compact congruence).

1. If  $D \subseteq [0, 1]^m$  is a finitely definable set, then  $\mathbb{I}(D) \subseteq \mathcal{F}_m \times \mathcal{F}_m$  is a finitely generated congruence.
2. If  $\theta \subseteq \mathcal{F}_m \times \mathcal{F}_m$  is a finitely generated congruence, then  $\mathbb{V}(\theta) \subseteq [0, 1]^m$  is a finitely definable set.

PROOF. To prove 1, recall that by Remark 4.2 there exist terms  $s, t \in \mathcal{F}_m$  such that  $D = \mathbb{V}(s, t)$ , and the latter equals  $\mathbb{V}(\langle\langle s, t \rangle\rangle)$  by Lemma 4.4. So  $\mathbb{I}(D) = \mathbb{I}(\mathbb{V}(\langle\langle s, t \rangle\rangle)) = \langle\langle s, t \rangle\rangle$ , where the last equality holds by Lemma 4.5. Item 2 follows at once from Lemma 4.4. ■

THEOREM 4.7. The adjunction  $\mathcal{V} \dashv \mathcal{I}$  in Theorem 2.9 restricts to an equivalence of categories between  $\text{MV}_{\text{fp}}$  and  $\text{D}_{\text{def } \mathbb{Z}}^{\text{op}}$ .

PROOF. If  $\mathcal{F}_m / \theta$  is finitely presented, so that  $\theta$  is finitely generated, then by 2 in Lemma 4.6,  $\mathbb{V}(\theta)$  is a finitely definable set. Further, the component  $\eta_{\mathcal{F}_m / \theta}$  of the unit at  $\mathcal{F}_m / \theta$  is an isomorphism, by Theorem 3.1, because  $\mathcal{F}_m / \theta$  is semisimple, by Lemma 4.5.

If  $D \subseteq [0, 1]^m$  is a finitely definable set, then it is a closed set (cf. the proof of Lemma 3.6), and therefore the component  $\epsilon_D$  of the co-unit at  $D$  is an isomorphism, by Theorem 3.1. By 1 in Lemma 4.6,  $\mathbb{I}(D)$  is finitely generated, hence  $\mathcal{I}(D) = \mathcal{F}_m / \mathbb{I}(D)$  is finitely presented. ■

**A concrete equivalent of  $\text{D}_{\text{def } \mathbb{Z}}$ .** The abstract category  $\text{D}_{\text{def } \mathbb{Z}}$  can be characterised in purely geometrical terms. This yields the geometric duality between finitely presented MV-algebras and rational polyhedra. As a general background reference on polyhedra, see [19].

A *convex combination* of a finite set of vectors  $v_1, \dots, v_u \in \mathbb{R}^m$  is any vector of the form  $r_1 v_1 + \dots + r_u v_u$ , for non-negative real numbers  $r_i \geq 0$  satisfying  $\sum_{i=1}^u r_i = 1$ . If  $S \subseteq \mathbb{R}^m$  is any subset, we let  $\text{conv } S$  denote the *convex hull* of  $S$ , i.e. the collection of all convex combinations of finite sets of vectors  $v_1, \dots, v_u \in S$ . A *polytope* is any subset of  $\mathbb{R}^m$  of the form  $\text{conv } S$ , for some finite  $S \subseteq \mathbb{R}^m$ , and a (*compact*) *polyhedron* is a union of finitely many

polytopes in  $\mathbb{R}^m$ . A polytope is *rational* if it may be written in the form  $\text{conv } S$  for some finite set  $S \subseteq \mathbb{Q}^d \subseteq \mathbb{R}^m$  of vectors with rational coordinates. Similarly, a polyhedron is *rational* if it may be written as a union of finitely many rational polytopes.

DEFINITION 4.8 (Cf. Definition 3.1 in [17]). Given a rational polyhedron  $P \subseteq [0, 1]^m$  and a continuous map  $\zeta = (\zeta_1, \dots, \zeta_n) : P \rightarrow [0, 1]^n$ , for  $n \geq 0$  an integer, we say that  $\zeta$  is a  $\mathbb{Z}$ -map if for each  $i = 1, \dots, n$ ,  $\zeta_i$  is *piecewise linear with integer coefficients*: in other words, if there is a finite number of (affine) linear polynomials with integer coefficients  $l_{i,1}, \dots, l_{i,j_i} : [0, 1]^m \rightarrow \mathbb{R}$  such that for every  $x \in P$  there is  $j \in \{1, \dots, j_i\}$  with  $\zeta_i(x) = l_{i,j}(x)$ . Finally, if  $Q \subseteq [0, 1]^n$  is a rational polyhedron, a function  $P \rightarrow Q$  is a  $\mathbb{Z}$ -map if it is the co-restriction to  $Q$  of a  $\mathbb{Z}$ -map  $P \rightarrow [0, 1]^n$ .

It is an exercise to show that the composition of  $\mathbb{Z}$ -maps between rational polyhedra in unit cubes is again a  $\mathbb{Z}$ -map. Moreover, identity maps on such rational polyhedra are obviously  $\mathbb{Z}$ -maps. Therefore, rational polyhedra lying in  $[0, 1]^m$ , for some integer  $m \geq 0$ , and  $\mathbb{Z}$ -maps between them, form a category; we denote it  $\mathbf{P}_{\mathbb{Z}}$ .

The key fact is that  $\mathbb{Z}$ -maps between rational polyhedra are precisely the definable maps.

LEMMA 4.9 (McNaughton's Theorem for rational polyhedra). *Let  $P \subseteq [0, 1]^m$  be a rational polyhedron, and let  $\lambda : P \rightarrow [0, 1]$  be a continuous function. Then  $\lambda$  is a  $\mathbb{Z}$ -map if, and only if,  $\lambda$  is a definable function.*

PROOF. A proof of the case  $P = [0, 1]^m$ , usually known as McNaughton's Theorem *tout court*, is in [7, 9.1.5]. For the generalisation to rational polyhedra, see [17, 3.2]. For references to other proofs, including McNaughton's original one, please see [7]. ■

Remark 4.10 (The Pierce-Birkhoff Conjecture). The notion of  $\mathbb{Z}$ -map in Definition 4.8 can be extended to arbitrary subspaces of the Tychonoff cube in the obvious manner. But then Theorem 4.9 will *fail* as soon as rational polyhedra are replaced by sufficiently general subspaces. Indeed, while it is always trivially true that definable functions are  $\mathbb{Z}$ -maps, there may be  $\mathbb{Z}$ -maps that are not definable. It is remarkable that this already occurs for real semialgebraic sets (see [3]) defined by polynomials with integer coefficients. Explicitly, let  $S \subseteq [0, 1]^2$  be the union of  $A = \{(x, y) \in [0, 1]^2 \mid y = 0\}$  and  $B = \{(x, y) \in [0, 1]^2 \mid y = x^2\}$ . Consider the (continuous) function  $\lambda : S \rightarrow [0, 1]$  given by  $\lambda(x, y) = x$  if  $(x, y) \in A$ , and  $\lambda(x, y) = y$  if  $(x, y) \in B$ . Then  $\lambda$  is a  $\mathbb{Z}$ -map, but contemplation of the properties of the singular point

$(0, 0) \in (A \cup B)$  will convince the reader that  $\lambda$  is not definable. The comparison between definable functions and  $\mathbb{Z}$ -maps on real semialgebraic sets in  $[0, 1]^m$  is closely related to versions of the *Pierce-Birkhoff conjecture* in real semialgebraic geometry; the interested reader is referred to [13] for further information.

LEMMA 4.11. *The category  $D_{\text{def } \mathbb{Z}}$  coincides with the category  $P_{\mathbb{Z}}$ .*

PROOF. By [15, Proposition 5.1],  $S$  is a rational polyhedron if, and only if, there is a  $\mathbb{Z}$ -map  $\zeta: [0, 1]^m \rightarrow [0, 1]$  vanishing precisely on  $S$ , that is, such that  $\zeta^{-1}(0) = S$ . If  $S \subseteq [0, 1]^m$  is finitely definable, it is of the form  $\mathbb{V}(R)$  for some finite  $R \subseteq \mathcal{F}_m \times \mathcal{F}_m$ . By Remark 4.2, we may assume that  $R$  is a singleton  $\{(s, 0)\}$ , so that  $S$  is the solution set over  $[0, 1]^m$  of the equation  $\zeta_s = 0$ , where  $\zeta_s$  is the function defined by  $s$ . By Lemma 4.9,  $\zeta_s$  is a  $\mathbb{Z}$ -map, and therefore, since  $S = \zeta_s^{-1}(0)$  by construction, we see that  $S$  is a rational polyhedron. Conversely, if  $S$  is a rational polyhedron in  $[0, 1]^m$ , there is a  $\mathbb{Z}$ -map  $\zeta: [0, 1]^m \rightarrow [0, 1]$  such that  $\zeta^{-1}(0) = S$ . By Lemma 4.9 there is a term  $s \in \mathcal{F}_m$  such that  $\zeta$  is the function defined by  $s$ , and therefore, since  $S = \mathbb{V}(s, 0)$ ,  $S$  is finitely definable. We have proved that  $D_{\text{def } \mathbb{Z}}$  and  $P_{\mathbb{Z}}$  have the same objects. The fact that they have the same morphisms is an immediate consequence of Lemma 4.9. ■

COROLLARY 4.12 (The duality theorem for finitely presented MV-algebras). *The adjunction  $\mathcal{V} \dashv \mathcal{I}$  in Theorem 2.9 restricts to an equivalence of categories between  $MV_{\text{fp}}$  and  $P_{\mathbb{Z}}^{\text{op}}$ .* ■

PROOF. Immediate consequence of Theorem 4.7 and Lemma 4.11. ■

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VINCENZO MARRA  
Dipartimento di Informatica e Comunicazione  
Università degli Studi di Milano  
Via Comelico 39-41  
20135 Milano, Italy  
`marra@dico.unimi.it`

LUCA SPADA  
Dipartimento di Matematica  
Università degli Studi di Salerno  
Via Ponte don Melillo  
84084 Fisciano (SA), Italy  
`lspada@unisa.it`