Classical statistical distributions can violate Bell's inequalities

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Abstract

We investigate two-particle phase-space distributions in classical mechanics characterized by a well-defined value of the total angular momentum. We construct phase-space averages of observables related to the projection of the particles' angular momenta along axes with different orientations. It is shown that for certain observables, the correlation function violates Bell's inequality. The key to the violation resides in choosing observables impeding the realization of the counterfactual event that plays a prominent role in the derivation of the inequalities. This situation can have statistical (detection related) or dynamical (interaction related) underpinnings, but non-locality does not play any role.

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Bell's theorem was originally introduced [1] to examine quantitatively the consequences of the Einstein-Podolsky-Rosen arguments [2] on the incompleteness of quantum mechanics. The core of the theorem takes the form of inequalities involving average values of two-particle observables. Bell showed that these inequalities must be satisfied by any theory containing additional local hidden variables. But as is well-known, quantum mechanical expectation values can violate the inequalities, and this violation has been experimentally verified with increasing precision in a high number of experiments [3]. Nowadays Bell's inequalities are generally taken (a few exceptions [4] aside) as revealing a fundamental contradiction between quantum mechanical predictions and locality [5], and the EPR setup is employed in quantum information as an instance of steering [6]. It would thus appear that particle classical mechanics – which only contains local dynamical variables – should trivially satisfy Bell-type inequalities. In this work however, we show that averages of measurements involving classical dynamical variables taken over specific statistical distributions (the classical analogs of quantum mechanical eigenstates) can lead to a violation of Bell's inequalities. This violation occurs in cases for which the term involving the simultaneous detection of counterfactual events – the main ingredient in the derivation of the Bell inequalities – becomes undefined, a situation that can have statistical (related to the detection process) or dynamical (interaction related) origins, as discussed in the two examples given in this work.

Before investigating the two-particle problem, we briefly expose the classical relation between the distribution of the angular momenta (to be employed below) and the corresponding particle distribution in configuration space (familiar from quantum-mechanics). Consider first a single classical particle and assume the modulus J of its angular momentum is fixed. The value of $\mathbf{J}$ then depends on the position of the particle in the phase-space defined by $\Omega = \{\theta, \phi, p_\theta, p_\phi\}$, where θ and ϕ refer to the polar and azimuthal angles in spherical coordinates and p_θ and p_ϕ are the conjugate canonical momenta. Let $\rho_z(\Omega)$ be the distribution in phase-space given by

$$\rho_{z_0}(\theta, \phi, p_\theta, p_\phi) = N\delta(J_z(\Omega) - J_{z_0})\delta(J^2(\Omega) - J_0^2). \tag{1}$$

ρ_{z_0} defines a distribution in which every particle has an angular momentum with the same magnitude, namely J_0, and the same projection on the z axis J_{z_0}. Hence ρ_{z_0} is the classical analog of the quantum mechanical density matrix $|jm\rangle\langle jm|$ since just like a quantum measurement of the magnitude j and z axis projection m of the angular momentum in such a state will invariably yield the eigenvalues of the operators $\hat{J}^2$ and $\hat{J}_z$, the classical measurement of these quantities when the phase-space distribution is known to be ρ_z will give J_0^2 and J_{z_0} [1]. Eq. (1) can be integrated over the conjugate momenta to yield the *configuration space* distribution

$$\rho(\theta,\phi) = N\left[\sin(\theta)\sqrt{J_0^2 - J_{z_0}^2/\sin^2(\theta)}\right]^{-1} \tag{2}$$

where we have used the defining relations $J_z(\Omega) = p_\phi$ and $J^2(\Omega) = p_\theta^2 + p_\phi^2/\sin^2\theta$. Further integrating over θ and ϕ and requiring the phase-space integration of ρ to be unity allows to set the normalization constant $N = J_0/2\pi^2$. There is of course nothing special about the z axis and we can define a distribution by fixing the projection J_a of the angular momentum on an arbitrary axis a to be constant (in this paper we will take all the axes to lie in the zy plane). Computing the distribution $\rho_{a_0} = \delta(J_a - J_{a0})\delta(J - J_0^2)$ is tantamount to rotating the coordinates towards the a axis in Eq. (2). Fig. 1 shows examples of configuration space particle distributions on the unit-sphere and gives for one plot the corresponding quantum mechanical angular-momentum eigenstate (the similarity is not accidental, as Eq. (2) is essentially the amplitude of the spherical harmonic in the semiclassical regime). We can also determine the average projection J_a on the a axis for a distribution of the type (2) corresponding to a well defined value of J_z:

$$\langle J_a\rangle_{J_{z_0}} = \int p_\phi \cos\theta_a \delta(J_z(\Omega) - J_{z0})d\Omega = J_{z_0}\cos\theta_a, \tag{3}$$

where θ_a is the angle $(\widehat{z,a})$ and the projection of the component of J_a on the y axis vanishes given the axial symmetry of the distribution.

The original derivation of the inequalities by Bell [1] involved the measurement of the angular momentum of 2 spin-1/2 particles along different axes. Here we will consider the fragmentation of an initial particle with a total angular momentum $\mathbf{J}_T = 0$ into 2 particles carrying angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$ (we will assume to be dealing with orbital angular momenta). Conservation of the total angular momentum imposes $\mathbf{J}_1 = -\mathbf{J}_2$ and $J_1 = J_2 \equiv J$. Quantum mechanically, this situation would correspond to the system being in the singlet

[1] In the general case, the classical analog of the eigenstate of a set of quantum mechanical operators $\{\hat{A}_1, \hat{A}_2 ...\}$ can be defined by finding the phase-space distribution invariant relative to the infinitesimal canonical transformations generated by the classical quantities $A_1(q,p), A_2(q,p), ...$; see A. Matzkin, in preparation.

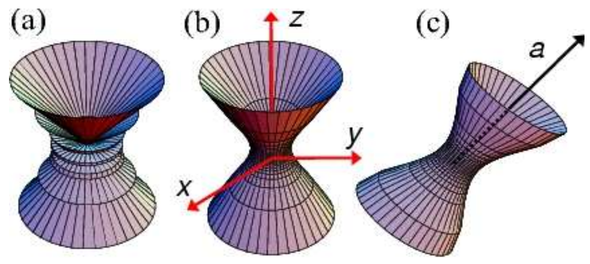

FIG. 1: Normalized angular distribution for a single particle in configuration space. (a) Quantum distribution (spherical harmonic $|Y_{JM}(\theta,\phi)|^2$). (b) Classical distribution $\rho_{z_0}(\theta,\phi)$ of Eq. (2). (c) Classical distribution ρ_{a_0} corresponding to a fixed value of J_a (here $\theta_a = \pi/4$). The angular momentum and the projection on the z [(a)-(b)] or a [(c)] axis is the same for the 3 plots ($J/\eta = 40$, with $\eta = \hbar$, and $M/J = 5/8$).

state arising from the composition of the angular momenta ($j_T = 0$, $m_1 = -m_2$). Classically the system is represented by the 2-particle phase space distribution

$$\rho(\Omega_1, \Omega_2) = N\delta(\mathbf{J}_1 + \mathbf{J}_2), \tag{4}$$

where N is again a normalization constant. This distribution will be employed for determining averages of observables related to the angular momenta of the two particles. Two examples will be studied.

In the first example, two types of detectors are employed: the first type gives a 'sharp' (S) measurement of J_{1a} only if J_{1a} is an integer multiple of some elementary gauge η, and gives 0 elsewhere. This detection can be represented by the phase-space quantity

$$S_a(\Omega_1) = J_{1a}(\Omega_1) \text{ if } \Omega_1 \in \Omega_{1k}, \tag{5}$$

$$S_a(\Omega_1) = 0 \text{ elsewhere}, \tag{6}$$

where Ω_{1k} are the parts of phase space where $J_{1a} = k\eta$ compatible with a detection (see Fig. 2(a)). The second detector gives a 'direct' (D) measurement of J_{2b} (the projection of $\mathbf{J}_2$ on an axis b). The corresponding phase-space function is

$$D_b(\Omega_2) = \mathbf{J}_2 \cdot \mathbf{b}. \tag{7}$$

In classical mechanics there is no natural unit for quantities having the dimension of an action, so J and η can be expressed in terms of arbitrary units, and any physical result will depend only on the ratio J/η. We will assume for definiteness that η is chosen so that the extremal values $\pm J$ can be reached. J/η must hence be either an integer or a half-integer, the extremal values in dimensionless units being given by $\pm L \equiv \pm J/\eta$. For example if $\eta = 2J$, the measurement can only yield the extremal values $L = \pm 1/2$ ($\eta = J$ allows to measure $\pm L = \pm 1$ and 0, $\eta = 2J/3$ allows $\pm L = \pm 3/2$ and $\pm(L-1) = \pm 1/2$ etc.). Note that the particle label 1 or 2 can be attached to the detectors: indeed, we will call '1' the particle detected by S and '2' the particle detected by D.

The classical average $E(a,b) = \langle S_a D_b \rangle$ for joint measurements over the statistical distribution ρ can be computed from

$$E(a,b) = \int S_a(\Omega_1) D_b(\Omega_2) \rho(\Omega_1, \Omega_2) d\Omega_1 d\Omega_2 \tag{8}$$

with Eqs. (4), (5) and (7). Given the characteristics (5)-(6) of the S detection, Eq. (8) is actually a discrete sum (over the parts of phase-space Ω_{1k} leading to the detection of $k\eta$) which can be written under the integral by introducing a delta function. Eq. (4) imposes $\theta_2 = \pi - \theta_1$ and $\phi_2 = \pi + \phi_1$, and Eq. (8) becomes

$$E(a,b) = \frac{1}{2} \sum_{k=-L}^{k=L} \int \left[L \cos\theta_1 \right] \delta\left(L\cos\theta_1 - k \right) \left[-L \cos\theta_1 \cos(\theta_b - \theta_a) \right] \sin\theta_1 d\theta_1, \tag{9}$$

where we have chosen the z axis to coincide with a to take advantage of the axial symmetry imposed by S_a; the $\frac{1}{2}$ prefactor is the only nontrivial normalisation factor (coming from the integration over θ_1). We obtain the average as

$$E(a,b) = -\frac{1}{6}(L+1)(2L+1)\cos(\theta_b - \theta_a), \tag{10}$$

which as expected depends solely on the ratio $J/\eta \equiv L$. The correlation function employed in Bell's inequality can be obtained in the standard (or CHSH) form [7]. We choose 4 axes a, b, a', b' (we can assume an S detector is placed along a and a', and a D detector along b and b') and determine the average values for each of the 4 possible combinations involving an S and a D detector. The correlation function C relating the average values obtained for different orientation of the detectors' axes is

$$C(a,b,a',b') = \left(|E(a,b) - E(a,b')| + |E(a',b) + E(a',b')| \right) (L)^{-2} \tag{11}$$

where we have divided by L^2 to obtain the CHSH correlation function in the standard form characterized by observed values bounded by ± 1. Here the detected values obey the

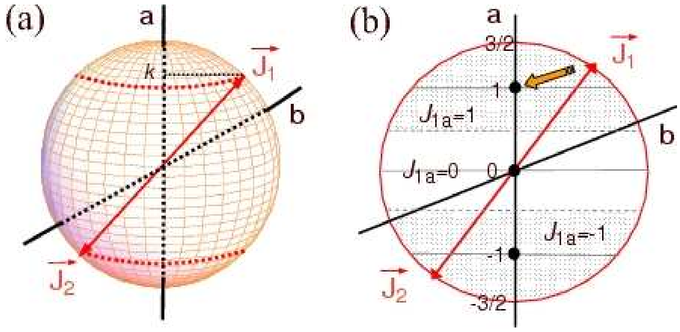

FIG. 2: Setups for the first (a) and second (b) examples investigated in this work. In (a) an S detector is placed along the a axis and a D detector along b. The angular momenta, originally distributed on the sphere, are constrained to move on the rings (red dotted) corresponding to fixed values of the projection on a. (b) shows the $(\widehat{z,y})$ plane for the $L = 1$ case (hence $J = 3/2$); the 3 zones correspond to the regions yielding one of the three possible measurement outcomes of J_{1a}: $-1, 0, 1$.

conditions $|S/L| \leq 1$ and $|D/L| \leq 1$, so that the usual derivation of the Bell inequalities would lead to

$$C(a, b, a', b') \leq 2. \tag{12}$$

The violation of this inequality by quantum-mechanical expectation values is generally taken as the main argument against the existence of local hidden-variables. By replacing Eq. (10) in Eq. (11), it can be seen that for $L = \frac{1}{2}$, 1 and $\frac{3}{2}$, there are several choices of the axes that lead to $C(a, b, a', b') > 2$. The maximal value of the correlation function corresponds to $C(0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}) = 4\sqrt{2}$ and $2\sqrt{2}$ for $L = \frac{1}{2}$ and 1 respectively [2].

The violation of the Bell inequality in this first example is due to the fact that we are only including in the statistics the *joint* measurements (for which both detectors click). However making an S-measurement along different orientations a and a' amounts to selecting different parts of phase-space. This can be seen by including the delta function accounting for the S measurement so as to define an effective phase-space density by $\tilde{\rho}(\Omega_1, \Omega_2) = \sum_k \delta(J_{1a} - k\eta)\, \rho(\Omega_1, \Omega_2)$. $\tilde{\rho}$ obviously depends on the orientation of the detectors, and the

[2] The reader familiar with the Bell inequalities for the quantum measurement of J_{1a} and J_{2b} will recognize the similarity of Eq. (10) with the quantum expectation value; the only difference is that the quantum expectation value is normalized respective to the number of possible outcomes $(2L+1)$ whereas here the normalization is relative to classical phase-space (namely the length $2L$ of the measurement axis).

quantity

$$\int S_a(\Omega_1) D_b(\Omega_1) S_{a'}(\Omega_1) D_{b'}(\Omega_1) \rho(\Omega_{1,}\Omega_2) d\Omega_1 d\Omega_2 \tag{13}$$

describing the average of counterfactual simultaneous measurements along the 4 axis can then vanish or become undefined. But this quantity is precisely the one that plays the central role in the derivation of the Bell inequality: in particular, the equivalence between the non-existence of the distribution function (13) and the violation of the inequality (12) is well-known [8]. It is noteworthy that if one includes the whole phase-space in the average (8), then it can be shown that $E(a,b)$ and $C(a,b,a',b')$ should be multiplied by the fraction of phase-space yielding joint measurements, and as a result Bell's inequality would not be violated (this is better seen if the delta function modeling the S-detector is replaced by a narrow ring having a finite surface [11]). Physically this objection makes sense provided one can envisage a particle analyzer able to detect the particles that have not been included in the statistics. This problem is well-known in quantum-mechanical contexts as an instance [9] of the so-called detection loophole pending on the experimental tests of Bell's inequalities.

Our second example goes further into the violation of Bell's inequalities by postulating a local interaction between the detector and the particle being measured: we then obtain a violation of the inequality for the entire ensemble of particles. Let us take two identical detectors T_1 and T_2 that give as only output the integer or half-integer values $k = L, L - 1, ... - L$ of the projection J_{1a} and J_{2b} of the angular momenta of the particles. We now choose $L = J/\eta - 1/2$, from which it follows that the maximal readout L is smaller than J; for notational simplicity we put $\eta = 1$ (so J, rather than J/η takes integer or half integer values). We further assume that there is an interaction between T_1 and particle 1 (and between T_2 and particle 2) affecting the angular momentum of the particle so that the transition $J_{1a} \to k$ is a physical process due to the measurement. We impose the following constraint on this process: given a statistical distribution ρ_1 for particle 1, the average $\langle J_{1a} \rangle_{\rho_1}$ over phase-space is the one given by averaging over the results obtained from the interaction-mediated measurement readouts. This constraint takes the form

$$\langle T_{1a} \rangle = \sum_{k=-L}^{L} k P(k, \rho_1) = \langle J_{1a} \rangle_{\rho_1} \tag{14}$$

where $P(k, \rho_1)$ is the probability of obtaining the reading k on the detector given the statistical distribution ρ_1. We will not be interested here in the details of the interaction yielding such probabilities; it will suffice for our purpose that a set of numbers $P(k,\rho_1)$ verifying Eq. (14) and obeying $\sum_k P(k,\rho_1) = 1$ can be obtained. We give explicit examples for 3 classical 1-particle distributions. (i) If ρ_1 is spherically symmetric, detection of a given k is equiprobable and we impose $P = 1/(2L+1)$. In this case, the simplest realization yielding these probabilities is to set

$$T_{1a} = k \text{ if } k - 1/2 < J_{1a} < k + 1/2. \tag{15}$$

One then cuts the sphere into $2L+1$ equal zones Ω_k centered on k (see Fig. 2(b)), and a direct calculation shows that Eq. (14) holds within each zone,

$$\langle T_{1a} \rangle_{\Omega_{1k}} = \langle J_{1a} \rangle_{\Omega_{1k}} = k. \tag{16}$$

(ii) If ρ_1 is of the type given by Eqs. (1)-(3) (axial symmetry with a given value of J_{1z}), we will take the $P(k,\rho_1)$ to be linked to the probability density $\chi(J_{1a}, J_{1z_0}, \theta_a)$ of finding a given value of J_{1a} if it is known that the projection of $\mathbf{J}_1$ on the z axis is J_{1z_0}. We set

$$P(k,\rho_1) = k^{-1} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} k' \chi(k', J_{1z_0}, \theta_a) dk' \tag{17}$$

for $|k| > 1$ (the bounds become 0 and $\pm\frac{3}{2}$ for $|k| = 1$ since $k = 0$ does not contribute). The normalized classical probability density

$$\chi(k, J_{1z_0}, \theta_a) = (\pi J)^{-1} \left[\left(1 - J_{1z_0}^2/J^2\right)\left(1 - k^2/J^2\right) - (\cos\theta_a - kJ_{1z_0}/J^2)^2 \right]^{-1/2} \tag{18}$$

is obtained from trigonometrical considerations [10]. χ verifies $\int k\chi dk = J_{z_0}\cos\theta_a$, ensuring that $P(k,\rho_1)$ and $\langle J_{1a} \rangle$ obey Eqs. (14) and (3) respectively. (iii) If ρ_1 is defined by a uniform density between 2 values of J_{1z}(rather than the single value J_{1z_0} of (ii)), say $J_{1z_0} - 1/2 < J_{1z} < J_{1z_0} + 1/2$, we can take $P(k,\rho_1)$ to be given again by Eq. (17) (meaning that the probability of obtaining the outcome k depends on the projection of $\mathbf{J}_1$ on the a axis if it is known that the mean projection of $\mathbf{J}_1$ when the particles belong to ρ_1 is J_{1z_0}).

The expectation value $E(a,b) = \langle T_{1a} T_{2b} \rangle$ for the 2 particle problem with the phase-space density ρ given by Eq. (4) is computed in the following way. The initial density ρ has spherical symmetry (case (i) in the preceding paragraph). We cut the sphere into $2L+1$ zones Ω_{1k} defined by Eq. (16). Within each zone, we have $T_{1a} = k$, and from the conservation of the total angular momentum we infer for the other particle that $k - 1/2 < -J_{2a} < k + 1/2$

(this falls in case (iii)). Hence,

$$E(a,b) = \sum_{k=-L}^{L} \int_{\Omega_{1k}} T_{1a}(\Omega_1) T_{2b}(\Omega_2) \rho(\Omega_1, \Omega_2) d\Omega_1 d\Omega_2, \tag{19}$$

becomes with the constraint (14)

$$\sum_{k=-L}^{L} k \int_{\cos^{-1}\frac{k+1/2}{J}}^{\cos^{-1}\frac{k-1/2}{J}} [-J \cos\theta_1 \cos(\theta_b - \theta_a)] \sin\theta_1 d\theta_1 = \frac{-L(L+1)}{3} \cos(\theta_b - \theta_a) \tag{20}$$

where we have used $J = L + 1/2$. The correlation function is again given by Eq. (11), since the maximum value detected by a T measurement is L, not J. The result given by Eq. (20) is familiar from quantum mechanics – it violates Bell's inequality for $L = 1/2$ with a maximal violation for $C(0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}) = 2\sqrt{2}$. As in the first example, violation occurs because the counterfactual term Eq. (13) is devoid of any meaning: a single particle's angular momentum cannot be measured simultaneously along different axes (since the interaction changes the phase-space distribution). Two characteristics of this interaction are worth mentioning: (i) the expectation values are independent from the form of the interaction: the derivation of $E(a,b)$ does not depend in any way on the $P(k,\rho)$'s. This ensures in particular that the total angular momentum is conserved on average. (ii) the interaction is local (T_1 affects only particle 1, T_2 affects particle 2), and the correlation between the two particles allowing to infer the distribution of one particle as a function of the measurement on the other is solely due to the conservation of the angular momentum.

The present results show that average values obtained from a specific classical phase-space distribution of particles can lead to a violation of Bell's inequalities. The necessary requirement is that the term accounting for the simultaneous measurement of two particles along four different axes does not exist, i.e. there is no 'element of reality' (in the EPR [2] sense) corresponding to this simultaneous event. In the first example reported in this work, conditioning the statistics to include only joint detections results in the sampling of incompatible regions of phase-space. More interestingly, in the second example the measurement process causes a local interaction that has no effect on the average values but affects the measurement outcome of the particles (and hence the correlation function). Our main conclusion, valid in both cases, is that the violation of the Bell inequalities can occur in classical mechanics without appealing to any nonlocal effects. Possible implications on the issue of local realism and non-locality in a quantum-mechanical context will be examined elsewhere [11].

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