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COMMUNICATIONS

COMPUTABLE BI-EMBEDDABLE CATEGORICITY

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We study the algorithmic complexity of isomorphic embeddings between computable structures. Suppose that L is a language. We say that L-structures $\mathcal{A}$ and $\mathcal{B}$ are *bi-embeddable* (denoted $\mathcal{A} \approx \mathcal{B}$) if there are isomorphic embeddings $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$ and $g\colon \mathcal{B} \hookrightarrow \mathcal{A}$. The systematic investigation of the bi-embeddability relation in computable structure theory was initiated by Montalbán [1, 2]: he proved that any hyperarithmetical linear order is bi-embeddable with a computable one. In [3], similar results were obtained for Abelian p-groups, Boolean algebras, and compact metric spaces. The paper [4] studies degree spectra with respect to bi-embeddability.

Definition 1. Let $\mathbf{d}$ be a Turing degree. We say that a computable structure $\mathcal{S}$ is $\mathbf{d}$-*computably bi-embeddably categorical* if, for any computable structure $\mathcal{A} \approx \mathcal{S}$, there are $\mathbf{d}$-computable isomorphic embeddings $f\colon \mathcal{A} \hookrightarrow \mathcal{S}$ and $g\colon \mathcal{S} \hookrightarrow \mathcal{A}$. The *bi-embeddable categoricity spectrum* of $\mathcal{S}$ is the set

$$CatSpec_{\approx}(\mathcal{S}) = \{\mathbf{d} : \mathcal{S} \text{ is } \mathbf{d}\text{-computably bi-embeddably categorical}\}.$$

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A degree $\mathbf{c}$ is the *degree of bi-embeddable categoricity* of $\mathcal{S}$ if $\mathbf{c}$ is the least degree in the spectrum $CatSpec_{\approx}(\mathcal{S})$.

Definition 1 is similar to the notions of categoricity spectrum and degree of categoricity which were introduced in [5]. The *categoricity spectrum* of a computable structure $\mathcal{S}$ is the set of all Turing degrees which are capable of computing isomorphisms among arbitrary computable copies of $\mathcal{S}$. The *degree of categoricity* of $\mathcal{S}$ is the least degree from the categoricity spectrum of $\mathcal{S}$.

Our first result gives examples of degrees of bi-embeddable categoricity. It shows that every degree of categoricity known in the literature [5, 6] can be realized as a degree of bi-embeddable categoricity. We make use of the following notion. A structure $\mathcal{A}$ is said to be *bi-embeddably trivial* (or *b.e. trivial* for short) if $\mathcal{B}$ and $\mathcal{A}$ are isomorphic for any $\mathcal{B}$ bi-embeddable with $\mathcal{A}$.

THEOREM 1. Let α be a computable nonlimit ordinal. Suppose that $\mathbf{d}$ is a Turing degree such that $\mathbf{d}$ is d.c.e. in $\mathbf{0}^{(\alpha)}$ and $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. There is a computable, bi-embeddably trivial structure $\mathcal{S}$ with degree of bi-embeddable categoricity $\mathbf{d}$.

Proof sketch. We build two b.e. trivial computable structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \cong \mathcal{B}$, $\mathcal{A}$ is $\mathbf{d}$-computably categorical, and any embedding from $\mathcal{A}$ into $\mathcal{B}$ must compute $\mathbf{d}$. Here we give a construction for the case where $\mathbf{d}$ is d.c.e. over $\mathbf{0}^{(2\beta+1)}$, where β is an infinite ordinal.

Ash's characterization of the back-and-forth relations for linear orders and his theorem on pairs of structures [7, Chaps. 11, 16] show that for any $\Sigma^0_{2\beta+1}$ set S, there is a computable sequence $(C_e)_{e\in\omega}$ of linear orders such that

$$C_e \cong \begin{cases} \omega^{\beta} \cdot 2 & \text{if } e \in S, \\ \omega^{\beta} & \text{if } e \notin S. \end{cases} \tag{$*$}$$

A relativized version of the argument from [5, Thm. 3.1] allows one to choose a set $D \in \mathbf{d}$ which is d.c.e. in $\mathbf{0}^{(2\beta+1)}$, and for any oracle X, we have

$$(\overline{D} \text{ is c.e. in } X) \;\Rightarrow\; D \leq_T X \oplus \mathbf{0}^{(2\beta+1)}.$$

The language of our structures $\mathcal{A}$ and $\mathcal{B}$ contains an equivalence relation $\sim$, a partial order $\leq$, a unary predicate T, and unary predicates P_e, where $e \in \omega$. Note that $D = U \setminus V$ for U and V c.e. in $\mathbf{0}^{(2\beta+1)}$, where $V \subset U$. We first describe the construction of $\mathcal{A}$. For every e, we choose elements a_e and b_e in $\mathcal{A}$, and for every P_e, $P_e(A)$ is infinite and includes a_e, b_e.

For a fixed e, we give a construction for the substructure on $P_e(A)$. We let $P_e(A)$ consist of two infinite equivalence classes (with respect to $\sim$) such that $a_e \not\sim b_e$. The two classes $[a_e]$ and $[b_e]$ will both contain pairs of linear orders, i.e., structures of the form (L_1, L_2) where L_1 and L_2 are linear orders (with respect to $\leq$), any $x \in L_1$ and $y \in L_2$ are incomparable, and $T([a_e]) = L_1$.

If $e = 2m$, then we encode the information whether or not m is an element of D in $P_e(A)$. There are three cases:

(1) if $m \notin U$, we build $T([a_e]), \neg T([a_e]), T([b_e]) \cong \omega^{\beta}$ and $\neg T([b_e]) \cong \omega^{\beta} \cdot 2$;

(2) if $m \in U \setminus V$, we build $T([b_e]) \cong \omega^{\beta}$ and $T([a_e]), \neg T([a_e]), \neg T([b_e]) \cong \omega^{\beta} \cdot 2$;

(3) if $m \in V$, we build $T([a_e]), T([b_e]), \neg T([a_e]), \neg T([b_e]) \cong \omega^\beta \cdot 2$.

Analyzing this construction, we see that

$$[a_e] \cong \begin{cases} (\omega^\beta \cdot 2, \omega^\beta \cdot 2) & \text{if } m \in U, \\ (\omega^\beta, \omega^\beta) & \text{if } m \notin U; \end{cases} \quad \text{and} \quad [b_e] \cong \begin{cases} (\omega^\beta \cdot 2, \omega^\beta \cdot 2) & \text{if } m \in V, \\ (\omega^\beta, \omega^\beta \cdot 2) & \text{if } m \notin V. \end{cases}$$

If $e = 2m+1$, then we let $[b_e] \cong (\omega^\beta, \omega^\beta \cdot 2)$, and $[a_e]$ is defined by setting

$$[a_e] \cong \begin{cases} (\omega^\beta \cdot 2, \omega^\beta \cdot 2) & \text{if } m \in \varnothing^{(2\beta+1)}, \\ (\omega^\beta, \omega^\beta) & \text{if } m \notin \varnothing^{(2\beta+1)}. \end{cases}$$

The existence of the uniformly computable sequence of structures $(C_e)_{e\in\omega}$ from $(*)$ implies that we can do the construction computably.

For $\mathcal{B}$, we again choose elements $\hat{a}_e$ and $\hat{b}_e$ for every e, and we build $\mathcal{B}$ like $\mathcal{A}$ with the difference that the roles of $\hat{a}_e$ and $\hat{b}_e$ are switched. Clearly, $\mathcal{B}$ and $\mathcal{A}$ are isomorphic and computable. It is not hard to show that they are b.e. trivial. Indeed, every embedding of $\mathcal{A}$ into a bi-embeddable copy $\hat{\mathcal{A}}$ must map elements in $P_e(A)$ to elements in $P_e(\hat{A})$, for every $e \in \omega$. Every $P_e(\hat{A})$ must have exactly two equivalence classes; otherwise $P_e(\hat{A}) \not\approx P_e(A)$. Moreover, the pairs of structures that we use are pairs of well-orders, hence these pairs are b.e. trivial.

Following the line of the proof in [8, Thm. 4], it is not hard to state that $\mathcal{A}$ is $\mathbf{d}$-computably categorical. It remains to show that $f \geq_T D$ for every $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$. We have $f \geq_T \mathbf{0}^{(2\beta+1)}$ because

$$m \in \varnothing^{(2\beta+1)} \Leftrightarrow f(a_{2m+1}) \sim \hat{b}_{2m+1} \quad \text{and} \quad m \notin \varnothing^{(2\beta+1)} \Leftrightarrow f(a_{2m+1}) \sim \hat{a}_{2m+1}.$$

Similarly, we obtain

$$m \notin U \setminus V \Leftrightarrow (f(a_{2m}) \sim \hat{a}_{2m}) \text{ or } (m \in V).$$

Hence $\overline{D}$ is c.e. in $f \oplus \mathbf{0}^{(2\beta+1)}$, so $D \leq_T (f \oplus \mathbf{0}^{(2\beta+1)}) \equiv_T f$.

The construction for the case $\alpha = 2\beta+2$ is nearly the same. The only difference is that in place of $(*)$, we use the following fact. For any $\Sigma^0_{2\beta+2}$ set S, there is a computable sequence $(C_e)_{e\in\omega}$ of linear orders such that

$$C_e \cong \begin{cases} \omega^{\beta+1} + \omega^\beta & \text{if } e \in S, \\ \omega^{\beta+1} & \text{if } e \notin S. \end{cases}$$

The proof for finite α can be obtained by minor modifications. □

The rest of the paper is devoted to bi-embeddable categoricity for structures from familiar algebraic classes. Recall that $\mathcal{A} = (A, E^2)$ is an *equivalence structure* if E is an equivalence relation on the domain of $\mathcal{A}$.

THEOREM 2 [9]. Any computable equivalence structure has degree of bi-embeddable categoricity $\mathbf{d} \in \{\mathbf{0}, \mathbf{0}', \mathbf{0}''\}$.

Note that a similar result for degrees of categoricity was proved by Csima and Ng (unpublished).

THEOREM 3. (a) A computable Boolean algebra is computably bi-embeddably categorical if and only if it is finite.

(b) A computable linear order is computably bi-embeddably categorical if and only if it is finite.

Note that Theorem 3 contrasts with the characterizations of computably categorical Boolean algebras [10, 11] and computably categorical linear orders [10, 12]—in particular, a computable Boolean algebra is computably categorical iff its set of atoms is finite.

An undirected graph is *strongly locally finite* if each of its components is finite. It is easy to show that every computable, strongly locally finite graph is $\mathbf{0}'$-computably categorical.

THEOREM 4. (a) There exists a computable, strongly locally finite graph which is not hyperarithmetically bi-embeddably categorical.

(b) The index set of $\mathbf{0}'$-computably bi-embeddably categorical, strongly locally finite graphs is Π^1_1-complete.

Proof. (a) Let $H \subseteq \omega^{<\omega}$ be a computable tree without hyperarithmetic paths. We build a strongly locally finite graph G_H such that the partial ordering under embeddability of its components is computably isomorphic to H.

For any $\sigma \in H$, G_H contains the component C_σ: a ray of length $|\sigma| + 1$ where the first vertex has a loop connected to it and the $(i+2)$th vertex for $i < |\sigma|$ has a cycle of length $\sigma(i)+2$ attached. Clearly, the partial ordering of the components is computably isomorphic to H by $C_\sigma \mapsto \sigma$. The graph G_H has a bi-embeddable copy $\tilde{G}$ that skips a fixed C_σ if σ lies on a path in H. Now consider embeddings $\mu\colon G_H \to G$ and $\nu\colon G \to G_H$. Then $C_\sigma \subset \mu(C_\sigma) \subset \nu(\mu(C_\sigma)) \subset \ldots$, and so there is $f \in [H]$ hyperarithmetic in $\mu \oplus \nu$. Hence, $\mu \oplus \nu$ itself cannot be hyperarithmetic.

(b) Let $(T_i)_{i\in\omega}$ be a uniformly computable sequence of trees such that T_i is well-founded iff $i \in \mathcal{O}$. For two strings σ and τ of the same length, we define $\sigma \star \tau = \sigma_0\tau_0\sigma_1\tau_1 \ldots \sigma_{|\sigma|-1}\tau_{|\tau|-1}$ and consider a sequence of trees $(S_i)_{i\in\omega}$, where

$$S_i = \{\xi : \xi \subseteq \sigma \star \tau,\ |\sigma| = |\tau|,\ \sigma \in T_i,\ \tau \in H\}.$$

Clearly, the sequence is uniformly computable, and S_i is well-founded iff $i \in \mathcal{O}$. Furthermore, no path in $[S_i]$ is hyperarithmetical. Using the same coding as above, we see that if $i \in \mathcal{O}$, then G_{S_i} is b.e. trivial and thus $\mathbf{0}'$-computably bi-embeddably categorical. If $i \notin \mathcal{O}$, then G_{S_i} is not $\mathbf{0}^{(\alpha)}$-computably bi-embeddably categorical for $\alpha < \omega_1^{\mathrm{CK}}$. □

Note that the index set of computably categorical structures is Π^1_1-complete [13]. We leave open whether a similar result can be obtained for computably bi-embeddably categorical structures.

REFERENCES

1. A. Montalbán, "Up to equimorphism, hyperarithmetic is recursive," *J. Symb. Log.*, **70**, No. 2, 360-378 (2005).
2. A. Montalbán, "On the equimorphism types of linear orderings," *Bull. Symb. Log.*, **13**, No. 1, 71-99 (2007).

3. N. Greenberg and A. Montalbán, “Ranked structures and arithmetic transfinite recursion,” *Trans. Am. Math. Soc.*, **360**, No. 3, 1265-1307 (2008).
4. E. Fokina, D. Rossegger, and L. San Mauro, “Bi-embeddability spectra and bases of spectra,” arXiv:1808.05451 [math.LO].
5. E. B. Fokina, I. Kalimullin, and R. Miller, “Degrees of categoricity of computable structures,” *Arch. Math. Log.*, **49**, No. 1, 51-67 (2010).
6. B. F. Csima, J. N. Franklin, and R. A. Shore, “Degrees of categoricity and the hyperarithmetic hierarchy,” *Notre Dame J. Formal Log.*, **54**, No. 2, 215-231 (2013).
7. C. J. Ash and J. F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy, Stud. Log. Found. Math.*, Vol. 144, Elsevier, Amsterdam (2000).
8. N. A. Bazhenov, “Effective categoricity for distributive lattices and Heyting algebras,” *Lobachevskii J. Math.*, **38**, No. 4, 600-614 (2017).
9. N. Bazhenov, E. Fokina, D. Rossegger, and L. San Mauro, “Degrees of bi-embeddability categoricity of equivalence structures,” *Arch. Math. Log.* (2018); https://doi.org/10.1007/s00153-018-0650-3.
10. S. S. Goncharov and V. D. Dzgoev, “Autostability of models,” *Algebra and Logic*, **19**, No. 1, 28-36 (1980).
11. J. B. Remmel, “Recursive isomorphism types of recursive Boolean algebras,” *J. Symb. Log.*, **46**, No. 3, 572-594 (1981).
12. J. B. Remmel, “Recursively categorical linear orderings,” *Proc. Am. Math. Soc.*, **83**, No. 2, 387-391 (1981).
13. R. G. Downey, A. M. Kach, S. Lempp, A. E. M. Lewis-Pye, A. Montalbán, and D. D. Turetsky, “The complexity of computable categoricity,” *Adv. Math.*, **268**, 423-466 (2015).