L&PS – Logic and Philosophy of Science
Vol. IX, No. 1, 2011, pp. 355-363

Becoming and the Algebra of Time

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ABSTRACT. The idea of becoming, namely that of a unique moving present constantly shifting from past to future, is often rejected by contemporary philosophers as a mere metaphor without any objective content. In this paper, a formal model is offered for temporal becoming, based on dynamical systems theory, thanks to which the dynamics of the transient present can be reduced to objective features such as the algebraic properties of the mathematical structure chosen to model time.

1. Introduction

Time passes. Or, at least, it seems to pass: the sands of time inexhaustibly recede from the unsettled future to the immutable past, only becoming tangible in the fleeting moment they are present to us. However, it has become a major trend in contemporary philosophy of science to deny the objectivity of the passage of time on the basis of contemporary space-time theories: since the rise of special relativity theory, time has been conceived as but one dimension of a four-dimensional differentiable manifold, whose points or events are given all at once, though they are partially ordered as of the before-after relation. The hypothesis that time might possess an internal dynamics – the idea of a *moving now*, or *temporal becoming* – is accordingly dismissed as a mere projection of human mind without any physical or theoretical import.

This work challenges this very charge of irrelevance, by showing that mathematical models of time display an essential feature – namely their alge-

Published by L&PS – Logic and Philosophy of Science
http://www2.units.it/ ~episteme — ISSN: 1826-1043

braic structure – on the basis of which time can be endowed with intrinsic dynamical properties.

2. Time Systems

Dynamical systems are classically conceived as *n*-dimensional differentiable manifolds or state spaces along with a family of continuous autonomous transformations, indexed by the set $\Re$ of time intervals. Giunti and Mazzola (2012) offer a more general definition of dynamical systems, in which time is only required to satisfy the algebraic features of a monoid, i.e. a semigroup with identity:

Definition 1.
Let $L = (T, +)$ *be a monoid with identity* 0*; the ordered pair* $(M, (g^t)^{t \in T})$ *is a* dynamical system *on* L*, denoted by* DS_L*, if and only if*
1. M *is a non-empty set;*
2. $(g^t)^{t \in T}$ *is a family of functions on* M*, indexed by* T*;*
3. for any $x \in M$*,*
(2.1) $g^0(x) = x$,
(2.2) *for any* $t, v \in T$ $\quad g^{t+v}(x) = g^t(g^v(x))$.

M is called the *state space* of the dynamical system, T its *time set* and L its *time model*; finally, for any $t \in T$, the transformation g^t is called a *state transition* of duration t.

Dynamical systems, so defined, are the least mathematical structure needed to model the evolution of deterministic systems, including systems with continuous state spaces and continuous time models (e.g. systems specified by ordinary differential equations), systems with continuous state spaces and discrete time models (e.g. systems specified by difference equations), and systems with discrete state spaces and continuous time models (e.g. cellular automata). In particular, monoids are the least mathematical structure one needs to provide physical time with, in order to describe deterministic physical processes.

Once dynamical systems are so reshaped, each time model $L = (T, +)$ can be associated with a dynamical system whose state space is identical to T, and whose family of state transitions is a left monoid action of L on itself. Mazzola and Giunti (2012) call systems of this kind *time systems*:

Definition 2.
Let $L=(T,+)$ *be a monoid with identity* 0*; the ordered pair* $(I,(\iota^t)^{t\in T})$ *is the* time system *of* L*, denoted by* TS_L*, if and only if*
(2.3) $I=T$;
(2.4) *for any* $i\in I$ *and any* $t\in T$, $\iota^t(i)=t+i$.

I is called the set of *instants* or *moments* of the time system while, for any $t\in T$, the transformation ι^t is called the *time transition* of duration t. It would be easy to prove that, for any monoid L, the time system of L is in fact a dynamical system on L whose state space is I and whose family of state transitions is $(\iota^t)^{t\in T}$.

Intuitively speaking, time systems describe the internal dynamics of time models: for any moment $i\in I$ and any duration $t\in T$, condition (2.4) demands that ι^t map i into the unique moment which is separated from i by a time lapse of duration t, namely the unique moment occurring at time $t+i$. For this reason, time systems offer a consistent mathematical representation of the passing of time from moment to moment, whose dynamical properties can be rigorously examined thanks to the conceptual tools of dynamical systems theory. Moreover, the dynamics of time systems is entrenched in the very algebraic structure of their time models – as we are now going to see.

In general, dynamical systems with different state spaces and on different time models may happen to describe the same deterministic system; in that case, we call them isomorphic:

Definition 3.
Let $DS_{L_1}=(M_1,(g^{t_1})^{t_1\in T_1})$ *and* $DS_{L_2}=(M_2,(g^{t_2})^{t_2\in T_2})$ *be dynamical systems on monoids* $L_1=(T_1,+)$ *and* $L_2=(T_2,\oplus)$ *respectively; a bijective function* $f:M_2\to M_1$ *is an* isomorphism *of* DS_{L_2} *in* DS_{L_1} *if and only if there exists a monoid isomorphism* $\rho:T_2\to T_1$ *of* L_2 *in* L_1 *such that, for any* $t_2\in T_2$ *and any* $x_2\in M_2$,
(2.5) $f(g^{t_2}(x_2))=g^{\rho(t_2)}(f(x_2))$.

In consequence, any two dynamical systems are isomorphic just in case there exists an isomorphism of either into the other. It can be proved that isomorphism is an equivalence relation on any given set of dynamical systems, preserving all their dynamical properties. For this reason, isomorphic dynamical systems are in all identical for the purposes of dynamical systems theory.

By definition, isomorphic dynamical systems possess isomorphic time models. Conversely, isomorphic time models possess isomorphic time systems as well:

Proposition 2.1.
Let L_1 be a monoid with time system TS_{L_1} and let L_2 be a monoid with time system TS_{L_2}; then any monoid isomorphisms ρ of L_2 in L_1 is an isomorphism of TS_{L_2} in TS_{L_1}.

Proof
Let $L_1 = (T_1, +)$ be a monoid with time system $TS_{L_1} = (I, (\iota^{t_1})^{t_1 \in T_1})$ and let $L_2 = (T_2, \oplus)$ be a monoid with time system $TS_{L_2} = (I, (\iota^{t_2})^{t_2 \in T_2})$. Let $\rho : T_2 \to T_1$ be a monoid isomorphism *of* L_2 *in* L_1. Hence, ρ is a bijection and there exists a monoid isomorphism, namely ρ itself, such that, for any $t_2 \in T_2$ *and any* $i_2 \in I_2$,

(2.6) $\rho(\iota^{t_2}(i_2)) = \rho(t_2 \oplus i_2) = \rho(t_2) + \rho(i_2) = \iota^{\rho(t_2)}(\rho(i_2))$.

Therefore, by Definition 3, ρ is an isomorphism between time systems.

□

In consequence, whenever two monoids are isomorphic, their time systems are isomorphic as well.

Symmetrically, each dynamical system can be associated with a unique monoid of transformations on its state space, called its *transition algebra*:

Definition 5.
Let $DS_L = (M, (g^t)^{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; the ordered pair $(H, \circ)$ is the transition algebra *of DS_L, denoted by TA_{DS_L}, if and only if*

(2.7) $H = \{h : \exists t \in T (h = g^t)\}$

and $\circ$ is the standard operation of function composition.

In general, transition algebras are homomorphic, but not isomorphic, to the time models of the associated dynamical systems. In the special case of time systems, however, transitions algebras and time models are identical up to isomorphism:

Proposition 2.2.
Every monoid is isomorphic to the transition algebra of its time system.

Proof
Let $L = (T,+)$ be a monoid with identity 0, let $TS_L = (I,(\iota^t)^{t \in T})$ be the timesystem of L and $TA_{TS_L} = (H, \circ)$ be the transition algebra of TS_L. Finally, let $\iota : T \to H$ be the family $(\iota^t)^{t \in T}$. Then:
• ι^0 maps identity element into identity element:
(2.8) $\iota(0) = \iota^0$;
• ι is structure-preserving: for any $t, v \in T$,
(2.9) $\iota(t+v) = \iota^{t+v} = \iota^t \circ \iota^v = \iota(t) \circ \iota(v)$;
• ι is surjective: by Definition 5, for any $h \in H$, $h = \iota^t$ for some $t \in T$; but, by hypothesis,
(2.10) $\iota(t) = \iota^t$;
• ι is injective: for any $t, v \in T$,

$$t \neq v$$

$$t + 0 \neq v + 0$$

(2.11) $\iota^t(0) \neq \iota^v(0)$

$$\iota^t \neq \iota^v$$

$$\iota(t) \neq \iota(v).$$

Hence, ι is a monoid isomorphism of L in TA_{TS_L} and, accordingly, L is isomorphic to TA_{TS_L} .

□

Finally, Proposition 2.1 and Proposition jointly 2.2 guarantee that

Proposition 2.3.
Every time system is isomorphic to the time system of its transition algebra.

Proposition 2.2 and Proposition 2.3 show that monoids and time systems are in reality alternative but equivalent ways of representing the same mathematical structure, whose essential properties are preserved independently of the chosen representation. In other words, demanding that physical time should be mathematically represented by a monoid is as much as demanding that physical time should be endowed with a proper dynamics while, conversely, demanding that the mathematical models of physical time should make sense of its passage is as much as demanding that time should be algebraically modeled by a monoid.

3. The Moving Present

Having identified the dynamics of time with the algebraic structure of its mathematical model is sufficient reason to reject the widespread conviction that the passage of time be a mere metaphor modern physics can simply dispense with. However, one may possibly object that in no way we proved that the dynamics of time systems is effectively the dynamics of a unique moment constantly shifting towards the future, as the ordinary conception of becoming would demand. The following discussion is dedicated to reply to this objection.

Proposition 2.2 and Proposition 2.3 established a one-to-one correspondence between the algebraic features of time models and the dynamical features of their time systems. One of the most interesting consequences of this result is that, due to its distinguishing algebraic properties, the identity element of any given monoid is endowed with exceptional dynamical features. In particular, the identity element is provided with an exceptional *orbit*:

Definition 6.
Let $DS_L = (M,(g^t)^{t\in T})$ *be a dynamical system on a monoid* $L = (T,+)$*; for any* $x \in M$*, the* orbit *of* x *is the set*

(3.1) $$orb(x) \overset{def}{=} \left\{ y \in M : \exists t \in T(y = g^t(x)) \right\}.$$

Intuitively speaking, the orbit of any given state x in the state space of a dynamical system includes all and only those states the system will evolve into if initially set in state x. In the case of time systems, the orbit of the identity element covers the entire state space; and therefore, the dynamics modeled by time systems is in fact the dynamics of their identity elements:

Proposition 3.1.
Let $TS_L = (I,(t^t)^{t\in T})$ *be the time system of a monoid* $L = (T,+)$ *with identity* 0*;*

then $orb(0) = I$.

Proof

Let $TS_L = (I, (\iota^t)^{t \in T})$ *be the time system of a monoid* $L = (T, +)$ *with identity* 0. Then, for any $t \in T$,

(3.2) $\iota^t(0) = t + 0 = t$,

and therefore

(3.3) $orb(0) = \{i \in I : \exists t \in T(\iota^t = i)\} = T = I$.

□

Proposition 3.1 guarantees that time systems do in fact represent the dynamics of a unique shifting moment; and, once again, this dynamics is rooted in the fundamental algebraic properties of time models. But how can this moment be identified with the moving present?

In the first place, we need to ground a formal definition of past, present and future times on the sole conceptual tools offered by dynamical systems theory. These definitions can be obtained out of the more general definitions of past and future states:

Definition 7.

Let $DS_L = (M, (g^t)^{t \in T})$ *be a dynamical system on a monoid* $L = (T, +)$ *with identity* 0*; for any* $x \in M$*, for any* $t \in T - \{0\}$*,*

• *the* t-future *of* x *is the set*

(3.4) $F^t(x) \overset{def}{=} \{y \in M : g^t(x) = y\}$;

• *the* t-past *of* x *is the set*

(3.5) $P^t(x) \overset{def}{=} \{y \in M : g^t(y) = x\}$.

Definition 8.

Let $DS_L = (M, (g^t)^{t \in T})$ *be a dynamical system on a monoid* $L = (T, +)$ *with identity* 0*; for any* $x \in M$*,*

• *the* future *of* x *is the set*

(3.6) $F(x) \overset{def}{=} \bigcup_{t \in T - \{0\}} F^t(x)$;

• *the* past *of* x *is the set (3.7)*

(3.7) $P(x) \overset{def}{=} \bigcup_{t \in T - \{0\}} P^t(x)$.

In general, the t-past set of any arbitrarily given state x includes all and only those states the system possibly displayed a non-null time interval t before entering state x; and, symmetrically, the t-future set of any arbitrarily given state x includes the unique state the system will evolve into a non-null time interval t after having been in state x. In the case of time systems, the t-past of any given moment i includes the states which preceded i by a non-null time lapse of duration t, while its t-future includes the moment which will follow i in a time lapse of the same duration: that is to say, the past and future sets of any moment i model its past and future *simpliciter*. By exclusion, the present of i should only include the image of i as of the time transition of null duration, namely i itself.

It would be easy to see that, given any time system $TS_L = (I, (\iota^t)^{t \in T})$, for any duration $t \in T$, the t-future set of the identity coincides with the present set of the unique moment $i \in I$ such that $i = t$. In other words, what moment counts as present at any time $i = t$ is exactly determined by the t-future evolution of the identity element.

4. Conclusion

Time systems offer a consistent representation of the internal dynamics of the mathematical representations of time, which is in fact equivalent to their algebraic structure. In addition, such a dynamics can be regarded as the dynamics of a unique ever-changing present, whose evolution coincides with the orbit of the identity element of the assumed time model. How this dynamical picture should be coordinated with physical time, or whether there are any metaphysical bases on which the physical present should be identified, is of course an open and very complex issue. But in any case, as long as our mathematical representations of physical time cannot dispense with the algebraic properties of closure, associativity and possession of an identity element that are typical of monoids, the idea of a moving present cannot be simply discarded as a mere projection of human consciousness without any theoretical counterpart.

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