Classical Motion and Instantaneous Velocity

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Abstract

The impetus theory of motion states that to be in motion is to have a non-zero velocity. The at-at theory of motion states that to be in motion is nothing over and above being at different places at different times, which in classical physics is naturally understood as the reduction of velocities to position developments. I first defend the at-at theory against the criticism that it renders determinism impossible. I then defend a novel impetus theory of motion that reduces positions to velocity developments. As these two accounts are, I claim, epistemically on par, I conclude that there is a surprising metaphysical underdetermination in our understanding of classical motion.

1 Introduction

There are two intuitively appealing ways to understand the nature of motion. One is that motion is nothing over and above the occupation, by an object, of different places at different times.[1] This is known as the *at-at theory of motion* (being *at* different places *at* different times) and is generally attributed to Russell, who explicitly argued for it (Russell, 1903, Ch. 54). One is inclined (because of the "nothing over and above" qualification) to say in this case that facts about motion are grounded in facts about the positions of objects; it is, in other words, usually understood as a reductionist account of motion. The other way of understanding the nature of motion is to say that motion is the process of moving, where moving is understood as the possession, by an object, of a non-zero instantaneous quantity of motion—a non-zero instantaneous velocity. I will call this the *impetus theory of motion*, essentially following the terminology in (Arntzenius, 2000). As velocity on this account is understood to be a basic quantity possessed by objects, it is evidently a non-reductive account of motion.

When stated thus, neither view would seem, on the face of it, to necessarily preclude the other. The possession of an instantaneous velocity is not obviously at odds with being at different places at different times; in other words, instantaneous velocities appear to be entirely consistent with the at-at theory of motion. There is a compelling argument, however, that the at-at theory implies that there are in fact no *truly* instantaneous velocities (Russell, 1903; Arntzenius, 2000; Albert, 2000). Briefly, the argument is that the explication of instantaneous velocity using calculus makes reference to non-instantaneous

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[1]This characterization makes reference to places in a way that is suggestive of a substantivalist interpretation of space. One could, with slight modifications, produce a relationist version as well.

position *developments* and hence velocity is not truly instantaneous. Although at each instant one can define an object's velocity *at* that instant, the velocity itself is fully grounded in the object's past and future behavior. Thus, according to this argument, the at-at theory of motion and the impetus theory of motion are incompatible theories of motion.

In a notable paper on the topic, Arntzenius claims significant drawbacks to both theories of motion. He claims that the at-at theory precludes the possibility of determinism by the way it defines velocity, whence he concludes that "surely the 'at-at' theory is wrong if it entails that" (Arntzenius, 2000). The impetus theory avoids this problem by reifying instantaneous velocities but does so at the ontological costs of adding fundamental quantities and necessarily imposing what I will call a kinematic constraint—an additional law of nature. Since the at-at theory is sufficient for describing motion without the inclusion of instantaneous velocities, the impetus theory should, it seems, by parsimony be rejected. Given these drawbacks one might find both theories unsatisfactory, as Arntzenius himself concludes, putting us in the rough when it comes to a theory of motion.[2]

In my view the lie is not so bad. My aim in this paper is to render satisfactory versions of both views, primarily against the criticisms of Arntzenius. There are various other criticisms of the at-at theory of motion as well, based mainly on the notion that the reductive account fails to do justice to the causal role of velocity (Tooley, 1988; Lange, 2005; Easwaran, 2014). As I think there is a natural extension of what I say here to those concerns (and for the sake of concision), I leave discussion of these concerns for another occasion and concentrate on Arntzenius's arguments.

The course of the paper is as follows. In §2 I rebut Arntzenius's criticisms of the at-at theory of motion, arguing that the at-at theory only precludes one particularly strong version of determinism and does not in fact do so "by logic and definition alone," as Arntzenius claims. In §3 I take up the impetus theory of motion. I do agree with Arntzenius's argument against the impetus theory of motion he discusses, which merely superadds instantaneous velocities to the at-at theory of motion, and rehearse this argument for convenience. There is, however, a subtle way to develop the impetus account to avoid any metaphysical profligacy. Arntzenius (2003) himself in fact suggests it. The idea is to treat velocity as basic (as the impetus account states) but define positions in terms of velocity developments through integration.[3] He quickly rejects it, albeit for what I will argue are poor reasons. I pick up the idea and develop it further, fully in parallel to the at-at theory, i.e. as an alternative reductive account, one that, however, instead reduces position facts to velocity facts rather than the other way around.

Thus, whereas Arntzenius concluded with two unsatisfying theories of classical motion, making a choice between them problematic, I conclude with two satisfactory views, which, however, also makes a choice between them problematic. I do not think the underdetermination can be easily broken, for example merely by intuitive considerations or preferences. Instead, the existence of these incompatible ways of understanding motion is in my view desirable, as it reveals surprising facts about the most familiar kind of physical motion.

2 The At-At Theory

The at-at theory of motion holds that what it is to be in motion is nothing over and above being *at* different places *at* different times. To add physical content to the account, it will be convenient to set it in the theoretical framework of classical mechanics. In classical mechanics motion is conventionally understood to be continuous and velocity is the time rate of change of position defined using calculus. Velocity is hence a derived quantity of motion depending only on objects' positions at different times. The standard presentation of classical mechanics therefore is naturally understood in the at-at way.

[2] Arntzenius also discusses the possibility of rejecting instants as a way of avoiding the difficulties he presents, but finds that route unsatisfying as well. The "no instants" view falls outside of the scope of this essay, so it will not be discussed further.

[3] A somewhat similar idea is also presented in (Tooley, 1988), although he does not take velocities as basic, instead basing them on position developments. Thus his view is essentially the superadditive impetus theory of motion.

2.1 Instantaneous Velocity as a Neighborhood Property

Let us begin with the claim that velocity on the at-at theory is not a truly instantaneous quantity (Russell, 1903; Albert, 2000; Arntzenius, 2000). Here is the typical calculus-based definition of "so-called" *instantaneous velocity*:

DEFINITION.

$$\dot{\mathbf{x}}(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} := \lim_{\Delta t \to 0} \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t},$$

where $\dot{\mathbf{x}}$ (the so-called instantaneous velocity) is a function of time t, $\mathbf{x}$ is the position as a function of time t, and Δt is a temporal interval.

Conceivably, position functions can take any functional form, as there is no a priori restriction on position functions such that position at a time must depend on the positions at other times. In other words, one may even allow discontinuous function, at least in principle. Thus, position is clearly a truly instantaneous property and, furthermore, is part of an object's instantaneous state—the complete description of all dynamical (changeable) properties of an object at an instant.

Is instantaneous velocity a truly instantaneous quantity and part of an object's instantaneous state? While plainly instantaneous velocity *is defined* at particular times t, Albert and Arntzenius argue that it is not truly a quantity of an object at an instant, given the above definition of velocity. As Albert says, "What needs to be kept in mind is just that there is all the difference in the world between being uniquely attachable *to* some particular time and being the component of the *instantaneous physical situation of the world* at that time!" (Albert, 2000, 17). So, although an instantaneous velocity can be defined for instants that satisfy the conditions of differentiability, it would be a mistake, according to Albert and Arntzenius, to say that this velocity is a quantity of an object at an instant.

As this is an essential point for what follows, I will expand on the reasoning a bit further. Consider, then, the case of average velocity, normally defined as follows:

DEFINITION.

$$\bar{\mathbf{v}}(t, \Delta t) := \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t}.$$

Of precisely what is an average velocity a property? In general an object in motion possesses an instantaneous velocity equal to the average velocity only at isolated instants (unless of course it is in uniform motion). One would not want to say that average velocity is a property of an object only at these points. One would also not want to say it is a property of the object at each instant in the interval, because average velocity in general changes depending on the chosen interval Δt (which is why it is included as an argument of the average velocity function). Evidently, then, average velocity should be understood as the property of an object *over an interval*, i.e. the interval between t and $t+\Delta t$.

Similar thinking suggests that instantaneous velocities are not truly instantaneous properties in the at-at theory. Refer to the definition of instantaneous velocity above. It necessarily makes reference to intervals (Δt) in the neighborhood of the given instant (limit point) t. Instantaneous velocity is therefore better thought of as what Arntzenius calls a *neighborhood property*. Neighborhood properties are properties possessed by an object not at an instant but in *neighborhoods* of an instant. It is crucial to understand that a neighborhood property is not to be thought of as a property of some *specific* interval around the instant, however, for it is not some interval to which one attributes the property but to the development of the function in the neighborhood of the instant.

To clarify the notion of a neighborhood property and the nature of the reference at work here, we should look to the familiar $\varepsilon - \delta$ definition of a limit:

DEFINITION. *The function f* ***approaches the limit*** *l* ***near*** *a means: for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $|f(x) - l| < \varepsilon$.*

To put this definition to work in the case of instantaneous velocity, I replace the symbols in the previous definition with the ones used above in the definition of instantaneous velocity, noting that the instantaneous velocity is equal to the limit of average velocities in the neighborhood of t, i.e. $\dot{\mathbf{x}}(t) = \lim_{\Delta t \to 0} \bar{\mathbf{v}}(t, \Delta t)$:

DEFINITION. *The function $\bar{\mathbf{v}}(t, \Delta t)$* ***approaches the limit $\dot{x}$ near*** *t means: for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all Δt, if $0 < |\Delta t| < \delta$, then $|\bar{\mathbf{v}}(t, \pm\Delta t) - \dot{\mathbf{x}}(t)| < \varepsilon$.*

It is worth unpacking the limit definition a bit in words. The idea is to pick any positive real number ε you like. If the instantaneous velocity exists at t, then what the definition requires is that one can always find some neighborhood around t, i.e. a temporal interval $t - \Delta t < t < t + \Delta t$, such that the difference between the average velocity over both sides of the interval and the instantaneous velocity is less than ε. Since the choice of ε is up to us, we can require the deviation between them to be as small as we like; if the instantaneous velocity exists, there will *still* be some choice of δ for which all neighborhoods of t up to $t \pm \delta$ will secure the required deviation (or less). Thus, the property of an instantaneous velocity does not depend on any particular neighborhood (one can always choose an ε which forces a choice of δ that "moves inside" of that neighborhood); it does, however, clearly depend on more than the position of the object at t to exist.[4]

An example should bring out some of the ramifications of these calculus considerations. Consider an object that is in uniform motion, which at some instant t jumps discontinuously to position L before returning to its previous inertial trajectory (position development). The limit of the object's position $\mathbf{x}(t)$ and its instantaneous velocity $\dot{\mathbf{x}}(t)$ are well-defined in this example. Because it is defined by way of t's neighborhood, the limit of its position is where it would have been had it not made the jump and its velocity is the velocity it was traveling at before and after the jump. The limit of position is thus a neighborhood property here, and it does not necessarily agree with the instantaneous position at time t. Similarly, its instantaneous velocity is a neighborhood property; it is not clear, however, what to say about the object's "actual" state of motion at its "jump" location, except perhaps that it is moving "discontinuously" (which still requires reference to its behavior at other times).[5]

Now suppose, as a second case, that the object jumped to another position $M \neq L$. In both of these cases the instantaneous positions of the two objects at time t differ but their "neighborhood positions" are identical. While it may sound strange to say that the object is at one place (in the usual sense) and, in another sense, not there, once one recognizes that neighborhood position is just a way of describing the continuity of some object's position development, it is clear that instantaneous velocity too is a way of describing an aspect of the object's position development, namely how fast it is changing.

Before turning to Arntzenius's critique of the at-at theory, I should mention the fact that other authors have made use of examples like these, where motion is discontinuous or not differentiable, in order to conjure intuitions about the nature of motion (Jackson and Pargetter, 1988; Tooley, 1988; Carroll, 2002). In general I will prefer to restrict attention in the following to classical motion, i.e. the kind of motion described by classical mechanics. In classical mechanics, as I noted above, motion is understood to be continuous. This assumption also permits one to take motion to be smooth (infinitely differentiable), since according to the (Stone-)Weierstrass theorem any continuous function can be approximated as closely as one likes by a smooth one over some closed interval (here, of time). I will make essential use of this assumption later, although I will occasionally consider cases where discontinuous motion is permitted.

2.2 Determinism and Motion

Arntzenius takes it to be an objectionable consequence of a theory of motion if by definition and logic alone determinism is rendered impossible. Let us see how this objection is raised against the at-at theory

[4] Cf. (Butterfield, 2006).

[5] Cf. (Jackson and Pargetter, 1988).

of motion.

The type of determinism Arntzenius has in mind is usually called *Laplacian determinism*, as it is evoked by Laplace in this oft-quoted passage:

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it...it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. (Laplace, 1902)

Albert (2000, 10) describes the present state of the world as the instantaneous states of every object in the world at the present instant, a description that accords with Laplace's. It is also the one that Arntzenius also adopts. These instantaneous states, along with the appropriate dynamical law (determined by the various classical forces at work), are the things one needs to know in classical mechanics to deduce the trajectories of all the objects in the world, if classical mechanics is deterministic in the Laplacian sense. If it is, then this is the sense in which Laplace's intelligence could know the movements of all objects for all times, past and future.

If, however, the at-at theory is correct and instantaneous velocity is not an instantaneous quantity in classical mechanics, then classical mechanics is not deterministic in the Laplacian sense.[6] On Albert's understanding of the present state of the world, the intelligence would be left scratching its head looking at its impoverished situational report, unable to determine anything whatsoever about the future or the past on its basis—all this, according to Arntzenius, because instantaneous velocity was simply *defined* in the way recently shown. Thus, we have the claim that determinism is rendered impossible by definition and logic alone.

The obvious first rejoinder to Arntzenius's objection is to say that our definition of determinism (or alternately, present state of the world) is mistaken. If we say this, though, then we finds ourselves in what may seem to be an uncomfortable dilemma over definitions. On the one hand, we can insist that the definition of instantaneous velocity is correct and, on the other, that Laplacian determinism (or present state of the world) is defined correctly. If one is unperturbed by giving up Laplacian determinism, then a simple solution is obviously at hand. Just give the Laplacian intelligence a mulligan and let it have access to neighborhood properties as well as the truly instantaneous state of all objects. In so doing we modify the notion of "present state of the universe" to include the neighborhood of the present, i.e. determinism comes to depend on both neighborhood and properly instantaneous properties.

Why, though, prefer one definition to the other? It seems to me that it is not so much a matter of preference, since it is not the case that definitions alone are not at issue. As Earman observes, for example, "we cannot begin to discuss the implications of physics for the truth of the doctrine of determinism until we know what determinism is; on the other hand, no precise definition can be fashioned without making substantive assumptions about the nature of physical reality" (Earman, 1986). As Earman rightly notes here, we have to make some substantive assumptions about the world even to begin evaluating the notion of determinism. So long as we are making substantive assumptions about the nature of physical reality, however, the issue at hand is at least not simply a matter of logic and definition alone.

What are the "substantive assumptions" underlying the definitions employed here? At first stroke they appear to depend on interpretive matters concerning laws of nature. One who thinks that the laws *produce* new physical situations from old ones may find any relaxation of the definition of determinism objectionable.[7] The assumption undergirding Laplacian determinism, from this point of view, is that the laws act on instantaneous physical states to produce new ones. If one thinks instead that laws are merely

[6] Although it no doubt courts confusion, I will continue to refer to instantaneous velocity as "instantaneous" even if it is not truly instantaneous, since that is the term usually used for it.

[7] Immediately preceding the quoted passage above, Laplace himself indicates this view when he says, "we ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow" (Laplace, 1902).

a *description* of regularities that can be gleaned from the physical facts, then relaxing the definition of determinism is entirely unproblematic. The state of the world does not have to be instantaneous; it may be whatever the most physically salient notion of determinism requires. In the case of classical mechanics, the relevant state of the world is then the neighborhood state of an instant.

That Arntzenius has the former point of view in mind follows from a further objection he makes. That objection is that assuming the standard definition of instantaneous velocity imposes *non-dynamical* constraints on evolution. This is objectionable, as he says, because

> ...surely our notion of a physical state is such that being in a particular physical state at some time does not by definition and logic alone put any constraints on what physical states the system can be in at other times. Physics may impose constraints on the possible developments of the physical states of systems, but surely logic and definition by itself should not do so. And that implies that neighborhood properties and neighbor hood states are not physical states, they are features of finite developments of physical states. (Arntzenius, 2000, 195)

If one has the idea that laws act on truly instantaneous states of affairs to produce new states of affairs, this concern might be quite reasonable. For this objection to be sustainable, though, it must be that Arntzenius is right about two things: that only physics can impose such constraints, and that physics does not impose the constraints that instantaneous velocity places on position developments.

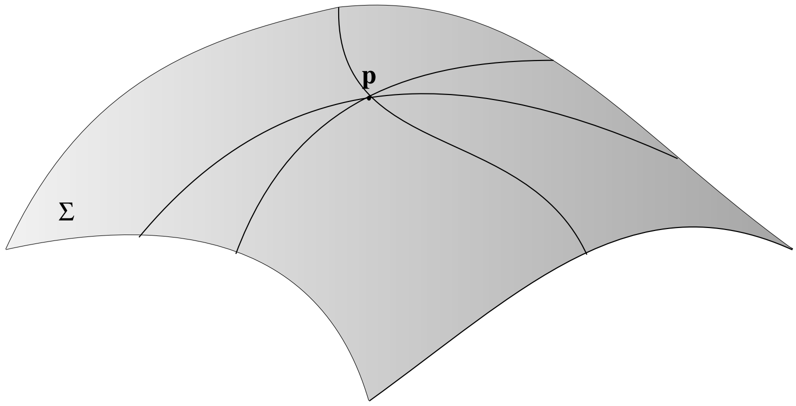

Figure 1: Some trajectories compatible with an object on surface Σ located at position p.

What are these constraints? Consider for simplicity the case of some object constrained to move on a surface Σ. At some time let the position of the particle be **p**. If position is the only instantaneous property of the object at that time, then *any* trajectory on the surface is kinematically possible (see Fig. 2.1), i.e. possible before consideration of constraints imposed by forces and the dynamical laws.[8] If one allows the instantaneous velocity to be part of the state of the particle, then the possession of a particular velocity, say **v**, restricts the possible position developments to the past and to the future (see Fig. 2.2). Only those developments which have **v** as their derivative at **p** are kinematically possible when both **p** and **v** are part of the state. Note that velocity imposes this constraint only in the neighborhood of **p**—for any point **q** in the neighborhood of **p** there exists some kinematically possible trajectory that passes through **q** from **p**—so it is in some sense a weak constraint but a constraint nonetheless.

[8]It is worth noting that the assumption of smooth trajectories also limits the possible trajectories considerably, although Arntzenius does not mention this assumption as similarly problematic, perhaps because he counts any position development whatever as physically admissible.

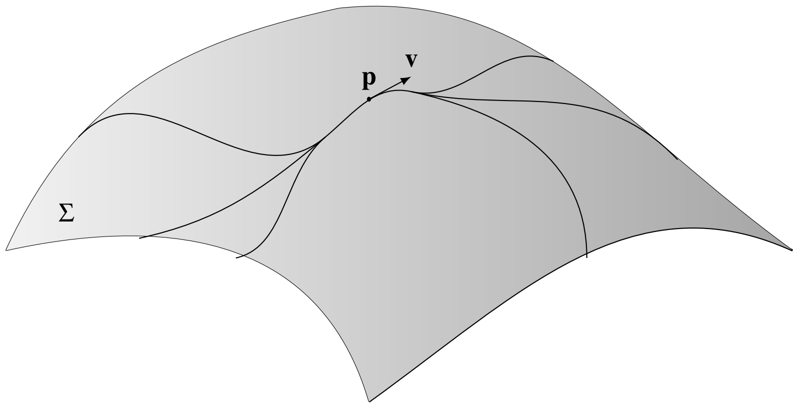

Figure 2: Trajectories compatible with an object on surface Σ located at position p with velocity $\vec{v}$.

There are, as I see it, two ways to respond to Arntzenius's charges.

First, we may allow that the kinematic constraint imposed by instantaneous velocity is actually a kind of "metaphysical" constraint. This constraint need not be imposed in such a way that instantaneous velocity becomes truly instantaneous. Rather it may be imposed in much the same way that dynamical constraints are imposed, namely via laws. For example, the requirement that trajectories be smooth might be considered a metaphysical law that is understood to be prior to physical law. Similarly, the kinematic constraint imposed by instantaneous velocity might be considered a metaphysical law.

Why think that this constraint is a metaphysical law? Perhaps because it expresses "what we mean" by velocity, or maybe because it is merely a pre-condition on any physics that would describe what we call classical motion. At the very least, the possibility of this view challenges the notion that only physics can constrain position developments as Arntzenius assumes. Either one accepts this possibility—physics is, after all, obviously committed to many metaphysical presuppositions already—or one has the burden of showing why there cannot be constraints like this.

On one popular view of metaphysics only metaphysical necessity can ground a metaphysical law of the kind I suggest. The exhibition of a possible world where that law does not hold is sufficient to undermine its lawhood. From this point of view one only has to consider worlds where there is discontinuous motion, as Tooley (1988) and Carroll (2002) do, to see that the metaphysical constraint imposed by instantaneous velocity does not hold. If one allows that metaphysical laws, like special science laws, do not need to constrain all metaphysical possibilities, then such thought experiments do not militate against this suggestion.

Nevertheless, there is a second way to respond that to some extent obviates the former. That way is to take the kinematic constraint imposed by instantaneous velocity to follow from the dynamical constraints imposed by the laws. The general form of such dynamical constraints in classical mechanics is given by Newton's Second Law:

$$\sum \mathbf{F} = m\mathbf{a};$$

that is, the instantaneous sum of forces on an object is equal to the mass of the object times its acceleration. The acceleration $\mathbf{a}$ is usually defined as the time derivative of the instantaneous velocity $\dot{\mathbf{x}}$; in other words, Newton's Second Law is

$$\sum \mathbf{F} = \ddot{\mathbf{x}},$$

where the acceleration $\ddot{\mathbf{x}}$ is the second time derivative of position.[9] To make acceleration the second time derivative of position, one must have a first time derivative of position, i.e. instantaneous velocity. Given that Newton's Second Law is a dynamical constraint of the kind that Arntzenius accepts as a physical constraint, it would seem to follow as *a matter of course* that the kinematic constraint imposed by instantaneous velocity is a physical constraint as well—it is, from this point of view, a constraint that simply follows from Newton's Second Law. The point is more transparent when one re-writes Newton's Second Law as two first order differential equations:

$$\sum \mathbf{F} = \dot{\mathbf{u}};$$
$$\mathbf{u} = \dot{\mathbf{x}},$$

where $\mathbf{u}$ is the instantaneous velocity. Here the kinematic constraint that is *part* of Newton's Second Law is explicit.

Let me summarize this section. I have argued that the objections raised by Arntzenius against the at-at theory of motion are unfounded. Although I agree with Arntzenius that instantaneous velocity is not truly instantaneous according to the at-at theory and is better understood as a neighborhood property, I disagree with the claim that instantaneous velocity threatens the determinism of classical mechanics.[10] If one does insist on holding onto the idea that only truly instantaneous properties can figure into the physical state of the world (or a system), then one must give up on the idea that the laws produce future states on this basis alone (in keeping with Laplacian determinism). But it is not true that this is a matter of logic and definition alone, as Arntzenius assumes. If one allows that the state of the world included neighborhood properties (arguably one may do this however one chooses to interpret laws of nature), then there is a clear and substantive sense in which classical mechanics is deterministic. This idea is motivated by attending to the kinematic constraints imposed by instantaneous velocities. If they are seen as either falling under a metaphysical law or following simply from the definition of Newton's Second Law, then they should be considered a part of the physically relevant state of the world.[11]

3 Impetus Theory of Motion

The impetus theory of motion holds that what it is to be in motion is possessing a non-zero quantity of motion, i.e. a non-zero velocity. In the context of classical dynamics this velocity must be a truly instantaneous velocity, just as position is taken to be truly instantaneous. However, since it is customary in classical dynamics to call the time rate of change of position, i.e. $\dot{\mathbf{x}}$, the instantaneous velocity, I will call the truly instantaneous quantity of motion of an object its *impetus* and denote it $\mathbf{v}$. To further avoid confusion I will usually henceforth refer to $\dot{\mathbf{x}}$ as the *kinematic velocity*. In this section I investigate the prospects of the impetus theory of motion in the context of classical motion. First, in agreement with Arntzenius (2000), I reject rescuing Laplacian determinism by simply supplementing classical dynamics with impetuses. I then raise a new challenge to the at-at theory of motion by re-conceiving classical mechanics in a way that takes velocity rather than position as basic and makes position a derived quantity.

3.1 Superaddition of Impetus to At-At Theory of Motion

I argued in the previous section that the Laplacian picture of determinism—a situation at a time being evolved deterministically forward by the laws of motion—is untenable on the at-at theory of motion. The

[9] It should not be necessary to rehearse the arguments of the previous subsection to see that acceleration is a neighborhood property like instantaneous velocity.

[10] Other threats to determinism, raised in (Earman, 1986) and (Norton, 2008), among others, remain standing of course.

[11] Responses to Arntzenius can also be found in (Butterfield, 2006) and (Smith, 2003a,b), although in the end they merely advise following the practice of physicists in including velocity as part of the physical state. This section improves on these responses by offering a philosophical rationale.

issue was that velocities are non-instantaneous on the at-at theory of motion while the laws of motion require truly instantaneous velocities in order to be deterministic (in the Laplacian sense). An obvious solution to this problem is to supplement the instantaneous properties of objects with a truly instantaneous velocity, i.e. an impetus. Then one may ostensibly maintain the definitions of determinism and physical state preferred by Albert and Arntzenius.

One is, however, quickly forced into difficulties with the impetus view when one considers the description of motion (Arntzenius, 2000). Insofar as objects follow continuous trajectories in classical mechanics, the kinematic velocity $\dot{\mathbf{x}}$ exists and correctly describes the time rate of change of position. Hence it is, in short, a velocity. If the impetus is to play a role in describing motion, then it must necessarily be equated with the kinematic velocity: $\mathbf{v} = \dot{\mathbf{x}}$. As the kinematic velocity and the impetus are both meant to be velocities describing motion, such a relation introduces a kind of *kinematic constraint*. They are to be considered, following the discussion in the previous section, as either metaphysically or physically necessary in classical mechanics. Yet, given the necessity of this kinematic constraint, impetuses appear to be incapable of doing any additional "work" in the theory over and above kinematic velocity. They are physically idle. As they are only introduced to soothe an acute metaphysical ailment, one should, by parsimony, preclude impetuses from classical ontology along with their associated kinematic constraints.

Although impetus is necessarily constrained to equal kinematic velocity for differentiable trajectories, perhaps one might think that the conceptual independence of the two velocities becomes manifest in a wider context, viz. one where trajectories are not continuous. This is not so. If position is not a differentiable function of time at some instants, then the impetuses must likewise be undefined at these instants, else they would be irrelevant to the description of actual motion. Suppose, for the sake of argument, that impetuses could disagree with the kinematic velocities. In the case of differentiable position functions the impetuses would not function as quantities of motion as intended, since they would give the wrong dynamical evolution of the objects' positions. In cases where the position functions results in undefined kinematic velocities at some instants, if impetuses were to possess a defined value, then they would also give the wrong dynamical evolution, as they would indicate a future motion that does not occur. Impetus is therefore, I claim, entirely expendable. Both velocities must agree when kinematic velocity is well-defined, whereas the kinematic velocity is not beholden to impetus in any physical way.

3.2 Position Reduced to Velocity Developments

Insofar as one makes the usual assumption that position is included in an object's instantaneous physical state, I have argued, in agreement with Arntzenius (2000), that the impetus view is superfluous and instantaneous velocity is not really instantaneous. I objected above, however, to the idea that there is something problematic about the at-at theory of motion vis-à-vis determinism, and so it seems that I have given reason to think that the at-at theory should be our preferred theory of classical motion. So much for a defense of orthodoxy. I now turn to a novel heretical challenge to the at-at theory. Rather than attempting to reveal the flaws of orthodoxy like other authors, I will defend an alternative conception of classical mechanics that naturally accords with the impetus theory of motion and not the at-at theory of motion. I claim that this alternative underdetermines our interpretation of classical motion, as only epistemically inaccessible facts could decide between them.

The incipient idea is to take velocity as a quantity of an object's instantaneous state and position as a quantity derived from velocity by integration: $\mathbf{x}(t) = \int \mathbf{v}(t)\,\mathrm{d}t$. This naturally leads to a kind of impetus theory, since velocity is now taken as a truly instantaneous quantity. In this case $\mathbf{v} = \dot{\mathbf{x}}$ not because they are equal by a kinematic constraint, but because the fundamental theorem of calculus applied to the definition of position just given informs us that the terms on both sides simply refer to the same property.

Arntzenius in fact briefly raises this possibility in his response to (Smith, 2003a) but quickly overrules it because there are "velocity developments that are incompatible with calculus"—in particular the "calculus definition of velocity" (Arntzenius, 2003, 282). In essence, he claims that one cannot allow arbitrary velocity functions (as one allows arbitrary position functions in the at-at view) because time derivatives of position functions cannot recover a (very) large class of these arbitrary velocity functions, viz. those functions which are not the derivatives of any function. Since this calculus definition of velocity would limit what functions could be used to describe velocity,"logic and definition alone would still imply constraints between instantaneous states at different times "(Arntzenius, 2003, 282).

An example will help clarify Arntzenius's objection. I will just use his example, the pathological function known as the Dirichlet function. Let $\mathbf{v}(t)$ be defined such that $\mathbf{v}(t) = 1$ for rational t and $\mathbf{v}(t) = 0$ for irrational t.[12] Now suppose that there is a position development $\mathbf{x}(t)$, the derivative $\dot{\mathbf{x}}(t)$ of which is this velocity development $\mathbf{v}(t)$. Then it must be the case that $\bar{\mathbf{v}}(t, \Delta t)$, i.e. $(\mathbf{x}(t+\Delta t) - \mathbf{x}(t))/\Delta t$, approaches the limit $\mathbf{v}$ near t. This cannot be the case, however, since for any ε such that $1 > \varepsilon > 0$ the inequality $|\bar{\mathbf{v}}(t, \pm\Delta t) - \dot{\mathbf{x}}(t)| < \varepsilon$ will not hold. This is because as one moves closer to t the Dirichlet function repeatedly jumps between 0 and 1 so that no limit point is ever approached. Since velocity functions with characteristics similar to the Dirichlet function cannot be recovered by differentiating position functions in this way, it seems that this definition of velocity imposes constraints on possible velocity developments merely by "logic and definition".

I say this is a bad argument. Before explaining why it is, though, some preliminaries are needed. Suppose that we do take pathological functions like the Dirichlet function seriously (for the moment) as physically possible velocity developments. In the impetus view presently under consideration, position is a derived property, as it is defined via integration. Here it becomes quite important precisely which notion of integration we use. The notion of integration familiar from basic calculus, Riemann integration, cannot be applied to the Dirichlet function. The function has no integral. Therefore an object with the Dirichlet velocity development has no position development when position is defined via Riemann integration.

Surely here is the problem, the absence of well-defined positions. Yet suppose that we allow everywhere continuous but nowhere differentiable position developments in the at-at theory of motion. Then an object whose position development is described by the Weierstrass function, the most well-known function of this kind, has no velocity development. Such a case, I submit, is no more strange than the previous. If pathological functions are a problem for the impetus theory, then they are for the at-at theory as well.

Arntzenius's objection, however, is not that the mere lack of a position development is problematic. Rather it is that there exist velocity developments which are not the derivatives of any position development according to the previously given definition of derivative. He claims that to assume this definition of velocity is to impose a constraint on future velocities at any given time (as discussed previously, he thinks that mere definition cannot constrain physical behavior). But why should we assume *this* definition of velocity in the present impetus theory of motion? Velocity is not defined in the impetus theory—it is taken as basic; position is derived and hence must be defined. Arntzenius's objection apparently depends, then, on duplicitously treating velocity as both basic and derived. If one were to do the same in the context of the at-at theory, i.e. treat position as both fundamental and derived, then one would have precisely the same problem: there exist position developments which are not the integrals of any velocity development. A position development following the Weierstrass function would be such a case. Hence there would be position developments that are incompatible with calculus, namely the calculus definition of position (via integration).

There are, however, alternative definitions of integration according to which functions like the Dirichlet function are in fact integrable. The well-known Lebesgue integral, for example, generalizes the notion of Riemann integration by utilizing a measure with respect to which integration is performed. Given a set X and a measure μ on the measurable subsets of X, the Lebesgue integral of a function f over the set $A \subset X$ is written $\int_A f \, \mathrm{d}\mu$. If we take the f to be the Dirichlet function, $X = \mathbf{R}$, $A = [0, 1]$, and μ the

[12]The particular details of this function are not visualizable at any scale. If one tries to plot it, it just looks like two lines at $\mathbf{v} = 1$ and $\mathbf{v} = 0$. The Thomae function is an alternative that is better able to be visualized.

standard Lebesgue measure associated with the real numbers, then

$$\int_A f \, \mathrm{d}\mu = 1.$$

Interestingly, the Lebesgue integral of the Dirichlet function over this finite interval is just a constant.

If we adopt this alternative definition of integration, then it appears we have a new problem: the usual calculus derivative of a constant is zero, which is obviously not the same as the Dirichlet function. Previously we had the fundamental theorem of calculus to guarantee a certain duality between differentiation and integration. To restore this kind of duality what we need to do is modify our definition of derivative to regain an analog of the fundamental theorem. The simplest way to do this is just to define differentiation in terms of Lebesgue integration, namely as

$$\lim_{A \to 0} \frac{1}{\mu(A)} \int_A f \, \mathrm{d}\mu.$$

The Lebesgue differentiation theorem guarantees that this derivative exists and, more importantly, equals f at almost every point in X. Thus if we take f to be the Dirichlet function, one of the possible derivatives of its integral is the Dirichlet function. While it is true that other derivatives are possible, viz. functions that differ in their values on a set of measure zero, this is no problem for an impetus theory of motion based on Lebesgue integration, since the velocity developments are basic and only positions are derived. Thus it is only necessary that we can recover the Dirichlet function by differentiation.

Still, recalling the concerns in the previous subsection, it does remain puzzling why an object moving along a constant derived trajectory would have such a complicated velocity development, as the non-zero velocity deviations appear to have no physical effect on the object's position. But by now this excursion into complicated mathematical analysis should appear rather strained. For the purposes of mechanics and motion, we would do well to heed the words of Poincaré from 1899:

> Logic sometimes makes monsters. For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. Indeed, from the point of view of logic, these strange functions are the most general; on the other hand those which one meets without searching for them, and which follow simple laws appear as a particular case which does not amount to more than a small corner.
>
> In former times when one invented a new function it was for a practical purpose; today one invents them purposely to show up defects in the reasoning of our fathers and one will deduce from them only that.
>
> If logic were the sole guide of the teacher, it would be necessary to begin with the most general functions, that is to say with the most bizarre. It is the beginner that would have to be set grappling with this teratologic museum. (as quoted in (Kline, 1972, 973))

The point to take away is that logical and mathematical possibility by no means correspond to metaphysical and physical possibility. When one reasonably restricts attention to suitably well-behaved functions to describe the motion of objects, one is able to keep the relevant physical notions in plain sight. That physics permits the reductive impetus theory of motion proposed here just as much as it permits the reductive at-at theory of motion discussed in the previous section. The wider context of mathematical functions is just a distraction—for the at-at theory as much as for the impetus theory.

Let us then turn to questions of ontology as a way of distinguishing this new impetus theory from the at-at theory. The at-at theory of motion takes position as basic and therefore is naturally understood as depending on the existence of space and time. Objects have a position in virtue of their embedding in space, at least when speaking in the substantivalist mode. What then does this impetus theory suggest? The impetus theory of motion takes velocity as basic, so, in analogy to the at-at theory, it is naturally understood as depending on the existence of some sort of *velocity* space. Objects have a velocity, one would say, in virtue of their embedding in velocity space. Space is a derivative concept (and perhaps emergent in some sense) in this context. Other ontologies apart from these are surely possible, e.g. relationalist alternatives, but let us concentrate on these naturally suggestive ones for the purpose of elucidating the reductive impetus theory of motion.

Granted, as spatial reasoning is so familiar, this velocital view is perhaps rather strange and unintuitive. A simple imaginative illustration may be helpful. Suppose that we are at the links looking at a golf ball in flight, and furthermore that we were able to perceive its instantaneous physical situation. Look at the ball. Is it moving? On the at-at theory we would certainly be able to say *where* it is but not whether it is *moving*, since it has no instantaneous property of velocity. If we knew the facts about the temporal neighborhood of that instant, then we could say whether it is moving or not. On the impetus theory the situation is just the opposite. In this case we would certainly be able to say whether the ball is *moving* or not but we could not say *where* it is. Similarly, if we knew the facts about the temporal neighborhood of that instant, then we could say quite well where the ball is. Noteworthy is the fact that the two views are not metaphysically intertranslatable given their different verdicts in this example—they say different things about the world at instants of time.

Now, it may have already occurred to some readers that there is at least one salient asymmetry between derivatives and integrals of the well-behaved functions used in mechanics: the derivatives of these functions are determined, whereas (indefinite) integrals are determined only up to an additive constant, the constant of integration. Thus, on the at-at theory, given a temporal neighborhoods, the velocities would be fully determined, whereas on the impetus theory, by contrast, given a temporal neighborhood, the positions would be determined only up to a constant.

Something must be said about this worry. Does this circumstance militate against the impetus theory? Well, the fundamental dynamical law in Newtonian mechanics, Newton's Second Law, at least does not by itself force a problem upon us. Re-expressing $\mathbf{F} = m\mathbf{a}$ in the impetus theory's terms yields $\mathbf{F} = m\dot{\mathbf{v}}$, which makes no necessary reference to position, so no reference to integrals. It is only when the forces depend on positions that the Second Law necessarily becomes a second order ordinary differential equation. Then the constants of integration become physically significant.

So consider, then, the example of Newton's Law of Universal Gravitation, which depends on relative positions to determine motion. For simplicity, consider the gravitational system of two objects and Newton's Law applied to one:

$$\mathbf{F}_{12} = Gm_1m_2\frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|^3},$$

where $\mathbf{F}_{ij}$ is the force on object i due to object j, G is Newton's gravitational constant, m_i is the mass of the object, and $\mathbf{x}_i$ is the position of the object i. In the impetus theory of motion the positions are derived quantities, so we would want to re-express Newton's Law of Universal Gravitation as follows:

$$\mathbf{F}_{12} = Gm_1m_2\frac{\int(\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t}{|\int(\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t|^3}$$

(where I ask the reader's forbearance in interpreting the precise meanings of terms in this equation). What the gravitational force is here clearly depends on what the integration constant is. Moreover, if one has the view that laws act on instantaneous states, then these integration constants are essential for securing determinism.

The reductive impetus theory of motion, however, like the at-at theory of motion, does not rescue Laplacian determinism. Instantaneous velocities in concert with Newton's Second Law cannot fully determine future and past states alone, just as instantaneous positions in concert with Newton's Second Law cannot. So, while the desire to save this kind of determinism motivated the consideration of impetuses above, it is certainly not a motivation to consider this impetus theory. But if one does not expect the impetus theory to rescue Laplacian determinism, then there is no problem with determining the integration constants.

An extended example would illustrate why, but for reasons of space I will instead give a slightly shorter argument that should be sufficiently compelling. First, recall that one can write Newton's Second Law as two first order ordinary differential equations: $\sum \mathbf{F} = \dot{\mathbf{v}}$ and $\mathbf{v} = \dot{\mathbf{x}}$. In the impetus theory the relevant equations would be written as follows: $\sum \mathbf{F} = \dot{\mathbf{v}}$ and $\mathbf{x} = \int \mathbf{v}\,\mathrm{d}t$. Note, however, that the fundamental theorem of calculus allows us to recover the first set from the second for the kinds of functions one considers in classical mechanics. In short, the impetus theory of motion relies on Newton's dynamical laws exactly as the at-at theory does.

Now, what these dynamical laws do is select, from the set of kinematically possible motions, the dynamically possible motions. By a motion I mean a specification of each object's position and velocity at each instant. In physically reasonable cases this evolution is deterministic, in the sense that these dynamically possible motions do not "cross" (do not ever share the same positions and velocities). It is not enough to pick out a particular motion to have just the velocities of each object at a particular instant, since many motions have the same velocities at an instant but different position specifications. It *is* enough, however, to specify the velocities of each object at two instants to pick out a particular motion, just as it is sufficient to specify the positions of each object at two instants to pick out a particular motion. Since the integration constants are required to fully specify the position developments (trajectories), it might appear, on the face of it, that some additional information is required. Picking out a particular motion, however, determines an entire position development, hence whatever physically relevant integration constants are included in those position developments. Thus there is no conceptual problem created by integration constants (just a slight practical complication with posing initial value problems).

Therefore I claim that the reductive impetus view is coherent and sufficiently dual to the at-at theory of motion to metaphysically underdetermine theory choice. The at-at theorist denies that kinematic velocity is truly instantaneous, giving a reductive account of kinematic velocity in terms of position developments. One adds impetuses at the cost of ontological redundancy. In the reductive impetus theory the roles of position and velocity are exactly reversed. By the same arguments one finds that velocities are truly instantaneous and that positions are not; the latter are instead reduced to velocity developments. One could add truly instantaneous positions back to try to regain Laplacian determinism, but the argument against them would mirror the argument against truly instantaneous velocities being added to the at-at theory of motion. Thus we are left with two alternatives: the at-at theory of motion and the reductive impetus theory of motion.

I claim, moreover, that there is no obvious, physically-grounded reason to prefer one over the other in the context of classical mechanics. Indeed, Carroll has argued that the at-at theory is irreproachable because of our epistemological limitations. As he says, knowing that the at-at theory is false would "require powers of discrimination well beyond us" (Carroll, 2002, 64), since any counter-example concerns goings-on at infinitesimal temporal intervals. If so, then the impetus theory of motion that I have presented is similarly irreproachable. There is at bottom nothing awry with the impetus version of reality; it describes a classical mechanics of the same kind as the at-at theory would, differing only on what is happening in epistemically inaccessible infinitesimal temporal intervals. So far as I can see, due to the symmetry of the two views there are no good reasons to favor one theory over the other, just long custom and convenience on which to base a preference for the at-at theory. The recognition that there are two viable interpretations of classical motion is, it strikes me, of considerable interest for what it suggests about the possibilities for the metaphysical grounds of motion (particularly vis-à-vis the spaces in which motion occurs).

4 Conclusion

I have defended two theories of motion, the well-known at-at theory and a distinctive impetus theory of motion. I claim that the choice between these two theories is epistemically underdetermined. It may indeed simply be a matter of convention whether one understands motion in one way or the other given the epistemic irreproachability of each theory. Assuming that there is some fundamental "space" that grounds the basic properties of the two theories, it would also perhaps then be a matter of convention whether one sees the world as fundamentally spatial or as fundamentally velocital, as they differ only in their instantaneous ontologies. Further investigation is required, however, to see if some significant difference between the theories can in fact be found.

To review, in the first part of the paper I defended the at-at theory of motion against the criticism that it makes determinism impossible by fiat. The impossibility of the Laplacian version of determinism is indeed a feature of classical mechanics, but it is not so merely by how velocity is defined. While I agree with the critics that velocities should not be thought of as really instantaneous in the at-at theory of motion, that they are so follows because of the nature of the laws, whether these laws be metaphysical laws or physical laws. The kind of determinism relevant to mechanics is not Laplacian determinism, then, but determinism based on states including instantaneous and neighborhood properties.

In the second part of the paper I considered two impetus theories of motion. I rehearsed the argument against an impetus theory of motion that makes velocity truly instantaneous by fiat. Kinematic velocity is a derived quantity in the at-at theory of motion, but one that fully accounts for motion without reference to some superadded impetus. Thus we are led to conclude that the impetus theory is superfluous. The second impetus theory of motion turns the at-at theory on its head by assuming that velocity is basic and truly instantaneous and position is derived. I defended this view from two major criticisms, namely that it is at odds with the definition of velocity, and that integration introduces undetermined constants of integration. I concluded that the this impetus theory of motion and the at-at theory are dual theories of motion, sufficiently symmetric that there is no evident way to decide between them as accounts of motion, at least on any epistemic grounds.

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