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Abstract

We show that without assuming completeness or continuity, a strongly independent preorder on a possibly infinite dimensional convex set can always be given a vector-valued representation that naturally generalizes the standard expected utility representation. More precisely, it can be represented by a mixture-preserving function to a product of lexicographic function spaces.

Keywords. Expected utility; discontinuous preferences; incomplete preferences; lexicographic representations.

JEL Classification. D81.

1 Introduction

The completeness axiom of expected utility theory was questioned even at its inception. Von Neumann and Morgenstern (1953) found it "very dubious", but claimed that without it, a vector-valued generalization of expected utility could be obtained, though they did not provide details. Likewise, the continuity axiom (or axioms, as there are several) has not received strong support. It is often presented as a 'merely technical' condition, adopted simply to underwrite a convenient representation theorem, and cases are commonly given where its status as normative requirement is quite doubtful.

This prompts the view that the essence of expected utility is the strong independence axiom. In this article we show that without assuming completeness or continuity, a strongly independent preorder on a possibly infinite dimensional convex set can always be given a vector-valued representation that naturally generalizes the standard expected utility representation. More precisely, we show that it can be represented by a mixture-preserving function to a product of lexicographic function spaces. Let us now explain what that means.

1.1 Product lexicographic representations

Let X be a nonempty convex set of any dimension,[1] and $\succsim_X$ a preorder (a reflexive, transitive binary relation) on X. Let V be a real vector space. A function $u\colon X \to V$ is *mixture preserving*

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[1]The dimension of X is defined as the dimension of the smallest affine space that contains it; equivalently, the dimension of the vector space $\mathrm{Span}(X - X)$.

(MP) if for all x, $y \in X$, and $\alpha \in (0,1)$,

$$u(\alpha x + (1-\alpha)y) = \alpha u(x) + (1-\alpha)u(y).$$

More generally, for any $x_1, \ldots, x_n \in X$ and positive numbers $\alpha_1, \ldots, \alpha_n$ summing to 1, it follows from the equation that $u(\sum \alpha_i x_i) = \sum \alpha_i u(x_i)$. The following is the abstract form of the standard expected utility representation.

> **(R)** There exists an MP function $u \colon X \to \mathbb{R}$ such that
>
> $$x \succsim_X y \iff u(x) \geq u(y).$$

It is well known that $\succsim_X$ satisfies R if and only if it is strongly independent, complete, and satisfies either one of the standard continuity axioms, the Archimedean condition or mixture continuity. If no continuity axiom is assumed, one must look for more general forms of expected utility representation. Following Hausner and Wendel (1952), one natural direction is to consider MP functions into a more general class of vector spaces.

Recall that a preordered vector space is a pair $(V, \succsim_V)$ where V is a vector space, and $\succsim_V$ is a preorder on V such that for any v, w, $u \in V$ and $\alpha > 0$, $v \succsim_V w$ implies $\alpha v + u \succsim_V \alpha w + u$. When $(V, \succsim_V)$ is a preordered vector space and $\succsim_V$ is a partial order (a preorder that is antisymmetric), we say $(V, \succsim_V)$ is a *partially ordered* vector space, and when $\succsim_V$ is a complete partial order, we say $(V, \succsim_V)$ is an *ordered* vector space and $\succsim_V$ is a vector order.

In general, a function $u \colon X \to V$ with values in a preordered vector space is a *representation* of the preorder $\succsim_X$ on X if and only if

$$x \succsim_X y \iff u(x) \succsim_V u(y).$$

Thus R says that $\succsim_X$ has an MP representation with values in $\mathbb{R}$.

Any function on X with values in a preordered vector space represents some preorder on X, and it is easy to see that, if the function is MP, then the preorder satisfies strong independence. A basic version of our main result says that strong independence is necessary *and sufficient* for the existence of an MP representation (Theorem 6 below).

However, we can interpret this result more concretely if we recall a standard construction of ordered vector spaces. Let $(\mathbb{J}, \geq_{\mathbb{J}})$ be an ordered set. The function space $\mathbb{R}^{\mathbb{J}}$ is a vector space, under the usual pointwise definition of function addition and scalar multiplication. Define the *lexorder* (lexicographic order) $\geq_{\text{lex}}$ on $\mathbb{R}^{\mathbb{J}}$ by $f \geq_{\text{lex}} g$ if and only if (firstly) either $f = g$ or there exists a $\geq_{\mathbb{J}}$-least $j \in \mathbb{J}$ such that $f(j) \neq g(j)$ and (secondly) for that least j, $f(j) > g(j)$. Then $(\mathbb{R}^{\mathbb{J}}, \geq_{\text{lex}})$ is a *lexicographic function space*. It is a partially ordered vector space. It is not an ordered vector space unless $\geq_{\mathbb{J}}$ is a well-order. But it has a natural ordered vector subspace. Let $\mathbb{R}^{\mathbb{J}}_{\text{wo}}$ be the subset of $\mathbb{R}^{\mathbb{J}}$ consisting of functions f such that $\{j \in \mathbb{J} \colon f(j) \neq 0\}$ is well-ordered by $\geq_{\mathbb{J}}$. Then $(\mathbb{R}^{\mathbb{J}}_{\text{wo}}, \geq_{\text{lex}})$ is an ordered vector space. As a special case, when $\mathbb{J}$ is a finite ordered set with n elements, we have $\mathbb{R}^{\mathbb{J}}_{\text{wo}} = \mathbb{R}^{\mathbb{J}} \cong \mathbb{R}^n$ with the standard lexorder. Unless otherwise stated, $\mathbb{J}$ is always an ordered set, with $\mathbb{R}^{\mathbb{J}}$, $\mathbb{R}^{\mathbb{J}}_{\text{wo}}$, and $\mathbb{R}^n$ always taken to be equipped with the lexorder. We say that an MP function with values in some $\mathbb{R}^{\mathbb{J}}_{\text{wo}}$ is *lexicographic MP*, or *LMP* for short.

We will be interested in the following property of $\succsim_X$:

> **(LR)** There exists an LMP representation $u \colon X \to \mathbb{R}^{\mathbb{J}}_{\text{wo}}$ of $\succsim_X$, for some ordered set $\mathbb{J}$.

Since the natural order and the lexorder on $\mathbb{R}$ coincide, LR is a natural generalization of R.

When X is a real vector space and $\succsim_X$ is a vector order, Hausner and Wendel (1952) show that $\succsim_X$ must satisfy LR. Our first main result generalizes this to the case where X is an arbitrary convex set and $\succsim_X$ is a complete strongly independent preorder. We also give minimality and uniqueness results not given by Hausner and Wendel.

If $\succsim_X$ satisfies LR it must be complete. The standard way of approaching representations of incomplete preorders is to seek a 'multi-representation'. That is a representation that consists of a set of functions, each representing a complete preorder, with the incomplete preorder characterized by some notion of 'agreement' among all the functions. It would be straightforward to state our main result in such terms, but since we are already dealing with vector-valued representations in the complete case, it seems more economical to stick within this approach in the general case.

Given an index set $\mathbb{I}$ and a family of preordered sets $\{(V_i, \succsim_{V_i}) : i \in \mathbb{I}\}$, we define the *product preorder* $\succsim_\Pi$ on $\prod_i V_i$ by $v \succsim_\Pi w$ if and only if $v_i \succsim_{V_i} w_i$ for all $i \in \mathbb{I}$. It is indeed a preorder. If all the V_i's are preordered (respectively, partially ordered) vector spaces, so is $\prod_i V_i$. In particular, suppose we have an index set $\mathbb{I}$ and for each $i \in \mathbb{I}$ an ordered set $\mathbb{J}_i$. Then we can form the partially ordered vector space $\prod_i \mathbb{R}^{\mathbb{J}_i}_{\text{wo}}$. We say an MP function with values in $\prod_i \mathbb{R}^{\mathbb{J}_i}_{\text{wo}}$ is *product LMP*, or *PLMP* for short.

The main property we are interested is the following.

> **(PLR)** There exists a PLMP representation $u \colon X \to \prod_i \mathbb{R}^{\mathbb{J}_i}_{\text{wo}}$ of $\succsim_X$, for some ordered sets $\mathbb{J}_i$ indexed by some set $\mathbb{I}$.

Since PLR and LR coincide when $\#\mathbb{I} = 1$, PLR is a natural generalization of LR, and therefore of R. Our main result is that for arbitrary convex X, strong independence is necessary and sufficient for $\succsim_X$ to satisfy PLR.

1.2 Motivation

The completeness axiom of expected utility has been amply criticized.[2] Instead we rehearse rationales for not assuming continuity, including one that is motivated by incompleteness. For brevity, we focus on normative aspects.[3]

For definiteness, consider two common continuity axioms.

> **(Ar)** For $x, y, z \in X$, if $x \succ_X y \succ_X z$, then $(1-\epsilon)x + \epsilon z \succ_X y$ for some $\epsilon \in (0, 1)$.

> **(MC)** For $x, y, z \in X$, if $\epsilon x + (1-\epsilon)y \succ_X z$ for all $\epsilon \in (0, 1]$, then $y \succsim_X z$.

For strongly independent preorders, Ar is equivalent to the standard Archimedean axiom, while MC is equivalent to the mixture continuity axiom of Herstein and Milnor (1953).

These and other continuity assumptions are often said to be 'technical conditions' that simplify the mathematics but are not entirely normatively compelling. For example, it is common to claim that when x is, for example, getting an extra dollar, y is the comfortable status quo, and z is being tortured to death, it is at least rationally permissible to prefer y to all mixtures of x and z, violating Ar.[4] Similar counterexamples apply to MC: now let x be the status quo, y be torture, and z torture plus a dollar.

Less direct arguments for not requiring continuity can also be given. For example, in game theory, dropping continuity has proved useful in the refinement of Nash equilibria. In ethics, it has been seen as a natural way of retaining the essentials of Harsanyi's utilitarianism while avoiding allegedly unwelcome implications such as the 'repugnant conclusion' of Parfit (1986).[5]

[2] See e.g. Sen (1970); Dubra, Maccheroni and Ok (2004).

[3] For an entry into empirical reasons for abandoning continuity, see Blume, Brandenberger, and Dekel (1989).

[4] See e.g. Kreps (1988); Gilboa (2009).

[5] See, respectively, Blume *et al* (1989) and McCarthy, Mikkola, and Thomas (2016).

A different rationale for not assuming continuity comes from the rejection of completeness. Given completeness, Ar and MC are equivalent for strongly independent preorders, and they are used more or less interchangeably. Indeed, they both seem to rest on a basic 'Archimedean intuition'. But when completeness fails for strongly independent preorders, Ar and MC together imply that comparability is an equivalence relation,[6] which is inconsistent with the examples that are typically used to motivate incompleteness. To strengthen this point, consider further

(Ar$^+$) For $x, y, z \in X$, if $x \succ_X y$, then $(1-\epsilon)x + \epsilon z \succ_X y$ for some $\epsilon \in (0, 1)$.

This modest strengthening of Ar rests on the same kind of Archimedean intuition as Ar and MC. But for strongly independent preorders, Ar$^+$ and MC together imply completeness. Since the case against completeness has seemed compelling to many writers, and since Ar$^+$ and MC rest on very similar Archimedean intuitions, one might conclude that the intuitions are unreliable.

1.3 Related literature

We are aware of two treatments of incomplete, strongly independent preorders. Suppose X is the set of probability measures on some finite set of consequences C. Assuming a weaker independence condition than strong independence, Fishburn (1982, Thm. 5.2) shows that there is an MP function $u\colon X \to \mathbb{R}^n$, with $n = \#C - 1$, such that

$$x \succ_X y \implies u(x) >_{\text{lex}} u(y).$$

However, such 'one-way representations' do not in general permit one to recover the preference relation. This limits their usefulness, and more recent focus has been on representations that fully characterize incomplete relations.[7] Thus under the same domain assumptions, Borie (2016) showed that a strongly independent preorder satisfies PLR; in addition, each $\mathbb{R}^{\mathbb{J}_i}_{\text{wo}}$ can be taken to be $\mathbb{R}^n$.

This result is appealingly simple, and applicable to many practical examples. A limitation, however, is that the domain assumptions exclude a vast range of approaches to the representation of risk and uncertainty. We give four examples. (i) In the case of objective risk, a typical setting takes X to be the set of Borel probability measures on a compact metric space. (ii) In the Anscombe-Aumann setting combining objective risk and subjective uncertainty, it is not uncommon to allow the set of 'roulette lotteries' to be Borel probability measures, or the set of states of nature to be infinite. (iii) In the Savage setting of subjective uncertainty, the set of states of nature is generally required to be infinite. (iv) Sometimes non-standard reals are used to represent infinitesimal probabilities; this is particularly natural given the association of failures of continuity with (relatively) infinite values.

Albeit with some extra work in the Savage setting,[8] each of these treatments of risk and uncertainty can be accommodated under the assumption that X is a convex set with no restrictions on its dimension. An analysis of strongly independent preorders on such an X therefore promises a wide range of applications. Moreover, there is not much cost to treating the general case as the main tools have long been known. An embedding technique that goes back to Stone (1949) reduces the general case to the case where X is a real vector space. This gives access to general structure theorems for ordered abelian groups and vector spaces, stemming from the Hahn embedding theorem. In particular, results of Hausner and Wendel (1952) and Conrad (1953) specialize to yield existence, minimality and uniqueness claims for order-isomorphisms into lexicographic function spaces; a simple extension argument then delivers our main result.

[6] This and the next claim is proved in McCarthy and Mikkola (2017).

[7] See Dubra *et al* (2004) and Evren (2014) for discussion.

[8] See e.g. Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003) for an explanation of how relatively modest assumptions allow the set of Savage acts to be treated as a convex set.

2 Main results

2.1 Results under completeness

For readability, we first present results that use completeness, and then generalize. Throughout, X is a nonempty convex set. When $\succsim_X$ is a preorder on X, the following is the central expected utility axiom.

Strong Independence (SI). For all $x, y, z \in X$ and $\alpha \in (0, 1]$,

$$x \succsim_X y \iff \alpha x + (1 - \alpha)z \succsim_X \alpha y + (1 - \alpha)z.$$

So $\succsim_X$ is an *SI preorder.* The following result generalizes the main theorem of Hausner and Wendel (1952) from vector spaces to arbitrary convex sets.

Theorem 1. *Let $\succsim_X$ be a preorder on X. Then $\succsim_X$ satisfies LR if and only if it is SI and complete.*

We now introduce three concepts. The 'lexical' order type of an SI preorder reflects its 'hierarchy of relative infinities'; a 'minimal' LMP function is an efficient representation of an SI preorder; it has what we call a 'functional' order type. We then show that an SI preorder can always be represented by a minimal LMP function, and the lexical and functional order types coincide.

Lexical order type. Suppose given a SI preorder $\succsim_X$ on X. Let $X^2_{++} := \{(x, y) \in X^2 \colon x \succ_X y\}$; it can be seen as the set of positive differences under $\succsim_X$. Define a binary relation $\ggcurly_X$ on X^2_{++} by $(x, y) \ggcurly_X (s, t)$ if and only if for all $\epsilon \in (0, 1]$, $\epsilon x + (1 - \epsilon)t \succ_X \epsilon y + (1 - \epsilon)s$. This can be seen as saying that according to $\succsim_X$, the positive difference between s and t is infinitesimal relative to the positive difference between x and w. Interpreted probabilistically, an arbitrarily small chance of x rather than y always outweighs a correspondingly almost certain chance of s rather than t.

By contrast, define $\approx_X$ on X^2_{++} by $(x, y) \approx_X (s, t)$ if and only if for some $\epsilon \in (0, 1)$, $(1 - \epsilon)x + \epsilon t \succ_X (1 - \epsilon)y + \epsilon s$ and $(1 - \epsilon)s + \epsilon y \succ_X (1 - \epsilon)t + \epsilon x$. This can be seen as saying that the two positive differences are comparable, or relatively finite. The relation $\approx_X$ is an equivalence relation on X^2_{++}. Let $\langle X^2_{++} \rangle := \{\langle (x, y) \rangle \colon (x, y) \in X^2_{++}\}$ be the corresponding partition of X^2_{++} into equivalence classes. Define a preorder $\gtrapprox_X$ on X^2_{++} by $(x, y) \gtrapprox_X (z, w)$ if and only if $(x, y) \ggcurly_X (z, w)$ or $(x, y) \approx_X (z, w)$. Define a partial order $\leq_X$ on $\langle X^2_{++} \rangle$ by $\langle (x, y) \rangle \leq_X \langle (z, w) \rangle$ if and only if $(x, y) \gtrapprox_X (z, w)$. If $\succsim_X$ is complete, $\leq_X$ is a total order (a complete partial order). When $\langle (x, y) \rangle <_X \langle (z, w) \rangle$, the relatively finite positive differences contained in the latter are infinitesimal relative to the relatively finite positive differences in the former. We define the *lexical* order type of $\succsim_X$ to be the order type of the poset $(\langle X^2_{++} \rangle, \leq_X)$.

Most of the notions just defined naturally generalize concepts introduced in Hausner and Wendel (1952) in the special case where X is a vector space and $\succsim_X$ is complete.

Minimality and functional order type. An LMP function $u' \colon X \to \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$ is a *restriction* of an LMP function $u \colon X \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ if $\varnothing \neq \mathbb{J}' \subseteq \mathbb{J}$, $u'(\cdot) = u(\cdot)_{|\mathbb{J}'}$, and u and u' represent the same (complete SI) preorder on X. It is a proper restriction if $\mathbb{J}' \subsetneq \mathbb{J}$. An LMP function is *minimal* if it has no proper restriction. It therefore provides an efficient representation of a preorder in that no element of its ordered set is inessential to the representation. A *minimal restriction* of an LMP function is a restriction that is minimal. The *functional* order type of an LMP function is the order type of the associated ordered set.

Theorem 2 (Minimality)**.** *(i) Every LMP function has a minimal restriction. Consequently, every complete SI preorder on a convex set can be represented by a minimal LMP function.*

(ii) The lexical order type of a complete SI preorder on a convex set is identical to the functional order type of each minimal LMP function that represents it.

(iii) Every total order type is the lexical order type of some complete SI preorder on a convex set.

The result provides a natural link between order theory and complete SI preorders. The proof provides an explicit way of defining a minimal restriction of any LMP function.

Theorem 3 (Uniqueness of minimal LMP representation)**.** *Minimal LMP functions* $u\colon X \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ *and* $u'\colon X \to \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$ *represent the same preorder on* X *if and only if there is an isomorphism* $F\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} \to \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$ *of ordered vector spaces and an element* x_0 *of* $\mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$ *such that* $u' = F \circ u + x_0$.

By analysing isomorphisms between lexicographic function spaces, we can give a more concrete criterion. A little roughly, such an isomorphism must be given by a lower-triangular matrix with positive entries on the diagonal. For the precise statement, let $\mathbb{R}^{\mathbb{J};j}_{\mathrm{wo}} = \{x \in \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} : x(i) = 0 \text{ for all } i <_{\mathbb{J}} j\}$.

Theorem 4. *A function* $F\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} \to \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$ *is an isomorphism of ordered vector spaces if and only if there exist (a) an order isomorphism* $f\colon \mathbb{J} \to \mathbb{J}'$*; (b) for each* $j \in \mathbb{J}$*, a positive real number* α_j*; and (c) for each* $j \in \mathbb{J}$*, a linear map* $F_j\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} \to \mathbb{R}$ *vanishing on* $\mathbb{R}^{\mathbb{J};j}_{\mathrm{wo}}$*, all satisfying*

$$F(x)(f(j)) = \alpha_j x(j) + F_j(x).$$

We also note without proof the well known special case in which $\mathbb{J}$ has one element:

Theorem 5. *A complete SI preorder satisfies Ar if and only if it has an LMP representation with values in* $\mathbb{R}$.

The other results will be proved in section 3.

2.2 General results

Theorem 6. *A preorder on a convex set* X *satisfies SI if and only if it has an MP representation in some preordered vector space.*

To get the result in terms of PLMP representations, say that a preorder $(Y, \succsim_2)$ *extends* a preorder $(Y, \succsim_1)$ if for all $x, y \in Y$, $x \sim_1 y \implies x \sim_2 y$, and $x \succ_1 y \implies x \succ_2 y$. The passage from the complete case to the general case rests on the following proposition. It is shown in Borie (2016); we give an alternative proof, discovered independently, that avoids a detailed construction.

Proposition 7. *Every SI preorder* $\succsim_X$ *on a nonempty convex set* X *is the intersection of its complete SI extensions. In other words,* $x \succsim_X y$ *holds if and only if* $x \succsim^{\mathrm{com}}_X y$ *for every complete SI preorder extending* $\succsim_X$.

The following is our main result.

Theorem 8 (PLR representation)**.** $\succsim_X$ *satisfies PLR if and only if it is strongly independent.*

Remark 9. It follows from Theorem 2 that, if $\succsim_X$ satisfies PLR, then then PLMP representation can always be chosen 'minimal' in the sense that the cardinality of the index set $\mathbb{I}$ is as small as possible, and, for each $i \in \mathbb{I}$, the LMP function $u_i\colon X \to \mathbb{R}^{\mathbb{J}_i}_{\mathrm{wo}}$ is minimal. Alternatively, if one is not interested in minimal representations, one can choose every $\mathbb{J}_i$ in the family to be one and the same ordered set $\mathbb{J}$. This is because there is in any case an ordered set $\mathbb{J}$ containing every $\mathbb{J}_i$ as a subset, making $\mathbb{R}^{\mathbb{J}_i}_{\mathrm{wo}}$ a subspace of $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$.

Remark 10. Given the view that SI is the essence of expected utility, it is natural to ask precisely what SI is contributing to Theorem 8. Say that a function, not necessarily mixture-preserving, is PL if it takes values in some product $\prod_i \mathbb{R}^{\mathbb{J}_i}_{\mathrm{wo}}$. Thus Theorem 8 says that any SI preorder on a convex set can be represented by a PL function that is in addition MP. Now products of lexicographic function spaces are quite specialized structures; one might anticipate that SI is responsible for a preorder being representable by any PL function, let alone one that is also MP. But this turns out to be false:

Theorem 11. *Every preorder has a PL representation.*

Thus the contribution of SI to Theorem 8 is precisely that the preorder can be represented by a PL function that is MP.

3 Proofs

3.1 Preliminaries

Throughout X is a nonempty convex set. Let V be the real vector space $\mathrm{Span}(X - X)$; this is a subspace of the vector space $\mathrm{Span}(X)$, so that the vector space operations on V are the restrictions of those on $\mathrm{Span}(X)$.

We now explain how to pass between X and V: that is, between SI preorders $\succsim_X$ on X and vector preorders $\succsim_V$ on V, and between MP functions representing $\succsim_X$ and linear functions representing $\succsim_V$. This will enable us to reduce our results to the case where X is a vector space. The following provides a useful way of representing elements of V.

Lemma 12. $V = \{\lambda(x - y) : \lambda \in (0, \infty), x, y \in X\}$.

This is Lemma 4.1 in McCarthy, Mikkola, and Thomas (2017).

Proposition 13. *There is a unique bijection between SI preorders $\succsim_X$ on X and vector preorders $\succsim_V$ on V such that*

$$x \succsim_X y \iff x - y \succsim_V 0 \quad \text{for all } x, y \in X. \tag{1}$$

Proof. It is straightforward to check that, given a vector preorder $\succsim_V$, (1) defines a unique SI preorder $\succsim_X$. Conversely, given an SI preorder $\succsim_X$, there is at most one vector preorder $\succsim_V$ satisfying (1). Indeed, given $v, w \in V$, we can write $v - w = \lambda(x - y)$ as in Lemma 12, and then (using the defining properties of a vector preorder) $v \succsim_V w \iff v - w \succsim_V 0 \iff \lambda(x - y) \succsim_V 0 \iff x - y \succsim_V 0 \iff x \succsim_X y$. Thus the relation $\succsim_V$ is completely determined by $\succsim_X$.

All that remains to be shown is that, given $\succsim_X$, there exists *at least* one $\succsim_V$ satisfying (1). To do this, define $\succsim_V$ by

$$v \succsim_V w \iff v - w = \lambda(x - y) \text{ and } x \succsim_X y \text{ for some } x, y \in X, \lambda > 0. \tag{2}$$

We have to check that the relation $\succsim_V$ defined in this way is a vector preorder satisfying (1).

It is clearly reflexive. For transitivity, suppose $v \succsim_V w$ and $w \succsim_V u$. Then for some $x, y, s, t \in X$, and $\lambda, \mu > 0$, $v - w = \lambda(x - y)$ and $w - u = \mu(s - t)$ with $x \succsim_X y$ and $s \succsim_X t$. The two equalities imply $v - u = \lambda(x - y) + \mu(s - t) = (\lambda + \mu)[(\frac{\lambda}{\lambda+\mu}x + \frac{\mu}{\lambda+\mu}s) - (\frac{\lambda}{\lambda+\mu}y + \frac{\mu}{\lambda+\mu}t)]$. Since $\succsim_X$ is SI, the two inequalities imply $\frac{\lambda}{\lambda+\mu}x + \frac{\mu}{\lambda+\mu}s \succsim_X \frac{\lambda}{\lambda+\mu}y + \frac{\mu}{\lambda+\mu}t$. By (2) these together imply $v \succsim_V u$, establishing transitivity. Finally, for any $v, w, u \in V$ and $\alpha > 0$: $v - w = \lambda(x - y)$ for some $\lambda > 0$ if and only if $(\alpha v + u) - (\alpha w + u) = \mu(x - y)$ for some $\mu > 0$. By (2), $v \succsim_V w \iff \alpha v + u \succsim_V \alpha w + u$, so $\succsim_V$ is a vector preorder.

We next claim that (1) holds. Clearly $x \succsim_X y$ implies $x - y \succsim_V 0$. Conversely, suppose $x - y \succsim_V 0$. Then for some $\lambda > 0$ and $s, t \in X$, $x - y = \lambda(s - t)$ and $s \succsim_X t$. The former yields $\frac{1}{\lambda+1}x + \frac{\lambda}{\lambda+1}t = \frac{1}{\lambda+1}y + \frac{\lambda}{\lambda+1}s$. By SI, $s \succsim_X t$ implies $\frac{1}{\lambda+1}x + \frac{\lambda}{\lambda+1}s \succsim_X \frac{1}{\lambda+1}x + \frac{\lambda}{\lambda+1}t$; substituting and applying SI again, we obtain $x \succsim_X y$, establishing (1). $\square$

To pass between MP functions on X and linear functions on V, we will appeal to the following general and useful result.

Theorem 14. *Let X be a nonempty set. Let Y, Z be vector spaces and let $f : X \to Y$, $g : X \to Z$ be such that $(f, g)(X) := \{(f(x), g(x)) : x \in X\}$ is convex.*

Then we have $g(x) = g(x') \implies f(x) = f(x')$ for all $x, x' \in X$ if and only if $f = Lg + y_0$ for some linear $L : Z \to Y$ and some $y_0 \in Y$.

Moreover, the restriction of L to $\operatorname{Span}(g(X) - g(X))$ is unique.

Proof. To take the last statement first, suppose that $f = Lg + y_0$ and also $f = L'g + y_0'$. Subtracting, we see that L and L' differ by the constant $y_0 - y_0'$ on $g(X)$, and therefore they are equal on $\operatorname{Span}(g(X) - g(X))$.

For the first statement, consider the condition

$$g(x) = g(x') \implies f(x) = f(x') \quad \text{for all } x, x' \in X. \tag{3}$$

It is clear that (3) holds if f is of the form $f = Lg + y_0$. For the converse, let $U := (f, g)(X)$; it is a convex set by assumption. Let $A := \operatorname{Span}(U - U)$; it is a linear subspace of $Y \times Z$. In light of Lemma 12 applied to U, the condition (3) is equivalent to the condition that A contains no elements of the form $(y, 0)$ with $y \neq 0$. Since A is a linear subspace, we find that

$$(y, z), (y', z) \in A \implies y = y'.$$

A is therefore the graph of a partial function L from Z to Y. By definition, the domain of L is the projection of A to Z, namely $\operatorname{Span}(g(X) - g(X))$, and L is characterized by the equation

$$A = \{(L(z), z) \colon z \in \operatorname{Span}(g(X) - g(X))\}.$$

Also, L is a linear function since A is a linear subspace. Extend L arbitrarily to a linear function from Z to Y. Fix $(y, z) \in U$ and set $y_0 = y - L(z)$. Then for any $x \in X$, we have $f(x) = L(g(x)) + y_0$. $\square$

Given a vector space W, say that two MP functions $u, u' \colon X \to W$ are *equivalent* if they differ by a constant; that is $u(\cdot) = u'(\cdot) + w_0$ for some $w_0 \in W$. Let $[u]$ be the equivalence class containing u.

Proposition 15. *Let W be a real vector space. There is a unique bijection between equivalence classes $[u]$ of MP functions $X \to W$ and linear functions $L \colon V \to W$ that satisfies*

$$u(x) - u(y) = L(x - y) \quad \textit{for all } u \in [u], x, y \in X. \tag{4}$$

Proof. Let $u \colon X \to W$ be an MP function. Fix $x_0 \in X$, and define $\iota \colon X \to V$ by $\iota(x) = x - x_0$. Then $(u, \iota)(X)$ is convex. Clearly $\iota(x) = \iota(x') \implies u(x) = u(x')$ for all $x, x' \in X$. By Theorem 14, there is a unique linear $L \colon V \to W$ satisfying $u = L\iota + w_0$ for some $w_0 \in W$, so (equivalently) satisfying (4). This L only depends on the equivalence class of u. Conversely, given a linear $L \colon V \to W$, define $u \colon X \to W$ by $u(x) = L(x - x_0)$; then u and L satisfy (4), establishing the bijection. $\square$

Proposition 16. *Suppose $\succsim_X$ and $\succsim_V$ are as in Proposition 13, satisfying* (1)*, and that* $[u]$ *and* L *are as in Proposition 15, satisfying* (4)*. Suppose also that the* W *of Proposition 15 is equipped with a vector preorder* $\succsim_W$*. Then each* $u \in [u]$ *is a representation of* $\succsim_X$ *if and only if* L *is a representation of* $\succsim_V$*.*

Proof. For right to left, suppose L represents $\succsim_V$. By (1) and (4), $x \succsim_X y \Leftrightarrow x - y \succsim_V 0 \Leftrightarrow L(x-y) \succsim_W 0 \Leftrightarrow u(x) \succsim_W u(y)$ for each $u \in [u]$. For left to right, suppose each member of $[u]$ represents $\succsim_X$. Given $v, w \in V$, by Lemma 12 choose $x, y \in X$, $\lambda > 0$ such that $v - w = \lambda(x-y)$. Then by (1) and (4) and the fact that $\succsim_W$ is vector preorder, $v \succsim_V w \Leftrightarrow v - w \succsim_V 0 \Leftrightarrow x - y \succsim_V 0 \Leftrightarrow x \succsim_X y \Leftrightarrow u(x) \succsim_W u(y) \Leftrightarrow u(x) - u(y) \succsim_W 0 \Leftrightarrow L(x-y) \succsim_W 0 \Leftrightarrow L(v-w) \succsim_W 0 \iff L(v) \succsim_W L(w)$, so L represents $\succsim_V$. □

3.2 Proof of Theorem 1

The left to right direction is clear from the fact that $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ is an ordered vector space. For right to left, let V and $\succsim_V$ be as in Proposition 13. Let $[v]_{\sim_V}$ denote the equivalence class of $v \in V$ under the equivalence relation $\sim_V$. Define a relation $\geq_Z$ on $Z \coloneqq V/{\sim_V}$ by

$$[v]_{\sim_V} \geq_Z [w]_{\sim_V} \iff v \succsim_V w.$$

Then $(Z, \geq_Z)$ is an ordered vector space. By Hausner and Wendel (1952, Thm. 3.1) there is an ordered set $\mathbb{J}$ and a linear $L' \colon Z \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ that represents $\geq_Z$. Then $L \colon V \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ given by $L(v) \coloneqq L'([v]_{\sim_V})$ is a linear function that represents $\succsim_V$. Let $[u]$ be as in Proposition 15 with $W = \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$. By Proposition 16, each member of $[u]$ is an MP function $X \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ that represents $\succsim_X$.

3.3 Proof of Theorem 2.

We first consider the case when $X = V$ is a vector space, and then show reduce the general case to this one.

Proposition 17. *Let* V *be a vector space and* $L \colon V \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ *be linear. Then* L *has a minimal restriction. Moreover,* L *is minimal if and only if, for every* $j \in \mathbb{J}$*, there is some* $v \in V$ *such that* j *is the least element of* $\mathbb{J}$ *with* $v(j) \neq 0$*.*

Proof. Let $\succsim_V$ be the vector preorder on V represented by L. For any $x \in \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ let $J(x)$ be the least element of $\mathbb{J}$ such that $x(J(x)) \neq 0$. Let $\mathbb{J}_* \coloneqq \{J(L(v)) : v \succ_V 0\}$. Define $L^* \colon V \to \mathbb{R}^{\mathbb{J}_*}_W$ by setting $L^*(v) \coloneqq L(v)|_{\mathbb{J}_*}$. We claim that L^* is a minimal restriction of L. It follows that L is minimal if and only if $\mathbb{J} = \mathbb{J}^*$, which is equivalent to the last claim of the proposition.

First we show that L^* is a restriction of L, i.e. for all $v, w \in V$, $L(v) \geq_{\mathrm{lex}} L(w) \Leftrightarrow L^*(v) \geq_{\mathrm{lex}} L^*(w)$. Defining $x = v - w$, and using the linearity of L and L^*, this rearranges to

$$L(x) \geq_{\mathrm{lex}} 0 \Leftrightarrow L^*(x) \geq_{\mathrm{lex}} 0 \quad \text{for all } x \in V. \tag{5}$$

Now, if $L(x) = 0$ then by construction $L^*(x) = 0$, so (5) holds. On the other hand, if $L(x) \neq 0$, then $L^*(x) \neq 0$, and indeed $J(L(x)) = J(L^*(x))$ and $L(x)(J(L(x))) = L^*(x)(J(L^*(x)))$. But $L(x) \geq_{\mathrm{lex}} 0 \Leftrightarrow L(x)(J(L(x))) \geq 0$, and similarly for L^* in place of L; so again (5) holds.

To show that L^* is minimal, suppose that $L' \colon V \to \mathbb{R}^{\mathbb{J}'}_W$ is a restriction of L^*, with $\mathbb{J}' \subsetneq \mathbb{J}_*$. Pick $j \in \mathbb{J}_* \setminus \mathbb{J}'$. By definition of $\mathbb{J}_*$ there is some $v_j \succ_V 0$ such that $J(L(v_j)) = j$. If $L'(v_j)(k) = 0$ for all $k \in \mathbb{J}'$ such that $j <_{\mathbb{J}} k$, we get $L'(v_j) = 0$, implying $v_j \sim_V 0$, a contradiction. Otherwise, pick the least $k \in \mathbb{J}'$ such that $j <_{\mathbb{J}} k$ and $L'(v_j)(k) \neq 0$. Since $\mathbb{J}' \subset \mathbb{J}_*$, we can pick $v_k \succ_V 0$

such that $J(L(v_k)) = k$. Then for some real $\alpha > 0$, $L'(\alpha v_k) >_{\text{lex}} L'(v_j)$, implying $\alpha v_k \succ_V v_j$. But $j <_{\mathbb{J}} k$ implies $L^*(v_j) >_{\text{lex}} L^*(\alpha v_k)$, hence $v_j \succ_V \alpha v_k$, a contradiction. Thus there can be no such L', so L^* is minimal. □

Proposition 18. *Suppose $[u]\colon X \to \mathbb{R}^{\mathbb{J}}_{\text{wo}}$ and $L\colon V \to \mathbb{R}^{\mathbb{J}}_{\text{wo}}$ correspond in the sense of Proposition 15. Then each $u \in [u]$ is minimal if and only if L is minimal.*

Proof. Let $\mathbb{J}' \subsetneq \mathbb{J}$. Define $u'\colon X \to \mathbb{R}^{\mathbb{J}'}_{\text{wo}}$ and $L'\colon V \to \mathbb{R}^{\mathbb{J}'}_{\text{wo}}$ by $u'(\cdot) \coloneqq u(\cdot)|_{\mathbb{J}'}$ and $L'(\cdot) \coloneqq L(\cdot)|_{\mathbb{J}'}$. Let $\succsim_X$ be the SI preorder on X represented by u (or any $u \in [u]$). Let $\succsim_V$ be the vector preorder on V given by Proposition 13. Then L represents $\succsim_V$ by Proposition 16.

Claim: u' (equivalently: each $u \in [u']$) represents $\succsim_X$ if and only if L' represents $\succsim_V$. This implies each $u \in [u]$ is minimal if and only if L is minimal, as needed.

To prove the claim, note that by (4), $u'(x) - u'(y) = L'(x - y)$ for all $x, y \in X$. For right to left, suppose L' represents $\succsim_V$. Fix $x, y \in X$. Then $x \succsim_X y \Leftrightarrow x - y \succsim_V 0 \Leftrightarrow L'(x - y) \geq_{\text{lex}} 0 \Leftrightarrow u'(x) - u'(y) \geq_{\text{lex}} 0$. This shows that u', and hence each $u \in [u']$, represents $\succsim_X$.

For left to right, suppose u' represents $\succsim_X$. Fix $v, w \in V$. By Lemma 12, choose $x, y \in X$, $\lambda > 0$ such that $v - w = \lambda(x - y)$. Then $v \succsim_V w \Leftrightarrow L(v - w) \geq_{\text{lex}} 0 \Leftrightarrow L(x - y) \geq_{\text{lex}} 0 \Leftrightarrow u(x) - u(y) \geq_{\text{lex}} \Leftrightarrow x \succsim_X y \Leftrightarrow u'(x) \geq_{\text{lex}} u'(y) \Leftrightarrow u'(x) - u'(y) \geq_{\text{lex}} 0 \Leftrightarrow L'(x - y) \geq_{\text{lex}} 0 \Leftrightarrow L'(v) \geq_{\text{lex}} L'(w)$. This shows that L' represents $\succsim_V$, establishing the claim. □

Proposition 19. *The functional order type of any minimal linear LMP function on a vector space is identical to the lexical order type of the (complete) vector preorder it represents.*

Proof. Let V be a vector space and $L\colon V \to \mathbb{R}^{\mathbb{J}}_{\text{wo}}$ be linear and minimal. Let $\succsim_V$ be the complete vector preorder L represents. Define a mapping $G\colon \langle V^2_{++} \rangle \to \mathbb{J}$ by $\langle (v, w) \rangle \mapsto J(L(v - w))$; recall that $J(L(v - w))$ is the least $j \in \mathbb{J}$ such that $L(v - w)(j) \neq 0$. It is sufficient to prove that G is an order isomorphism.

Now $(v, w) \approx_V (s, t)$ implies that for some $\epsilon \in (0, 1)$, $(1 - \epsilon)L(v - w) >_{\text{lex}} \epsilon L(s - t)$ and $(1-\epsilon)L(s-t) >_{\text{lex}} \epsilon L(v-w)$, so we must have $J(L(s-t)) = J(L(v-w))$. Similarly, $(v, w) \succ\!\succ_V (s, t)$ implies that for all $\epsilon \in (0, 1)$, $\epsilon L(v - w) >_{\text{lex}} (1 - \epsilon)L(s - t)$, so $J(L(v - w)) < J(L(s - t))$. The first of these implies that G is well-defined, and since $\leq_V$ is a total order, together they imply that for all $(v, w), (s, t) \in V^2_{++}$, $\langle (v, w) \rangle \leq_V \langle (s, t) \rangle \Leftrightarrow J(L(v - w)) \leq_{\mathbb{J}} J(L(s - t))$. Thus G is an order embedding.

It remains to show that G is surjective. Recall from Proposition 17 that, since L is minimal, every $j \in \mathbb{J}$ is of the form $J(L(v))$ for some $v \succ_V 0$. But then $j = G(\langle (v, 0) \rangle)$. □

We omit proof of the following lemma, which is straightforward from (1) and the definitions of the inequalities.

Lemma 20. *(i) Suppose $\succsim_X$ and $\succsim_V$ correspond as in Proposition 13. For $x, y, s, t \in X$, $(x, y) \gtrapprox_X (s, t)$ if and only if $(x - y, 0) \gtrapprox_V (s - t, 0)$.*

(ii) Let $(V, \succsim_V)$ be a preordered vector space. Then for $(v, w) \in V^2_{++}$, $(v, w) \approx_V (v - w, 0)$, and for $(v, 0) \in V^2_{++}$ and $\lambda > 0$, $(v, 0) \approx_V (\lambda v, 0)$.

Proposition 21. *Suppose $\succsim_X$ and $\succsim_V$ correspond as in Proposition 13. Then they have identical lexical order types.*

Proof. Consider the map $\langle (x, y) \rangle_X \mapsto \langle (x - y, 0) \rangle_V$. We claim that this is an order isomorphism $\langle X^2_{++} \rangle \to \langle V^2_{++} \rangle$, which is sufficient for the result. By Lemma 20(i), the mapping is well-defined and an order embedding. Given $\langle (v, w) \rangle_V \in \langle V^2_{++} \rangle$, by Lemma 12 and (1), $v - w = \lambda(x - y)$ for some $x \succ_X y$. By Lemma 20(ii), $\langle (v, w) \rangle_V = \langle (v - w, 0) \rangle_V = \langle (x - y, 0) \rangle_V$, so the mapping is onto, hence an order isomorphism. □

Proof of Theorem 2. (i) Fix an LMP function $u\colon X \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$. Let $\succsim_X$ be the complete SI preorder it represents. Let $\succsim_V$ be the vector preorder corresponding to $\succsim_X$, as in Proposition 13. Let $L\colon V \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ be the linear function corresponding to $[u]$, in the sense of Proposition 15. By Proposition 17, L has a minimal restriction $L^*\colon V \to \mathbb{R}^{\mathbb{J}^*}_{\mathrm{wo}}$ for some $\mathbb{J}^* \subseteq \mathbb{J}$. Let $u^*\colon X \to \mathbb{R}^{\mathbb{J}^*}_{\mathrm{wo}}$ be the MP function correspond to L^* in the sense of Proposition 15. By Proposition 16 u^* represents $\succsim_X$, and by Proposition 18, this implies u^* is minimal. Therefore u^* is a minimal restriction of u. The second sentence of (i) then follows from Theorem 1.

(ii) Let $\succsim_X$ be a complete SI preorder on X, and $u\colon X \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ a minimal LMP function that represents it. Let $\succsim_V$ be the vector preorder corresponding to $\succsim_X$ as in Proposition 13, and $L\colon V \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ the linear function corresponding to u as in Proposition 15. By Proposition 16, L represents $\succsim_V$. By Proposition 18, L is minimal. Obviously, the functional order type of u is identical to the functional order type of L, namely the order type of $\mathbb{J}$. By Proposition 19, the functional order type of L is identical to the lexical order type of $\succsim_V$; by Proposition 21, the latter is identical to the lexical order type of $\succsim_X$. Since u was arbitrary, this establishes the result.

(iii) Let $\mathbb{J}$ be an ordered set of a given total order type. Then $(\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}, \geq_{\mathrm{lex}})$ is such a preorder. □

3.4 Proof of Theorem 3

To prove this and Theorem 4, we will refer to some terminology and results from Conrad (1953). A real vector space is what he would call an 'abelian operator group with operator domain $\mathbb{R}$'. $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ is what he calls a 'Γ-group', and indeed a 'Γ-sum', for $\Gamma := \mathbb{J}$. Recall that $J(x)$ is our notation for the least element of $\mathbb{J}$ such that $x(J(x)) \neq 0$. A linear map $L\colon V \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ allows us to define a 'Γ-valuation' on V, by setting $\Gamma^v = \{J(L(v))\}$ for any non-zero $v \in V$, and this makes V also into a Γ-group (Conrad's Theorem 1.1). It also makes L into a homomorphism between Γ-groups.

A 'c-subgroup' of $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ is a linear subspace S such that, for any $j \in \mathbb{J}$, there is an $x \in S$ with $J(x) = j$. With all this in mind, Conrad's III(a) on his p. 15 unpacks to:

> **Conrad Result A.** If $L, L'\colon V \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ are linear embeddings of a vector space V onto c-subgroups of $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ such that $J \circ L = J \circ L'$, then there is a linear isomorphism $F\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ such that $J = J \circ F$, and such that $L' = F \circ L$.

(The conditions involving J ensure that L, L', and F are Γ-homomorphisms.) Moreover, Conrad's Corollary on his p. 14 entails

> **Conrad Result B.** If $F\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ is a linear embedding onto a c-subgroup of $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ such that $J = J \circ F$, then F is an isomorphism.

Now back to the proof of Theorem 3. It is obvious that u and u' represent the same preorder on X if they are related in the stated way. To prove the converse, let L and L' be the linear maps corresponding to $[u]$ and $[u']$ in the sense of Proposition 15. By Propositions 17 and 18, L and L' are both minimal representations of the same vector preorder on V. Because u, u', L, L' are related by equation (4), it suffices to find an isomorphism $F\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} \to \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$ of ordered vector spaces such that $L' = F \circ L$.

Quotienting V by $\ker L = \ker L'$, we can assume that L and L' are embeddings. Moreover, Proposition 19 shows that we can identify $\mathbb{J}$ with $\mathbb{J}'$, and, according to the proof of that proposition, do so in such a way that, for any $v \in V$, $J(L(v)) = J(L'(v))$.

Proposition 17 tells us that, since $L\colon V \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ is minimal, for every $j \in \mathbb{J}$ there is some $v \in V$ such that $j = J(L(v))$. In Conrad's terminlogy, this means that L and L' map V onto

c-subgroups of $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$. Result A stated above then says that that there is a linear isomorphism $F\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} \to \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} = \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$ such that $L' = F \circ L$ and such that $J = J \circ F$.

It remains to show that this F is order-preserving. Given $x \in \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$, it suffices to show that $x \geq_{\mathrm{lex}} 0$ if and only if $F(x) \geq_{\mathrm{lex}} 0$. We can choose $v \in V$ such that $J(L(v)) = J(x)$ and (then rescaling as necessary) $L(v)(J(x)) = x(J(x))$. This means that $J(L(v) - x)) >_{\mathbb{J}} J(x)$. It follows that $J(F(L(v)-x)) >_{\mathbb{J}} J(F(x))$, and therefore that $F(L(v))(J(F(x))) = F(x)(J(F(x)))$. Therefore $x \geq_{\mathrm{lex}} 0 \iff L(v) \geq_{\mathrm{lex}} 0$, and $F(x) \geq_{\mathrm{lex}} 0 \iff F(L(v)) = L'(v) \geq_{\mathrm{lex}} 0$. Since $L(v) \geq_{\mathrm{lex}} 0 \iff L'(v) \geq_{\mathrm{lex}} 0$, we find that $x \geq_{\mathrm{lex}} 0 \iff F(x) \geq_{\mathrm{lex}} 0$, as desired.

3.5 Proof of Theorem 4

Suppose that $F\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} \to \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$ is a function with the stated form. Using f to identify $\mathbb{J}$ and $\mathbb{J}'$, this means

$$F(x)(j) = \alpha_j x(j) + F_j(x).$$

Such an F is clearly a linear, order-preserving embedding. Moreover, F is an isomorphism of $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$ onto a c-subgroup of $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$. By Conrad's Result B stated above, F must be a linear isomorphism. Thus F is an isomorphism of ordered vector spaces.

Conversely, suppose that $F\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} \to \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$ is an isomorphism of ordered vector spaces. Let us show that it has the stated form. Recall that, for any $x \in \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$, we write $J(x)$ for the least $j \in \mathbb{J}$ such that $x(j) \neq 0$; similarly, for $x \in \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}}$, $J'(x)$ is the least $j' \in \mathbb{J}'$ such that $x(j') \neq 0$. Since F is order-preserving, it must be the case that $J(x) \geq_{\mathbb{J}} J(y) \implies J'(F(x)) \geq_{\mathbb{J}'} J'(F(y))$. Therefore there is a unique order-preserving function $f\colon \mathbb{J} \to \mathbb{J}'$ characterised by the property that, for any $x \in \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}$, $f(J(x)) = J'(F(x))$. Since F is an isomorphism, f is an order-isomorphism.

Now, for $j \in \mathbb{J}$, define $W_j = \{x \in \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} : J(x) >_{\mathbb{J}} j\} \subset \mathbb{R}^{\mathbb{J},j}_{\mathrm{wo}} = \{x \in \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}} : J(x) \geq_{\mathbb{J}} j\}$. The function $x \mapsto x(j)$ is a linear map $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}/W_j \to \mathbb{R}$, restricting to an isomorphism $\mathbb{R}^{\mathbb{J},j}_{\mathrm{wo}}/W_j \to \mathbb{R}$. Similarly for $j \in \mathbb{J}'$ define $W'_j = \{x \in \mathbb{R}^{\mathbb{J}'}_{\mathrm{wo}} : J'(x) >_{\mathbb{J}} j\} \subset \mathbb{R}^{\mathbb{J}',j}_{\mathrm{wo}}$.

The definition of f entails that L maps W_j isomorphically onto $W'_{f(j)}$ and $\mathbb{R}^{\mathbb{J},j}_{\mathrm{wo}}$ isomorphically onto $\mathbb{R}^{\mathbb{J}',f(j)}_{\mathrm{wo}}$. Therefore $x \mapsto L(x)(f(j))$ is a linear function $\mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}/W_j \to \mathbb{R}$. Any such function is of the form $x \mapsto \alpha x(j) + F_j(x)$, for some real α and some $F_j\colon \mathbb{R}^{\mathbb{J}}_{\mathrm{wo}}/\mathbb{R}^{\mathbb{J},j}_{\mathrm{wo}} \to \mathbb{R}$. It is clear that α must be positive in order for L to be order-preserving.

Remark 22. Conrad (1953) proves a closely related result as his Theorem 5.2.

3.6 Proof of Theorem 6

It is straightforward to check that given an MP representation of $\succsim_X$ with values in a preordered vector space, $\succsim_X$ must satisfy SI. For the converse, suppose that $\succsim_X$ is an SI preorder. By Proposition 13 there is a corresponding vector preorder on the vector space $V = \mathrm{Span}(X - X)$, satisfying (1). Choosing any $x_0 \in X$, we have an MP function $u\colon X \to V$, $u(x) = x - x_0$. In light of (1), this is a representation of $\succsim_X$.

3.7 Proof of Proposition 7

Given a preorder $(Y, \succsim_Y)$, write $x \curlywedge_Y y$ to mean neither $x \succsim_Y y$ now $y \succsim_Y x$.

Proposition 23. *If $(V, \succsim_V)$ is a preordered vector space, then $v \succsim_V w$ if and only if $v \succsim_V^{\mathrm{com}} w$ for all complete vector preorders $\succsim_V^{\mathrm{com}}$ extending $\succsim_V$.*

Lemma 24. *Suppose given $v_0 \in V$ such that $v_0 \curlywedge_V 0$. Then there exists a complete vector preorder $\succsim_V^{\mathrm{com}}$ extending $\succsim_V$ such that $v_0 \prec_V^{\mathrm{com}} 0$.*

Proof. Let us show that there exists a vector preorder $\succsim'_V$, not necessarily complete, extending $\succsim_V$ and such that $v_0 \prec'_V 0$. In fact, we can define $\succsim'_V$ by the rule:

$$w \succsim'_V 0 \iff \exists \lambda \geq 0 : w + \lambda v_0 \succsim_V 0.$$

Then, by Zorn's Lemma, a maximal such extension $\succsim_V^{\text{com}}$ exists. This extension must be complete, since otherwise we could find a further extension using the same trick. □

Proof of Proposition 23. Suppose that $v \succsim_V w$. Then $v \succsim_V^{\text{com}} w$, for all complete vector preorders $\succsim_V^{\text{com}}$ extending $\succsim_V$, by definition of 'extending'. Conversely, suppose that $v \succsim_V^{\text{com}} w$, or equivalently $v_0 := v - w \succsim_V^{\text{com}} 0$, for all such $\succsim_V^{\text{com}}$. We cannot have $v_0 \prec_V 0$, for that would require $v_0 \prec_V^{\text{com}} 0$. Nor can we have $v_0 \curlywedge_V 0$: by Lemma 24, we would then have some $\succsim_V^{\text{com}}$ with $v_0 \prec_V^{\text{com}} 0$. Therefore we must have $v_0 \succsim_V 0$, hence $v \succsim_V w$, as desired. □

Proof of Proposition 7. This is immediate from Propositions 13 and 23. □

3.8 Proof of Theorem 8

Suppose that $\succsim_X$ satisfies PLR, or more generally that $\succsim_X$ has an MP representation in a preordered vector space. Then it is easy to check that $\succsim_X$ satisfies SI.

So, conversely, suppose that $\succsim_X$ satisfies SI. Let $\mathbb{I}$ be the set of its complete SI extensions. For each $i \in \mathbb{I}$, let $u_i \colon X \to \mathbb{R}^{\mathbb{J}_i}_{\text{wo}}$ be a LMP representation of the ith complete SI extension; it exists by Theorem 1. Taking all the u_i together, we obtain a PLMP function $u \colon X \to \prod_i \mathbb{R}^{\mathbb{J}_i}_{\text{wo}}$. We have $u(x) \succsim_\Pi u(y)$ if and only if $u_i(x) \geq_{\text{lex}} u_i(y)$ for all i, or, equivalently, if and only if $x \succsim_X^{\text{com}} y$ for every complete SI extension of $\succsim_X$. Applying Proposition 7, we find that $u(x) \geq_P u(y)$ if and only if $x \succsim_X y$. Thus u is a PLMP representation of $\succsim_X$.

3.9 Proof of Theorem 11

Our proof extends that of Chipman (1960, Theorem 3.1) to the case when $\succsim_X$ is incomplete.

Let $(X, \succsim_X)$ be a preordered set. Define $U_x := \{y \in X \mid y \succsim_X x\}$, and let $\mathcal{U} := \{U_x \mid x \in X\}$.

Let λ be the least ordinal whose cardinality is equal to that of $\mathcal{U}$. Since λ is an ordinal, it is a well-ordered set; so under the lexorder $\geq_{\text{lex}}$, $\mathbb{R}^\lambda = \mathbb{R}^\lambda_{\text{wo}}$.

Arbitrarily index members of $\mathcal{U}$ with ordinals less than λ, so that $\mathcal{U} = \{U_\beta \colon \beta < \lambda\}$. Define a function $f \colon X \to \mathbb{R}^\lambda$ by

$$f(x)(\beta) = \begin{cases} 1, & \text{if } x \in U_\beta; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that if $\succsim_X$ is complete, then f represents $\succsim_X$.

But in general, let Σ be the group of permutations on λ. For $\sigma \in \Sigma$, define $\sigma f \colon X \to \mathbb{R}^\lambda$ by $(\sigma f)(x)(\beta) = f(x)(\sigma^{-1}\beta)$. Finally, define $F \colon X \to \prod_{\sigma\in\Sigma} \mathbb{R}^\lambda$ by $F(x)_\sigma = (\sigma f)(x)$. We claim that when $\prod_{\sigma\in\Sigma} \mathbb{R}^\lambda$ is equipped with the product order $\succsim_\Pi$, the PL function F represents $\succsim_X$. Equivalently,

$$x \succsim_X y \iff (\sigma f)(x) \geq_{\text{lex}} (\sigma f)(y) \quad \forall \sigma \in \Sigma. \tag{6}$$

To see this, note that for all $\sigma \in \Sigma$, $x, y \in X$, $x \succ_X y \implies (\sigma f)(x) >_{\text{lex}} (\sigma f)(y)$ and $x \sim_X y \implies (\sigma f)(x) = (\sigma f)(y)$. Suppose $x \curlywedge_X y$. Then there is a least ordinal $\gamma < \lambda$ such that $f(x)(\gamma) = 1$ and $f(y)(\gamma) = 0$, and a least ordinal $\delta < \lambda$ such that $f(x)(\delta) = 0$ and $f(y)(\delta) = 1$. Let $\sigma' \in \Sigma$ be the permutation $(\gamma\delta)$. If $\gamma < \delta$, then $f(x) >_{\text{lex}} f(y)$ but $(\sigma' f)(y) >_{\text{lex}} (\sigma' f)(x)$. Similarly, if $\delta < \gamma$, then $f(y) >_{\text{lex}} f(x)$ but $(\sigma' f)(x) >_{\text{lex}} (\sigma' f)(y)$. These observations establish (6).

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