Evolution Nodes in Newtonian Supertasks

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ABSTRACT: The present article provides an analysis of the instants of a system that performs a Newtonian supertask. For each instant it studied the possibility of the system having, from the instant in question, more than one possible course of evolution; i.e. the possibility of it being an evolution node. This analysis shows that some supertasks presented as deterministic in Pérez Laraudogoitia (2007) are in fact indeterministic and specifies the difficulties ahead in showing the radical indeterminism suggested by Atkinson & Johnson (2009).

Keywords: critical instant, evolution node, failure of energy conservation, indeterminism, Newtonian Mechanics, supertasks.

1. *Introduction*

Two main anomalies are discussed in the literature on Newtonian supertasks: indeterminism and the failure of energy conservation. The prevailing situation is that both appear together; when indeterminism is present, energy conservation fails. Indeed, the only set of examples that generates indeterminism without the failure of energy conservation is the one presented in Pérez Laraudogoitia (2008). The present article is motivated by the possibility of the inverse situation: is the failure of energy conservation possible in deterministic supertasks? Since indeterminism apparently may (or may not) emerge in the same supertask, depending on the instantaneous state of the system performing it, the issue needs to be raised in more precise terms. To begin with, we need to establish a precise terminology to specify the problem and the particular aspect of it we want to discuss.

First, by *Newtonian supertask* we mean an infinite sequence of perfectly elastic binary collisions, between extended bodies or between point particles, which occurs within a finite time interval and within the theoretical framework of Newtonian mechanics. In particular, we focus on supertasks with order type ω or ω^*. For illustrative purposes, we refer essentially to the behaviour of the supertask presented in Pérez Laraudogoitia (1996), which we call ST. Initially, at time $t = 0$, we have an infinite number of particles arranged in a one-dimensional space (the x-axis, say), so that each particle P_n (with $n \in \{0, 1, 2, 3, \dots\}$) is initially at $x = 1/2^n$, in an inertial frame of reference in which P_0 has a velocity $v > 0$ while the remaining particles are at rest. If we assume that all particles have the same mass and interact only through perfectly elastic collisions, then we have a Newtonian supertask: P_0 collides with P_1, and the first particle comes to rest while the second acquires velocity v; some time later, P_1 collides with P_2, and the first particle comes to rest while the second acquires velocity v; in general, and following a strict successively sequence, every P_n in motion with v collides with P_{n+1} at rest, and the first particle comes to rest while the second acquires velocity v. When such a sequence of collisions concludes, at $t = 1/v$, every particle will be at rest.

By *critical instant* we mean an instant of time t for which there is at least one open neighbourhood $(t - \delta, t + \delta)$, with δ arbitrarily small, in which at least one Newtonian supertask of order type ω or ω^* is performed. Stated more intuitively, a critical instant is the first instant in which a supertask of order type ω has been performed, or is the last instant in which a supertask of order type ω^* has not yet been performed. Thus, for ST, as presented above, $t = 1/v$ is a critical instant.

By *non-conservative critical instant* we mean a critical instant in which the total energy of the system is different from the energy in at least one of the two intervals that make up its punctured neighbourhood $(t - \delta) \cup (t + \delta)$. Intuitively, a non-conservative critical instant is the first instant in which a supertask of order type ω has been performed and the total energy of the system is different from the energy during the performance of that supertask; or is the last instant in which a supertask of order type ω^* has not yet been performed and the total energy of the system is different from the energy during the performance of that particular supertask. Clearly, the critical instant $t = 1/v$ in ST is a non-conservative critical instant: in it the energy is null due to the rest of every particle, whereas, during the performance of the supertask, the energy is not null due to the motion of one of the particles.

By *evolution node* we mean an instant at which the system, from that instant on, has at least two different possible courses of evolution. Intuitively, at an evolution node the course of evolution bifurcates; at that bifurcation, the system has to choose only one course from several options. In the system that performs ST, for instance, every instant (before, during, and after the performance of the supertask) is an evolution node. To see this, we need only note that the temporal inversion of ST is also a possible process within the framework of Newtonian mechanics. This reversed process is a supertask of order type ω^*, starting from a non-conservative critical instant in which all the particles are at rest. So, the system which performs ST will have in every instant an infinite subset of particles at rest capable of performing, at any time, the temporal inversion of ST. Hence, such a subset always has a bifurcation with at least two courses of evolution: one is to continue with the evolution described for ST and the other is to perform the temporal inversion of ST. It is clear that, in the system performing ST, every instant is an evolution node.

Having established this terminology, the problem of the forms in which indeterminism is related to the failure of energy conservation in Newtonian supertasks can be precisely stated. Note, for example, that the non-conservative critical instant of ST is also an evolution node, i.e. in that instant there is a failure of energy conservation at the same time the system is set for an indeterministic evolution. This is in fact the situation for all the non-conservative supertasks precisely described to date in the literature. But is it possible for a non-conservative critical instant of some supertask not to be an evolution node? In precise terms, the question may be stated as follows:

(A) Are there Newtonian supertasks with non-conservative critical instants that are not evolution nodes?

Although our main concern is to answer (A), this article addresses a more specific problem of particular relevance to (A). We perform an analysis that focuses on an example of a supertask of order type ω that culminates in a non-conservative critical instant, which, unlike the critical instant of ST, is not manifestly an evolution node. The example is just a variation of ST. In the first section this example, which we call STMNC, is presented in detail. The article subsequently focuses on a question of particular relevance to (A):

(B) Which instants of a system that performs a Newtonian supertask are evolution nodes?

Question (B), always in reference to STMNC and similar examples (specifically, those that initially have an infinite subset of particles at relative rest, and whose total mass is finite), is addressed in two parts. To begin with, in the second section we analyse the instants prior to their critical instant for the supertasks with order type ω. We show that, for STMNC and other processes presented in the literature as not manifestly indeterministic (specifically, the ones presented in Pérez Laraudogoitia (2007, pp. 726-8)), the instants prior to its critical instant are actually evolution nodes, and therefore those processes are indeterministic. In the third section, for the supertasks with order type ω, we analyse the instants subsequent to their critical instant, as well as the critical instant itself. Under the general model designed by Atkinson and Johnson (2009), the analysis sheds some light on the difficulties to be faced in proving the instants in question are evolution nodes.

2. A non-conservative and not manifestly indeterministic supertask

Let us consider the same initial state of the system that performs ST with the following modifications. For the sake of simplicity, we suppress particle P_0 and consider that, initially, particle P_1 is the only one in motion. Additionally, we assume that each particle P_n has a mass $m_n = \frac{1}{(n+1)^2}$, so the total mass of the system is $\frac{\pi^2}{6} - 1$, since it is well known that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Thus, the mass ratio γ_n between every pair of adjoining particles P_n and P_{n+1} is $\gamma_n = \frac{m_{n+1}}{m_n} = \frac{(n+1)^2}{(n+2)^2}$. This system clearly performs a supertask too, which we will call STMNC. P_1, which has velocity v_1', will collide with P_2, which is at rest; P_1 thereby will acquire velocity v_1'' which, since the mass of P_1 is greater than that of P_2, will be smaller than velocity v_2' acquired by P_2. Some time later, P_2, now with velocity v_2', will collide with P_3, which is at rest; P_2 will thereby acquire velocity v_2'', which, since the mass of P_2 is greater than that of P_3, will be smaller than the velocity v_3' acquired by

P_3. In general, P_n, with velocity $v_n{}'$, will collide with P_{n+1}, which is at rest; P_n will thereby acquire velocity $v_n{}''$, which, since the mass of P_n is greater than that of P_{n+1}, will be smaller than the velocity $v_{n+1}{}'$ acquired by P_{n+1}. To be sure this sequence of collisions occurs, we assume that $v_{n+1}{}'' < v_{n+2}{}''$; so the system performs a supertask, since P_1 with $v_1{}''$ will at some instant reach the initial position occupied by the accumulation point of the set of particles, which implies that, at that instant, the motion will have already propagated to all the particles under the infinite sequence of collisions.

With this, under momentum and energy conservation in the collision of P_n (initially moving with $v_n{}'$) with P_{n+1} (initially at rest), we know that the particles will acquire respectively velocities

$$v_n{}'' = \tfrac{1-\gamma_n}{1+\gamma_n} v_n{}' \qquad (1)$$

and

$$v_{n+1}{}' = \tfrac{2}{1+\gamma_n} v_n{}' \qquad (2)$$

By recursion, from (2)

$$v_{n+1}{}' = 2^n v_1{}' \prod_{k=1}^{n} \tfrac{1}{1+\gamma_k}\,. \qquad (3)$$

Thus, from (1) and (3), each particle P_{n+1} will end up with a velocity

$$v_{n+1}{}'' = \frac{1-\gamma_{n+1}}{1+\gamma_{n+1}} 2^n v_1{}' \prod_{k=1}^{n} \tfrac{1}{1+\gamma_k}\,. \qquad (4)$$

Now, to ensure that the sequence of collisions actually occurs in the way we have specified, we need $v_{n+1}{}'' < v_{n+2}{}''$, i.e. taking (4) in account, we need $(1 - \gamma_{n+1})(1 + \gamma_{n+2}) < 2(1 - \gamma_{n+2})$, which is equivalent to

$$(3 - \gamma_{n+1})\, \gamma_{n+2} < 1 + \gamma_{n+1}. \qquad (5)$$

For the case of STMNC, it is clear this inequality holds. Substituting the mass ratio, which is $\gamma_n = \frac{(n+1)^2}{(n+2)^2}$, (5) is equivalent to $0 < 1$, an inequality that holds independently of the value of n.

Now we need to prove that the critical instant of STMNC is a non-conservative critical instant; i.e. that the system experiences a loss of energy whenever the supertask STMNC is performed. We know the initial kinetic energy is $\frac{1}{2} m_1 v_1{}'^2$. We know also, by recursion in the mass ratio, that $m_{n+1} = m_1 \prod_{k=1}^{n} \gamma_n$. Then, considering (4), the final kinetic energy is given by

$$T_f = \frac{1}{2} m_1 v_1{}'^2 \left(\frac{1-\gamma_1}{1+\gamma_1}\right)^2 + \frac{1}{2} m_1 v_1{}'^2 \sum_{n=1}^{\infty} \left[\frac{2^{2n}(1-\gamma_{n+1})^2}{(1+\gamma_{n+1})^2} \prod_{k=1}^{n} \tfrac{\gamma_k}{(1+\gamma_k)^2} \right] \qquad (6)$$

From (6), it is seen that we can express the loss of energy with the following inequality:

$$\frac{1}{2}m_1 v_1'^2\left(\frac{1-\gamma_1}{1+\gamma_1}\right)^2+\frac{1}{2}m_1 v_1'^2\sum_{n=1}^{\infty}\left[\frac{2^{2n}(1-\gamma_{n+1})^2}{(1+\gamma_{n+1})^2}\prod_{k=1}^{n}\frac{\gamma_k}{(1+\gamma_k)^2}\right]<\frac{1}{2}m_1 v_1'^2$$

which, for our model, is equivalent to

$$\sum_{n=1}^{\infty}\left[\frac{2^{2n}(1-\gamma_{n+1})^2}{(1+\gamma_{n+1})^2}\prod_{k=1}^{n}\frac{\gamma_k}{(1+\gamma_k)^2}\right]<1-\left(\frac{1-\gamma_1}{1+\gamma_1}\right)^2=\left(\frac{12}{13}\right)^2 \qquad (7)$$

Now, the mass ratios indicate that the product $\prod_{k=1}^{n}\frac{\gamma_k}{(1+\gamma_k)^2} = \prod_{k=1}^{n}\frac{(k+1)^2(k+2)^2}{(2k^2+6k+5)^2}$. Expressed in this way, it is clear that each term $\frac{(k+1)^2(k+2)^2}{(2k^2+6k+5)^2} < \frac{(k+1)^2(k+2)^2}{(2k^2+6k+4)^2} = \frac{1}{2^2}$; so we know that the product $\prod_{k=1}^{n}\frac{\gamma_k}{(1+\gamma_k)^2} < \frac{1}{2^{2n}}$. Therefore, from the sum in (7) (the left side), each term $\frac{2^{2n}(1-\gamma_{n+1})^2}{(1+\gamma_{n+1})^2}\prod_{k=1}^{n}\frac{\gamma_k}{(1+\gamma_k)^2} < \frac{(1-\gamma_{n+1})^2}{(1+\gamma_{n+1})^2}$. Since, under our mass ratios, $\frac{(1-\gamma_{n+1})^2}{(1+\gamma_{n+1})^2} = \frac{(2n+5)^2}{(2n^2+10n+13)^2} < \frac{(2n+5)^2}{(2n^2+9n+10)^2} = \frac{1}{(n+2)^2}$, we finally find that (7) holds:

$$\sum_{n=1}^{\infty}\left[\frac{2^{2n}(1-\gamma_{n+1})^2}{(1+\gamma_{n+1})^2}\prod_{k=1}^{n}\frac{\gamma_k}{(1+\gamma_k)^2}\right]<\sum_{n=1}^{\infty}\frac{1}{(n+2)^2}=\sum_{n=3}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}-\frac{5}{4}<\left(\frac{12}{13}\right)^2$$

Hence, the critical instant of STMNC is a non-conservative critical instant.

At first sight, the non-conservative critical instant of STMNC seems not to be an evolution node, which would suggest an affirmative answer to (A). As the next section shows, this is not an obstacle to STMNC being an indeterministic process, i.e. for some other instants to be evolution nodes. And as the section also shows, this is especially relevant for systems presented as deterministic in the literature.

3. The evolution nodes prior to the critical instant

Now we show that every instant of STMNC prior to its critical instant is an evolution node. To this end, we designed a general model of supertasks in which all their particles ended at rest.

We called the model Gγ1, which is a supertask consisting of an infinite sequence with order type ω of perfectly elastic collisions between point particles, as in ST and STMNC, but, unlike ST and STMNC, with the added assumption that all the particles may initially be in motion. We assumed that the mass ratio between adjoining particles $\gamma_n = m_{n+1}/m_n \neq m_{n+2}/m_{n+1} = \gamma_{n+1}$. We also reduced our model to cases where $\gamma_n < 1$ for all $n \in \{0, 1, 2, 3, 4, \ldots\}$. At the initial instant $t_0{'} = 0$, the first particle P_0 is in motion at $x_0{'} = 0$ at velocity $v_0{'} > 0$, whereas each particle P_{n+1} (with $n \in \{0, 1, 2, 3, 4, \ldots\}$) is located at position x_{n+1}, such that $x_{n+1} < x_{n+2} < x_{n+3} < \ldots$, and has a velocity v_{n+1}, with $v_{n+1} > v_{n+2} > v_{n+3} > \ldots$ Some time later, at the instant $t_1{'} > t_0{'}$, P_0 collides with P_1 at $x_0{''} = x_1{'} = 1/2$, where P_0 acquires velocity $v_0{''} = 0$ while P_1 acquires velocity $v_1{'}$. After that, at the instant $t_2{'} > t_1{'}$, P_1 collides with P_2 at $x_1{''} = x_2{'} = 3/4$, where P_1 acquires velocity $v_1{''} = 0$ while P_2 acquires velocity $v_2{'}$. In general, at the instant $t_{n+1}{'}$, P_n collides with P_{n+1} at $x_n{''} = x_{n+1}{'} = 1 - 1/2^{n+1}$, where P_n acquires velocity $v_n{''} = 0$ while P_{n+1} acquires velocity $v_{n+1}{'}$. This sequence of collisions occurs within a finite interval of time (demonstrated below), successfully performing a supertask in which, in its final state, all the particles are at rest.

Having presented this general model of supertasks, we specified the velocities of the particles required for the evolution described. From linear momentum and energy conservation in an elastic collision, we know the following system of equations that relates the velocities involved in each of the collisions described in the previous paragraph (collision in which initially P_n has velocity $v_n{'}$ and P_{n+1} velocity v_{n+1}, to end up respectively with $v_n{''} = 0$ and $v_{n+1}{'}$):

$$0 = \frac{1-\gamma_n}{1+\gamma_n} v_n{'} + \frac{2\gamma_n}{1+\gamma_n} v_{n+1} \qquad (8)$$

$$v_{n+1}{'} = \frac{2}{1+\gamma_n} v_n{'} - \frac{1-\gamma_n}{1+\gamma_n} v_{n+1} \qquad (9)$$

From (9), $v_{n+1} = \frac{2}{1-\gamma_n} v_n{'} - \frac{1+\gamma_n}{1-\gamma_n} v_{n+1}{'}$, which, substituted in (8), leads to

$$v_{n+1}{'} = \frac{1+\gamma_n}{2\gamma_n} v_n{'} \qquad (10)$$

Now, from (10) it is clear that

$$\begin{aligned}
v_1{'} &= \frac{1+\gamma_0}{2\gamma_0} v_0{'} \\
v_2{'} &= \left(\frac{1+\gamma_1}{2\gamma_1}\right)\left(\frac{1+\gamma_0}{2\gamma_0}\right) v_0{'} \\
v_3{'} &= \left(\frac{1+\gamma_2}{2\gamma_2}\right)\left(\frac{1+\gamma_1}{2\gamma_1}\right)\left(\frac{1+\gamma_0}{2\gamma_0}\right) v_0{'} \\
&\vdots \\
v_{n+1}{'} &= \frac{1}{2^{n+1}}\left(\frac{1+\gamma_n}{\gamma_n}\right)\cdots\left(\frac{1+\gamma_0}{\gamma_0}\right) v_0{'}. \qquad (11)
\end{aligned}$$

On the other hand, from (8) we know that

$$v_{n+1} = -\frac{1-\gamma_n}{2\gamma_n} v_n{}' \qquad (12)$$

Then, considering (11) and (12), it is also easy to see that

$$v_1 = -\frac{1-\gamma_0}{2\gamma_0} v_0{}'$$
$$v_2 = -\frac{1-\gamma_1}{2\gamma_1}\left(\frac{1+\gamma_0}{2\gamma_0}\right) v_0{}'$$
$$v_3 = -\frac{1-\gamma_2}{2\gamma_2}\left(\frac{1+\gamma_1}{2\gamma_1}\right)\left(\frac{1+\gamma_0}{2\gamma_0}\right) v_0{}'$$
$$\vdots$$
$$v_{n+2} = -\frac{1}{2^{n+2}}\frac{1-\gamma_{n+1}}{\gamma_{n+1}}\left(\frac{1+\gamma_n}{\gamma_n}\right)\cdots\left(\frac{1+\gamma_0}{\gamma_0}\right) v_0{}'. \qquad (13)$$

Now, so that $v_{n+1} > v_{n+2} > v_{n+3} > \ldots$, we must add a restriction (that limits our general model a little more, but by no means threatens our aim here). From (13) we know that every velocity v_{n+1} moves particle P_{n+1} towards the left (as $\gamma_n < 1$, therefore $v_{n+1} < 0$), and to ensure that $v_{n+1} > v_{n+2}$, it suffices that $|v_{n+2}| > |v_{n+1}|$; which implies that

$$\frac{|v_{n+2}|}{|v_{n+1}|} = \frac{(1-\gamma_{n+1})(1+\gamma_n)}{2\gamma_{n+1}(1-\gamma_n)} > 1$$

which is equivalent, since $\gamma_n < 1$, to the following inequality:

$$\gamma_{n+1} < \frac{1+\gamma_n}{3-\gamma_n}. \qquad (14)$$

This is the restriction we must add to ensure that $v_{n+1} > v_{n+2} > v_{n+3} > \ldots$ in the systems this general model covers.

We then verified that, independently of the position x_{n+1} every particle P_{n+1} needed to take to move from x_{n+1} to $x_{n+1}{}'$ at velocity v_{n+1}, the sequence of collisions is actually performed as specified above in the description of the general model; that is to say, that P_{n+1} (travelling at v_{n+1}) collides with P_n (travelling at $v_n{}'$) before colliding with P_{n+2} (travelling at v_{n+2}). First, since P_{n+1} travels from x_{n+1} at $t_0{}' = 0$ to $x_{n+1}{}'$ at $t_{n+1}{}'$ at a constant velocity v_{n+1}, we know that $v_{n+1} = \frac{x_{n+1}{}' - x_{n+1}}{t_{n+1}{}'}$, from which it is clear that

$$x_{n+1} = x_{n+1}{}' - t_{n+1}{}' v_{n+1}. \qquad (15)$$

But we previously specified that $t_{n+1}{}'$ was the instant in which particle P_{n+1} acquires velocity $v_{n+1}{}'$ after colliding with P_n, which then comes to rest. We also specified that this collision occurred at

$$x_{n+1}{}' = 1 - 1/2^{n+1}. \qquad (16)$$

Thus, each particle P_n will travel the distance from x_n' to x_{n+1}' at a constant velocity v_n', during the time interval $t_{n+1}'-t_n'=\frac{x_{n+1}'-x_n'}{v_n'}=\frac{1}{2^{n+1}v_n'}$. This leads to

$$
\begin{aligned}
&t_1'=\frac{1}{2v_0'}\\
&t_2'=\frac{1}{2v_0'}+\frac{1}{2^2v_1}\\
&t_3'=\frac{1}{2v_0'}+\frac{1}{2^2v_1'}+\frac{1}{2^3v_2'}\\
&\vdots\\
&t_{n+1}'=\frac{1}{2v_0'}+\frac{1}{2^2v_1'}+\cdots+\frac{1}{2^{n+1}v_n'} \qquad (17)
\end{aligned}
$$

Now, we want to prove that P_{n+1} (which departs from x_{n+1} at $t_0' = 0$ at v_{n+1}) collides with P_n at x_{n+1}' and at t_{n+1}' before colliding with P_{n+2} (which departs from x_{n+2} at $t_0' = 0$ at v_{n+2}) at a hypothetical position x_h and at a hypothetical instant t_h. Clearly, this hypothetical instant can be expressed in two ways:

$$t_h=\frac{x_h-x_{n+1}}{v_{n+1}} \qquad (18)$$

$$t_h=\frac{x_h-x_{n+2}}{v_{n+2}}\,. \qquad (19)$$

From the system (18) (19), the hypothetical position is $x_h=\frac{v_{n+2}x_{n+1}-v_{n+1}x_{n+2}}{v_{n+2}-v_{n+1}}$, and the hypothetical instant is

$$t_h=\frac{x_{n+1}-x_{n+2}}{v_{n+2}-v_{n+1}}\,. \qquad (20)$$

Taking the symbols specified, what we wish to prove is simply that

$$t_{n+1}' < t_h. \qquad (21)$$

If we substitute (15) in (20), (21) can be rewritten as:

$$t_{n+1}'<\frac{(x_{n+1}'-x_{n+2}')+(t_{n+2}'v_{n+2}-t_{n+1}'v_{n+1})}{v_{n+2}-v_{n+1}} \qquad (22)$$

As we are only interested in cases where $|v_{n+2}| > |v_{n+1}|$, and we know, from (13), that these two velocities are negative, (22) can be multiplied by $(v_{n+2} - v_{n+1})$ (a factor that switches the direction of the inequality), to obtain that

$$(t_{n+1}' - t_{n+2}')v_{n+2} > x_{n+1}' - x_{n+2}'. \qquad (23)$$

From (16) we know that $x_{n+1}{'} - x_{n+2}{'} = -\frac{1}{2^{n+2}}$, and from (17) that $t_{n+1}{'} - t_{n+2}{'} = -\frac{1}{2^{n+2} v_{n+1}{'}}$. If this is substituted in (23), and we multiply both sides by (-2^{n+2}), then (23) is equivalent to the following inequality:

$$\frac{v_{n+2}}{v_{n+1}{'}} < 1$$

If we substitute (12) in v_{n+2}, this inequality can be rewritten as

$$-\frac{1-\gamma_{n+1}}{2\gamma_{n+1}} < 1$$

which is equivalent to

$$\gamma_{n+1} > -1. \qquad (24)$$

But by definition $\gamma_{n+1} > 0$, so (23) always holds. Therefore (22) always holds too; in other words, every particle P_{n+1} collides with P_n before it can collide with P_{n+2}.

To complete our demonstration that the general model designed here corresponds to systems that perform a supertask successfully, we must show that the interval of time in which the entire sequence of collisions occurs is finite. This interval of time can be expressed as the limit of $t_{n+1}{'}$ (as written in (17)) when $n \to \infty$:

$$t_T{'} = \frac{1}{2v_0{'}} + \frac{1}{2^2 v_1{'}} + \frac{1}{2^3 v_2{'}} + \cdots + \frac{1}{2^{n+1} v_n{'}} + \cdots \qquad (25)$$

and (25) in turn can be rewritten as

$$t_T{'} = \frac{1}{2v_0{'}}\left(1 + \frac{v_0{'}}{2v_1{'}} + \frac{v_0{'}}{2^2 v_2{'}} + \cdots + \frac{v_0{'}}{2^n v_n{'}} + \cdots\right). \qquad (26)$$

It is clear that for condition

$$v_0{'} < v_1{'} < \ldots < v_n{'} \qquad (27)$$

to hold, it suffices that $v_n{'} < v_{n+1}{'}$ for every n, which, substituting (10), is equivalent to

$$\gamma_n < 1. \qquad (28)$$

As our general model is confined precisely to cases in which (28) holds, all the systems the model covers satisfy (27). It is clear, then, that the following inequality also holds:

$$t_T{'} = \frac{1}{2v_0{'}}\left(1 + \frac{v_0{'}}{2v_1{'}} + \frac{v_0{'}}{2^2 v_2{'}} + \cdots + \frac{v_0{'}}{2^n v_n{'}} + \cdots\right) < \frac{1}{2v_0{'}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots\right);$$

since from (27) and (28) we know that $\frac{v_0{'}}{v_n{'}} < 1$, which implies that each term of the sum expressed inside the parenthesis on the left side is less than its respective term of the sum expressed inside the parenthesis on the right side of the inequality (except the first one, which is equal). As we know, the sum of the right side is convergent. Hence this inequality shows that the time (expressed with (26)) the system takes to perform the supertask is finite.

In short, all the supertasks covered by the general model Gγ1 can be described as follows. To begin with, they have an infinite number of particles P_0, P_1, P_2, …, P_n, … in one-dimensional space (the x-axis). For every n (with $n \in \{0, 1, 2, 3, \ldots\}$), there is a mass ratio $\gamma_n = m_{n+1}/m_n$, which relates the mass m_n of particle P_n to the mass m_{n+1} of the adjoining particle P_{n+1}, such that $\gamma_n < 1$, $\gamma_n \neq \gamma_{n+1}$ and $\gamma_{n+1} < \frac{\gamma_n+1}{3-\gamma_n}$. Initially, at time $t_0' = 0$, particle P_0 is in motion at a constant velocity $v_0' > 0$ and is located at $x_0' = 0$, whereas each of the other particles P_{n+1} moves at a constant velocity v_{n+1} (expressed in (13)) at x_{n+1} (expressed in (15)), with $x_{n+1} < x_{n+2} < x_{n+3} < \ldots$ and $v_{n+1} > v_{n+2} > v_{n+3} > \ldots$ (this inequality considers the negative sign of each velocity, which moves the particles leftwards, so $|v_{n+1}| < |v_{n+2}| < |v_{n+3}| < \ldots$). Thus, at time t_{n+1}' (expressed in (17)), particle P_n collides with particle P_{n+1} at $x_n'' = x_{n+1}' = 1 - 1/2^{n+1}$, where P_n acquires velocity $v_n'' = 0$ while P_{n+1} acquires velocity v_{n+1}' (expressed in (11)). So, from the instant t_T' on (expressed in (25) or in (26))–the critical instant of the supertask–the infinite sequence of collisions will have been performed, and each particle P_n will be at rest in its final position $x_n'' = 1 - 1/2^{n+1}$.

The critical instant t_T' is clearly a non-conservative critical instant. Prior to it, an infinite set of particles is in motion, whereas from that instant on every particle is at rest. It is equally obvious that the critical instant and its subsequent instants are evolution nodes. The process reversed in time is also a possible process, thus the infinite set of particles at rest has at least two possible evolutions: to continue at rest or to perform the temporal inversion of Gγ1.

This feature of Gγ1 is of special relevance to STMNC: among the cases Gγ1 covers, there is one whose set of particles coincides exactly with the set of particles of STMNC. This has an important consequence, namely, that as long as the infinite sequence of collisions of STMNC has not finished, there will be an infinite subset of particles at rest located exactly as an infinite subset of Gγ1 in its critical instant. Therefore, such a subset of particles has two possible evolutions: either to remain at rest until the performance of STMNC sets the particles in motion, or to perform the corresponding inverse process of Gγ1. In other words, bearing in mind that before STMNC finishes its performance, there is an infinite subset of particles at rest that corresponds exactly to the final state of rest of an infinite subset of particles of Gγ1, and to the initial state of rest of an infinite subset of particles of the process Gγ1 reversed in time, the system that performs STMNC is indeterministic as long as the supertask has not yet been finished. Put more concisely, every instant prior to the critical instant of STMNC is an evolution node.

It is easy to verify that the subset of particles of STMNC at rest corresponds to the subset of particles at the critical instant of one of the systems that Gγ1 covers. First, every particle P_n of STMNC still at rest is located at $x_n = 1 - 1/2^n$, a position which corresponds to the initial position of the reversed processes of Gγ1 (the fact that the trigger particle has an index $n = 1$ instead of $n = 0$ is not important). Second, it is clear that for STMNC $\gamma_n < 1$, since $\gamma_n = \frac{(n+1)^2}{(n+2)^2}$ $\forall n$. Third and last, the restriction expressed with the inequality in (14) must hold; indeed, if we substitute the mass ratios of STMNC in (14), we find that the inequality is equivalent to $0 < 1$, which is an inequality that holds independently of the value of n. Definitively, every instant prior to the critical instant of STMNC is an evolution node.

Indeterminism generated by Gγ1 also has consequences for systems that have been presented in literature as not manifestly indeterministic. Specifically, in Pérez Laraudogoitia (2007, pp. 726-8) a general model of supertasks is presented with the same initial conditions as ST, the only modification being that the mass ratios between adjoining particles are different from each other. In precise terms, the mass ratios this general model covers is given by the following expression:

$$\gamma_{n+1} = \frac{2a - 1 + \gamma_n}{2a + 1 - \gamma_n} \qquad (29)$$

with $a < 1$. The evolution of all the supertasks included is, indeed, very similar to that of STMNC. Initially, one particle approaches an infinite set of particles at rest. Under the sequence of collisions, each particle, one by one, acquires motion. Once the supertask is performed, each particle P_n has a velocity $v_n'' < v_{n+1}''$. The major difference between these processes and STMNC is that their critical instant is not a non-conservative critical instant; the energy is the same before, during, and after the performance of the supertask.

The indeterminism in the supertasks covered by Pérez Laraudogoitia's general model may be shown in the same way as it is in STMNC. Just as an infinite subset of particles of STMNC can spontaneously perform the temporal inversion of its correspondent process of Gγ1 before STMNC is finished, the subset of particles still at rest in the supertasks of Pérez Laraudogoitia's general model (before they finish) can perform the temporal inversion of their corresponding processes of Gγ1.

Verifying that the systems included in Pérez Laraudogoitia's general model are among the systems covered by Gγ1 is a simple task. First, it is obvious that each particle P_n of this general model can be initially located at $x_n = 1 - 1/2^n$, the position that corresponds to the initial position in the reversed processes of Gγ1. Second, (29) corresponds to the function $f(x) = \frac{2\alpha-1+x}{2\alpha+1-x}$, which has two fixed points: $x = 1$, which is an attractor when $a > 1$ and a repellor when $a < 1$, and $x = 2a - 1$, which is a repellor when $a > 1$ and an attractor when $a < 1$. As the general model explicitly includes only the cases in which $a < 1$ and with finite mass, taking some $\gamma_n > 1$ would imply, by the repulsion of $x = 1$, the total mass to be infinite; then, necessarily $\gamma_n < 1$. Third and last, the restriction expressed with the inequality in (14) must hold; indeed, if we substitute the mass ratios (29) in (14), we find that the inequality is equivalent to $\gamma_n < 1$, which, as we have just seen, holds. With this, all supertasks included in Pérez Laraudogoitia's general model are clearly indeterministic processes and, specifically, every instant prior to their critical instant is an evolution node.

It is interesting to note that the results obtained in this section under the general model Gγ1 could not have been obtained under the general model presented in Pérez

Laraudogoitia (2007, pp. 725-6)[1] nor under the general model presented in Atkinson & Johnson (2009, pp. 946-50). All three generate indeterminism in infinite systems of particles under the final state of relative rest to which these particles come. Nevertheless, the last two, the only ones available until now in the literature, impose an initial state in which all the particles except one are at rest. This forces these general models to cover systems in which mass ratios of adjoining particles cannot correspond to those of STMNC or of those analysed in Pérez Laraudogoitia's model.

4. Are the critical instant and its subsequent instants evolution nodes?

We analysed the possibility of the critical instant of STMNC (and of similar systems) and its later instants being evolution nodes. The general model Gγ1 designed in the previous section does not work for this purpose, because at the instants we want to analyse we do not have an infinite set of particles at rest (not at least under the most evident evolution). The general model recently presented in Atkinson & Johnson (2009, pp. 942-6, *not* the same as the one mentioned in the previous section) can help us to approach our aim. Even so, we confined our analysis to a specific range of systems included in the general model designed by Atkinson and Johnson, largely because it ranges so widely that little can be said of what interests us here (which is not to say that such breadth of scope is without interest).

The idea on which this model is based, and which suggests that the critical instant of STMNC (and similar supertasks) and its later instants are evolution nodes, is the following. Take STMNC by way of example. The supertask starts with a first particle in motion and the other particles at rest. Could the final state of STMNC also be the result of the performance of a supertask starting from some other initial state? The answer, as we will see below, is affirmative (at least for the range of cases we cover, including STMNC). In other words, the temporal inversion of STMNC (and similar supertasks) is seen to be an indeterministic process because, starting from the critical instant, it has at least two possible courses of evolutions: the performance of the temporal inversion of STMNC or the performance of the temporal inversion of the supertask with a different initial state.

For a more intuitive understanding of this possible kind of indeterminism, it is a good idea to go back to ST and Gγ1. To begin with, note that ST is not only indeterministic due to the infinite subset of particles at relative rest capable of spontaneously performing the reversed process of ST. It is also indeterministic because, in this reversed process of ST, velocity v can take any value. So, in addition to the two possibilities already mentioned (performing the temporal inversion of ST or remaining at rest), there is in fact an infinite number of possible evolutions: the spontaneous performance of the temporal inversion of ST can occur with any value for velocity v. The same kind of indeterminism is also found, in precise terms, in the processes that Gγ1

[1] Note that, though they belong to the same paper, this general model is not the same as the one analysed in previous paragraphs. The one mentioned here covers a range of supertasks in which the particles end up with velocities $v_n'' = v_{n+1}''$ whereas the one analysed in previous paragraphs covers a range in which particles end up with velocities $v_n'' < v_{n+1}''$.

covers. Here the possible reversed processes may be identified with the value of v_0' (on which the other velocities depend). As (16) makes clear, the positions of particles at rest before the spontaneous performance of the reversed processes do not depend on v_0'. Therefore, v_0' can take any value (given the location of the particles, the direction of this velocity will always be the same).

Although Atkinson and Johnson's general model enables us to prove (as we show below) the existence of indeterministic evolutions of some systems that perform the reversed processes of the supertasks with order type ω whose initial state consists of one particle in motion while the other particles are at rest (STMNC included), we have not yet proved, strictly speaking, the existence of indeterministic evolutions of the direct processes of the same supertasks (not, at least, from their critical instant on). The only thing that has been proved (which is, of course, of great value) is the existence of infinite solutions for the velocities that particles can take by means of the infinite sequence of perfectly elastic collisions; that there is an infinite set of initial velocities that would lead the particles to the same unique set of final velocities, if the sequence of collisions were achieved. But this does not imply that the sequence of elastic collisions required is feasible or consistent. Indeed, Atkinson and Johnson (2009, p. 956) acknowledge that their general model does not specify the positions particles can take. And as we shall also see, positions play an important role in the difficulties encountered in proving indeterminism in these systems. Not all positions are valid, and it may be that none is.

In short, under this possible form of indeterminism we cannot be sure that the critical instant of supertask STMNC (the process not reversed in time) is an evolution node. Even so, Atkinson and Johnson's idea certainly helps to clarify the issue. Specifically, we are looking to answer the following question:

> (C) Does the state of the system STMNC (and similar supertasks with order type ω) in its critical instant correspond exactly to the state of some other supertask of order type ω^* in its critical instant, in such a way that the critical instant of STMNC (of the process not reversed in time) is an evolution node?

Clearly, an affirmative answer would tip the answer to (A) to the negative. Here, for the cases we cover, we specify the difficulties involved in giving an affirmative answer.

Our analysis only covers the cases included in the following general model, which we call Gγ2. In this model, the mass ratio of each pair of adjoining particles is such that $\gamma_n = m_{n+1}/m_n \neq m_{n+2}/m_{n+1} = \gamma_{n+1}$. Additionally, to verify that the chain of collisions is strictly sequential, we limited our model to the cases in which every $\gamma_n < 1$ and the final velocities were $v_0'' < v_1'' < v_2'' < \ldots < v_n'' < \ldots$

Our reference case consists of a supertask in which, initially, P_0 is in motion at velocity $v_0' = \Omega > 0$ approaching P_1 which is at rest, as are all the other particles, located such that, taken together, they possess an accumulation point. Generally, we know, by linear momentum and energy conservation in an elastic collision between particles P_n and P_{n+1}, that velocities resulting from the collision can be expressed in terms of the velocities before the collisions and the mass ratio, as follows:

$$v_n'' = \frac{1-\gamma_n}{1+\gamma_n} v_n' + \frac{2\gamma_n}{1+\gamma_n} v_{n+1} \qquad (30)$$

$$v_{n+1}' = \frac{2}{1+\gamma_n} v_n' - \frac{1-\gamma_n}{1+\gamma_n} v_{n+1} \qquad (31)$$

As, for the reference case, $v_{n+1} = 0\ \forall n$ and $v_0' = \Omega$, then, from (30),

$$v_0'' = \tfrac{1-\gamma_0}{1+\gamma_0}\Omega \qquad (32)$$

while, from (31), we know that

$$v_{n+1}' = \frac{2^{n+1}}{(1+\gamma_n)(1+\gamma_{n-1})\cdots(1+\gamma_0)}\Omega$$

which, considering (31) and (32), leads us to

$$v_n'' = \frac{2^n(1-\gamma_n)}{(1+\gamma_n)(1+\gamma_{n-1})\cdots(1+\gamma_0)}\Omega \qquad (33)$$

From (33) we find that the condition imposed (namely, that $v_n'' < v_{n+1}''$), is equivalent to

$$\gamma_{n+1} < \frac{1+\gamma_n}{3-\gamma_n} \qquad (34)$$

which is the same as restriction (14) obtained for Gγ1. So, the general model Gγ2 also only covers cases where this relation between mass ratios holds.

This is so far the entire model for the reference case. Now, what we need are the other initial conditions that lead the system to the same final state, for the critical instant of the temporal inversion of this supertask to be an evolution node. So, from (31)

$$v_1' = \frac{2}{1+\gamma_0} v_0' - \frac{1-\gamma_0}{1+\gamma_0} v_1$$

$$v_2' = \frac{2}{1+\gamma_1}\left(\frac{2}{1+\gamma_0} v_0' - \frac{1-\gamma_0}{1+\gamma_0} v_1\right) - \frac{1-\gamma_1}{1+\gamma_1} v_2$$

$$v_3' = \frac{2}{1+\gamma_2}\left(\frac{2}{1+\gamma_1}\left(\frac{2}{1+\gamma_0} v_0' - \frac{1-\gamma_0}{1+\gamma_0} v_1\right) - \frac{1-\gamma_1}{1+\gamma_1} v_2\right) - \frac{1-\gamma_2}{1+\gamma_2} v_3$$

$$\vdots$$

$$v_{n+1}{}' = \frac{2^{n+1}}{(1+\gamma_n)(1+\gamma_{n-1})\cdots(1+\gamma_0)} v_0{}' - \left(\frac{2^n(1-\gamma_0)}{(1+\gamma_n)(1+\gamma_{n-1})\cdots(1+\gamma_0)} v_1 \right.$$
$$\left. + \frac{2^{n-1}(1-\gamma_1)}{(1+\gamma_n)(1+\gamma_{n-1})\cdots(1+\gamma_1)} v_2 + \cdots + \frac{2(1-\gamma_{n-1})}{(1+\gamma_n)(1+\gamma_{n-1})} v_n + \frac{(1-\gamma_n)}{(1+\gamma_n)} v_{n+1} \right).$$

(35)

On the other hand, from (30) we know that $v_{n+1} = \frac{1+\gamma_n}{2\gamma_n} v_n{}'' - \frac{1-\gamma_n}{2\gamma_n} v_n{}'$, which, substituting (32) in it, means that

$$v_1 = \frac{1-\gamma_0}{2\gamma_0} (\Omega - v_0{}') \qquad (36)$$

and substituting (33) and (35), that

$$v_{n+2} = \frac{2^n(1-\gamma_{n+1})}{\gamma_{n+1}(1+\gamma_n)(1+\gamma_{n-1})\cdots(1+\gamma_0)} (\Omega - v_0{}') + \frac{1-\gamma_{n+1}}{2\gamma_{n+1}} \left(\frac{2^n(1-\gamma_0)}{(1+\gamma_n)(1+\gamma_{n-1})\cdots(1+\gamma_0)} v_1 \right.$$
$$\left. + \frac{2^{n-1}(1-\gamma_1)}{(1+\gamma_n)(1+\gamma_{n-1})\cdots(1+\gamma_1)} v_2 + \cdots + \frac{2(1-\gamma_{n-1})}{(1+\gamma_n)(1+\gamma_{n-1})} v_n + \frac{(1-\gamma_n)}{(1+\gamma_n)} v_{n+1} \right)$$

(37)

Now, to obtain a general expression of v_{n+3} from (37) is a simple step. If we separate the addend which is multiplied by v_{n+2} in this expression of v_{n+3}, it is clear that the rest of the expression is a function of v_1, v_2, …, and v_{n+1}, as is the general expression of v_{n+2} (37). Indeed, now it is also easy to see that it may be rewritten exclusively in terms of v_{n+2}, since it contains exactly the terms of v_{n+2}, so that v_{n+3} may be written, now with the addend rewritten exclusively in terms of v_{n+2} plus the addend originally separated, as follows:

$$v_{n+3} = \frac{2\gamma_{n+1}(1-\gamma_{n+2})}{\gamma_{n+2}(1+\gamma_{n+1})(1-\gamma_{n+1})} v_{n+2} + \frac{(1-\gamma_{n+2})(1-\gamma_{n+1})}{2\gamma_{n+2}(1+\gamma_{n+1})} v_{n+2} = \frac{(1-\gamma_{n+2})(1+\gamma_{n+1})}{2\gamma_{n+2}(1-\gamma_{n+1})} v_{n+2}$$

(38)

Moreover, from (36) and (37), $v_2 = \frac{(1-\gamma_1)(1+\gamma_0)}{2\gamma_1(1+\gamma_0)} v_1$; so then we know, better than (38), that

$$v_{n+2} = \frac{(1-\gamma_{n+1})(1+\gamma_n)}{2\gamma_{n+1}(1-\gamma_n)} v_{n+1} \qquad (39)$$

Therefore, with (36) and (39), we can already express the initial velocity for every particle P_{n+1} in terms of the initial velocity of particle P_0 in the reference case, Ω, and of the velocity of the same particle in each of the remaining cases, $v_0{}'$:

$$v_1 = \frac{1-\gamma_0}{2\gamma_0}(\Omega - v_0')$$

$$v_2 = \frac{(1-\gamma_1)(1+\gamma_0)}{2^2\gamma_1\gamma_0}(\Omega - v_0')$$

$$v_3 = \frac{(1-\gamma_2)(1+\gamma_1)(1+\gamma_0)}{2^3\gamma_2\gamma_1\gamma_0}(\Omega - v_0')$$

$$\vdots$$

$$v_{n+2} = \left(\frac{1-\gamma_{n+1}}{2^{n+2}\gamma_{n+1}}\right)\left(\frac{1+\gamma_n}{\gamma_n}\right)\left(\frac{1+\gamma_{n-1}}{\gamma_{n-1}}\right)\cdots\left(\frac{1+\gamma_0}{\gamma_0}\right)(\Omega - v_0') \tag{40}$$

From (40) $v_0' \geq \Omega$ is obviously convenient if the system is to perform a supertask successfully. Since every mass ratio $\gamma_n < 1$, the other factors expressed in (40) –all those different from $(\Omega - v_0')$– are greater than zero. Then, if $v_0' > \Omega$, the sign of v_{n+1} is minus. Considering this, it also helps if $v_{n+1} > v_{n+2} > v_{n+3} > \ldots$, but since every velocity has a minus sign, it suffices to know that $|v_{n+1}| < |v_{n+2}| < |v_{n+3}| < \ldots$, which will hold when $|v_{n+1}/v_{n+2}| < 1\ \forall n$. But, from (40), we know that this inequality is equivalent to $\gamma_{n+1} < \frac{1+\gamma_{n+1}}{3-\gamma_{n+1}}$, which is the same as (34), so it holds for the entire range of systems this general model covers. Thus, we may be sure it is possible for every pair of adjoining particles to reach each other at some instant in such a way that the infinite set of particles performs a consistent and strictly sequential supertask.

To see that this is feasible, it is not necessary to specify the positions in which particles must locate initially. It is sufficient to show that there are initial conditions (positions included) consistent with the sequence of collisions we have specified. The initial conditions that show this are precisely the ones that are most interesting here: the temporal inversion of the final states that correspond to some time after the completion of the reversed supertasks. Let us look, then, to the reversed processes. Clearly, if the initial positions of particles (for the direct process) have an accumulation point, then the reference case $v_0' = \Omega$ leads consistently to the sequence of collisions the model specifies; its temporal inversion is also, then, a process consistent with the reversed sequence of collisions. Now, if we assume the state of the critical instant of the reference case reversed in time as an initial condition of the temporal inversion of the remaining cases (supertasks, now with order type ω^*, and with any value $v_0' \geq \Omega$), and assume likewise that the particles spontaneously begin to collide with the reversed pattern specified for Gγ2, then P_{n+1} must acquire velocity $-v_{n+1}$ (that is, the inverse of the one expressed in (40), which turns out to be positive) before P_n acquires velocity $-v_n$ (which also turns out to be positive). Since $-v_n < -v_{n+1}$ and P_n acquires $-v_n$ at some x-position which is less than the x-position at which P_{n+1} acquires $-v_{n+1}$, then P_n cannot reach P_{n+1} again. This is so, of course, without loss of generality: during the reversed process, once a particle acquires its final velocity (the reversed initial velocity (40)), it will not collide with any other particle. Thus, the free path particles have during the reversed process for any $v_0' \geq \Omega$ is exactly the same free path they have during the direct process. Therefore, although we do not know accurately which initial positions particles in Gγ2 should take, we are sure that there are some sets of positions consistent with the strict sequence of collisions specified.

To sum up, all supertasks covered by general model Gγ2 may be described as follows. Take an infinite number of particles P_0, P_1, P_2, …, P_n, … in one-dimensional space (the x-axis) in that order: the position of each P_n is always an x-position less than the x-position at which each P_{n+1} is located (except when they collide, when they have the same position). Initially particle P_0 moves at velocity $v_0' \geq \Omega$ (any real value greater than zero) whereas each of the other particles is moving at velocity v_{n+1} as expressed in (40). So P_1 will collide only once with P_2, some time later P_2 will collide only once with P_3, and, in general and sequentially, P_n will collide only once with P_{n+1}. When the supertask is performed successfully, each particle will have a velocity v_n'' as expressed in (33). Now, velocities as expressed in (33) are exclusively functions of Ω. Thus, for each Ω taken, there is an infinite number of possibilities (any $v_0' \geq \Omega$) in which the particles have the precise velocities and positions in order to perform some different sequential supertask with the same final state.

Once we have this general model, we may return to the supertask STMNC. Clearly, considering what we have just obtained, the temporal inversion of STMNC is an indeterministic process. In precise terms, the critical instant of the temporal inversion of STMNC is an evolution node. Nevertheless, we cannot assure the same for the direct process. We are not yet in a position to give an affirmative answer to (C).

In any case, under the indeterminism generated by Gγ2, there are clearly obstacles to the critical instant of STMNC (for the direct process) and its subsequent instants being evolution nodes. In the states after the performance of STMNC (and similar supertasks) particles have velocities $v_1'' < v_2'' < v_3'' < \ldots < v_n'' < \ldots$; if the temporal inversion of some other supertask with order type ω has such velocities in its initial state, then, from (33), the final velocities of its direct process must be

$$v_n'' = -\frac{2^n(1-\gamma_n)}{(1+\gamma_n)(1+\gamma_{n-1})\cdots(1+\gamma_0)}\Omega \tag{41}$$

Therefore, in such a supertask

$$v_1'' > v_2'' > v_3'' > \ldots > v_n'' > \ldots \tag{42}$$

From the procedure followed to obtain (40), the initial velocities required for the particles to acquire such a final state, by a strict sequence of binary collisions, are

$$v_1 = -\frac{1-\gamma_0}{2\gamma_0}(\Omega + v_0')$$
$$v_{n+2} = -\left(\frac{1-\gamma_{n+1}}{2^{n+2}\gamma_{n+1}}\right)\left(\frac{1+\gamma_n}{\gamma_n}\right)\left(\frac{1+\gamma_{n-1}}{\gamma_{n-1}}\right)\cdots\left(\frac{1+\gamma_0}{\gamma_0}\right)(\Omega + v_0') \tag{43}$$

If we assume that the sequence of collisions occurs in such a way that each particle P_n continues at its final velocity v_n'' until the end of the supertask, then, from (43), we can use the same argument given for Gγ2 to show the existence of initial conditions (positions included) that lead to the successful performance of the supertask. Nevertheless, the problem with a supertask with such features is that considering the particles continue at velocity v_n'' until the end of the supertask is an unwarranted assumption.

Let us see why. From (41) and (42) it is clear that, as v_{n+1} is greater in magnitude than v_n, particle P_{n+1} might collide with P_n before the rest of the particles could reach the stipulated final velocities; in other words, the stipulated final state of velocities the particles sequentially acquire could suddenly be lost before this final state is definitively achieved, through some collision between particles that have already obtained their stipulated final velocity. Of course, this situation could be solved by adding a certain distance between the particles. The problem with this suggestion is that adding distance between particles should set the system capable of performing the supertask successfully; but this is not however certain to do so. Thus, a difficulty needs to be solved: the positions of particles must be such that the system keeps the final velocities sequentially acquired (expressed in (41)) until the supertask is completely performed, with the certainty that this supertask can in fact be completed.

There is one more difficulty. Suppose that the difficulty expressed in the previous paragraph is solved, and that we know specifically a valid range of initial positions with their corresponding final velocities for such a supertask. Does some set of these valid final positions for such a supertask correspond exactly to some set of valid final positions for supertasks of the same kind of STMNC (the ones that initially have an infinite subset of particles at relative rest and whose total mass is finite)? Clearly, we need an affirmative answer to show the indeterminism Gγ2 suggests is present in supertasks of the same kind of STMNC. Nevertheless, it may well be that the correct answer is a negative.

Definitively, we cannot be sure that under the mechanism of indeterminism specified by Atkinson and Johnson the system that performs STMNC is indeterministic once the supertask is finished, that some instants from its critical instant on are evolution nodes. This very interesting issue remains to be settled.

5. Conclusion

Although not an exhaustive answer to (B), the answer we give is relevant. We saw that for the supertask STMNC (and for supertasks of the same kind analysed herein) every instant prior to its critical instant is an evolution node. This is not only important to (A), but also to some supertasks presented in the literature as presumably deterministic, since it revealed that these supertasks are in fact indeterministic. On the other hand, the incomplete part of our answer is for the critical instant and instants coming after it. We remain unsure about whether they are, or are not, evolution nodes. Nevertheless, the difficulties that have to be solved to show they are evolution nodes, for a range of cases, have been specified, at least under the indeterminism generated by the general model Gγ2. To solve these difficulties would tip the answer to (A) to the negative, and would make Atkinson and Johnson's suggestion, namely, that indeterminism in supertasks is very radical, a fact.

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