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Quantum Analog of the Black- Scholes Formula(market of financial derivatives as a continuous weak measurement)

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Abstract: We analyze the properties of optimum portfolios, the price of which is considered a new quantum variable and derive a quantum analog of the Black-Scholes formula for the price of financial variables in assumption that the market dynamics can by considered as its continuous weak measurement at no-arbitrage condition.

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1. Introduction

Most of the modern market models are based on classical representation of its state. It is assumed that this state is characterized by a set of parameters (in general case by vector $\vec{u}$), and its dynamics is characterized by the trajectory $\vec{u}(t)$. Each of the state parameters has a well determined value, and inaccuracy of its measurement is determined only by imperfection of means of measurement used. At the same time the influence of measurement procedure on the state of the studied object is neglected.

Similar approximation in classical physics proved to be unacceptable for analysis of experiments with micro objects, resulting in the advent of quantum mechanics 100 years ago. Recently a great number of publications (and even books [1]) dedicated to the development and analysis of quantum economic models has appeared. This tendency

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is justified by the fact that the main property of quantum systems – the unforeseeable influence of measurement on system state, is fully realized here. It is due to the fact that traders are guided more by current dynamics of price formation, rather than by its objective factors in choosing the strategy. On the other hand, it is their strategies, which determine the results of "measurement" of assets price.

Several attempts of generalization of the Black-Scholes-Merton formula have been recently made for calculating the "fair" price of financial derivatives using the quantum-mechanical formalism. Let us discuss some of them. In the paper [2] the differential equation of the classical Black-Scholes-Merton model is represented in the form of analogue of the Schrödinger equation. At the same time, in the obtained equation the imaginary unit is not present, thus it cannot be considered a quantum equation. It is virtually the same classical equation, which can give classical solutions in the simplest cases. The advantages of using quantum-mechanical formalism become apparent in consideration of volatility as a random variable, but this generalization actually describes classical (in physical meaning) market states.

The quantum nature of price dynamics can principally occur, in the same way as in physics, due to simultaneous immeasurability of variables characterizing the classical state of the system. For a strict foundation of the necessity of using the quantum formalism in economics (and in particular in the Black-Scholes-Merton model), one must formally determine the measurement procedure for the share price and the price of its financial derivative, and to demonstrate the non-commutativity of these procedures. We are planning to represent the review of results obtained in this sphere and our own substantiation of the inevitability of quantum-mechanical description of market dynamics [3,4] in our next publication.

For the present, we postulate the simultaneous immeasurability of share price and the velocity of its variation, as it has been made i.e. in [5] or [6].

In paper [5] the authors suggest to add to the Brownian motion B_t of the directly observed share price a random process Y_t, related to the influence of factors, which are not simultaneously observed with the factors included in B_t. However, in this case the mechanism of occurrence of both the first and the second random processes is not discussed. Due to this limitation, the model [5] should be considered phenomenological.

We would like to note that the orthodox quantum theory describes the processes of two principally incompatible types. It is the evolution of a closed quantum system, which is described by the action of a unitary operator on the state vector; and the variation procedure (interaction with a classical detector), which is described by the projection operator. In the framework of this formalism we can describe the market dynamics only as a series of instant destructive measurements (collapses of state). It has been shown in [7], that the attempt to describe the continuous observation of the quantum system in the framework of the orthodox approach results in contradictions (Zeno quantum effect).

In the information model of the financial market's state one can assume that the behavior of each trader is determined by the set of possible alternative scenarios of price dynamics. Then the economic analog of the collapse can be treated as a calculation and declaration of price, as a result of which each of the traders obtains objective information. On the basis of this information, a number of possible alternatives are reduced in his mind. Accordingly, the whole market state changes in general. This variation is described in the framework of the information approach of the quantum Bayes rule [8] and is unambiguously determined by the information obtained as a result of the measurement. It has been shown in [9] that the use of quantum Bayes rule is mathematically equivalent to the projective postulate in the orthodox approach; however it allows generalizations for the used operators. In particular, operators corresponding to obtaining various results in the process of measurement of a certain value are not necessarily orthogonal. This corresponds to the case of quantum measurement and allows avoiding contradictions in description of continuous weak quantum measurements.

There are a number of reasons for using this generalization for modeling the dynamics of the financial market.

1. The interval of price declaration in the financial market δt (up to 1 minute) is negligibly small compared to the characteristic time periods of the external factors' effect.

2. The velocity of price information obtained by the traders is finite in the limit of $\delta t \to 0$

Due to this fact the traders only partially "trust" the price information contained in each separate declaration. At the same time the information, sufficient for a substantial variation of a trader's state, is accumulated during a certain period of time, substantially exceeding δt.

3. Simultaneous calculation and declaration of prices corresponding to non-commuting variables (i.e. prices for a share and its futures) is possible.

In the present paper we are discussing the market dynamics as a process of continuous observation of market state variables. For simplicity we shall assume that the market state corresponds to a pure quantum state. This means that all traders possess the same information both on declared prices and on external factors influencing the prices. In terms of quantum mechanics we can say that all the traders in each moment of time are "identically prepared". A significant feature of the quantum-mechanical model of continuous measurement is the existence of two complementary sources of stochasticity of measurement results. The first of them is related to the external factors and is described in the Hamiltonian as a potential component random in time. The second source is related to the principal quantum unpredictability of the observation results. This means that even in a market fully isolated from the influence of external factors the stochasticity will occur as a consequence of influence of the results of continuous market state observation. It is interesting to note that in a mathematical expression this random factor can be represented as an imaginary component in the Hamiltonian, which corresponds to the conclusion made in [5] on the grounds of other considerations.

2. Modification of Classical Derivation of the Black-Scholes Formula

Let us comment the main assumptions used in classical derivation and their quantum interpretation.

1. Like in a classical model we assume ideality of traders, i.e. we consider that they possess full and identical information, on the basis of which they build their quantum strategies. In the classical model this means that all traders trade identical optimum portfolios at identical price. In the quantum model this assumption allows considering the condition of the whole market $|\psi\rangle$ as a pure and not mixed quantum state. Further it will be appropriate considering a more general model, using the apparatus of density matrix.

2. Continuous time corresponds to performance of continuous weak quantum measurements which are not limited to a series of destructing measurements.

3. $r = const$ - means that the market of bonds is considered as a classical factor without account of the feedback (influence of the market of shares and options on the price of bonds is not taken into account). It is equivalent to the classical description of the controlling external influence on the quantum system. The dependency $r(t)$ can be set randomly. Moreover, we can also generalize the model and take account of the influence of "observation" (declared prices) result on this dependency in the framework of a particular classical model of formation of bank interest rate.

4. A priori set law of share price evolution

$$\frac{dS}{dt} = \mu S + \sigma S \cdot R(t), \tag{1}$$

where $R(t)$ is the uncorrelated Gauss stochastic noise with zero main value transforms into a stochastic differential equation of market state dynamics at continuous weak measurement of "optimum" portfolio price.

$$\frac{d\,|\psi\rangle}{dt} = \left[-\frac{i}{\hbar}\hat{H} - K(\hat{A}-c)^2\right]|\psi\rangle + \sqrt{2K}(\hat{A}-c)\,|\psi\rangle\,\frac{dw}{dt}, \tag{2}$$

where $\hat{A}$ is the operator of the measured variable, $c = \langle\psi|\,\hat{A}\,|\psi\rangle$ is its mathematical expectation, H is the Hamilton operator, the function of which we will discuss later, K is the parameter of fuzziness of measurement. In order to retain vector norm $|\psi\rangle$,$(\langle\psi| + \langle d\psi|)\,(|\psi\rangle + |d\psi\rangle) = \langle\psi \mid \psi\rangle$ it is sufficient to take $dw^2 = dt$ [10]. Let us note that we are considering an idealized model, in which the stochastic nature of price dynamics is caused by its constant weak measurement, unlike the classical model, in which it is set a priori. Classical external influences, also causing price fluctuations, are not considered here, though formally they can be accounted as a random component of the Hamilton operator. The result of price measurement is considered its declaration at the current moment of trades. On the one hand, it contains information about the market status, as it is calculated in accordance with certain rules on the basis of submitted bids.

On the other hand, it influences the market state, because on the basis of obtain information the traders submit their bids. As the speed of obtaining the information is limited, in the model of continuous market at decreasing of the time step of discretization, the degree of uncertainty of the obtained results correspondingly increases. It is the main source of stochasticity in the model of continuous weak measurement.

5. Absence of riskless arbitrating capabilities in the classical model of optimum portfolio price is set as $dV/V = rdt$. In the quantum model the same condition must be satisfied for the mathematical price expectation of measured variable A. Otherwise, it will be possible to obtain a (generally) riskless gain by exchanging the corresponding securities for bonds. If the portfolio price is "measured", the operator $\hat{A} \equiv \hat{V}$ corresponds to this procedure, influencing the traders' state in accordance with their strategies and rules of price determination. Then we obtain the condition

$$\frac{d(\ln \langle V \rangle)}{dt} = \frac{1}{\langle V \rangle} \cdot \frac{d \langle V \rangle}{dt} = r, \quad \text{where} \quad \langle \psi | \hat{V} | \psi \rangle = \langle V \rangle \tag{3}$$

At the same time it is however possible to measure directly the speed of changing of the logarithmic price of the optimum portfolio (in practice in means conclusion of contracts with more complex structure, in which the payment is agreed depending on this parameter). In this case we obtain the ratio

$$\left\langle \frac{d \ln V}{dt} \right\rangle = r, \quad \text{and} \quad \hat{A} \equiv \left(\frac{d \ln \hat{V}}{dt} \right) \tag{4}$$

Let us note that in the classical limit both these variants, correspond to the same formula. However, the influence of measurement on market state in the quantum model makes them different. In the present paper we limit ourselves to the consideration of the first variant.

6. In the classical model the condition of optimality of portfolio means that its structure $V_{opt} = -f + (\partial f/\partial S)S$ ensures riskless condition at any related share and financial derivative price variations. With such a structure the risk turns into 0. However, in the quantum mechanical model the function $f(S,t)$ exists only for mathematical expectations of corresponding prices, which are calculated as average quantum mechanical values of results of weak measurements. In this connection the risk value remains non-zero, and the portfolio structure is to minimize it.

As a measure of risk as one of the possibilities we use the value of dispersion of optimum portfolio price distribution (in the classical model it turns into 0). Then

$$\sigma_V = \langle V^2 \rangle - \langle V \rangle^2 = \langle (-f + kS)^2 \rangle - \langle (-f + kS) \rangle^2 \to \min . \tag{5}$$

Assuming that it is the continuous function of parameter k, determining the portfolio structure, we can write down the optimality condition in the following form $\partial \sigma_V / \partial k = 0$. From it we can obtain:

$$k_{opt} = \frac{\langle f \cdot S \rangle - \langle f \rangle \cdot \langle S \rangle}{\sigma_S} . \tag{6}$$

The same as in the classical model, the parameter k can depend on time. In the quantum mechanical model the ratio, determining the optimum portfolio structure connects corresponding operators, rather than the measured values of the variables. Due to incommutability of operators $\hat{S}$ and $\hat{f}$:

$$\langle f \cdot S \rangle = \left[\langle \psi | \hat{f}\hat{S} | \psi \rangle + \langle \psi | \hat{S}\hat{f} | \psi \rangle \right] /2 \tag{7}$$

3. The Black-Scholes Quantum Formula for a Specific Financial Derivative

The first step in the simplest variant of classical derivation using Ito formula is obtaining of financial derivative price dynamics equation. This allows further excluding the random variable R(t) from classical equations for dynamics S and f.

For dynamics of average values of quantum variables with account of (2) we obtain:

$$\frac{d\langle S \rangle}{dt} = \langle \psi | \hat{S}\hat{B} + \hat{B}\hat{S} | \psi \rangle + \langle \psi | \hat{S}\hat{C} + \hat{C}\hat{S} | \psi \rangle R(t) \tag{8}$$

and

$$\frac{d\langle f \rangle}{dt} = \langle \psi | \hat{f}\hat{B} + \hat{B}\hat{f} | \psi \rangle + \langle \psi | \hat{f}\hat{C} + \hat{C}\hat{f} | \psi \rangle R(t) + \left\langle \frac{\partial f}{\partial t} \right\rangle , \tag{9}$$

where

$$\hat{B} = \left[-\frac{i}{\hbar}\hat{H} - k(\hat{V} - c)^2 \right] ; \tag{10}$$

$$\hat{C} = \sqrt{2k}(\hat{V} - c) \tag{11}$$

Excluding from them the random factor R(t), we obtain a quantum analog of the classical formula:

$$\frac{d\langle f \rangle}{dt} - \left\langle \frac{\partial f}{\partial t} \right\rangle - \left\langle \hat{f}\hat{B} + \hat{B}\hat{f} \right\rangle = \frac{\left\langle \hat{f}\hat{C} + \hat{C}\hat{f} \right\rangle}{\left\langle \hat{S}\hat{C} + \hat{C}\hat{S} \right\rangle} \cdot \left[\frac{d\langle S \rangle}{dt} - \left\langle \hat{S}\hat{B} + \hat{B}\hat{S} \right\rangle \right] \tag{12}$$

For shortening the notation we shall further in this paper represent the symmetrized product of operators of type $\left(\hat{f}\hat{C} + \hat{C}\hat{f} \right) /2$ as an ordinary product.

For further use it necessary to substitute the expressions for operators $\hat{B}$ and $\hat{C}$ in the decisive form. In this case a significant difference from the classical derivation method occurs. The point is that the share price dynamics in the classical model is considered set a priori, while in the quantum model it depends on the "measured" value. In our case this is price of optimum portfolio. Therefore at substituting of operators we should take account of its structure, set by the quantum expression for k_{opt}. Besides, in cases when a different quantum variable, other than the optimum portfolio price, is being "measured", we should use operator $\hat{A}$ instead of $\hat{V}$.

At the next stage of the classical derivation the condition of no-arbitrage is used, from which an additional connection of variables S and f is obtained. In the quantum case from (3) we obtain

$$\frac{d\langle f\rangle}{dt} - r\langle f\rangle = -k_{opt}\left(\frac{d\langle S\rangle}{dt} - r\langle S\rangle\right) \tag{13}$$

Despite the "similarity" of both formulas (12,13), unlike the classical analog they include different proportionality coefficients between $\frac{d\langle f\rangle}{dt}$ and $\frac{d\langle S\rangle}{dt}$. They become identical and equal $\frac{\partial f}{\partial S}$ either in the classical limit, or at a certain value$c = \langle\psi|\,\hat{V}\,|\psi\rangle$. In this case, the same as in the classical case, we can exclude the total time derivatives and obtain the following system:

$$\left[\begin{array}{l} \frac{\langle\hat{f}\hat{C}\rangle}{\langle\hat{S}\hat{C}\rangle} = k_{opt} \\ \frac{r\langle\hat{f}\rangle - \left\langle\frac{\partial\hat{f}}{\partial t}\right\rangle - \langle\hat{f}\hat{B}\rangle}{\langle\hat{f}\hat{C}\rangle} = \frac{r\langle\hat{S}\rangle - \langle\hat{S}\hat{B}\rangle}{\langle\hat{S}\hat{C}\rangle} \end{array}\right. \Rightarrow \left[\begin{array}{l} \frac{\langle\hat{f}\hat{C}\rangle}{\langle\hat{f}\hat{S}\rangle - \langle\hat{f}\rangle\langle\hat{S}\rangle} = \frac{\langle\hat{S}\hat{C}\rangle}{\langle\hat{S}^2\rangle - \langle\hat{S}\rangle^2} \\ \left\langle\frac{\partial\hat{f}}{\partial t}\right\rangle + r\left\langle\hat{S}\right\rangle\frac{\langle\hat{f}\hat{C}\rangle}{\langle\hat{S}\hat{C}\rangle} + \frac{\langle\hat{f}\hat{B}\rangle\langle\hat{S}\hat{C}\rangle - \langle\hat{S}\hat{B}\rangle\langle\hat{f}\hat{C}\rangle}{\langle\hat{S}\hat{C}\rangle} = r\left\langle\hat{f}\right\rangle \end{array}\right. \tag{14}$$

This system is in fact a quantum analog of the classical formula

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_S^2 S^2\frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} = rf \tag{15}$$

The additional equation occurs due to the fact that in our case the dynamics of price average value is not set a priori, it is the result of procedure of its continuous measurement and influence of Hamiltonian. However, if we assume that not the price of optimum portfolio in one of the variants (3, 4), but the price of share is measured, then, by substituting S instead of $\hat{V}$ in (10) and (11), we obtain an identity from the first condition, and from the second condition we obtain

$$\left\langle\frac{\partial\hat{f}}{\partial t}\right\rangle + r\left\langle\hat{S}\right\rangle\frac{\left\langle\hat{f}\hat{S}\right\rangle - \left\langle\hat{f}\right\rangle\left\langle\hat{S}\right\rangle}{\left\langle\hat{S}^2\right\rangle - \left\langle\hat{S}\right\rangle^2} + \frac{\left\langle\hat{f}\hat{B}\right\rangle\left\langle\hat{S}\hat{C}\right\rangle - \left\langle\hat{S}\hat{B}\right\rangle\left\langle\hat{f}\hat{C}\right\rangle}{\left\langle\hat{S}\hat{C}\right\rangle} = r\left\langle\hat{f}\right\rangle \tag{16}$$

The result of solution of the obtained system with account of boundary conditions connecting the share prices and the financial derivative in the moment of closing the contract will be a functional dependency of operator$\hat{f}$ on the operator $\hat{S}$ and time t. However, for this purpose it is necessary to explicitly draw out the expression for the Hamiltonian. In a number of works dedicated to the analysis of quantum economic phenomena the Hamiltonian is used without sufficient grounds for the particle in the potential field. In physics, the form of Hamiltonian is determined by the general requirements connected with the homogeneity and isotropy of space and the principle of relativity [11]. We assume that in the same manner in economic models the form of Hamiltonian should also be determined by the type of symmetries set by the formal rules of trading. Our further research will be dedicated to the analysis of these properties and derivation of formulas for the Hamiltonian in various economic systems. Therefore, in the present paper we limited ourselves to the derivation of Black- Scholes quantum formula in the general form.

Let us note that the obtained system of operator equations does not include the market states, described by the wave function, it only includes the individual parameter of measurement weakness k. It follows from the general considerations of invariance of description with respect to the selection of the discretization step δt that $k \approx (\delta t)^{-1/2}$. Besides, the functional dependence $\hat{f}(\hat{S}, t)$ for the operator of the financial derivative price measurement is determined only by the terms and conditions of the contract and by the form of the Hamiltonian.

As a conclusion let us note the specific features of the financial market's dynamics in the framework of the suggested approach. In the discussed examples we assumed the price of optimum portfolio V to be continuously measured.

As this value represents a linear combination of the share price and the price of futures, in the orthodox approach these prices, being non-commuting variables, do not have a defined value. At the same time, in the real market both these prices (and the portfolio price as their linear combination) can be declared simultaneously. In the framework of the theory of continuous weak quantum measurement this contradiction is resolved due to the fact that the operators of weak measurement are not orthogonal.

Probabilistic distribution of the price of optimum portfolio V is unambiguously determined by the value of the wave function at the current moment of time. On receiving a next random value of V_j the market state described by the wave function also changes in a random manner. At the next step we have a new distribution of probabilities for V_j. As a result a random process of portfolio price variation is induced by the continuous weak measurement even without external random factors (their presence can be taken into account by a formal addition of a random additive component to the Hamiltonian). At the same time the new value of k_{opt} must be recalculated at each step. Thus, we obtain a realization of a random process, in which the probabilistic distributions of the share price and the financial derivative at each moment of time eliminate the possibility of arbitraging.

Let us note, that in the process of deriving the quantum Black-Scholes-Merton formula we have excluded the random factor, related to the observation of the market portfolio price, only for the dynamics of the mean values $\langle S \rangle$ and $\langle f \rangle$. In this case the absence of risks for the optimum portfolio means that the variation of the mean quantum-mechanical (expected) price of the portfolio does not depend on the results of random measurement. However, they affect its structure, which must be recalculated at each step as in the classical case. As the measured portfolio price at each step is different from the average, risk cannot be avoided for a specific realization. It can only be minimized for the optimum strategy.

References

[1] Belal E. Baaquie, *Quantum Finance: Path Integrals and Hamiltonians for Options and Interest Rates* (Cambridge University Press, 2004)

[2] Belal E. Baaquie, J. Phys. I France. **7,** 1733 (1997)

[3] S.I. Melnyk, I.G. Tuluzov, A.N. Omelyanchouk, Physics of Mind and Life, Cosmology and Astrophysics, **6,** 48 (2006)

[4] I.G. Tuluzov, S.I. Melnik, A.N. Omelyanchouk, Applied Statistics. Actuarial and Financial Math, **1-2**, 180 (2005)

[5] W. Segal, I. E. Segal, *The Black-Scholes pricing formula in the quantum context*, Proc. Natl. Acad. Sci. USA, vol. 95, pp. 4072-4075, March 1998, Economic Sciences. Centro Vito Volterra, Universit'a di Roma Torvergata, via Columbia 2, 00133

[6] Accardi, L.; Boukas, A. (2007). *The quantum Black-Scholes equation*; GJ-PAM; 2 (2): 155-170.

[7] J. P. Singh and S. Prabakaran, EJTP **3**, No. 11 (2006) 11-27

[8] Filippo Neri. Phys. Rev. A **72**, 062306 (2005) *(6 pages)*

[9] R. Schack, T. A. Brun, and C. M. Caves, Phys. Rev. A 64, 014305 (2001)

[10] M.B. Mensky *Continuous quantum measurement and path integrals,* Bristol and Philadelphia: IOP Publishing, (1993)

[11] R. Feynmann, A. Hibs, *Quantum Mechanics and Trajectory Integrals* (Mir, Moscow, 1968)