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What Frege Meant When He Said: Kant is Right about Geometry†

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This paper argues that Frege's notoriously long commitment to Kant's thesis that Euclidean geometry is synthetic *a priori* is best explained by realizing that Frege uses 'intuition' in two senses. Frege sometimes adopts the usage presented in Hermann Helmholtz's sign theory of perception. However, when using 'intuition' to denote the source of geometric knowledge, he is appealing to Hermann Cohen's use of Kantian terminology. We will see that Cohen reinterpreted Kantian notions, stripping them of any psychological connotation. Cohen's defense of his modified Kantian thesis on the unique status of the Euclidean axioms presents Frege's own views in a much more favorable light.

1. Introduction

There is a well-known and apparently inexplicable fact concerning Gottlob Frege. From 1873 to 1925, he seems consistently to endorse Kant's thesis that Euclidean geometry is a body of synthetic *a priori* propositions grounded on pure intuition. In *The Foundations of Arithmetic*, for instance, Frege writes: 'In calling the truths of geometry synthetic *and a priori*, he [Kant] revealed their true nature. And this is still worth repeating, since even to-day it is often not recognized' [Frege, 1980, pp. 101–102]. Writing to David Hilbert fifteen years later, Frege still maintains that an 'intuition of space' is the 'nonlogical basis' of geometric axioms [Frege, 1971, p. 9].

One of the reasons Frege's commitment to Kant's thesis is so puzzling has to do with nineteenth-century advances in developing non-Euclidean geometries and in perceptual psychology, advances which many of his

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colleagues thought undermined Kant's account. Alberto Coffa describes Frege's intellectual milieu as follows:

> By the second half of the nineteenth century, people began to wonder first about the exclusive necessity of Euclidean geometry and then about the role of intuition in *any* geometry, Euclidean or otherwise. As neo-Kantians were forced to address this issue more extensively, it slowly emerged that the master's silence was not a sign of unspoken wisdom. Revealingly, the cleverest among neo-Kantians silently pushed Kant's pure intuition to the corner of their doctrine of geometry.... [Coffa, 1991, pp. 42–43].

Frege's continual insistence that Euclidean geometry is the unique body of *a priori* geometric truths grounded on intuition would thus align him with only the most orthodox and dogmatic of the neo-Kantians. This is especially problematic for those who take Frege to be the paradigmatic realist. It is not surprising, then, that one finds Frege scholar Matthias Schirn, when defending the 'common view that Frege was a realist with respect to abstract objects', simultaneously having to explain away those remarks on geometry with 'a Kantian ring' [Schirn, 1996b, p. 119; 1996a, p. 21].

A second problem is reconciling what Frege says about intuition with his anti-psychologism. Consider these remarks on intuition:

> What is objective...is subject to laws, what can be conceived and judged, what is expressed in words. What is purely intuitable is not communicable. [Frege, 1980, p. 35]

> It is in this way that I understand objective to mean what is independent of our sensation, intuition, and imagination, and of all construction of mental pictures out of memories of earlier sensations.... [*ibid.*, p. 36].

Here Frege places intuition in the category of non-objective psychological entities. These passages are immediately followed with his description of a subjective idea: 'An idea in the subjective sense is what is governed by psychological laws of association; it is of a sensible pictorial character.... Subjective ideas are often demonstrably different in different men.' Intuitions are thus subjective ideas, 'which are often demonstrably different in different men', and not to be confused with objective ideas, which must be 'the same for all' [*ibid.*, p. 37]. If this is precisely and exclusively what Frege means by 'intuition', then when he subsequently criticizes Hilbert's axiomatization of geometry because it fails to adhere to the traditional understanding of geometric axioms as expressing 'basic facts of our intuition', Frege would be faulting Hilbert for forgetting that geometric axioms are laws describing essentially private incommunicable mental states [Frege, 1971, pp. 7, 25, 27]. Anyone familiar with Frege's mantra that discovering the grounds of truth for a proposition is not a matter for psychological investigation should balk at the suggestion that he considered the whole of geometry to be an exception to the rule. Therefore, a closer examination of his use of the term 'intuition' is called for.[1]

My aim in this paper is to show that Frege attaches at least two very different meanings to the term 'intuition' . This will require connecting Frege's remarks on intuition and geometry to a debate between Hermann von Helmholtz and Hermann Cohen, founder of the Marburg school of neo-Kantianism. We will see that Helmholtz interpreted Kant's thesis that Euclidean geometry is synthetic *a priori* in such a way that it was rendered extremely vulnerable to any new developments in perceptual psychology. He then appealed to the reigning nineteenth-century theory of sense perception in an effort to refute Kant's thesis. Cohen and his mentoree, Paul Natorp, charged Helmholtz with having misread Kant's notion of pure intuition by giving it a decidedly psychological cast. Once this notion was properly understood, they argued, Helmholtz's argument from perceptual psychology was clearly beside the point. After presenting the Helmholtz-Marburger exchange, I will argue that one of the meanings that Frege attaches to the term 'intuition' indicates that he is endorsing the Cohen-Natorp account of geometry. If I am right, then Frege's expressed views on geometry not only cohere with his anti-psychologism, but also affirm it.

This is not to say that recognizing Frege holds a Marburgian view on the nature of geometry resolves all of the tensions previously mentioned, at least not to everyone's satisfaction. Aligning Frege with Cohen and Natorp may still trouble those who insist on reading him as a realist. In this paper, I do not attempt to reconcile what Frege says about geometry with other remarks most often cited as evidence for his

[1] Schirn [1996a] also notes the problem with making Frege's remarks about the subjectivity of intuitions square with his remarks about intuition as the source of geometric knowledge. Schirn does not offer any solution to this problem. He suggests, however, that Frege may have thought the purely subjective intuitive content associated with geometric propositions could somehow be dislodged from their objective content. Mark Wilson [1997] makes a similar suggestion. In both cases, it is assumed that Frege is using the term univocally throughout his writings. See footnote 17 below, for more on Wilson's proposal for resolving the tension between the objectivity of geometry and the subjectivity of intuition.

realism.[2] But I do want to point out that the idealism espoused by the Marburgers is not the same as that represented in the philosophies of Fichte, Hegel, or even Kant. While other German idealisms portray the thinking, knowing, or willing subject as imposing certain necessary structural elements on the parade of undifferentiated phenomena in order for the world to appear to us as an objectively given something, Cohen and Natorp assign this essential constituting task to the sciences. Because the Marburgers look to the sciences, rather than to ordinary sense experience, as delivering our most epistemically privileged representations of the world, their views have been likened to those later expressed by Willard Quine and Wilfred Sellars: 'Natorp shares with both of them a reliance on the sciences for our premium representations of the world' [Kolb, 1981, p. 245].[3]

Cohen and Natorp can still be broadly construed as idealists, however, since they refuse to acknowledge any meaningful notion of the world or of an object outside of these scientific representations. Neither of them considers it the task of the physicist (or the mathematician for that matter) to describe a world conceived of as a thing-in-itself, a thing that is out there and is what it is entirely independently of our tools for cognizing it.[4] Rather, the physicist's task is to describe a world conceived of as the subject matter of mathematical physics, which requires making whatever is out there conform to certain mathematical and logical practices. I have argued elsewhere that Frege's views on geometry, on the distinction between concepts and objects, and on those whom he pejoratively called 'formalists' suggest that he is better read as a Marburgian sort of idealist than as the paradigmatic realist.[5]

Since I also devote little space here to defending the Marburgers' account of geometry, the reader might still be left wondering why Frege adhered to a position that, according to Coffa, aligns him with the least

[2] See [Merrick, 2004, §III.C.2], where I argue that the textual evidence usually adduced in support of Frege's realism should be interpreted more narrowly, strictly as criticizing definitional procedures which, according to Frege, involved a fallacious inference from logical consistency to mathematical existence. See also [Carl, 2001, pp. 3–10], for his argument that Frege's remarks on the objectivity and independence of mathematical entities do not commit him to Platonism.

[3] Richardson [2003] also presents a much needed English summary of Cohen's [1871], *Kants Theorie der Erfahrung*. He shows Cohen's interest in specifying a new meaning of 'experience', one which references the term to 'the empirical knowledge of the world of Newtonian science' [Richardson, 2003, p. 60]. See also [Friedman, 2000a, pp. 32–37], which presents the main features of Cohen's Marburg School of neo-Kantianism, in contrast to Wilheln Windelband's Southwest School.

[4] With this loose characterization of a thing-in-itself, I hope to capture the notion of realism which has usually been associated with Frege. See, for example, [Schirn, 1996b, pp. 119–120] and [Carl, 2001, p. 4].

[5] See [Merrick, 2004], §III.A.2 and my Conclusion.

clever of the neo-Kantians. But notice that this assessment of the Marburgian position comes after general relativity has supplanted Newtonian physics as our best theory of the macro features of the natural world. We will see that Cohen, Natorp, and Frege all took it for granted that Newton's physics, if not the final say on what the general features of the world are like, was at least a necessary precursor in arriving at that final say. Even Helmholtz and others skeptical of the Kantian thesis did not anticipate the kind of paradigm shift in mathematical physics that would occur in 1919, when general relativity was considered entirely confirmed: '[T]he founders of "Non-Euclidean geometry" have never maintained its objective truth ...' [Helmholtz, 1996, p. 685]. And it was not until 1921 that word of this confirmation forced Cohen's student, Ernst Cassirer, to admit that the Kantian thesis was now rendered untenable:

> The *factum* of geometry has lost its unambiguous definiteness; instead of the one geometry of Euclid, we find ourselves facing a plurality of equally justified geometrical systems, which all claim the same intellectual necessity, and which, as the example of the general theory of relativity seems to show, can rival the system of classical geometry in their applications, in their fruitfulness for physics. [Cassirer, 1953, p. 353]

Though the Marburgers had long acknowledged the logical consistency ('intellectual necessity') of non-Euclidean geometries, it was the 'fruitfulness for physics' criterion that they appealed to in conferring special status on Euclid's geometry.

So, to challenge my argument based on the claim that I am ascribing to Frege an unreasonable view of geometry, one must show that it was an unreasonable view to hold at the time Frege held it. Suffice it to say that I do not consider the position which Cohen and Natorp articulated in opposition to Helmholtz as unsophisticated as Coffa seems to suggest. Ultimately, however, I leave it to the reader to decide whether the Marburgian view of geometry was a reasonable one, focusing instead on convincing her that Frege did, in fact, accept it.

2. Helmholtz's Sign Theory of Perception and its Challenge to Kant's Thesis[6]

By 1870, Helmholtz had helped put the finishing touches on a theory describing the psycho-physiological processes whereby the initial effects

[6] Given the purposes of this paper, my exposition of Helmholtz's sign theory (*Zeichentheorie*) of perception and its purported epistemological implications highlights

registering on our sense organs are combined so as to produce a collection of sensory inputs capable of having some representational content. Helmholtz and his colleagues maintained that raw sense data should be thought of as uninterpreted signs, by which they meant that the initial effects on our sense organs are neither spatially nor temporally ordered, but only differ in terms of their quality and intensity. They also stressed that raw sensory inputs provide us with almost no information about the nature of the external stimulus giving rise to them:

> What physiological investigations now show is that the deeply incisive difference [in the qualities of sensations] does not depend, in any manner whatsoever, upon the kind of the external impression whereby the sensation is excited, but is determined alone and exclusively by the sensory nerve upon which the impression impinges. [Helmholtz, 1996, p. 693]

Helmholtz cites experimental results demonstrating that the excitation of the optic nerve produces similar light effects, regardless of whether the nerve is excited by 'objective light' (aether vibrations impinging on it) or an electric current being passed through the eye or by applying pressure on the eyeball [*ibid.*]. Furthermore, the same external stimuli will produce radically different effects, depending on which sensory nerve is excited: 'The same aether vibrations as are felt by the eye as light, are felt by the skin as heat' [*ibid.*, p. 694].

Uninterpreted sensory qualia do, however, eventually become sensory signs or symbols, capable of alerting us to the presence of a certain type of stimulus: 'Inasmuch as the quality of our sensation gives us a report of what is peculiar to the external influence by which it is excited, it may count as a *symbol* for it' [*ibid.*, p. 695]. How does this transition from uninterpreted sign to sensory symbol occur? Regularly occurring qualitative distinctions are registered on memory, leaving traces that are subsequently associated with one another and with certain spatiotemporal characteristics. Helmholtz describes this process as a series of

those features of Helmholtz's account which provided the fodder for Cohen's and Natorp's critique, as well as those helpful in teasing out Helmholtz's use of Kantian terminology, *e.g.*, 'intuition'. For expositions of Helmholtz's theory from a broader, less jaundiced, perspective, see [Hatfield, 1990] and [Patton, 2004]. See also [Friedman, 1997], who presents a more nuanced exposition than the one presented here. For instance, Friedman tracks how Helmholtz's conception and use of the principle of causality shifted from his 1855 lecture 'On human vision' to his 1878 'Facts in perception.'

unconscious inductive inferences operating on memory traces, rather than on what is expressible in words or propositions:

> There appears to me in reality only a superficial difference between the inferences of logicians and those inductive inferences whose results we recognize in the intuitions of the outer world we attain through sensations. The chief difference is that the former inferences are capable of expressions in words, while the latter are not, because instead of words they deal only with sensations and memory-images of sensations. [Helmholtz, 1868, p. 217] (retranslated and cited in [Hatfield, 1990, p. 201])

The conclusion of this series of unconscious inductive inferences is a sensible perception with some represented content, *e.g.*, my perception of a book to the left of me.

To get a better picture of how these subconscious inductive inferences work, Helmholtz provides the following example. The perceiving subject experiences brightness, upon moving her eye the brightness fades, moving it back the brightness intensifies. He likens these eye movements to an 'experiment through which' we arrive at law-like generalizations regarding the spatial and temporal characteristics of the qualia [Helmholtz, 1996, p. 706]. In this case, the generalization arrived at is 'if I move my eyes to the right, I will experience brightness' and 'if I move my eyes to the left, I will experience darkness'.[7] Through this process of experimentation and unconscious inductive inferences, the alternating sensory signs, brightness/darkness, come to be interpreted as 'an enduring existence of different things at the same time one beside another':

> One does not yet need to think of substantial things as what are here supposed to exist one beside another. 'To the right it is bright, to the left it is dark' ... could for example be said at this stage of knowledge, with right and left being only names for certain eye movements ... [*ibid.*, p. 698]

Once these visual cues become similarly associated with certain tactile cues, *etc.*, the subject infers the sensible image of a luminous object over there in front of me and to the right.

Helmholtz then applied the sign theory of perception in evaluating various Kantian doctrines. The fact that the represented content of our

[7] Remember, however, that these experiments and inductive generalizations are taken to be unconscious and sub-linguistic, *i.e.*, entirely private and inexpressible.

perceptual experience is partially determined both by our mental and physical constitution and by the unconscious inferences which we are innately predisposed to make purportedly confirmed the Kantian thesis that we do not perceive things as they are in and of themselves.[8] Indeed, this is the main point of characterizing perceptions as signs, rather than images, of external stimuli [Helmholtz, 1996, pp. 695–696]. Helmholtz and his colleagues maintained that nineteenth-century perceptual psychology sided with Kant on this point, thus challenging an underlying assumption of the naïve materialist philosophy fashionable in Germany since 1850 [Poma, 1997, p. 3].

Helmholtz also presented his notion of an interpreted perceptual sign as the scientifically upgraded model of Kant's earlier notion of an intuition as the sensible representation of a particular given object:

> I believe the resolution of the concept of intuition into the elementary processes of thought to be the most essential advance in the recent period. This resolution is still absent in Kant, which is something that then also conditions his conceptions of the axioms of geometry as transcendental propositions. Here it was especially the physiological investigations on sense perceptions which led us to the ultimate elementary processes of cognition. These processes had to remain still unformulable in words, and unknown and inaccessible to philosophy, as long as the latter investigated

[8] In his account of Helmholtz's sign theory, Friedman highlights Helmholtz's rejection of nativist theories of perception in favor of the 'learned or acquired' process whereby perceptual signs allow for interpretation [Friedman, 1997, p. 31]. Helmholtz himself often claims that he is presenting an empiricist theory of perception, in contrast to the nativist theories of his predecessors or colleagues. I agree that Helmholtz clearly rejects nativist perceptual theories which treat the spatiotemporal ordering of perceptions as innate in the sense of being entirely due to the constitution of our sense organs or, as he puts it, arriving 'ready-made' without a process of unconscious experimentation and inductions involving memory traces [Helmholtz, 1996, p. 705]. However, I also believe that it is a fair reading of [Helmholtz, 1876; 1878b] to say that the unconscious processes whereby raw sensations attain their most primitive spatiotemporal characteristics are presented as processes towards which we are innately predisposed. And more importantly for our purposes, we will see that this is how Cohen reads Helmholtz. Besides, it would certainly be a mistake to interpret Helmholtz as claiming that this primitive spatiotemporal form is learned or acquired from experiencing particular sensibly perceived objects, since he maintains that there are unconscious inferences yielding 'initial, original facts' of spatial ordering which are necessary for and prior to perceiving objects in the outer world [Helmholtz, 1996, p. 706]. The fact that Friedman himself is not ascribing a straightforwardly empiricist view of perception to Helmholtz is especially evident in [Friedman, 2000b], where he likens Helmholtz's account of unconscious inferences operating on the data received from voluntary bodily motion to Kant's account of the transcendental synthesis of the imagination.

> only cognitions finding their expression in language. [Helmholtz, 1996, pp. 712]

Apparently, Helmholtz thought that Kant equated full-fledged intuitive representations with the raw input of sensibility. Hatfield confirms this: '[Helmholtz] believed, for Kant spatial intuitions were unanalyzable into more primitive components and arose without the activity of the understanding' [Hatfield, 1990, p. 203].[9]

According to Helmholtz, Kant mistakenly assumed that the initial effects registering on our sensible faculty are immediately invested with certain spatiotemporal characteristics and that these spatial characteristics are Euclidean. As the sign theory of perception has shown, however, these effects are not initially spatial at all. Helmholtz argues, however, that the theory supports the idea that some minimal spatial relationships are imposed on the field of perceptual experience in virtue of the subject's psychophysiology. Thus, Kant was correct in thinking that a spatial form is given to us prior to our perception of empirical entities, but wrong in thinking that this spatial form possessed the particular properties described by the Euclidean axioms: 'But space may very well be a form of intuition in the Kantian sense, and yet not necessarily involve the [Euclidean] axioms' [Helmholtz, 1996, p. 686].

Throughout his argument that the theory of sensible signs is incompatible with the claim that Euclidean geometry characterizes the spatial form identical with our pure form of intuition, Helmholtz tends to equate the terms '*a priori*' and 'pure' with innate.[10] As a consequence, he

[9] Of course, Helmholtz is mistaken. For Kant, full-fledged intuitive representations result from a process whereby the raw inputs of sensibility are combined in accordance with elements and functions that must be contributed by the understanding. Furthermore, despite [Friedman, 2000b]'s comparison of Helmholtz's sign theory with Kant's transcendental synthesis of the imagination, I maintain that Kant would not have considered his intuitive representations to be the result of 'the elementary processes of cognition' described here by Helmholtz. As we have seen, Helmholtz describes this process as an unconscious psychological operation on the memory traces of sensations. By contrast, Kant describes the 'threefold synthesis' responsible for obtaining intuitions of spatiotemporal objects as not only requiring some minimal level of consciousness, but also as distinct from the psychological process of associating recollected sensations which Berkeley and Hume had previously articulated in their account of sense perception (see [Kant, 1965, A97–A130], [Hatfield, 1990, pp. 32-42], and [Merrick, 2004, §§I.A.1, I.B.3]. To my mind, Helmholtz's sign theory of perception bears more resemblance to the Berkeley-Hume account than to any of Kant's various accounts of the syntheses necessary for obtaining intuitions of spatially extended entities.

[10] Note, for example, his complaint that defenders of Kant's thesis on Euclidean geometry never posed the question 'whether visual estimation is innate and given *a priori* or whether it is not acquired too' [Helmholtz, 1996, p. 718]. And in a later essay, where he is more concerned with refuting nativist theories of perception than with articulating his theory *vis-à-vis* its relation to Kant's doctrines, Helmholtz writes that the 'simplest,

sets out to settle the question whether the Euclidean axioms are *a priori* propositions grounded on pure intuition by asking whether the axioms express 'initial, original facts of our perception' that a subject could become acquainted with independently of experiencing any externally located stimuli [Helmholtz, 1996, p. 706]. It is consistent with the theory, says Helmholtz, to assume that we come equipped with the capacities to distinguish the changes in sensory qualia that can be induced by 'the impulses of our will' and the changes in qualia over which we have no control [*ibid.*, p. 697]. He then defines 'spatial relation' as a relationship that we alter in an immediate manner by the impulses of our will.[11] As previously mentioned, Helmholtz thinks of these alternating state changes as a kind of experiment, which can lead us to intuit a particular spatial configuration, *e.g.*, to the left and to the right. He also maintains that these experiments can be initiated voluntarily and independently of an actual encounter with an external object [*ibid.*, p. 698].

Reflecting on everything that nature has endowed us with, Helmholtz concludes that a subject could become acquainted with the spatial relationship denoted by 'one beside another' prior to, and as a necessary causal condition of, perceiving more complex phenomena, *e.g.*, the chair is beside the table. 'One beside another' thus qualifies as an original fact of spatial perception because the subject 'knows' this fact 'without yet having previously gained any understanding of the external world' and because all empirically observed phenomena will conform to this basic spatial relationship [*ibid.*, p. 698]. Thus there may be some legitimacy in identifying such original facts of with 'the general form of spatial intuition [which] is transcendentally given' [*ibid.*, p. 700]. However, the spatial relations described by the Euclidean axioms, the parallel postulate in particular, are not given along with these original facts. First of all, there is no evidence that these highly specific spatial relationships could be read off of sensory state-changes induced by impulses of the will. Secondly, the research in non-Euclidean geometries conducted by Gauss, Riemann, Lobatschevsky, Beltrami, and Helmholtz himself indicates that some non-Euclidean spaces can be intuited. Kant's thesis that Euclidean geometry is necessarily and universally true concerning the entities inhabiting the domain of our perceptual experience is thus refuted by current research in sense perception and by recent mathematical developments [*ibid.*, p. 701].

most important visual images for a human infant . . . are not given to the child, a priori and independently of experience by some innate mechanism' [Helmholtz, 1894, pp. 505–506].

[11] One of the consequences that Helmholtz draws from this definition is that we also have *a priori* knowledge that space is our form of outer sense: 'In this case space will also appear to us—imbued with the qualities of our sensations of movement—in a sensory manner, as that through which we move, through which we gaze forth' [1996, p. 697].

3. The Marburgers' De-Psychologized Kantian Theses

The fact that Helmholtz interpreted Kant's thesis that Euclidean space is our pure form of intuition as the claim that Euclidean space is the spatial form of external perception for which we are innately predisposed is not all that surprising, given the prevailing hermeneutics applied to Kant's *Critique* at the time. Prior to Cohen's [1871] *Kant's Theory of Experience*, it was common to see Kant's arguments that space and time are *a priori* subjective forms of intuition as his weighing in on the rationalists' side of the debate over innate ideas. This, in turn, encouraged the idea that Kant's theses could be bolstered or refuted by drawing from empirical psychology [Hatfield, 1990, p. 111]. With the publication of this book, Cohen began what would become a career-long project of presenting a thoroughly de-psychologized version of Kantian epistemology. The first step was to prove that 'Kant had overcome the pre-critical disjunction: Innate or acquired?', and thus, the terms '*a priori*' and 'innate' should not be treated as synonyms in an appropriate lexicon of Kantian terminology [Cohen, 1871, p. 87].[12]

According to Cohen, his colleagues failed to appreciate the distinctively Kantian notion of apriority because they confused Kant's transcendental investigation into the conditions accounting for the knowledge embodied in our scientific practices with a Lockean type of investigation into 'soul-apparatus' required for perceptual experience [Cohen, 1883, pp. 4–5]:

> [The transcendental method] does not investigate the principles of human reason, but it seeks the scientific validation of the foundation of the sciences . . . What makes them sciences? What is the source of the character of their universality and necessity? From which concept do we derive their worth as knowledge, valid within their area? What characteristics and methods of knowledge clarify those historical facts of knowledge, the sciences, is a methodological question, this is the question of the sciences, when they are compelled to consider their own principles. This, and nothing else, is the transcendental question (Cohen's 1881 preface to Lange's *History of Materialism*, cited in [Kluback, 1987, p. 11]).

[12] In the first edition of *Kants Theorie der Erfahrung*, Cohen is explicitly confronting J. F. Fries and especially J. F. Herbart. Helmholtz's *The Origin and Meaning of Geometrical Axioms* and 'The facts in perception' did not appear until 1878. However, given the nature of Cohen's criticisms, it is clear they are also applicable to Helmholtz's reading of Kant. And in second edition [1885], Helmholtz is specifically and similarly accused.

The consequence of Kant's Copernican revolution in the field of epistemology, claimed Cohen, is that the object under investigation is no longer experience understood in the usual sense, namely, the experience of objects presented to us in sense perception. Instead, Kant introduced a 'new concept of experience', experience 'as it occurs in unbroken progress in genuine science, genuine human traffic, in all genuine culture'.[13]

Recognizing that Kant is operating with a new concept of experience explains why the post-Copernican notions of '*a priori*' and 'pure' should be seen as transcending the innate-*vs*-acquired debate. The task of the Kantian epistemologist is not to determine which features of perceptual experience are imposed by the subject's cognitive apparatus and which are the result of her interaction with the world. Rather, the task is to uncover those elements which contribute the universality and necessity characteristic of the propositions asserted within the sciences: 'Experience itself becomes concept, which we must build up in pure intuition and thought... We build up a concept of experience as a synthetic unity of experiences, according to transcendental principles' [Cohen, 1871, p. 104]. What 'we call *a priori*' are those 'constructive elements' needed to generate the subject matter of a discipline, especially the subject matter of Newtonian physics [*ibid.*]. For the Marburgers, Kant's most significant insight was recognizing that the generality, necessity, and cognitive significance appropriate for truly scientific concepts and propositions can only be secured when practitioners, mathematicians and natural scientists alike, construct their respective domains of '*objective unities*' in accordance with *a priori* principles [Natorp, 1981, p. 263].[14]

Given the Marburgian interpretation, Kant's thesis that Euclidean geometry is a body of synthetic *a priori* propositions is immune to any new developments in the area of perceptual psychology. Euclidean space is an *a priori* or pure form of space because everything that can be known about this spatial structure is obtained from a set of fundamental principles laid down by the practitioners of geometry, without having to import any empirically verified, contingent matter of fact:

> Here the form is *a priori* because it is definable *a priori* in accordance with general principles. [Cohen, 1871, p. 97]

[13] This account of the Marburgian interpretation of Kant's Copernican turn is taken from Paul Natorp, as cited in [Kohnke, 1991, p. 181].

[14] See also [Cohen, 1871, pp. 10–13]. Here Cohen presents Kant's argument, in the preface of the second edition of the *Critique*, that neither mathematics nor physics attained the secure path of a scientific inquiry until each abandoned the practice of passively reading off an object's properties in favor of constructive methods employing *a priori* concepts.

> These general principles, however, are the axioms of geometry. The common bond of the axioms makes up the meaning and content of the spatial form. [Cohen, 1885, p. 238]

They maintained that at the core of each science was a collection of 'basic concepts and propositions' articulating the 'primary objective content assumptions' for that discipline [Natorp, 1981, p. 252]. When it comes to geometry, these 'objective content assumptions' are expressed in the Euclidean axioms. The axioms express the objective and cognitively significant meaning of the basic concepts, *e.g.*, point, line, and straight line, from which all other properly geometric notions are derived. Within the context of the traditional synthetic method of Euclidean geometry, the axioms also include postulates stating the possibility of constructing the basic elements, *e.g.*, an infinitely extended straight line, from which all other Euclidean constructions are obtained. So, given this context, the Marburgers are quite accurate in saying that the axioms contain 'the meaning and content' from which the entire spatial form of Euclidean geometry is built up.[15]

The Marburgers also insisted that the basic propositions lying at the foundation of a discipline were not themselves subject to any additional external justification:

> We become certain of the truth within the proper internal network of the science, developed from primary objective content assumptions as they are formulated in the basic concepts and propositions of that science.... The mathematician or physicist who truly grasps the nature of his science will find it superfluous to seek the grounds for the laws of truth for his knowledge in psychology. He will in principle deny such a search; he recognizes only the laws of his own science, not an alien science, as the judge of truth. [Natorp, 1981, p. 252]

The fundamental concepts and propositions at the core of each science serve as the final arbiters of truth and fix the objective meaning of the primitive terms it employs. One cannot ask for further justification of these basic constitutive elements without running the risk of slipping into psychologism, substituting 'the laws of psychic life' for 'laws of truth', and trading away the objective subject-matter of the sciences for the subjective content of consciousness [*ibid.*,

[15] For more on the nineteenth-century debate over the use of analytic versus synthetic proof techniques in geometry, see [Tappenden, 1995] and [Wilson, 1997].

pp. 249–252].[16] Deciding which propositions express the basic objective content assumptions of a science is a task internal to that science: 'Which axioms the geometry has to accept and how to formulate the same is a matter for geometry' [Cohen, 1885, p. 228]. In short, Helmholtz's strategy of letting the findings of perceptual psychology decide whether Euclidean geometry is a body of *a priori* truths with respect to all things spatial constitutes a violation of the autonomy and sphere of governance accorded to geometry as a science [Natorp, 1981, p. 252].

So far, however, nothing I have said warrants the Marburgian claim that the spatial form derived from Euclid's axioms is the unique pure spatial form. By 1870, it was well-known that non-Euclidean geometries had been developed and apparently engendered no logical contradictions. It was also known that non-Euclidean geometries could be modeled in Euclidean three-space. What right, then, did they have for conferring special status on Euclidean geometry as the geometry containing propositions that are necessarily and universally valid concerning the domain of spatial entities? For any axiomatized geometric system would have its own corresponding sphere of objective content as circumscribed relative to its basic concepts and propositions. And since the meaning and validity of geometric notions is, on their account, determined solely in reference to a particular set of axioms, what sense does it make to ask which geometry, Euclidean or non-Euclidean, qualifies as the true description of anything spatial? Euclidean geometry will be universally and necessarily true of its pure spatial form, and non-Euclidean geometries will be universally and necessarily true of theirs as well.

It is at this stage in their defense of Kant's thesis that the Marburgers rely upon the crucial fact that Newtonian physics was the governing paradigm of the natural sciences. Remember, experience in the

[16] L. Anderson [2005] notes that the pejorative use of 'psychologism' was not coined by the Marburgers, but by Wilhelm Windelband in 1884. He further notes, however, that the two main lines for attacking psychologism were already in place by the 1870s. Anderson claims that the first, which stressed the objectivity of *a priori* mathematical and/or logical laws, was subsequently developed by Frege, as evidenced by his critique of Husserl. The second, which stressed the normativity of *a priori* laws, was developed by Cohen, as evidenced by his 1871 *Kants Theorie der Erfahrung* and later works. Anderson thus distinguishes Frege's and the Marburgers' notion of apriority, at least in terms of how this notion functions within their respective critiques of psychologism. In addition, [Burge, 2001] argues that Frege's notion of apriority differs significantly from Kant's, bearing much more of similarity to the notion of apriority advanced by traditional rationalists, *e.g.*, Leibniz. To articulate, and to defend fully, my thesis that Frege shares the Marburgian view on the apriority of Euclidean geometry in response to Anderson and Burge would take us too far afield. However, to the extent that the argument presented in this paper is compelling, it would seem to challenge any effort to distance Frege's conception from that of the Marburgers, as well as any attempt to align the former with that of traditional rationalism.

post-Copernican sense is 'a concept', which the scientist must 'build up in pure intuition and thought', thereby presenting 'a synthetic unity of experiences, according to transcendental principles' [Cohen, 1871, p. 104]. For a pure spatial form to be designated by the expression 'pure intuition' it is not enough to show that this form is definable in accordance with a set of general principles, arbitrarily chosen by the practitioners of geometry. It must also be shown that this spatial form and its constituting elements are required for constructing experience as it presents itself in the unbroken progress of the natural sciences since the seventeenth century:

> This maturity of philosophy, [Kant's Copernican turn], has come with the maturity of science which starts with Galileo and which ends with Newton. Since Newton, there exists a science which is built upon principles and is conscious of its foundations and preconditions, and which progresses according to the mathematical method. Only now was there presented an object towards which the transcendental question of the possibility of *a priori* knowledge could be directed. [Cohen, 1883, p. 7]

> Newton made geometry a part of mechanics, not in the raw empirical sense but in the transcendental sense. The concept of pure intuition designates this inner condition as a condition of pure experience. [Cohen, 1885, p. 230]

Galileo and Newton were the first to transcribe 'the rhapsody of perception' into the law-like unity of inter-subjectively valid representations of objects captured in the propositions of natural science [Cohen, 1871, p. 101]. This transcription required rejecting the representation of entities given to us in 'direct sensual appearance', in favor of a representation of natural phenomena conforming to mathematical and logical forms [Natorp, 1981, p. 255]. Because the natural phenomena of Newtonian physics are represented in conformity to the axioms, concepts, and constructive method of traditional Euclidean geometry, Euclidean space qualifies as the spatial form designated by 'pure intuition'. To say that Euclidean space is the form of pure intuition is to say that 'the form of [Euclidean] geometry is essential for the natural sciences' [Cohen, 1885, p. 238].

It is irrelevant whether or not non-Euclidean geometries are logically consistent systems or even if perceptual psychology indicates that non-Euclidean spaces can be visualized:

> He [Helmholtz] says: 'If spaces of a different kind are imaginable then one could disprove that the axioms of

> geometry are necessary consequences of an *a priori* given transcendental form of our intuitions in Kant's sense.' ... If those spaces are now imaginable and even intuitable, then a lot is still missing until they can be considered products of pure intuition as well; for that, their scientific fertility has to be proven or, if not yet ripened, they have to be anticipated and designated as leading prospects. [Cohen, 1885, p. 231]

To challenge the claim that Euclidean geometry is the body of universal and necessary truths about anything scientifically recognized as having spatial characteristics, Helmholtz and others must show 'that these [non-Euclidean] spaces are necessary and fruitful to deepen' our understanding of experience as it occurs in the sciences as a whole [*ibid*., p. 230]. Up until 1919, this was something that only Euclidean geometry had achieved.

Finally, let me stress the significant differences in the Marburgers' and Helmholtz's use of the term 'intuition'. For Helmholtz, a full-fledged intuition is a completely interpreted sensory sign, resulting in an image of a particular sensed entity with spatial and temporal characteristics, *e.g.*, a book presented now and to the left of me. A *pure* intuition of a spatial configuration would be the perception of a spatial relationship that a subject could obtain simply in virtue of the various capacities and dispositions she was born with, *e.g.*, one beside another. For the Marburgers, by contrast, the notion of pure intuition has nothing whatsoever to do with the cognitive capabilities of a knowing or sensibly perceiving agent:

> Intuition and thought are abbreviations for scientific methods, methods which are so independent from the special content of inquiry that they refer back to, that they can supply the general pre-conditions for all scientific inquiry. [Cohen, 1883, p. 3]

> Intuition does not signal a vague faculty of knowledge, nor an unjustified given; rather, it signals the act, that is, the constructive method by which mathematics reaches its knowledge: '*A priori* space is not physical space, neither is it geometrical space, in the exact sense, but merely the process of production and formation of the latter. This is the meaning of space as pure intuition'. [Poma, 1997, pp. 50–51]

'Pure intuition' refers to a method, *i.e.*, the method of generating and cognizing the spatial structure(s) studied within geometry and, more importantly, the spatial structure presupposed by the natural sciences. This method includes the basic concepts, axioms, and procedure of traditional Euclidean geometry, rather than those employed in non-Euclidean geometries. On the Marburgers' thoroughly de-psychologized reading of Kantian epistemology, the notion of space as a pure intuition and as a source of *a priori* knowledge is completely severed from the notion of a sensible faculty or the form of that faculty. The *a priori* source of knowledge identified as pure intuition consists of the axioms, basic spatial configurations and method of traditional Euclidean geometry: 'This is the meaning of space as pure intuition.'

We also saw that the Marburgers resist the idea that the Euclidean axioms stand in need of any external justification. The truth of the axioms does not stem from a purported fact that they describe the form of our sensibility nor that they describe the properties of an 'unjustified given', such as a pre-Copernican notion of physical space or of a metaphysical geometric structure. To the extent that the axioms are susceptible to a justification, it would appear to be an indirect kind of justification. That is, in order to account for the accumulation of knowledge concerning abstract spatial configurations represented in the long tradition of Euclidean geometry and concerning natural phenomena represented in the sciences since the seventeenth century, we must acknowledge the set of Euclid's axioms, rather than a set denying the parallel postulate, to be universally and necessarily true concerning everything spatial. For a Marburger, to say that the Euclidean axioms express basic facts of pure intuition is not to say that they *describe* the spatial relationships of either a subject's form of external perception or things-in-themselves. Rather, it is to say that they *prescribe* certain spatial relationships that correspond to the objective and cognitively significant meaning of the terms 'point', 'line', and 'straight line' as used within the sciences.

This is not to say that the Marburgers would reject any research involving non-Euclidean geometries, especially if that research shed light on the space-relations of Euclidean geometry. When criticizing Helmholtz, Cohen demands that developing new geometric structures should, at the very least, expand and deepen our understanding of geometry [Cohen, 1885, p. 203]. He also suggests that the *a priori* elements composing the foundation of geometry will be modified as science progresses; the 'foundation that forms the treasure house of the content of knowledge' can be 'eternally increased' ([Cohen, 1876], cited in [Kluback, 1987, p. 13]). Hermann Hankel [1867] claimed that the synthetic treatment of projective geometry contributed to our knowledge of Euclidean relationships. It would seem, then, that Cohen could endorse this development as a legitimate trajectory for geometric research, so long as a tight connection with traditional Euclidean geometry is maintained.[17]

[17] [Wilson, 1997] argues that appreciating Frege's endorsement of these developments in projective geometry can help relieve some of the tension between his remarks

Cohen's [1876] concern is that mathematicians adopt methods which preserve the continuity of the subject matter of geometry and add to our knowledge of that subject matter: 'all new foundations ought to be presented as deepening the old ones' [*ibid.*, p. 13]. Arbitrarily negating Euclid's parallel postulate, emptying geometric terms of their traditional meanings, and investigating any logically consistent system of geometric axioms threatens the status of geometry as a science. From our vantage point, one can see Cohen urging mathematicians to engage in practices that could have forestalled Thomas Kuhn's [1970] claim that the actual practice of scientists undermines the notion of science as a cumulative process of gaining knowledge about a somewhat fixed domain of entities.

4. Frege's Distinct Uses of 'Intuition' and Defense of Kant's Thesis[18]

I will now present the textual evidence that Frege operates with two different meanings for the term 'intuition'. The first is the notion of an intuition as a sensory representation of a particular entity; the second is the notion of intuition as the pure source of geometric knowledge. I will also argue that what Frege means when using 'intuition' in the first sense is one of Helmholtz's perceptual signs and what he means when using 'intuition'

on the subjectivity of intuitions and the objectivity of geometric propositions. According to Wilson, Frege aligned himself with early nineteenth-century geometers, who sought to mimic the generality of analytic geometry, while maintaining a connection to the method and semantics of traditional Euclidean geometry. This motivation encouraged renewed interest in projective geometry, which led to the subsequent recognition of duality principles. In projective three-space, the principle states that any true proposition still holds when 'point' is substituted for 'plane' and *vice versa*. Knowledge of duality principles can then explain how one could insist on the subjectivity of intuitive representations, without calling into question the purported objective content expressed in geometric propositions. The principles guarantee that there is some shared objective content expressed in geometric propositions, despite the fact that two people's intuitive representations may differ. While I agree that this resolution finds substantial textual support in §26 of the *Foundations*, I do not believe that it satisfactorily addresses all of the problems arising from Frege's use of 'intuition'. First of all, Wilson concedes that, for Frege, geometric propositions possessing cognitive significance must be derived from basic facts of Euclidean geometry, which are originally given to us in intuition [Wilson, 1997, p. 129]. If we assume that 'intuition' here refers to a capacity for visualizing spatial configurations, then Frege seems oblivious to nineteenth-century research in perceptual psychology and its challenge to such a view. For my further elaboration on this point, see p. 22 of this text. Secondly, there are passages where Frege locates intuition, the source of geometric knowledge, as falling within the 'domain of the objective' [Frege, 1979, p. 273]. See also pp. 67–68 of this text, where I cite textual evidence that Frege aligned intuition with the sense (objective content) of geometric propositions.

[18] Although I will use the terms 'sense' and 'meaning' throughout this section, I do not intend the reader to assume that I am using them in Frege's technical sense, unless specifically noted.

in the second sense is precisely what the Marburgers intended by designating Euclidean space as the method of pure intuition, the *a priori* source of geometric knowledge.

Frege's use of the term 'intuition' to denote the sensory representation of a particular occurs almost exclusively in the *Foundations* and here he self-consciously adheres to the characterization of an intuitive representation as it was presented in Kant's *Critique*. Consider his objection to Hermann Hankel's use of the expression 'a pure intuition of magnitude', where the magnitude that Hankel is envisioning is something that is 'valid for magnitudes in every field' [cited by Frege, 1980, p. 18]. Frege responds that while it is perfectly reasonable to talk about a '*concept* of magnitude' representing 'all the different things that are called magnitudes', it is incoherent to talk about an *intuition* of a magnitude in general:

> I cannot even allow an intuition of 100,000, far less of number in the general, not to mention magnitude in general. We are all too ready to invoke inner intuition, whenever we cannot produce any other ground of knowledge. But we have no business in doing so, to lose sight altogether of the sense of the word 'intuition'. [*ibid*., pp. 18–19]

According to Frege, the meaning of the word 'intuition' was first fixed by Kant: 'Kant in his *Logic* defines it as follows: "An intuition is an *individual* idea (*repraesentatio singularis*), a concept is a *general* idea (*repraesentatio per notas communes*) or an idea of *reflexion* (*repraesentatio discursive*)" ' [*ibid*.].[19] Frege acknowledges that, in the *Logic*, there is 'no mention of any connexion' between sensibility and intuition, but that Kant explicitly made this connection in the *Critique*. He then cites the passage from *Transcendental Aesthetic* [Kant, 1965, A19/B34]. Here sensibility is defined as our capacity for receiving representations through being affected by some indeterminate something, and Kant insists that sensibility 'alone provides us with *intuitions*' [Frege, 1980, p. 19]. Given that this detour into Kant scholarship is supposed to undermine Hankel's claim, we can take Frege's suppressed conclusion to be the following: an intuitive representation of a magnitude is simply too particular and too bound up with what is sensibly perceptible to do the work that Hankel

[19] Frege is quoting here from Kant's views on logic as they were presented and edited by his student, Gottlob Benjamin Jasche in an 1800 publication entitled *Kant's Logic*. Currently, Kant scholars refer to this work as the *Jasche Logic* in order to stress the point that we cannot simply assume that these were in fact Kant's views on logic. However, this was the text that Frege and his colleagues understood as representing Kant's thoughts on the subject.

expects of it. An intuition in this sense cannot serve as a general idea capable of representing all particular magnitudes from every scientific field.

Notice too that when Frege uses the term 'intuitions' to designate sensory representations, he is thinking of them as the images presenting themselves to a subject through sense perception or the imagination, with all the particularity that that entails. The fact that Frege is thinking of intuitions in this manner is hinted at in his rebuttal to Hankel. It comes out even more starkly, however, in his critique of Kant's view of arithmetic:

> Kant thinks he can call on our intuition of fingers or points for support, thus running the risk of making these propositions appear to be empirical, contrary to his own expressed opinion; for whatever our intuition of 37863 fingers may be, it is at least certainly not pure. Moreover, the term 'intuition' seems hardly appropriate, since even 10 fingers can, in different arrangements, give rise to different intuitions. And have we, in fact, an intuition of 135664 fingers or points at all? [Frege, 1980, p. 6]

Frege's point is that the intuition of four strokes (or fingers) arranged like '1111' is different from the intuition of '11 11'. Furthermore, our capacity for forming images delimits the scope of entities that can be intuited. If we cannot form the image of $1000^{1000^{1000}}$ strokes, dots, *etc.*, we cannot have an intuition of $1000^{1000^{1000}}$ [*ibid.*, p. 101].[20]

So, the first meaning that Frege attaches to the term 'intuition' is one that he takes to be equivalent to the Kantian notion of an intuition as a sensory representation of a particular, which, in turn, is interpreted as the image presented to us via our sensory apparatus.

Furthermore, we know that Helmholtz had introduced his notion of a perceptual sign as the nineteenth-century model of what Kant originally intended by the term 'intuition'. Now let us look again at Frege's distinction between objective and subjective ideas, and his rationale for treating intuitions as subjective ideas:

> What is objective ... is subject to laws, what can be conceived and judged, what is expressible in words.

[20] Michael Resnik [1980] notes that the argument that Frege provides here against the claim that arithmetic is based on intuition would seem to undermine a Kantian view of geometry as well, since we presumably cannot form the image of a chiliagon, a polygon with 1000 sides. Reading remarks such as Resnik's first prompted the question leading up to this paper: can we provide a more charitable interpretation of Frege's views on geometry by showing that he must not be operating with just one notion of intuition throughout his writings?

> What is purely intuitable is not communicable. [Frege, 1980, p. 35]

> I understand objective to mean what is independent of our sensation, intuition and imagination, and of all construction of mental pictures out of memories of earlier sensations.... [*ibid.*, p. 36]

> An idea in the subjective sense is what is governed by the psychological laws of association; it is of a sensible, pictorial character... Subjective ideas are often demonstrably different in different men, objective ideas are the same for all... It is because Kant associated both meanings with the word ['idea'] that his doctrine assumed such a very subjective, idealist complexion, and his true view was made so difficult to discover. [*ibid.*, p. 37]

Remember too that, according to the sign theory of perception, the represented content of sensory perceptions is the construction of mental images by means of the subject's unconscious inductive processing of memories of earlier sensations. Moreover, this process is purportedly not expressible in words and neither is the represented content, at least not at the initial stages. This explains why Frege classified intuitions as subjective ideas. Intuitions are sensible signs, and sensible signs do not meet Frege's criteria for objective ideas.

This is also explains why Frege must now use the term 'object', rather than 'intuition', to denote the objective ideas or meanings corresponding to the singular terms employed in a propositional context.[21] The term 'intuition' had been forever corrupted by those, like Helmholtz, who privileged the psychological reading of Kant's epistemology. By tossing Helmholtz's perceptual signs into the bin of subjective ideas and arguing that an intuition in this sense cannot be the objective meaning of the term 'point' or 'line',[22] Frege shows himself to be in agreement with the Marburgers that we ought not look to psychology for supplying the objectively valid meaning of geometric terms. It also suggests that he agrees that our epistemically privileged representations of entities should not be those corresponding to 'direct sensual appearance', but rather to the presentation of entities as mediated by the propositions of a science. Let us turn now to Frege's second notion of intuition.

[21] See the footnote to [Frege, 1980, p. 37].

[22] See [Frege, 1980, pp. 35–36].

The second sense that Frege attaches to the term 'intuition' is as the ultimate ground of geometric knowledge. This usage is scattered throughout his writings:

> [T]he truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy . . . [Euclidean space is] the only one whose structures we can intuit . . . [Frege, 1980, p. 20]

> I call axioms propositions that are true but that are not proved because our understanding of them derives from that nonlogical basis which may be called an intuition of space. [Frege, 1971, p. 9]

> From the geometrical source of knowledge flow the axioms of geometry . . . Yet here one has to understand the word 'axiom' in precisely its Euclidean sense . . . I cannot emphasize strongly enough that I only mean axioms in their original Euclidean sense, when I recognize a geometrical source of knowledge in them. [Frege, 1979, p. 273]

> I have had to abandon the view that arithmetic does not need to appeal to intuition either in its proofs, understanding by intuition the geometrical source of knowledge, that is, the source from which flow the axioms of geometry. [*ibid.*, p. 278]

It would be a mistake to read Frege's claim that we can only intuit the structures of Euclidean space as an assertion about our perceptual capabilities and then to assume that this psychological fact is what he presents as the ultimate justification for the truth of Euclidean geometry. First of all, Frege was certainly aware that such a claim was not supported by perceptual psychology.[23] Secondly, Frege did not think the space(s) constituting the subject matter of Euclidean geometry was sensibly intuited at all: 'Even the objects of geometry, points, straight lines, surfaces, *etc.* cannot really be perceived by the senses' [*ibid.*, pp. 265–266]. Thirdly, there is no textual evidence that Frege thought we were

[23] Based on his references to memory traces of sensations and his close relationship with Ernst Abbe, who was deeply engaged with research in the theory of ophthalmic instrument construction, we can assume that Frege was well aware of current developments in perceptual psychology.

capable of a non-sensory, purely intellectual, intuition of abstract geometric structures.[24]

Given that Frege identifies the Euclidean axioms, and only those axioms, as intuition *qua* source of geometric knowledge and given his familiarity with Cohen's notion of pure intuition as a method,[25] I suggest another possible reading of his claim '[Euclidean space] is the only one whose structures we can intuit'. Because the Euclidean axioms are the source of geometric knowledge and because the truths of Euclidean geometry are necessarily valid concerning all that is spatially intuitable, we cannot obtain an immediate, objectively valid, cognitively significant, representation of something as a *spatial* something unless it is represented in conformity with the axioms, basic elements, and synthetic method of

[24] Even Matthias Schirn, one who strongly supports reading Frege as a mathematical platonic realist, rejects the idea that Frege would have relied upon intellectual intuition to account for our cognitive access to abstract mathematical entities: 'If Frege had been confronted with the postulation of a special faculty of mathematical intuition *à la* Gödel, which is supposed to allow us direct cognitive access to the remote realm of abstract objects, he would probably have stigmatized it as a devastating "irruption of psychology into logic" ' [Schirn, 1996b, p. 118].

[25] We know that Frege was aware of Cohen's de-psychologized reading of Kant's 'pure intuition' as a method by 1885, at the very latest, because this is when he published a review of Cohen's [1883] *Das Princip der Infinitesimal-Methode und seine Geschichte*, which contains this reading. Noting the date when Frege read Cohen's *Princip* may also explain why, after 1884, Frege became increasingly more careful in distinguishing his use of 'intuition' to denote the psychological imaging of a particular from his use of 'intuition' to denote the objective (non-psychological) source of geometric knowledge. In the 1879 preface to *Begriffsschrift*, for instance, Frege explains that his logicist project required developing a symbolism capable of representing proofs insuring that nothing 'intuitive' could slip in unawares. However, it is unclear whether Frege's concern is with the particularity of intuitions, given that logical laws must 'transcend all particulars', or that appeals to intuition import something from the geometrical source of knowledge [Frege, 1970, p. 5]. §23 of *Begriffsschrift* suggests that his primary interest in excluding intuition is to show that arithmetical objects and concepts can be represented without having to borrow anything from sense perception or geometry, the non-logical sources of knowledge [*ibid*., p. 55]. And this is made explicit in an article written sometime between 1924 and 1925: 'I have had to abandon the view that arithmetic does not need to appeal to intuition either in its proofs, understanding by intuition the geometrical source of knowledge, that is, the source from which flow the axioms of geometry' [Frege, 1979, p. 278]. Here Frege stresses that intuition, in the second sense, has nothing do with anything psychological and everything to do with the traditional Euclidean axioms: '. . . I cannot emphasize strongly enough that I only mean axioms in the original Euclidean sense, when I recognize a geometrical source of knowledge in them' [*ibid*., p. 273]. Frege does refer to the source of geometric knowledge as a faculty in his 1873 doctoral dissertation. However, to my knowledge, all subsequent references to intuition, in the second sense, are completely de-psychologized. The fact that Frege never explicitly states that he is appropriating Cohen's use of 'intuition' is not so surprising, since Frege is notorious for failing to acknowledge any debts to his contemporaries.

traditional Euclidean geometry. If this reading is the correct one, then Frege is not looking to psychology to provide the ground for geometric knowledge or for an external justification of the Euclidean axioms. Instead, similar to the Marburgers, he is claiming that the axioms articulate the basic objective content assumptions for the science of geometry and serve as general pre-conditions for what is considered spatial within the natural sciences. Thus, they function as prescriptive norms, governing and correcting our sense perception of spatial relationships.

To see if this reading can be sustained, we first need to ask whether Frege did, as a matter of fact, think the Euclidean axioms expressed the objective meaning and content from which the entire subject matter of geometry is derived. And the answer is yes:

> We cannot very well define an angle without presupposing knowledge of what constitutes a straight line. To be sure . . . we shall always come upon something which, being a simple, is indefinable, and must be admitted to be incapable of further analysis. And the properties belonging to these ultimate building blocks of a discipline contain, as it were *in nuce*, its whole contents. In geometry, these properties are expressed in the axioms insofar as they are independent of one another. [Frege, 1971, p. 143]

According to Frege, there are certain geometric notions, *e.g.*, straight line, that are essentially defined by the set of Euclidean axioms: 'Their sense [objective meaning] is indissolubly bound up with the axiom of the parallels' [Frege, 1979, p. 247]. All other properly defined, cognitively meaningful, geometric terms, *e.g.*, 'angle' or 'triangle' must be composed out of these basic constituents. We know, therefore, that the Euclidean axioms are universally and necessarily true concerning spatial entities and relations, since they constitute part of the meaning of geometric terms:

> When a straight line intersects one of two parallel lines, does it always intersect the other? . . . I can only say: so long as I understand the words 'straight line', 'parallel', and 'intersect' as I do, I cannot but accept the parallels axiom. If someone does not accept it, I can only assume that he understands these words differently [*ibid.*]

For Frege, as for the Marburgers, to claim that intuition is the *a priori* source of geometric knowledge is to affirm the special role of Euclid's axioms in legislating the inter-subjectively binding and cognitively significant meaning of geometric terms.[26]

The second question is whether Frege thought that pure intuition, *i.e.*, the axioms, concepts, and spatial configurations of traditional Euclidean geometry, are tools necessarily employed in the natural sciences as well. Do they possess the requisite generality to function as preconditions for all scientific inquiry? Again, the answer is yes. Frege maintains that the

[26] Frege scholars will shudder at the thought that Frege recognized Euclid's axioms as 'defining' basic geometric terms. One worry is that Frege's main charge against Hilbert and his defender, Korselt, is that they confuse axioms with definitions. Another is that this characterization of Euclid's axioms makes them sound awfully similar to what Carnap later described as meaning postulates, implying that Frege took geometric axioms to be analytic, which he expressly denied [Frege, 1980, pp. 101–102]. In response, let me clarify my position by introducing Kant's distinction between two types of definitions. The first is 'a purely verbal definition' of a concept or what he also describes as the definition of 'an arbitrarily invented concept' [Kant, 1965, A593/B621 and A729/B757]. The only restriction on definitions of this type is that the resulting concept exhibit logical consistency. However, this is not enough to insure the possibility of applying the concept to any object which might be given to us. Therefore, while definitions of this sort can ground logically necessary judgments, both concepts and judgments may still lack genuine cognitive content. In contrast, 'real definitions' must associate the concept-word with some 'clear property' insuring the possibility of a determinative application to the entire domain of spatiotemporal objects [*ibid.*, A242]. Providing real definitions for mathematical concepts requires presenting a rule-guided procedure (schema) that not only captures the meaning of a purely mathematical notion, but also accounts for the applicability of the concept to empirical phenomena. Kant considered the traditional Euclidean axioms and constructive definitions to be exemplars of real definitions, guaranteeing the application and relatively wide scope of validity for geometric concepts.

My point here is that Frege, like the Marburgers, shares Kant's view that the traditional Euclidean axioms and postulates functioned as real definitions for basic geometric concepts. I argue, on pp. 71–72 of this text, that Frege's charge against Hilbert and Korselt is not that they are treating Euclid's axioms as definitions *per se*, but that they are depriving the axioms of their status as real definitions in Kant's sense and substituting purely verbal definitions for arbitrarily invented concepts. Deciding whether Hilbert is actually guilty of this charge lies beyond the scope of this paper. Given this interpretation, is Frege obliged to recognize the Euclidean axioms as analytic? No, not if we apply Frege's criteria for distinguishing analytic from synthetic propositions. Frege never characterizes analytic propositions as those which are true by virtue of the meaning of their constituent terms. As Michael Beaney points out, Frege could not endorse this notion of analyticity without threatening his logicist agenda: '[I]t is by no means clear that his logicist definitions and axioms embody the sameness of sense that would be seem to be a condition of their "analyticity" ' [Beaney, 1997, p. 25]. Instead, analytic propositions are those whose scope of validity is maximally general and whose truth can be established by drawing solely from logical laws and definitions [Frege, 1980, p. 4]. Since my reading does not force Frege to claim that the Euclidean axioms have a scope of validity equivalent to arithmetic propositions or that they are derivable from logic alone, then it similarly does not force him to claim that the axioms are analytic. I am grateful to William Demopoulous and two anonymous referees for urging me to address these concerns.

domain over which the propositions of a discipline are considered valid is 'determined by the nature of its ultimate building blocks' [Frege, 1971, p. 143]. He reminds his mathematical colleagues that the axioms of traditional Euclidean geometry include postulates guaranteeing the possibility of performing the 'the simplest procedures', *e.g.*, that a straight line may be drawn from any to point to another, from which all other geometric constructions are obtained [Frege, 1979, pp. 206–207]. This postulated possibility of generating a geometric line is not a claim about us, about what we can actually draw or are psychologically capable of:

> But what in actual fact is this drawing a line? It is not, at any rate, a line in the geometrical sense that we are creating when we make a stroke with a pencil. And how in this way are we to connect a point in the interior of Sirius with a point in Rigel? Our postulate cannot refer to any such external procedure. It refers rather to something conceptual. But what is here in question is not a subjective, psychological possibility, but an objective one. [*ibid.*, p. 207]

A postulate is a fundamental truth that asserts the possibility of an objective procedure whereby a geometric entity can be constructed. He then argues that postulates assert 'the existence of something with certain properties' [*ibid.*]. The ultimate building blocks of Euclidean geometry are, therefore, the appropriately idealized spatial configurations described in the postulates and conforming to the other axioms. Since these building blocks in nature 'ultimately are spatial configurations', the boundary of Euclidean geometry 'will be restricted to what is spatial' [Frege, 1971, p. 143]. This includes anything considered to be a conceptually idealized spatial configuration within the natural sciences as well, such as the line connecting a point in the interior of Sirius with a point in Rigel.

So, according to Frege, the fundamental concepts and propositions of Euclidean geometry serve as the final arbiters of truth and fix the objective meaning of the terms employed in geometry as a science. Furthermore, the Euclidean postulates assert the objective possibility of constructing the basic spatial configurations, with specific properties, which the geometer relies on to develop her subject matter, *i.e.*, the space(s) investigated in geometry proper. Finally, the spatial entities and relations expressed by Euclid's axioms and postulates are of such a nature that they can serve as pre-conditions for the investigation of spatial entities in other scientific disciplines. It is for these reasons, I maintain, that Frege insists that it is only 'axioms in their original Euclidean sense' that should be identified with intuition, conceived of as 'the geometrical source of knowledge' [Frege, 1979, p. 273].

For Frege then, as for the Marburgers, the notion of intuition as the source of geometric knowledge is not bound up with the subject's capacity for sense perception. Instead, 'intuition', in the second sense, refers to a method, the axioms, fundamental concepts, and constructive procedures of traditional Euclidean geometry. These, in turn, govern the perceptual observations made within a scientific context:

> In order to know the laws of nature we need perceptions that are free from illusion. And so, on its own, sense perception can be of little use to us, since to know the laws of nature we also need the other sources of knowledge, logical and geometrical. Thus we can only advance step by step—each extension in our knowledge of the laws of nature providing us with a further safeguard against being deceived by the senses and the purification of our perceptions helping us to a better knowledge of the laws of nature . . . We need perceptions, but to make use of them, we also need the other sources of knowledge. Only all taken in conjunction make it possible for us to penetrate ever deeper into mathematical physics. [Frege, 1979, p. 268]

On Frege's account, Kant's thesis that Euclidean geometry is a body of *a priori* truths, whose ultimate source of justification is pure intuition, is not a thesis about which truths describe the spatial form imposed by our sensory apparatus, but a thesis about which geometric propositions and concepts serve as prescriptions for picking out empirically observable points, straight lines, *etc*. And to say that the source of geometric knowledge is intuition is simply to say that these prescriptive notions are Euclidean ones.

To reject Kant's thesis is thus tantamount to acknowledging that the accumulation of knowledge represented in the long tradition of Euclidean geometry and the knowledge of nature that we thought we had achieved since Galileo and Newton first mathematized the natural sciences was itself an illusion:

> The question at the present time is whether Euclidean or non-Euclidean geometry should be struck off the role of the sciences and made to line up as a museum piece alongside alchemy and astrology . . . That is the question. Do we dare to treat Euclid's elements, which have exercised unquestioned sway for 2000 years, as we have treated astrology? It is only if we do not dare to do this that we can put Euclid's axioms forward as propositions that are neither false nor doubtful. [Frege, 1979, p. 169]

In other words, when pushed to give a justification for the truth of the Euclidean axioms, what Frege offers is an indirect type of justification. If geometry as it has been practiced to date counts as a science and is thus understood as contributing to our knowledge of the world, then we must acknowledge the truth of these axioms.

I began this paper by mentioning Frege's complaint that Hilbert seemed to forget that geometric axioms must express basic facts of our intuition. We are now in a position to understand better the nature of that complaint. I have argued that when Frege says that Euclidean axioms assert basic facts of our intuition, he means that they prescribe the inter-subjective, cognitively significant, meanings of geometric terms and characterize the nature of the basic spatial elements from which all properly geometric entities are constructed. Frege is pleased, therefore, whenever Hilbert characterizes the axioms of his system as 'certain basic and interconnected facts of our intuition', which 'geometry requires . . . for its consequential construction' [Frege, 1971, p. 25].[27] And he worries whenever Hilbert seems to deviate from this characterization of the axioms, as evidenced by the latter's ambiguous use of the terms 'point' or 'straight line':

> [I]t also is unclear what you [Hilbert] call a point. One first thinks of points in the sense of Euclidean geometry, and is confirmed in this by the proposition that the axioms express basic facts of our intuition. Later on, however (p. 20), you conceive of a pair of numbers as a point. [*ibid.*, pp. 6–7]

Hilbert responds to Frege's charge of equivocation by pointing out that he is intentionally not saddling his primitive geometric notions with any previously understood meaning or reference: 'I do not want to presuppose anything as known' [*ibid.*, p. 11]. The task of the geometer, claims Hilbert, is simply to develop a formal system of inter-related concepts, held together by necessary logical relations. Whether the term 'point' refers to Euclid's extensionless spatial entities, an ordered pair of numbers, or a chimney sweep is irrelevant: 'the base elements can be construed as one pleases' [*ibid.*, p. 13].

Hilbert thus divests geometric terms of the meaning and reference which they have within the context of traditional Euclidean geometry and this is the crux of Frege's complaint:

> We are easily misled by the fact that the words 'point', 'straight line', *etc.* have already been in use for a long time.

[27] Here Frege is quoting from the description of the axioms that Hilbert presents in the introduction and §1 of his *Foundations*.

> But just imagine the old words completely replaced by new ones especially invented for this purpose, so that no sense is as yet associated with them. And now ask whether everyone would understand the Hilbertian axioms and definitions in this form. [Frege, 1971, p. 60]

For Frege and the Marburgers, the axioms/postulates of Euclid's *Elements* express the objective content assumptions for geometry insofar as it qualifies as a science, a body of cumulative knowledge. Hilbert willfully deviates from the traditional usage and appears uninterested in explaining how his schema of formal concepts can be applied: 'I do not know how, given your definitions, I could decide the question of whether my pocketwatch is a point' [Frege, 1971, p. 18]. Frege concludes that Hilbert's is a system of pseudo-concepts, pseudo-propositions and pseudo-axioms, lacking in 'thought-content', 'sense', and 'knowledge' [*ibid.*, pp. 27, 85]. To the extent that developers of a geometric system are successful in detaching their basic concepts and propositions from the ultimate source of validity, namely, the method of Euclidean geometry, they are essentially detaching their system from intuition, the source of geometric knowledge. We can safely assume, then, that Frege is taking a swipe at Hilbert, when he subsequently writes of those whose 'recent works have muddied the waters', potentially contaminating the geometric source of knowledge, by attaching 'a different sense to the sentences in which the axioms have been handed down to us' [Frege, 1979, p. 273].

5. Conclusion

The purpose of this paper was to resolve certain tensions regarding Frege's commitment to a Kantian view of geometry by paying careful attention to his use of the term 'intuition' and showing how his usage maps onto the debate between Helmholtz and the Marburgers. We saw that Frege uses 'intuition' to refer to the images presented to us via sense perception and that here he is adopting Helmholtz's use of the term. By insisting on the subjectivity of intuitions in the Helmholtzian sense, Frege is simply affirming his general thesis that psychology provides no assistance in determining the objective meaning or ground for scientific concepts and propositions. We also saw that when Frege uses 'intuition' to denote the ground of geometric knowledge what he means is what the Marburgers meant, that the Euclidean axioms prescribe the cognitively significant meaning of basic geometric terms and function as the final arbiters of truth for geometry as a science. Frege's claim that intuition is the ground of geometry is thus perfectly compatible with his tirades against psychologism and his own remarks on the subjectivity of intuition.

I also hope to have gone some way in convincing the reader that the Marburgers' modified Kantian thesis that Euclidean geometry was a body of synthetic *a priori* truths could be reasonably held until word spread of the 1919 confirmation of Einstein's general relativity. And so, let me conclude with the following reason for aligning Frege with the Marburgers' rendering of Kant's thesis: it allows for a more charitable interpretation of Frege's oft-stated and notoriously late commitment to it. Consider Schirn's final assessment of Frege's position:

> I leave it to the reader to judge whether Frege's characterization of our knowledge of Euclidean geometry as synthetic *a priori* must be regarded as a retrograde step, especially in light of the work of Riemann, Helmholtz and other contemporaries. [Schirn, 1996a, p. 27]

Schirn goes on to complain that Frege never presents an argument for the thesis and 'seems to take it more or less for granted' [*ibid.*]. In making this assessment, Schirn assumes that whenever Frege speaks of intuition as the source of geometric knowledge, he is referring to something psycho-physiological, 'a faculty for visualizing geometric configurations' [*ibid*, p. 21]. He then presents Helmholtz's objections to Kant's thesis, showing that Frege's position is unwarranted. Given my argument, Frege did not need to defend a Kantian view on the nature of Euclidean geometry against Helmholtzian objections because the Marburgers had already rendered it immune to such objections. The only thing that Frege, as well as Cohen and Natorp, can be charged with taking for granted is that physics would continue to build on the mathematical foundations of Newton's *Principia*.

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