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HOW TO APPLY MATHEMATICS

ABSTRACT. This paper presents a novel account of applied mathematics. It shows how we can distinguish the *physical content* from the *mathematical form* of a scientific theory even in cases where the mathematics applied is indispensable and cannot be eliminated by paraphrase.

1. INTRODUCTION

The philosophical literature offers two competing views about the role of mathematical objects in physical theories. On the one hand, there are the nominalists, who deny that there are any mathematical objects, and who argue that the scientific use of mathematics is merely an abbreviative manner of speaking. Physical theories could be formulated without referring to mathematical objects, they claim, but it is usually more convenient not to do so, and to pretend that they exist. The second view is advocated by W.V. Quine and Hilary Putnam (1971), who reject the nominalist's position as untenable, and argue that there is no principled difference between the theoretical roles of mathematical and physical objects. On their view, scientific theories are about numbers and functions as much as they are about electrons and quarks. Mathematics and physics have a joint subject matter.

My complaint is that neither of these two views provides an account on which physics genuinely *applies* mathematics. According to the nominalist, there is nothing to be applied, and the only sense that Quine and Putnam can make of application is one in which mathematics applies physics as much as physics applies mathematics. The aim of this paper is to show that we can do better. There is room for a third kind of view, which provides a more plausible account of what mathematics is contributing to physical theory.

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2. THE INDISPENSABILITY ARGUMENT

Let me begin by taking a closer look at Putnam's Indispensability Argument.[1] The official aim of this argument is to refute the nominalist by showing that most physical theories cannot even be formulated without mathematics. But the argument would also show that there is no logically significant difference between the theoretical contributions of mathematical and physical objects. And that would rule out *any* view that tries to assign different roles to them, not only mathematical nominalism.

Putnam's argument begins by considering Newton's theory of gravitation. This theory claims that if two objects a and b have masses m_a and m_b kilogram, respectively, and if the distance between them is d meters, then they attract each other with a force of

$$G\frac{m_a m_b}{d^2}\text{ newton.}$$

Here G is the gravitational constant. Putnam then asks us to give an account of the logical structure of this theory, subject to the following two assumptions:

A1: Only accounts in extensional first-order languages are acceptable.

A2: There are only finitely many physical objects.

If we accept A1 then there is only one way to render the quantification over masses and distances used in Newton's theory. With second-order quantification over properties unavailable, we would have to treat masses and distances as *relations* between physical objects and numbers. For example,

Object a has mass n kilogram

would have to be analysed in terms of a "mass in kilogram"-relation M as

(1) Man.

We could then mimic second-order quantification over mass properties by using first-order quantification over the corresponding numbers.[2]

In (1), the number n is being used as a mere "index" for a mass ascription, and there seems to be no principled reason why some other object could not perform the same role. The mathematical nominalist might therefore try to use suitably chosen physical objects for the same purpose.[3] But if we grant Putnam's second assumption

A2 then there are not enough of them to do the job. There are uncountably many possible masses that need to be taken care of, and too few physical objects to label all of them.

Hence if Newton's theory is a representative case of a physical theory – and there seems to be no reason to deny this – then the only way we can regiment such theories in an extensional first-order language is by treating mathematical objects as relata of physical objects. Any acceptable formalisation of such theories would thus have to contain atomic formulae of the form

$$Rp_1 \dots p_k m_1 \dots m_l.$$

Here R is a $(k + l)$-ary relation, the *p*s are terms for physical objects (variables or individual constants) and the *m*s are terms for mathematical objects.[4]

Let us call an atomic formula *mixed* if both k and l are greater than zero. The Indispensability Argument then shows that physical theories *essentially* contain mixed atomic formulae: they have no alternative formulation that lacks them. This serves to refute the mathematical nominalist by showing that mathematics is ineliminable from physical theories, but it would also prevent us from making any significant distinction between the theoretical contributions of mathematical and physical objects. For in mixed atomic formulae, mathematical and physical objects are logically on a par.

Putnam might insist that there is still an important difference: mathematical theories can be stated independently of any physical claims, but physical theories cannot be stated independently of any mathematical claims. But this would presuppose that logical constructs of mixed atomic formulae count as physical rather than mathematical, and there is nothing in the logical structure of mathematics and physics that licenses this claim. The situation is perfectly symmetrical, and we could equally well count such claims as part of mathematics. We could of course still make sense of application in terms of the possession of a joint subject matter, but that would mean that physics applies mathematics as much as mathematics applies physics, and that doesn't seem right.

Mathematical nominalists usually respond to Putnam's argument by denying either one of his two premises. For example, Field (1980) denies the assumption A2 that there are only finitely many physical objects. He regards spacetime regions – of which there are infinitely many – as physical objects, and argues that we can use them to give a nominalist reconstruction of physical theories. Hellman (1989) and Chihara (1990) make use of the second strategy, and develop nominalist accounts of science that rely on rejecting Putnam's first-orderisability requirement A1.[5]

Like Hellman and Chihara, I want to reject A1, but I will do so for different reasons. In this paper, I will present three arguments against the first-orderisability of scientific theories that are independent of the aversion to abstracta that motivates the nominalist. The view of applied mathematics that I will end up defending is explicitly committed to the existence of mathematical objects. So while the nominalist introduces richer logical resources with the aim of *eliminating* mathematical objects, my project is to employ them to give a better account of the *contribution* mathematics makes to physical theory.

3. MATHEMATICS AND INFINITY

Suppose physics were finite. That is, suppose there were only finitely many physical objects, finitely many spacetime points, and finitely many physical properties. In this case, the physical history of the world could be completely described by a *state description*: a long, but finite, conjunction that lists what location and properties every object possesses at any given time. We could then easily give a mathematics-free first-order characterisation of any physical theory by forming the finite disjunction of all the state descriptions that it counts as nomically possible. In this way, mathematics could be eliminated from any finite physical theory.[6]

This does not refute Putnam's argument, because our current physics tells us that we are not in the finite case. There are infinitely many properties of spatial location, infinitely many properties of mass, infinitely many properties of charge, and so on. And in the case of Newton's theory of gravitation, it is precisely the mathematics needed to deal with the infinitely many possible masses and distances of the two particles that resists first-order elimination. Since physics is infinite, a state description would have to be an infinitely long conjunction, and first-order languages do not allow this.

However, this indispensability is clearly a fact about the conceptual resources of first-order languages, not about physics. If we were gods and spoke infinite languages then we could do without mathematics even in the infinite case, by forming the infinite disjunction of the appropriate infinite state descriptions. This would permit us to get by without ever mentioning mathematical objects, but our physical theories would still have to talk about electrons and quarks, for otherwise they would not be *physical* theories.

Mathematics plays such a crucial role in physics because there is a mismatch between the complexity of physical phenomena and our expressive resources (as tentatively characterised by formal first-order languages). Without the expressive power of mathematics, we cannot deal with the infinite systems that physics is concerned with. This might establish that mathematics is indispensable in our scientific theories, but it surely does not show that the theoretical contributions of mathematical and physical objects are the same. On the contrary: if the main purpose of applied mathematics is to boost our expressive capacities, then that is an important respect in which the theoretical role of mathematical objects differs from that of physical ones.

Of course, there are also some similarities between mathematical and physical objects. Both serve to simplify our theories. Theories that postulate physical objects are simpler than purely phenomenological ones, and finite physical theories that apply mathematics are often simpler than those that do not. But we do not want to have an account of applied mathematics that only focuses on these similarities. We also want to account for the differences, and Putnam denies us the logical space to do so.

That no first-order formulation of a scientific theory is able to adequately represent their different theoretical contributions does not change the fact that mathematical and physical objects enter scientific theory for different reasons. All it shows is that the Indispensability Argument cuts the other way. Rather than establishing that there is no difference between the theoretical roles of mathematical and physical objects, it merely demonstrates that Putnam's first-order framework lacks the expressive resources to account for the ideology-boosting role that mathematical objects play in physical theory.[7]

4. MASS PROPERTIES

Let us take another look at the mass attributions discussed earlier. In (1), we spelled out a's having mass n kilogram in terms of its bearing the relation M to the number n. Independently of the considerations presented in the previous section, this seems implausible: its mass is surely an *intrinsic* property of a physical object, not a relation it bears to a number.

To see how richer logical resources permit a better account, let us start by adding property quantifiers to our language. We could then say that an object a has mass n kilogram just in case it has a property X such that anything with X bears M to n:

$\exists X[\forall z(Xz \supset Mzn) \wedge Xa]$

This correctly attributes a non-relational property to a, but still has the disadvantage that many properties that are not masses satisfy the open sentence in the scope of the existential quantifier. For example, if only objects with electron charge had electron mass then any object with electron charge would bear M to whatever the number of electron mass is. So there is no guarantee that the property attributed to a is a mass (rather than a charge).

Some of these problems can be overcome by using a counterfactual conditional '$\Box\!\!\rightarrow$' instead of a material one:

$\exists X[\forall z(Xz \Box\!\!\rightarrow Mzn) \wedge Xa]$

Even if all objects with electron charge had electron mass, it would not be true of all objects that they would have electron mass if they had electron charge. This is an improvement, but this proposal still does not do well with properties that logically entail the possession of a given mass. The property of being red *and* having electron mass is not a mass property even though any object that has it is guaranteed to have electron mass.

To exclude such "conjunctive" properties, we need to work a little harder. First, let us define a partial ordering on properties via

$X \leq Y \equiv_{\mathrm{def}} \Box\forall z(Xz \supset Yz).$

Then the *maximal* property X that satisfies an open sentence $\Phi[X]$ is

$\mu X\,\Phi[X] =_{\mathrm{def}} \imath Y \forall Z(\Phi[Z] \supset Z \leq Y),$

where '$\imath$' is a definite property description operator. Given these definitions, we could then identify mass n kilogram with the maximal property X such that, if anything were X, it would bear M to n:

(2) $[\mu X \forall z(Xz \Box\!\!\rightarrow Mzn)]a.$

The property attributed to a is now guaranteed to be a mass property, and (2) has the added advantage of being of predicate-object form. The expression within the square brackets picks out a non-relational physical property of mass that then gets attributed to the physical object a.

Apart from accounting for mass properties, the regimentation (2) also tells us something about the mass-in-kilogram relation. It grants that there is a relation such as M, but insists that it is internal. If we hold all mathematical properties fixed then whether or not an object a bears M to n only depends on what non-relational mass property a possesses.

Giving a precise account of what it is for a relation to be internal (or a property to be intrinsic) is a notoriously difficult question, and nothing said here amounts to a solution to this problem.[8] But we do not actually need a general account of intrinsicness. In the special case of M, we can offer

$$(3) \qquad Mxy \equiv [\mu X \forall z (Xz \mathrel{\Box\!\!\rightarrow} Mzy)]x$$

as a formalisation of the claim that Mxy holds in virtue of x's mass, and this turns out to capture *enough* of what it is for M to be internal to serve our current purposes.

The claim (3) permits us to convert a mixed atomic formula (the 'Mxy' on the left-hand side) into an expression that merely attributes a physical property to a physical object. This feature will become important in the next section, when we have to face the task of distinguishing the physical content of a theory from its mathematical form. Anticipating the role they will play in that context, let me call statements of this type *separation postulates*.

My proposal is that what we have just said about mass ascriptions is true about applied mathematics in general. The relations that physical objects bear to mathematical objects are *always* internal. Whether or not such relations obtain only depends on the purely physical properties and relations of the physical relata, and on the purely mathematical properties and relations of the mathematical relata. To implement this proposal in the current framework, we just need to take whatever first-order characterisation of a theory Putnam would choose and supplement it with suitable separation postulates. For any mixed relation R we would add:

$$(4) \qquad Rp_1 \ldots p_k m_1 \ldots m_l \equiv [\mu Y \forall z_1 \ldots z_k (Yz_1 \ldots z_k \mathrel{\Box\!\!\rightarrow} Rz_1 \ldots z_k m_1 \ldots m_l)]p_1 \ldots p_k$$

As before, the ps are terms for physical objects, and the ms and zs are terms for mathematical ones.

The separation postulate (4) claims that, if we hold the mathematical properties of the ms fixed, the truth of '$Rp_1 \ldots p_k m_1 \ldots m_l$' only depends on whether the relation Y holds of the ps. It thus ensures that R is an internal relation between the *aggregate* of the ps and the *aggregate* of the ms. But R need not be internal as a relation between the ps taken separately, nor would we want it to be. For example, we might formalise "The distance between a and b is n metres" as '$Dabn$'. The corresponding separation postulate

$$Dabn \equiv [\mu Y \forall xy (Yxy \mathrel{\Box\!\!\rightarrow} Dxyn)]ab$$

would then claim that whether or not *Dabn* holds only depends on the mathematical properties of *n* (its position in the series of real numbers) and the physical properties of *a* and *b* taken together. But the relation thus attributed to *a* and *b* is their spatial distance, and that is a paradigm example of an *external* relation.

By using a logically more complex expression on the right-hand side of our separation postulates, we could analyse *Y* into more basic physical relations and properties, and thus specify what particular contribution each of the *p*s would need to make. These issues would have to be attended to in any comprehensive account of scientific theories, but the details of this would depend on the specific *R* under consideration. Let me therefore simplify matters by sticking to separation postulates of the simple form (4).

5. PHYSICAL CONTENT

So far, I have given two arguments against Putnam's first-orderisability thesis: the cardinality argument in Section 2 and the considerations concerning mass properties in Section 3. My third, and final, objection is that Putnam's view conflicts with scientific practice.[9]

The way physicists individuate their theories, two formulations of the same theory can apply different mathematics. For example, Newton's *Principia* employs geometric reasoning while Lagrange's *Mécanique analytique* uses the calculus of variations. But, in spite of their mathematical differences, physicists take them to express the *same* theory of classical mechanics. By contrast, scientists *never* identify theories that mention different physical objects but apply the same mathematics. Both the theory of radioactive decay and the theory of population growth use exponential functions, but nobody regards them as alternative formulations of the same theory.

This asymmetric way of treating mathematical and physical objects in theory individuation presupposes that they contribute to physical theory in significantly different ways (which Putnam denies). To vindicate scientific practice in this respect, we would need a way of distinguishing the *physical content* of a theory (what it says about physical objects) from the *mathematical form* in which it is conveyed. In that case, we could say that scientists identify theories with the same physical content. But it is far from obvious whether this form/content distinction can be drawn. It is natural to suppose that the distinction can only be made for theories that make *dispensable* use of mathematics. Two theories could then be said to have the same physical content if and only if they have the same mathematics-free paraphrase. But to extend this to the general case would require that applied mathematics is *always* eliminable, and that is precisely what the Indispensability Argument denies.

However, there is a different way of making the distinction that does not rely on paraphrastic eliminability. We can use the constructions we developed in the previous section to isolate the physical content of a theory even in cases where the mathematics applied is ineliminable. Call a sentence *purely physical* if it is a logical construct of atomic formulae that have no mathematical relata other than within the scope of property descriptions. Now say that two theories have the same physical content just in case they entail the same purely physical claims.[10]

The way Putnam regiments them, most scientific theories entail *no* purely physical claims and thus have zero physical content according to this definition. But our separation postulates (4) permit us to extract the contribution mixed atomic formulae make to a theory's physical content, and thus effect the desired separation from its mathematical form. Hence what makes the mathematical and the physical so inextricably intertwined in Putnam's case is merely his refusal to admit the logical resources needed to pry them apart.

Following Quine (1953a), say that a theory T is *ontologically committed* to an object x if and only if the theory logically entails x's existence. Say that a theory T is *physical* just in case it has non-trivial physical content. In these terms, we can then offer the following notion of application:[11]

Definition. A physical theory T *applies* an object x if and only if (i) T is ontologically committed to x, and (ii) there is a theory T' that is not committed to x's existence, but which has the same physical content as T.

It may well happen that a theory T applies mathematical objects without there being a theory T' with the same physical content that is not committed to any mathematical objects. All that is required by our definition of application is that T' be committed to mathematical objects *other than* the ones that T is committed to. To say that a scientific theory applies mathematics in the above sense does not entail that the theory could be formulated without reference to mathematical objects. We can have application without eliminability.

If a theory T applies x in this sense then the only function of T's reference to x is to help express its physical content. Our definition thus correctly assigns to mathematical objects the ideology-boosting role that we identified in Section 2. But all of this only works if there are indeed mathematical objects. The numbers and functions referred to by the separation postulates (4) need to exist for the expressions within the square brackets to pick out the relations that we want them to pick out. We identified a theory's physical content with only a proper part of its deductive closure. The rest of the theory, which makes ample reference to mathematical objects, is not superfluous. It plays an essential role in expressing its physical content. Hence this account of applied mathematics does nothing to advance the case of the mathematical nominalist.

6. CONCLUSION

The primary aim of the mathematical nominalist was to support his ontological views by showing that we could in principle eliminate all reference to mathematical objects from our physics. I not sure this part of his project can succeed, but I think the nominalist was right about one thing: that the primary role of applied mathematics is that of an ideology-boosting representational aid. But I think he underestimated the extent to which mathematics expands our expressive resources. In some cases, applying mathematics permits us to formulate propositions that we could not express otherwise, and is for that reason ineliminable.

In my view, Quine and Putnam get the ontology right – there really are mathematical objects – but they fail to give an acceptable account of what they are doing in physical theories. The account presented here tries to do better, by showing that we can separate a theory's physical content from its mathematical form without having to assume the feasibility of a nominalist elimination project. To spell out this view, we had to admit property quantifiers and modal operators, and we also had to accept the separation postulates. But that's fine: the richer logical resources are needed elsewhere in philosophy, anyway, and the separation postulates not only do the job, they are also true.

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NOTES

[1] Putnam (1971, Chapter 5). A similar point is made in Quine (1953b, 1981a), but Putnam gives what I regard as the more compelling version of the argument.
[2] We could also reveal the role of the unit kilogram by treating it as a further relatum, as I did in Meyer (2002). To do so might be helpful in explaining how our measurement practices succeed in determining the relation M. But since that is not a question I am interested in here, let me simplify matters by suppressing the contribution of the unit mass.
[3] This is the project in Goodman and Quine (1947).
[4] Resnik (1990) and others have claimed that in modern physics, and especially in quantum mechanics, no clear distinction between physical and mathematical objects can be made. I disagree, for reasons similar to those advanced in Peressini (1998).
[5] For a survey of mathematical nominalism, see Burgess and Rosen (1997).
[6] I do not mean to suggest that a mathematics-free formulation of finite physics would be *better* than one that uses mathematics. The latter kind of formulation is likely to be simpler and more elegant. The point of the present considerations is to get clearer about what makes mathematics indispensable in science, not what makes it merely convenient.
[7] Here I am using "ideology" in the sense of Quine (1951).
[8] For further details and references, see the symposium on defining "intrinsic" in *Philosophy and Phenomenological Research* 63.2 (2001).
[9] Maddy (1992) and Azzouni (1997) present related criticisms of the Indispensability Argument.
[10] Theories with the same physical content are, roughly, what Rosen (2001) calls "nominalistically equivalent". Rosen defines the notion, but does not offer an account of applied mathematics.
[11] For a rival "Fregean" account of application, see Steiner (1998).

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