What is Nominalistic Mereology?

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Abstract Hybrid languages are introduced in order to evaluate the strength of "minimal" mereologies with relatively strong frame definability properties. Appealing to a robust form of nominalism, I claim that one investigated language $\mathcal{H}_m$ is maximally acceptable for nominalistic mereology. In an extension $\mathcal{H}_{gem}$ of $\mathcal{H}_m$, a modal analog for the classical systems of Leonard and Goodman (J Symb Log 5:45–55, 1940) and Leśniewski (1916) is introduced and shown to be complete with respect to 0-deleted Boolean algebras. We characterize the formulas of first-order logic invariant for $\mathcal{H}_{gem}$-bisimulations.

Keywords Non-classical logic · Formal mereology · Formal ontology · Hybrid logic

1 Introduction

Consider the first-order (FO) language $\mathcal{L}$ whose non-logical symbols consist of just a single binary predicate $\leq$ and countably many constants. The quantifier-free *sentential* fragment $\mathcal{L}^-$ of $\mathcal{L}$ is a weak mereological language whose formulas are capable of describing finite part-to-whole structures with named elements. A baseline theory of the part relation therefore emerges in an exceedingly weak fragment of first-order logic (FOL). $\mathcal{L}^-$ is "minimal" in the sense that it does not contain open formulas defining further relations in terms of $\leq$.

Obvious reasons for selecting a FO mereologic will include considerations of frame definability and axiomatizability of infinite structures. However, as we will

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see, there are various fragments of $\mathcal{L}$ sufficiently strong to define the traditional mereological classes of frames. And it is therefore of interest to identify a minimal theory of parthood between $\mathcal{L}^-$ and $\mathcal{L}$ with sufficiently strong frame definability properties. Our goal will be to identify languages with a certain balance: those with the expressive power to define every traditional mereological class of frames but lacking the power to define relations in terms of $\leq$ not required to define those frame classes.

Our primary motivation for investigating these languages concerns issues in metaphysics and ontology. Over the last century, many rather influential philosophers have claimed or suggested that the ground structure of reality can be described in mereological terms (see e.g. [10, 11, 20, 22, 37]). Many of these authors, especially Leśniewski and Lewis, would even suggest that the logical conceptions we deploy to represent mereological patterns in the physical world need not imply the reality of abstract set-theoretic constructions.

Let us look back in time. The formal turn in mereology would have two primary causes. Firstly, many mathematicians and philosophers became frustrated with the imprecision of natural language as a medium for working. And secondly, intrigue in formalization was spurred by a desire to elucidate the contrasts between concrete structures and sets. Both were true of Stanisław Leśniewski. His obsession with rigor and his disavowal of the conception of distributive class espoused by Russell and Whitehead led him to formalize a highly sophisticated theory of parts. Ultimately, his goal was to provide a formal mereology with the power and scope to serve as a foundation for mathematics free of the antinomies of naïve set theory.

At least since the time of Leśniewski's seminal work, the theory of nominalism has maintained close theoretical ties with mereology. *Nominalism* is the view that (1) abstract (non-spatiotemporal) objects do not exist, and (2) spatiotemporal (so-called *concrete*) individuals exist and are the only existing things. The combined vision of nominalism and formal mereology will then be one of identifying a nominalistically acceptable theory of parthood. But how far-reaching should this vision be conceived?

Leśniewski espoused a nominalism of the most radical form. He rejected so-called *general objects* and platonic universals. His distaste for abstract objects was no doubt partly due to his aversion to set theory. But even in general in his writings he explicitly rejected anything non-concrete or non-particular. Fascinatingly, he thought of his own logical systems in this way. For Leśniewski, formal languages are aggregates of concrete "marks." Some of these are printed on paper in books and articles. Others are verbal utterances, chalk marks on blackboards, and pixelated patches on computer screens. In philosophy, this theory of formal languages is known as *inscriptionalism*. The radical nature of inscriptionalism can be seen by comparing it with the typical, modern conception of formal languages. If a logical system is a concrete assemblage of particular physical inscriptions, then it cannot be infinite. Formal languages must literally *grow* in various locations as new formulas are written. Commitments to infinite sets of formulae were to be relinquished by invoking the notion of equiformity (see [40, 41]). Two syntactical objects are equiform if they are graphically identical. And two sequences of marks are the same formula if they are equiform. Still, the description of formal languages as concrete entities closed under equiform structures will succeed only if the equiformity relation is concrete. And this appears *prima facie* implausible.

For by interpreting the equiformity as its set-theoretic extension or as a type of concrete "general object" or universal clearly violates Leśniewski's robust theory of nominalism. And this problem suggests that formal languages are best understood in the modern way as sets containing each grammatically well-formed object.

For Leśniewski, even mathematical objects are concrete. They are "extensional fusions" and not classes in the modern sense containing the null set. Interestingly, his systems contained various higher-order quantifiers and variables. Quine describes in *The Time of My Life* [36] that he and Leśniewski spent long evenings disputing whether the use of these items committed Leśniewski to the reality of abstract sets. Perhaps unsurprisingly, Quine thought that it did, and Leśniewski thought otherwise.

More modern conceptions of mereo-nominalism, especially in the field of philosophical ontology, are less far-reaching. Quine thought part-to-whole structures figured centrally in the ground conception of reality (see in particular [35]). Famously, he argued for a relaxed version of nominalism [34] nowadays called *class nominalism*: existent objects are either concrete, physical entities or *abstract sets*, no exceptions. He offered and indispensability argument for sets. Referents of terms for sets must be presumed in order to make sense of our best, regimented theories of physical science. It is puzzling that Quine rejected second and higher-order logics, since only these would have *captured* the expression of his set-theoretic commitments.

Nominalistic theories espousing the existence of sets are known as those of *class nominalism*. The theory amounts to a compromise: intensional entities are rejected in favor of sets. For any predicate '*P*', the property or relation denoted by '*P*' is the set of tuples the predicate defines. Like Quine, David Lewis also held a type of class nominalism. One important difference between their versions is that, for Lewis, sets containing only concrete members (so called *impure sets*) are concrete ([24, p. 83] and see also [25, p. 59]). This interpretation of impure sets has one obvious advantage for the nominalist. For, if impure sets are concrete, there will be no worry that quantification over them implies ontological commitments to abstract items. And thus without violating nominalism, highly complicated part-to-whole structures are captured by single sentences. But this boon hinges solely on the idea that impure sets are concrete. I am inclined to the view that all sets are abstract. Firstly, in research in fields of pure mathematics, (for example set theory, number theory, real analysis and so on) mathematicians clearly do not study located particular things. That they must would render the nature of their research either inaccurate or inexplicable. Secondly, given a traditional understanding of sets, even impure classes with concrete objects contain items—like the null set—which seem to defy an interpretation sympathetic to nominalism. Finally, the *number* of impure sets is far greater than the number of concrete particulars. Every theoretical division of a single particular concretum, say *x* (i.e. every *set* of concrete objects whose fusion is *x*), will be a unique object distinct from *x*. But according to any strict form nominalism, just the concrete fusion *x* exists, and the divisions of a single object are to be rejected. Only the concretum is clearly *located* in spacetime, and whatever divides it seems to change its nature entirely.

The intensional nature of properties and relations also poses problems for the class nominalist. Properties are, in general, intensional objects. For example, consider Quine's often cited example that the property of being a (well-formed) creature with

a kidney and the property of being a (well-formed) creature with a heart are distinct but nonetheless co-extensional. So the extensions of properties will be insufficient surrogates for the various features to which we make mention. Therefore the indispensability of intensional notions for describing the world will make adopting any form class nominalism problematic. And without admitting them into the ontology, there will be no way to trace the multitude of distinctions we make back to the world. Thus I am inclined that ontological commitments arise by the veracious ascription of features to objects. Discerning the accurate features for reality's description just is the determination of real structure. So like Armstrong [1, 2], Lewis [22], and many others, I view the need for predicates as entailing ontological commitments. And like Quine, I think the indispensability of higher-order quantifiers and variables implies the reality of sets. Ascribing properties and relations implies that the devices "carve at the joints of reality", to repeat a phrase from Plato. Making real logical or conceptual distinctions implies ontological distinctions.

Thus Quine's rejection of properties and relations is intriguing. Quine allows himself the full resources of FOL but gives predicates no representation in his ontology. In short, he holds that we can ascribe properties and relations without these denoting real features. And even the mereologist, with his sparsest of ontologies, has a problem. According to any robust form of nominalism, an object's features will be dispensable outright or in favor of impure sets. So part-to-whole instances, conceived as multiply repeated links, will denote no objects whatsoever. This is obviously unacceptable. One and the same part-to-whole relation links multiple pairs of objects in separate instances. So it is no wonder why formal mereologics devoid of abstract items have never been devised: they simply do not exist.

There is an important connection between nominalistic ontology and formal mereology concerning the issue of *capturing infinite structures*. To capture structures of arbitrary size requires the adopted formal language have strong enough expressive power. But if the use certain nominalistically unacceptable formal devices is both necessary and ontologically committing, this will pose a serious problem for the nominalist. Consider the well known example of the principle of unrestricted fusion: the property that any set of objects no matter how scattered or random combines to form an individual. For example, Quine, Lewis, Sider, and many others hold that any set of material objects unrestrictedly combine [24, 35, 37]. Interestingly, Quine's regimented language of FOL will fail to capture infinite unrestrictedly fused structures up to isomorphism. Consider the class of infinite complete Boolean algebras. Arguably, these structures represent the structural features of our most intimate representation of spatial deconstructions. But there does not exist a countably infinite complete Boolean algebra [17]. And by the downward Löwenheim Skolem theorem, any FO theory has a countable model. This little lesson is important and often overlooked in metaphysics: infinite sets must be countenanced before they can be fused. Thus if we wish to capture completed infinite structures, we must have the facility of a higher order logic.

In metaphysics there has always been the goal of identifying more fundamental properties and relations. More explanatorily powerful concepts are preferred. Hodgepodge relations derivable from more powerful ones are rejected. Nevertheless, only what is necessary to capture the envisaged structures will be adopted. The entire

point of selecting the most explanatorily powerful parameters in formal theories is to achieve a certain ontological parsimony. Goodman and Quine's purpose in [12] is a case in point. And Lewis' notion of natural properties in [22] is also an important example. And recently, Sider's *Four dimensionalism* [37] and *Writing the Book of the World* [39] represent attempts to identify fundamental structure.

Robust nominalistic aspirations aside, an interesting theoretical question therefore presents itself: what would a *maximally nominalistically acceptable* formal language and logic for mereology be? What is "the best we can do"? That is, if, in the strictest sense, the project fails as I have claimed, how far can we minimize commitments? I suggest a type of *logical incrementalism*. The approach is based on the familiar method of "guarding" FO quantifications in modal logic. Logical resources are selected piecemeal by restricting patterns of quantification. All formulas will be built up by nesting particular types of quantified formulas closed under the Boolean operations.

2 Axiom Systems and Starting Points

Interest in minimal mereologies is motivated by the arithmetical properties FO mereologics. Relevant FO theories include those of various partial orders. On the one hand, we have traditional theories of objects containing types of non-complementary remainders (see for example [47, Section 3.1]). On the other, we have models of *General Extensional Mereology* which Tarski showed [43] are essentially Boolean algebras (BAs). In a seminal result, Tarski demonstrated that the FO theory of BAs is decidable [43]. For the classification of BAs up to elementary equivalence, Tarski [44] (see also [9], Chapter 2 in [13] and Chapter 7 in [28]) provided the structural criteria of elementarily equivalent BAs in terms of algebraic invariants as well as axiomatizations for each class. The determination of the simplest forms of such axiom systems in the sense of syntactic complexity is given by Wasziewicz [48]. And finally, Kozen [18] showed that the elementary theory of BAs is $\leq$-log-complete for the Berman complexity class $\bigcup_{c<\omega} STA(*, 2^{cn}, n)$, the class of sets accepted by alternating Turing machines running in time $2cn$ for some constant c and making at most n alternations on inputs of length n. The theory is therefore computationally equivalent to the FO theory of real addition with order. However in the context of ideological parsimony and for many practical applications like research over finite models, the strong expressive properties of FO theories will be superfluous. And from the nominalistic standpoint we should make a clean break between, on the one hand, pure arithmetical expressivity and reasoning and, on the other, reasoning about concrete individuals and their parts.

Modal frameworks present themselves as alternative starting points. Currently, all investigated modal mereologies are couched in modal languages with propositional variables $\{p, q, r...\}$. Each proposition symbol represents either a subset of an implied domain of items or a multiply instantiatable simple property capable of being true at various "worlds" or "states". Let us consider some examples that surpass our intended borders. In Vakarelov [45], "A modal logic of set relations" the author investigates a modal logic whose atomic formulae range over subsets of the domain. Goranko and

Vakarelov [14] also introduce a language with a set-theoretic semantics interpreted over powerset algebras whose modalities represent membership relations implicit in the Boolean set-operations. Balbiani et al. [4] provide a modal logic based on membership modalities with a topological interpretation. Other approaches extend $\mathcal{L}$ with relations that are not FO-definable in terms of $\leq$. For instance, in the spirit of Whitehead's [49] original motivations, Nenov and Vakarelov [29] introduce a modal mereotopology in a language with both parthood and contact modalities. Similarly in Kontchakov et al. [16] the authors investigate spatial constraint languages with equality, contact and connectedness predicates, as well as Boolean operations on regions, interpreted over low-dimensional Euclidean spaces. The investigations of formal mereology, however, begin from the standpoint of constraining our list of primitives. And this will imply that the baseline system contain no background topological and set-theoretic notions.

Stone's representation theorem implies that any logic sound and complete with respect to arbitrary BAs will also be so with respect to a set algebra. So in the context of characterization, many extensional mereological structures are not mathematically divorced from set-theoretic constructions. However, given the standard semantics of modal languages, proposition symbols correspond to either sets or properties of states. Thus the modal mereologics currently investigated will not have Kripke models whose terms denote only individual elements.

Any mereology must contain devices referring to traditional mereological properties and relations. Consider the property of being an *atom*. The notion of an indivisible object has figured centrally in metaphysics and mereology since the time of Democritus. Indeed to this day, there are prominent philosophers who still hold some version of strong atomism (see e.g. [46]) which is the view that all (or, for van Inwagen, nearly all) objects are atoms. In general, issues concerning the atomicity of physical objects and space are some of the most important in present philosophical debates (see e.g. [15, 26, 27, 38, 50]). The distinction between atomic and atomless objects is also important in both the fields of formal mereology and BAs. Much of the axiomatic variability and computational complexity of various lattices and BAs boils down to statements expressing the addition and relative subtraction of *atomic* objects. And Tarski's description of elementary invariants in his proof of the decidability of BAs centrally concerns the status of atomic properties of BAs.

Our baseline mereology will be couched in a hybrid modal language $\mathcal{H}_m$. $\mathcal{H}_m$ is an improper extension of Arthur Prior's nominal tense language [30–33]. $\mathcal{H}_m$ contains no propositional symbols but instead:

(a)   atomic symbols called *nominals* which name elements of the domain,
(b)   operators for part $[\geq]$ and extension $[\leq]$ and their inverses $[\bar{\geq}]$ and $[\bar{\leq}]$, and
(c)   an atom constant α expressing that the present state is an indivisible object.

Thus in hybrid logics we make reference to objects almost solely by nominal designation. And it is in virtue of this feature that we eliminate pure arithmetical expressions.

In $\mathcal{H}_m$ one can define the existence modality and capture finite named structures in the relevant signatures up to isomorphism. And modal operators for the standard mereological relationships are definable over nominals. $\mathcal{H}_m$-formulas correspond to

formulas in FO logic with maximally one free variable and represent complex properties containing named individuals related part-to-whole. One of the most interesting features of the languages we study is that each has the ability to represent indexical, egocentric facts. These form a basis for an egocentric mereologic in the sense of Prior's 1968 "Egocentric Logic" [31]. $\mathcal{H}_\mathsf{m}$ also has rather strong frame definability properties. It has the expressive power to define the entire range of classes of supplementary partial orders up to that of extensional mereological structures (cf. [47, Section 3.1]). As for axiomatics, in the principal result of the paper we show that there is a $\mathcal{H}_\mathsf{m}$-axiomatization of the class of 0-deleted BAs. Moreover, the selection of $\mathcal{H}_\mathsf{m}$ is well motivated. For we will demonstrate that in various fragments of $\mathcal{H}_\mathsf{m}$, nominal mereological operators are inexpressible. But one deficiency $\mathcal{H}_\mathsf{m}$ shares with FOL is that it cannot define the class of unrestrictedly fused structures or complete BAs. In contrast, in $\mathcal{H}_\mathsf{o}$—the extension of $\mathcal{H}_\mathsf{m}$ with proposition symbols—there are single formulas defining these classes. We will show that many algebraic notions are expressible in $\mathcal{H}_\mathsf{o}$. Nonetheless, $\mathcal{H}_\mathsf{o}$ has many expressive limitations. For example, there are not arbitrary mereological operators for the traditional mereological relationships of disjointness, proper part, proper extension, complement, and so on. Thus we also consider a master extension $\mathcal{H}_\mathsf{gem}$ of $\mathcal{H}_\mathsf{o}$ with these operators. In $\mathcal{H}_\mathsf{gem}$ alternative notions of extensional fusion are expressible and a hybrid axiom system corresponding to *General Extensional Mereology* in [21] is obtainable. Finally, we characterize the fragment of FOL of our selected languages.

3 Strong Hybrid Languages for Mereology

Let $\Phi = \{p, q, r, ...\}$ be a countably infinite set of atomic formulas whose members are called *propositional variables*. Let $\Omega = \{\mathbf{1}, \mathbf{0}, i, j, k, ...\}$ be a countably infinite set of atomic formulas whose members are called *nominals*. We call α the *atom constant*. We first define our master hybrid mereological language $\mathcal{H}_\mathsf{gem}$. $\mathcal{H}_\mathsf{gem}$ is defined recursively:

$$\phi := \top \mid i \mid p \mid \alpha \mid \neg\phi \mid \phi \wedge \psi \mid$$
$$[\leq]\phi \mid [\geq]\phi \mid [\overline{\leq}]\phi \mid [\overline{\geq}]\phi \mid [<]\phi \mid [>]\phi \mid [\sim]\phi \mid [\overline{\mathsf{O}}]\phi$$

The fragment $\mathcal{H}_\mathsf{m}$ of $\mathcal{H}_\mathsf{gem}$ is the recursively defined language whose operators are $[\leq], [\geq], [\overline{\leq}], [\overline{\geq}]$, and whose atomic symbols are just the nominals, $\top$, and α. If a formula contains no proposition symbols, we call it *pure*. Thus each $\mathcal{H}_\mathsf{m}$-formula is pure. $\mathcal{H}_\mathsf{o}$ is the extension of $\mathcal{H}_\mathsf{m}$ with proposition symbols. The language obtained by extending $\mathcal{H}_\mathsf{gem}$ with the second-order (SO) propositional quantifier $\forall p.\phi$ we call *extended SO propositional hybrid logic* ($ESOPHL$). Our hybrid languages are interpreted on models. Henceforth an *extended hybrid frame* (*frame* for short) is a tuple $(W, \leq, \overline{c})$ where W is a non-empty domain of elements, $\leq$ is a binary relation on W, and $\overline{c}$ is a sequence of distinguished elements of W. V is a *hybrid valuation* if it is a function with domain $\Omega \cup \Phi$ such that $\forall i \in \Omega$, $V(i)$ is a singleton subset of W, and $\forall p \in \Phi$, $V(p)$ is a subset of W. V is a *pure hybrid valuation* if it is a function with domain Ω such that $\forall i \in \Omega$, $V(i)$ is a singleton subset of W. Let

$\mathcal{M} = (W, \leq, \overline{c}, V)$ and $(W, \leq, \overline{c})$ be a frame. If V is a pure hybrid valuation we call $\mathcal{M}$ an *pure extended hybrid model* (or *pure model*), and if V is not pure, we call $\mathcal{M}$ an *extended hybrid model* (or *model* for short).

Definition 1 (Models of Mereological Type) A *m-frame* is a frame $(W, \leq, 1)$ where $1 \in W$ is the only distinguished element called *the top*. A *m-model* is a tuple $\mathcal{M} = (\mathcal{F}, V)$ such that $\mathcal{F}$ is a m-frame, and V is a hybrid valuation. A *GEM-model* is a m-model $(\mathcal{F}, V)$ where $\mathcal{F}$ is a 0-deleted Boolean algebra (*0/BA*). A hybrid model $\mathcal{M} = (\mathcal{F}, V)$ is *of Boolean type* if $\mathcal{F}$ is a frame $(W, \leq, 1, 0)$ such that 1 and 0 are the only distinguished elements, and V is a hybrid valuation. A *BA-model* is a 5-tuple $(W, \leq, 1, 0, V)$ where $(W, \leq, 1, 0)$ is a BA and V is a hybrid valuation.

Interpret '$x \leq y$' as 'x is a part of y' and '$x < y$' as '$x \leq y \wedge x \neq y$'. In mereology, the overlap relation $x\mathbf{O}y$ is definable by $\exists z(z \leq x \wedge z \leq y)$, disjointness $x\mathbf{D}y$ by $\neg x\mathbf{O}y$, and the complement relation $\sim w = v$ by $\forall u \in W(u \leq w \leftrightarrow u\mathbf{D}v)$. w *is an atom* (notation: $At(w)$) if and only if $\forall v(v \leq w \rightarrow v = w)$.

Definition 2 (Truth) Let $\mathcal{M} = (W, \leq, \overline{c}, V)$ be a model and $w \in W$. Then

$$
\begin{aligned}
&(W, \leq, 1, V), w \models \top &&\Longleftrightarrow w = w \\
&(W, \leq, 1, V), w \models i &&\Longleftrightarrow \{w\} = V(i) \\
&(W, \leq, 1, V), w \models \mathbf{c} &&\Longleftrightarrow w = c \\
&(W, \leq, 1, V), w \models \alpha &&\Longleftrightarrow \forall v(v \leq w \rightarrow v = w) \\
&(W, \leq, 1, V), w \models \neg\varphi &&\Longleftrightarrow (W, \leq, 1, V), w \not\models \varphi \\
&(W, \leq, 1, V), w \models \varphi \wedge \psi &&\Longleftrightarrow (W, \leq, 1, V), w \models \varphi \wedge (W, \leq, 1, V), w \models \psi \\
&(W, \leq, 1, V), w \models [\leq]\varphi &&\Longleftrightarrow \forall v \in W\ (w \leq v \Longrightarrow (W, \leq, 1, V), v \models \varphi) \\
&(W, \leq, 1, V), w \models [\geq]\varphi &&\Longleftrightarrow \forall v \in W\ (v \leq w \Longrightarrow (W, \leq, 1, V), v \models \varphi) \\
&(W, \leq, 1, V), w \models [<]\varphi &&\Longleftrightarrow \forall v \in W\ (w < v \Longrightarrow (W, \leq, 1, V), v \models \varphi) \\
&(W, \leq, 1, V), w \models [>]\varphi &&\Longleftrightarrow \forall v \in W\ (v < w \Longrightarrow (W, \leq, 1, V), v \models \varphi) \\
&(W, \leq, 1, V), w \models [\sim]\varphi &&\Longleftrightarrow \forall v \in W\ (v =\sim w \Longrightarrow (W, \leq, 1, V), v \models \varphi) \\
&(W, \leq, 1, V), w \models [\overline{\leq}]\varphi &&\Longleftrightarrow \forall v \in W\ ((W, \leq, 1, V), v \models \varphi \Longrightarrow w \leq v) \\
&(W, \leq, 1, V), w \models [\overline{\geq}]\varphi &&\Longleftrightarrow \forall v \in W\ ((W, \leq, 1, V), v \models \varphi \Longrightarrow v \leq w) \\
&(W, \leq, 1, V), w \models [\overline{\mathbf{O}}]\varphi &&\Longleftrightarrow \forall v \in W\ ((W, \leq, 1, V), v \models \varphi \Longrightarrow w\mathbf{O}v) \\
&(W, \leq, 1, V), w \models \forall p.\phi &&\Longleftrightarrow \forall S \subseteq W\ (W, \leq, 1, V)V[p \longmapsto S], w \models \phi
\end{aligned}
$$

where $V[p \longmapsto S]$ is obtained by changing the valuation V such that $V(p) = S$.

For any $i \in \Omega$, if $\{w\} = V(i)$ we say w *is the denotation of i*. Note that under the interpretation, α is true at $w \in \mathcal{M}$ if w has no proper parts. Observe that $[\overline{\leq}]$, $[\overline{\geq}]$ and $[\overline{\mathbf{O}}]$ are obvious inverses of their respective relations. In addition to the boxes $[\leq], [\geq], [<], [>], [\sim], [\overline{\leq}], [\overline{\geq}], [\overline{\mathbf{O}}]$ we will also make heavy use of their diamond duals $\langle\leq\rangle, \langle\geq\rangle, \langle<\rangle, \langle>\rangle, \langle\sim\rangle, \langle\overline{\leq}\rangle, \langle\overline{\geq}\rangle, \langle\overline{\mathbf{O}}\rangle$. Each of the latter is an existential operator. For example, $\mathcal{M}, w \models \langle\overline{\geq}\rangle\phi \Longleftrightarrow \exists v \in \mathcal{M}\ v \not\leq w \wedge \mathcal{M}, v \models \neg\phi$. Clearly any $\mathcal{H}_{\mathsf{gem}}$-formula is equivalent to a FO formula with maximally one free variable in a language whose vocabulary contains a countably infinite set of unary predicates and constants. And it is also obvious that each ESOPHL-formula is equivalent to a SO formula with maximally one free variable.

Definition 3 (Translation) The *standard translation* of ESOPHL-formulas is given in the following, where $i \in \Omega$ and $p \in \Phi$:

$$
\begin{aligned}
ST_x(\top) &= x = x \\
ST_x(i) &= i = x \\
ST_x(p) &= Px \\
ST_x(\alpha) &= \forall y (y \leq x \rightarrow y = x) \\
ST_x(\neg\varphi) &= \neg ST_x(\varphi) \\
ST_x(\varphi \wedge \psi) &= ST_x(\varphi) \wedge ST_x(\psi) \\
ST_x([\leq]\varphi) &= \forall y (x \leq y \rightarrow ST_y(\varphi)) \\
ST_x([\geq]\varphi) &= \forall y (y \leq x \rightarrow ST_y(\varphi)) \\
ST_x([<]\varphi) &= \forall y (x < y \rightarrow ST_y(\varphi)) \\
ST_x([>]\varphi) &= \forall y (y < x \rightarrow ST_y(\varphi)) \\
ST_x([\overline{\leq}]\varphi) &= \forall y (ST_y(\varphi) \rightarrow y \leq x) \\
ST_x([\overline{\geq}]\varphi) &= \forall y (ST_y(\varphi) \rightarrow x \leq y) \\
ST_x([\overline{O}]\varphi) &= \forall y (ST_y(\varphi) \rightarrow x O y) \\
ST_x([\sim]\varphi) &= \forall y (y =\sim x \rightarrow ST_y(\varphi)) \\
ST_x(\forall p.\varphi) &= \forall P (ST_x(\varphi))
\end{aligned}
$$

where y is a variable that has not been used so far in the translation.

3.1 Expressivity

Logical Expressions in $\mathcal{H}_{\mathsf{m}}$ Consider the semantics of the existential operator E of extended modal languages, its dual, and the well-known @-operator of hybrid logic:

$$
\begin{aligned}
\mathcal{M}, w &\models \mathsf{E}\phi \iff \exists v \in |\mathcal{M}| \; \mathcal{M}, v \models \phi \\
\mathcal{M}, w &\models \mathsf{A}\phi \iff \forall v \in |\mathcal{M}| \; \mathcal{M}, v \models \phi \\
\mathcal{M}, w &\models @_i\phi \iff \mathcal{M}, v \models \phi \wedge V(i) = \{v\}
\end{aligned}
$$

Proposition 1 ($\mathsf{E}\phi$ and $@_i\phi$) *The existence and @-operator are definable in $\mathcal{H}_{\mathsf{m}}$.*

Proof The $\mathcal{H}_{\mathsf{m}}$-formula $\langle\leq\rangle\phi \vee \langle\overline{\leq}\rangle\neg\phi$ defines $\mathsf{E}\phi$. And $\mathsf{E}(i \wedge \phi) \leftrightarrow @_i\phi$. $\qquad\square$

Mereological expressions and operators We will now see that $\mathcal{H}_{\mathsf{gem}}$ is expressively optimal as a modal mereological language. In the base framework, we have part, extension, proper part, proper extension, and complement. And we will show that the other important concepts are definable from these. However $\mathcal{H}_{\mathsf{gem}}$ is too strong to serve as a "nominalistic" mereological language. We will show that in $\mathcal{H}_{\mathsf{m}}$ there are operators relating the "present location" to fusions and products of nominals. And therefore $\mathcal{H}_{\mathsf{m}}$ will emerge as as an optimal nominalistic alternative to $\mathcal{H}_{\mathsf{gem}}$. It approximates the expressive power of $\mathcal{H}_{\mathsf{gem}}$ with pure operators of the form $[R](i, ...,)$ where R is a mereological relation and $i_1...i_n$ are nominals. Consider the FO definable relations in Table 1. The Tarski fusion relation corresponds to a notion of fusion found in [42] and [23]. It is an extensional formulation: x and y have an upper bound all parts of which overlap either x or y. If an extensional formulation is not desired, a relaxed version is GEM fusion: there is something that overlaps

exactly those things that overlap either x or y. GEM fusion is the formulation that best reflects the notion of fusion found in standard treatments of mereology like in [21] and [7].

Consider then, the following operators which, in addition to those we have as primitives, should be required to round out our list of mereological operators. In particular, note that we have nominal binary formulations of the various fusion relations found in Table 1 as well as generalized operators which are true at the fusion of objects meeting a certain condition ϕ.

$$
\begin{aligned}
\mathcal{M}, w \models \langle O \rangle \phi & \Longleftrightarrow \exists v \in \mathcal{M} \ w O v \wedge \mathcal{M}, v \models \phi \\
\mathcal{M}, w \models \langle D \rangle \phi & \Longleftrightarrow \exists v \in \mathcal{M} \ w D v \wedge \mathcal{M}, w \models \phi \\
\mathcal{M}, w \models \langle + \rangle (i, j) & \Longleftrightarrow \exists u, v \in \mathcal{M} \ u + v = w \ \wedge \mathcal{M}, \\
& \quad w \models \phi \wedge V(i) = \{u\} \wedge V(j) = \{v\} \\
\mathcal{M}, w \models \langle +_a \rangle (i, j) & \Longleftrightarrow \exists u, v \in \mathcal{M} \ u +_a v = w \wedge \mathcal{M}, \\
& \quad w \models \phi \wedge V(i) = \{u\} \wedge V(j) = \{v\} \\
\mathcal{M}, w \models \langle +_b \rangle (i, j) & \Longleftrightarrow \exists u, v \in \mathcal{M} \ u +_b v = w \ \wedge \mathcal{M}, \\
& \quad w \models \phi \wedge V(i) = \{u\} \wedge V(j) = \{v\} \\
\mathcal{M}, w \models \langle \times \rangle (i, j) & \Longleftrightarrow \exists u, v \in \mathcal{M} \ u \times v = w \ \wedge \mathcal{M}, \\
& \quad w \models \phi \wedge V(i) = \{u\} \wedge V(j) = \{v\} \\
\mathcal{M}, w \models \langle \bigoplus \rangle \phi & \Longleftrightarrow \bigoplus \phi = w \wedge \mathcal{M}, w \models \phi \\
\mathcal{M}, w \models \langle \bigoplus_a \rangle \phi & \Longleftrightarrow \bigoplus_a \phi = w \wedge \mathcal{M}, w \models \phi \\
\mathcal{M}, w \models \langle \bigoplus_b \rangle \phi & \Longleftrightarrow \bigoplus_b \phi = w \wedge \mathcal{M}, w \models \phi \\
\mathcal{M}, w \models \langle \bigotimes \rangle \phi & \Longleftrightarrow \bigotimes \phi = w \wedge \mathcal{M}, w \models \phi
\end{aligned}
$$

Each of these is definable in $\mathcal{H}_{\mathsf{gem}}$ and several of them are expressible in $\mathcal{H}_{\mathsf{m}}$.

Proposition 2 (Mereological Operators) *Let $i, j \in \Omega$ and $\phi \in \mathcal{H}_{\mathsf{gem}}$. The operator expressions $\langle O \rangle \phi$, $\langle D \rangle \phi$, $\langle + \rangle (i, j)$, $\langle +_a \rangle (i, j)$, $\langle +_b \rangle (i, j)$, $\langle \times \rangle (i, j)$, $\langle \bigoplus \rangle \phi$, $\langle \bigoplus_a \rangle \phi$, $\langle \bigoplus_b \rangle \phi$, and $\langle \bigotimes \rangle \phi$ are definable in $\mathcal{H}_{\mathsf{gem}}$. And if ϕ is a $\mathcal{H}_{\mathsf{m}}$-formula, $\langle O \rangle \phi$, $\langle D \rangle i$, $\langle + \rangle (i, j)$, $\langle +_a \rangle (i, j)$, $\langle \times \rangle (i, j)$, $\langle \bigoplus \rangle \phi$, $\langle \bigoplus_a \rangle \phi$, $\langle \bigotimes \rangle \phi$, and $[\sim] i$ are definable in $\mathcal{H}_{\mathsf{m}}$.*

Table 1 FO standard mereological sums and products

Relations	FO Expression	Abbreviation
Supremum	$\forall w (x \le w \wedge y \le w \leftrightarrow z \le w)$	$x + y = z$
Tarski fusion	$x \le z \wedge y \le z \wedge \forall w (w \le z \rightarrow w O x \vee w O y)$	$x +_a y = z$
GEM fusion	$\forall w (z O w \leftrightarrow (x O w \vee y O w))$	$x +_b y = z$
Product	$\forall w (w \le x \wedge w \le y \leftrightarrow w \le z)$	$x \times y = z$
General supremum	$\forall w (z \le w \leftrightarrow \exists v (v \le w \wedge \phi(v)))$	$\bigoplus \phi = z$
General fusion a	$\forall w ((\phi(w) \rightarrow w \le z) \wedge (w \le z \rightarrow \exists v (v O w \wedge \phi(v))))$	$\bigoplus_a \phi = z$
General fusion b	$\forall w (z O w \leftrightarrow \exists v (\phi(v) \wedge v O w))$	$\bigoplus_b \phi = z$
General product	$\forall w (w \le z \leftrightarrow \exists v (w \le v \wedge \phi(v)))$	$\bigotimes \phi = z$

Proof

$$\mathcal{M}, w \models \langle O \rangle \phi \iff \mathcal{M}, w \models \langle \geq \rangle \langle \leq \rangle \phi$$

$$\mathcal{M}, w \models \langle D \rangle i \iff \mathcal{M}, w \models \neg \langle O \rangle i$$

$$\mathcal{M}, w \models \langle D \rangle \phi \iff \mathcal{M}, w \models \neg [\overline{O}] \phi$$

$$\mathcal{M}, w \models \langle + \rangle (i, j) \iff \mathcal{M}, w \models \langle \geq \rangle i \wedge \langle \geq \rangle j \wedge [\overline{\leq}](\langle \geq \rangle i \wedge \langle \geq \rangle j)$$

$$\mathcal{M}, w \models \langle +_a \rangle (i, j) \iff \mathcal{M}, w \models \langle \geq \rangle i \wedge \langle \geq \rangle j \wedge [\geq](\langle O \rangle i \vee \langle O \rangle j)$$

$$\mathcal{M}, w \models \langle +_b \rangle (i, j) \iff \mathcal{M}, w \models \langle O \rangle i \wedge \langle O \rangle j \wedge [\overline{O}](\langle O \rangle i \wedge \langle O \rangle j)$$

$$\mathcal{M}, w \models \langle \times \rangle (i, j) \iff \mathcal{M}, w \models \langle \leq \rangle i \wedge \langle \leq \rangle j \wedge [\overline{\geq}](\langle \leq \rangle i \wedge \langle \leq \rangle j)$$

$$\mathcal{M}, w \models \langle \oplus \rangle \phi \iff \mathcal{M}, w \models [\overline{\geq}]\langle \leq \rangle \phi \wedge [\geq]\langle \leq \rangle \phi$$

$$\mathcal{M}, w \models \langle \oplus_a \rangle \phi \iff \mathcal{M}, w \models [\overline{\geq}]\phi \wedge [\geq]\langle O \rangle \phi$$

$$\mathcal{M}, w \models \langle \oplus_b \rangle \phi \iff \mathcal{M}, w \models [\geq][\leq]\langle O \rangle \phi \wedge [\overline{O}]\langle O \rangle \phi$$

$$\mathcal{M}, w \models \langle \otimes \rangle \phi \iff \mathcal{M}, w \models [\overline{\leq}]\langle \geq \rangle \phi \wedge [\leq]\langle \geq \rangle \phi$$

$$\mathcal{M}, w \models [\sim] i \iff \mathcal{M}, w \models [\geq]\langle D \rangle i \wedge [\overline{\geq}]\langle D \rangle i$$

$\square$

3.2 Boolean Algebraic Notions, Filters, and Object-to-Property Relations

In order to match the expressivity over BAs, we must extend $\mathcal{H}_m$, $\mathcal{H}_o$, and $\mathcal{H}_{gem}$ with expressions which allow for the presence of the bottom 0. We will define three analogous languages $\mathcal{H}_{ba}$, $\mathcal{H}_{bo}$, and $\mathcal{H}_{bem}$ suitable for reasoning over BAs. Let $(W, \leq, 1, 0, V)$ be a BA-model. Firstly, to the vocabularies of each new language we add the nominal **0** to denote the bottom whose satisfaction condition is given by

$$(W, \leq, 1, 0, V), w \models \mathbf{0} \iff w = 0.$$

Operators in $\mathcal{H}_{ba}$ and $\mathcal{H}_{bo}$ are those in $\mathcal{H}_m$ and $\mathcal{H}_o$, respectively, and will be given the same interpretation. The hybrid language $\mathcal{H}_{bem}$ will be an analog to $\mathcal{H}_{gem}$ and its operators will be given the same interpretation as those in $\mathcal{H}_{gem}$ with the exception of $[\sim]$ and $[\overline{O}]$. The mereological rendering of the overlap relation $\exists z(z \leq x \wedge z \leq y)$ will not capture the intended meaning, for over BAs, as every object dominates 0, we have $\forall x \forall y (x O y)$. Boolean algebraic overlap $x V y$ is defined by $\exists w(w \neq 0 \wedge w \leq x \wedge w \leq y)$. The inverse $[\overline{O}]$ operator will therefore be eliminated for the inverse operator $[\overline{V}]$ whose satisfaction condition is given by

$$(W, \leq, 1, 0, V), w \models [\overline{V}]\varphi \iff \forall v \in W \ (W, \leq, 1, 0, V), v \models \varphi \Rightarrow w V v.$$

In $\mathcal{H}_{bem}$, the operator $[\sim]$ will be exchanged for $[\complement]$ where

$$(W, \leq, 1, 0, V), w \models [\complement]\varphi \iff \forall v \in W \ (v = \complement w \Longrightarrow (W, \leq, 1, 0, V), v \models \varphi).$$

The $\mathcal{H}_m$ interpretation α is also inappropriate over BAs. Any element w of a BA is an *atom* if and only if $w \neq 0 \wedge \forall v((v \leq w \wedge v \neq 0) \rightarrow v = w)$. The semantics for α is then changed to $(W, \leq, 1, 0, V), w \models \alpha \iff w \neq 0 \wedge \forall v((v \leq w \wedge v \neq$

$0) \to v = w)$. It is natural (especially in the context of physical objects) that 0 not be construed as a part of objects. Over BAs, this motivates a meta-analysis of the parthood relation as a subset of the dominance relation: $\sqsubseteq = (\leq /\{(0, v) \mid 0 \leq v \text{ and } v \in W\})$ (Table 2). We therefore arrive at three new languages $\mathcal{H}_{ba}$, $\mathcal{H}_{bo}$, and $\mathcal{H}_{bem}$ which correspond to suitable Boolean analogs for $\mathcal{H}_m$, $\mathcal{H}_o$, and $\mathcal{H}_{gem}$, respectively. Consider the following notions analogous to the mereological relations of the last section.

$$
\begin{aligned}
\mathcal{M}, w &\models \langle\sqsubseteq\rangle\phi &&\Longleftrightarrow \exists v \in \mathcal{M}\ w \sqsubseteq v \wedge \mathcal{M}, v \models \phi \\
\mathcal{M}, w &\models \langle\sqsupseteq\rangle\phi &&\Longleftrightarrow \exists v \in \mathcal{M}\ v \sqsubseteq w \wedge \mathcal{M}, v \models \phi \\
\mathcal{M}, w &\models \langle\sqsubset\rangle\phi &&\Longleftrightarrow \exists v \in \mathcal{M}\ w \sqsubset v \wedge \mathcal{M}, v \models \phi \\
\mathcal{M}, w &\models \langle\sqsupset\rangle\phi &&\Longleftrightarrow \exists v \in \mathcal{M}\ v \sqsubset w \wedge \mathcal{M}, v \models \phi \\
\mathcal{M}, w &\models \langle\mathsf{V}\rangle\phi &&\Longleftrightarrow \exists v \in \mathcal{M}\ w\mathsf{V}v \wedge \mathcal{M}, v \models \phi \\
\mathcal{M}, w &\models \langle\mathsf{S}\rangle\phi &&\Longleftrightarrow \exists v \in \mathcal{M}\ \neg w\mathsf{V}v \wedge \mathcal{M}, w \models \phi \\
\mathcal{M}, w &\models \langle\sqcup\rangle(i, j) &&\Longleftrightarrow \exists u, v \in \mathcal{M}\ u \sqcup v = w \wedge \mathcal{M}, \\
& && \quad w \models \phi \wedge V(i) = \{u\} \wedge V(j) = \{v\} \\
\mathcal{M}, w &\models \langle\sqcap\rangle(i, j) &&\Longleftrightarrow \exists u, v \in \mathcal{M}\ u \sqcap v = w \wedge \mathcal{M}, \\
& && \quad w \models \phi \wedge V(i) = \{u\} \wedge V(j) = \{v\} \\
\mathcal{M}, w &\models \langle\textstyle\sum\rangle\phi &&\Longleftrightarrow \textstyle\sum \phi = w \wedge \mathcal{M}, w \models \phi \\
\mathcal{M}, w &\models \langle\textstyle\prod\rangle\phi &&\Longleftrightarrow \textstyle\prod \phi = w \wedge \mathcal{M}, w \models \phi
\end{aligned}
$$

Proposition 3 (Boolean Operators) *Let $i, j \in \Omega$ and $\phi \in \mathcal{H}_{ba}$. The operators $\langle\sqsubseteq\rangle\phi$, $\langle\sqsupseteq\rangle\phi$, $\langle\sqsubset\rangle\phi$, $\langle\sqsupset\rangle\phi$, $\langle\mathsf{V}\rangle\phi$, $\langle+\rangle(i, j)$, $\langle\times\rangle(i, j)$, $\langle\sum\rangle\phi$, and $\langle\prod\rangle\phi$ are*

Table 2 FO relations over BAs and standard FO Boolean operations

Relations	FO Expression	Abbreviation
Part	$x \leq y \wedge x \neq 0$	$x \sqsubseteq y$
Extension	$x \leq y \wedge x \neq 0 \wedge y \neq 0$	$x \sqsupseteq y$
Proper Part	$x \leq y \wedge x \neq y \wedge x \neq 0$	$x \sqsubset y$
Proper Extension	$x \geq y \wedge x \neq y \wedge x \neq 0 \wedge y \neq 0$	$x \sqsupset y$
Overlap	$\exists w(w \neq 0 \wedge w \leq x \wedge w \leq y)$	$x\mathsf{V}y$
Disjoint	$\neg x\mathsf{V}y$	$x\mathsf{S}y$
Supremum	$\forall w(x \leq w \wedge y \leq w \leftrightarrow z \leq w)$	$x \sqcup y = z$
Infimum	$\forall w(w \leq x \wedge w \leq y \leftrightarrow w \leq z)$	$x \sqcap y = z$
Complement	$\forall w((x \vee y = w \to w = 1) \wedge (x \wedge y = w \to w = 0))$	$x = \complement y$
G. Sum	$\forall w(z \leq w \leftrightarrow \exists v(v \leq w \wedge \phi(v)))$	$\sum \phi = z$
G. Product	$\forall w(w \leq z \leftrightarrow \exists v(w \leq v \wedge \phi(v)))$	$\prod \phi = z$

definable in $\mathcal{H}_{\mathsf{ba}}$ *and thus in* $\mathcal{H}_{\mathsf{bo}}$ *and* $\mathcal{H}_{\mathsf{bem}}$. *And if* $\phi \in \mathcal{H}_{\mathsf{gem}}$, $\langle\mathsf{S}\rangle\phi$ *is definable in* $\mathcal{H}_{\mathsf{bem}}$.

$$
\begin{aligned}
\mathcal{M}, w &\models \langle\sqsubseteq\rangle\phi &&\Longleftrightarrow \mathcal{M}, w \models \langle\leq\rangle\phi \wedge \neg\mathbf{0} \\
\mathcal{M}, w &\models \langle\sqsupseteq\rangle\phi &&\Longleftrightarrow \mathcal{M}, w \models \langle\geq\rangle\phi \wedge \neg\mathbf{0} \\
\mathcal{M}, w &\models \langle\sqsubset\rangle\phi &&\Longleftrightarrow \mathcal{M}, w \models \langle\leq\rangle\phi \wedge \neg(i \vee \mathbf{0}) \\
\mathcal{M}, w &\models \langle\sqsupset\rangle\phi &&\Longleftrightarrow \mathcal{M}, w \models \langle\geq\rangle\phi \wedge \neg(i \vee \mathbf{0}) \wedge \neg\mathbf{0} \\
\mathcal{M}, w &\models \langle\mathsf{V}\rangle\phi &&\Longleftrightarrow \mathcal{M}, w \models \langle\geq\rangle(\neg\mathbf{0} \wedge \langle\leq\rangle\phi) \\
\mathcal{M}, w &\models \langle\mathsf{S}\rangle i &&\Longleftrightarrow \mathcal{M}, w \models \langle\mathsf{V}\rangle i \\
\mathcal{M}, w &\models \langle\mathsf{S}\rangle\phi &&\Longleftrightarrow \mathcal{M}, w \models \neg[\overline{\mathsf{V}}]\phi \\
\mathcal{M}, w &\models \langle\mathsf{C}\rangle i &&\Longleftrightarrow \mathcal{M}, w \models (\neg(\mathbf{0} \vee \mathbf{1}) \rightarrow [\geq]\langle\mathsf{S}\rangle i \wedge [\overline{\geq}]\langle\mathsf{S}\rangle i) \\
& && \qquad \wedge\, \mathbf{0} \leftrightarrow @_i\mathbf{1} \wedge \mathbf{1} \leftrightarrow @_i\mathbf{0} \\
\mathcal{M}, w &\models \langle\sqcup\rangle(i, j) &&\Longleftrightarrow \mathcal{M}, w \models \langle\geq\rangle i \wedge \langle\geq\rangle j \wedge [\overline{\leq}](\langle\geq\rangle i \wedge \langle\geq\rangle j) \\
\mathcal{M}, w &\models \langle\sqcap\rangle(i, j) &&\Longleftrightarrow \mathcal{M}, w \models \langle\leq\rangle i \wedge \langle\leq\rangle j \wedge [\overline{\geq}](\langle\leq\rangle i \wedge \langle\leq\rangle j) \\
\mathcal{M}, w &\models \langle\textstyle\sum\rangle\phi &&\Longleftrightarrow \mathcal{M}, w \models [\overline{\geq}]\langle\leq\rangle\phi \wedge [\geq]\langle\leq\rangle\phi \\
\mathcal{M}, w &\models \langle\textstyle\prod\rangle\phi &&\Longleftrightarrow \mathcal{M}, w \models [\overline{\leq}]\langle\geq\rangle\phi \wedge [\leq]\langle\geq\rangle\phi
\end{aligned}
$$

Properties and Filters A *filter F* of a partially ordered set $(P, \leq)$ is a set such that (i) $\forall x, y \in F$, $\exists z \in F$ such that $z \leq x$ and $z \leq y$ and (ii) $\forall x \in F \forall y \in P$, $(x \leq y \Rightarrow y \in F)$. The dual of notion of filter is the *ideal*, the concept obtained by replacing $\leq$ with $\geq$ in the definition above. A filter is *principal* if it is a filter generated by a single element, i.e. an upward closed set of the form $\{x \mid x \geq y\}$ for some element y. And an ideal is a *principal* if it is a downward closed set of the form $\{x \mid x \leq y\}$ for some element y.

Proposition 4 *In* $\mathcal{H}_{\mathsf{bo}}$ *the following are expressible.*

(1) *p is upward closed:* $\qquad\qquad$ $\mathsf{A}(p \rightarrow [\leq]p)$
(2) *p is downward closed:* $\qquad\quad$ $\mathsf{A}(p \rightarrow [\geq]p)$
(3) *p is the principal filter generated by i:* $\quad$ $@_i[\leq]p \wedge \mathsf{A}(p \rightarrow \langle\geq\rangle i)$
(4) *p is the principal ideal generated by i:* $\quad$ $@_i[\geq]p \wedge \mathsf{A}(p \rightarrow \langle\leq\rangle i)$
(5) *i is the fusion of p:* $\qquad\qquad\quad$ $\mathsf{A}(p \rightarrow \langle\leq\rangle i) \wedge @_i[\geq]\langle\mathsf{O}\rangle p$
(6) *i is the product of p:* $\qquad\qquad$ $\mathsf{A}(p \rightarrow \langle\geq\rangle i) \wedge @_i[\leq]\langle\mathsf{O}\rangle p$

4 Frame Definability

We say φ *is valid on model* $\mathcal{M}$ (notation: $\mathcal{M} \models \varphi$), if $\forall w \in \mathcal{M}$, $\mathcal{M}, w \models \varphi$. We say φ *is valid on a frame* $\mathcal{F}$ (notation: $\mathcal{F} \models \varphi$) if φ is valid on $\mathcal{M} = (\mathcal{F}, V)$ for any hybrid valuation V. If Σ is a set of $\mathcal{H}_{\mathsf{gem}}$-formulas, $\mathcal{F} \models \Sigma \Longleftrightarrow \forall\phi \in \Sigma\ (\mathcal{F} \models \phi)$. Let K be a class of frames. We say $\Sigma \subseteq \mathcal{H}_{\mathsf{m}}$ *defines* K if, for all frames $\mathcal{F}$, $\mathcal{F} \in \mathsf{K} \Longleftrightarrow \mathcal{F} \models \Sigma$. If $\Sigma = \{\phi\}$, for some single $\phi \in \mathcal{H}_{\mathsf{gem}}$, we say that ϕ *defines* K. A model is *named* if every element in $\mathcal{M}$ is the denotation of some nominal (i.e. $\forall w \in W$, $\exists i \in \Omega$ where $V(i) = \{w\}$). A frame $\mathcal{F}$ is *named* if every element in $\mathcal{F}$ is distinguished. A *closed formula* is one which is equivalent to a FO sentence

under the standard translation. Many formulas of $\mathcal{H}_{\text{gem}}$ (e.g. $@_i \langle \leq \rangle j$) are closed. A $\mathcal{H}_{\text{gem}}$-*theory* is a set of closed $\mathcal{H}_{\text{gem}}$-formulas. A $\mathcal{H}_{\text{gem}}$-*theory T of a frame* $\mathcal{F}$ is the set of closed $\mathcal{H}_{\text{gem}}$-formulas such that $\mathcal{F} \models T$. We say that a $\mathcal{H}_{\text{gem}}$-theory T is κ-categorical if there is, up to isomorphism, exactly one frame $\mathcal{F}$ of size κ such that $\mathcal{F} \models T$.

Proposition 5 *Any $\mathcal{H}_{\text{m}}$-theory of a finite named frame such that each distinguished element is interpreted by a nominal is m-categorical.*

Proof Let $\mathcal{F} = (W, \leq, c_1...c_n)$ be a finite named frame of size $m \leq n$. By assumption the elements of $\mathcal{F}$ are distinguished and interpreted by nominals $i_1...i_n$. Now $\mathcal{H}_{\text{gem}}$-formulas of the form $\mathsf{A}(\bigvee_{x=1}^{k} i_x) \wedge (\bigwedge_{1 \leq l \neq h \leq m} \neg @_{i_l} i_h) \wedge (@_{i_v} j_1 \wedge ... \wedge @_{i_w} j_{k-m})$ express that there are m distinct objects such that maximally $k - m$ of them $i_v...i_w$ with multiple names. The relation $\leq^{\mathcal{F}}$ can be diagrammed in an obvious way as a conjunction of sentences of the form $@_{i_x} \langle \leq \rangle i_y$ and $@_{i_x} \langle \not\leq \rangle i_y$. And each such formula will be in the hybrid theory of $\mathcal{F}$. And thus the hybrid theory of $\mathcal{F}$ is m-categorical. $\square$

Proposition 6 (Local and Global Correspondence on Models) *For all $\mathcal{H}_{\text{gem}}$-formulas φ, hybrid models $\mathcal{M}$, states $w \in \mathcal{M}$ (i) $\mathcal{M}, w \models \varphi \Longleftrightarrow \mathcal{M} \models ST_x(\varphi)[w]$. And* (ii) $\mathcal{M} \models \varphi \Longleftrightarrow \mathcal{M} \models \forall x \, ST_x(\varphi)$.

Proof (i) By an easy induction on the complexity of ϕ. (ii) An easy consequence of (i). $\square$

Proposition 7 *In $\mathcal{H}_{\text{m}}$, the atomic and atomless classes frames are definable.*

Proof $\mathsf{A}\langle \geq \rangle \alpha$ expresses $\forall x \exists y (y \leq x \wedge At(y))$. $\mathsf{A}\neg \alpha$ expresses that there exists no atom. $\square$

Proposition 8 *$\mathcal{H}_{\text{m}}$ lacks the finite model property.*

Proof The closed formula $\mathsf{A}(\langle \geq \rangle \top \wedge \neg \alpha)$ expresses $\forall x (\exists y (y \leq x) \wedge \neg At(x))$. $\square$

Lemma 1 (Frame Definability via Pure Formulas) *Each pure formula of $\mathcal{H}_{\text{gem}}$ defines an elementary class of frames.*

Proof Assume that $\mathcal{F} \models \phi$ where $\phi \in \mathcal{H}_{\text{gem}}$. Observe that ϕ contains some finite number n of nominals $i_1, ..., i_n$. To indicate this, we write $\phi(i_1, ..., i_n)$.

$\mathcal{F} \models \phi \Longleftrightarrow (\mathcal{F}, V) \models \phi(i_1, ..., i_n)$ *for any hybrid valuation V*
$\Longleftrightarrow (\mathcal{F}, V) \models \forall x \, ST_x(\phi(i_1, ..., i_n))$ *for any hybrid valuation V [by Proposition 6]*
$\Longleftrightarrow (\mathcal{F}, V) \models \forall x \, ST_x(\phi(i_1/x_1, ..., i_n/x_n))[s(x_1), ..., s(x_n)]$ *for any FO variable assignment s and where $x_1, ..., x_n$ are variables not occurring in $\forall x \, ST_x(\phi(i_1, ..., i_n))$*
$\Longleftrightarrow \mathcal{F} \models \forall x_1...\forall x_n (\forall x \, ST_x(\phi(i_1, ..., i_n))[i_1/x_1, ..., i_n/x_n])$. *Since $ST_x(\phi(i_1, ..., i_n))$ is FO, $\forall x_1...\forall x_n (\forall x \, ST_x(\phi(i_1, ..., i_n))[i_1/x_1, ..., i_n/x_n])$ is FO.*

$\square$

Traditional mereological frame conditions expressed in the language of FO logic are displayed at the top of Table 3. By Lemma 1 *pure* $\mathcal{H}_{\mathsf{gem}}$-formulas always define FO frame conditions. Thus an immediate consequence of this lemma, the standard translation, and global correspondence is that if ϕ and ψ are pure subformulas appearing in those $\mathcal{H}_{\mathsf{gem}}$-formulas at the bottom of Table 3, then each defines the corresponding first-order condition at the top of the table. We will see that this makes our task of identifying corresponding logics easy. Similar to Lemma 1, what the next lemma shows is that when pure hybrid formulas are used as axioms they are immediately complete with respect to the frames they define. We say ψ is a *pure instance* of ϕ if ψ is obtained from ϕ by uniformly substituting nominals for nominals.

Lemma 2 *Let $\mathcal{M} = (\mathcal{F}, V)$ be a named model and ϕ a pure formula. Suppose that for all pure instances ψ of ϕ, $\mathcal{M} \models \psi$. Then $\mathcal{F} \models \phi$.*

Proof Assume $\phi \in \mathcal{H}_{\mathsf{m}}$ and $\mathcal{M}$ is a named model. Suppose for all pure instances ψ of ϕ, $\mathcal{M} \models \psi$. Let $\mathcal{M} \models \phi(i_1...i_n)$ where $i_1...i_n$ are the nominals in ϕ. By Proposition 6 we have $\mathcal{M} \models \forall x\, ST_x(\phi(i_1...i_n))$. $\mathcal{M} \models (\forall x\, ST_x(\phi(i_1...i_n)))[i_1/j_1...i_n/j_n]$ by assumption for any $j_1...j_n \in \Omega$. As $\mathcal{M}$ is fully named, $\mathcal{M} \models \forall x_1...\forall x_n \forall x\, ST_x(\phi(x_1...x_n))$. And as the latter is a nominal-free closed formula, $\mathcal{F} \models \forall x_1...\forall x_n \forall x\, ST_x(\phi(x_1...x_n))$. Thus for any valuation V, $\mathcal{F}, V \models \phi(i_1...i_n)$. ▢

Let us consider three very significant classes of structures in the literature of formal mereology. Let $\mathcal{MM}$ be the class of partial orders closed under supplementation—the class of *minimal mereological frames* [48, Section 3.1; 7]. Let $\mathcal{EM}$ be class of partial orders closed under strong supplementation—the class of *extensional mereological frames* [48, Section 3.1; 7]. Let $\mathcal{GEMS}$ be the class of GEMS structures closed under unrestricted fusion a $\mathsf{E}\phi \rightarrow \mathsf{E}\langle\bigoplus_a\rangle\phi$ where ϕ is pure.

Theorem 1 *(i) $\mathcal{MM}$ and $\mathcal{EM}$ are definable in $\mathcal{H}_{\mathsf{m}}$. (ii) $\mathcal{GEM}$ is definable in $\mathcal{H}_{\mathsf{gem}}$.*

Proof (i) Straightforward (see Table 1). (ii) Let GEM be the $\mathcal{H}_{\mathsf{gem}}$-formulas defining partial orders, extensionality, unit, and the unrestricted fusion a schema: $\mathsf{E}\phi \rightarrow \mathsf{E}\langle\bigoplus_a\rangle\phi$ where ϕ is any pure formula of $\mathcal{H}_{\mathsf{gem}}$. The instances of this schema include: $\mathsf{E}(\langle\geq\rangle i \vee \langle\geq\rangle j) \rightarrow \mathsf{E}\langle\bigoplus_a\rangle(\langle\geq\rangle i \vee \langle\geq\rangle j)$, $\mathsf{E}(\langle\leq\rangle i \wedge \langle\leq\rangle j) \rightarrow \mathsf{E}\langle\bigoplus_a\rangle(\langle\leq\rangle i \wedge \langle\leq\rangle j)$, $\mathsf{E}\langle\mathsf{D}\rangle i \rightarrow \mathsf{E}\langle\bigoplus_a\rangle\langle\mathsf{D}\rangle i$, $\mathsf{E}\top \rightarrow \mathsf{E}\langle\bigoplus_a\rangle\top$. ($\Rightarrow$) So let $\mathcal{F} \models \Sigma$. Then for any named model $\mathcal{M} = (\mathcal{F}, V)$, $\mathcal{M} \models \Sigma$. By Lemma 2 $\mathcal{F}$ must be an extensional partial order with unit closed under finite a fusions, finite products, complements, and unrestricted fusion a. Note that $\mathcal{M} \models \neg\mathsf{E}\langle\sim\rangle\langle\bigoplus_a\rangle\top$. Thus $\mathcal{F}$ is a 0-deleted BA closed under unrestricted fusion a. ($\Leftarrow$) Next suppose $\mathcal{B}$ is a 0-deleted BA and let V be hybrid valuation. Obviously $(\mathcal{B}, V) \models \Sigma$ since any such structure is an extensional partial order with unit closed under finite a unrestricted a-fusions, finite products, and complements. ▢

Table 3 $\mathcal{H}_{\text{gem}}$ formulas defining the standard mereological frame conditions

Principle	FO Axiom
Reflexivity	$\forall x(x \leq x)$
Transitivity	$\forall x \forall y \forall z(x \leq y \wedge y \leq z \rightarrow x \leq z)$
Antisymmetry	$\forall x \forall y(x \leq y \wedge y \leq x \rightarrow x = y)$
Weak company	$\forall x \forall y(x < y \rightarrow \exists z(z < y \wedge z \neq x))$
Strong company	$\forall x \forall y(x < y \rightarrow \exists z(z < y \wedge z \nleq x))$
Supplementation	$\forall x \forall y(x < y \rightarrow \exists z(z \leq y \wedge z\mathsf{D}x))$
Strong supplementation	$\forall x \forall y(y \nleq x \rightarrow \exists z(z \leq y \wedge z\mathsf{D}x))$
Extensionality	$\forall x \forall y(x = y \leftrightarrow \forall z(z < x \leftrightarrow z < y))$
Complementation	$\forall x \forall y(y \nleq x \rightarrow \exists z \forall w(w \leq z \leftrightarrow (w \leq y \wedge w\mathsf{D}x)))$
Density	$\forall x \forall y(x < y \rightarrow \exists z(x < z < y))$
Atomicity	$\forall x \exists y(y \leq x \wedge At(x))$
Atomlessness	$\forall x \exists y(y < x)$
Unit	$\forall y(y \leq 1)$
Bottom	$\forall y(0 \leq y)$
Bound	$\forall x \forall y(\xi(x, y) \rightarrow \exists z(x \leq z \wedge y \leq z))$
Supremum existence	$\forall x \forall y(\xi(x, y) \rightarrow \exists z(x + y = z))$
Fusion existence a	$\forall x \forall y(\xi(x, y) \rightarrow \exists z(x +_a y = z))$
Fusion existence b	$\forall x \forall y(\xi(x, y) \rightarrow \exists z(x +_b y = z))$
Product existence	$\forall x \forall y(\xi(x, y) \rightarrow \exists z(x \times y = z))$
Strong ψ-bound	$(\exists w \phi(x) \wedge \forall w(\phi(w) \rightarrow \psi(w))) \rightarrow \exists z(\phi(w) \rightarrow w \leq z)$
Strong ψ-supremum a	$(\exists w \phi(x) \wedge \forall w(\phi(w) \rightarrow \psi(w))) \rightarrow \exists z(z = \bigoplus \phi)$
Strong ψ-fusion a	$(\exists w \phi(x) \wedge \forall w(\phi(w) \rightarrow \psi(w))) \rightarrow \exists z(z = \bigoplus_a \phi)$
Strong ψ-fusion b	$(\exists w \phi(x) \wedge \forall w(\phi(w) \rightarrow \psi(w))) \rightarrow \exists z(z = \bigoplus_b \phi)$
Strong ψ-product	$(\exists w \phi(x) \wedge \forall w(\phi(w) \rightarrow \psi(w))) \rightarrow \exists z(z = \bigotimes \phi)$
Unrestricted fusion a	$\exists w \phi(w) \rightarrow \exists z(z = \bigoplus_a \phi)$
Unrestricted fusion b	$\exists w \phi(w) \rightarrow \exists z(z = \bigoplus_b \phi)$
	$\mathcal{H}_{\text{gem}}$ Axiom
Reflexivity	$i \rightarrow \langle \leq \rangle i$
Transitivity	$\langle \leq \rangle \langle \leq \rangle i \rightarrow \langle \leq \rangle i$
Antisymmetry	$i \rightarrow [\leq](\langle \leq \rangle i \rightarrow i)$
Weak company	$@_i \langle < \rangle j \rightarrow \mathsf{E}(\langle < \rangle j \wedge \neg i)$
Strong company	$@_i \langle < \rangle j \rightarrow \mathsf{E}(\langle < \rangle j \wedge \langle \overline{\leq} \rangle \neg i)$
Supplementation	$@_i \langle < \rangle j \rightarrow \mathsf{E}(\langle \leq \rangle j \wedge \langle \mathsf{D} \rangle i)$
Strong supplementation	$@_j \langle \overline{\leq} \rangle \neg i \rightarrow \mathsf{E}(\langle \leq \rangle j \wedge \langle \mathsf{D} \rangle i)$
Extensionality	$@_i j \leftrightarrow \mathsf{A}(\langle < \rangle i \leftrightarrow \langle < \rangle j)$
Complementation	$@_j \langle \overline{\leq} \rangle \neg i \rightarrow \mathsf{E}([\geq](\langle \leq \rangle j \wedge \langle \mathsf{D} \rangle i) \wedge [\overline{\geq}](\langle \leq \rangle j \wedge \langle \mathsf{D} \rangle i))$
Density	$@_i \langle < \rangle j \rightarrow \mathsf{E}(\langle < \rangle j \wedge \langle > \rangle i)$
Atomicity	$\mathsf{A}\langle \geq \rangle \alpha$
Strong Atomicity	$\mathsf{A}\alpha$
Atomlessness	$\mathsf{A}\neg \alpha$

Table 3 (continued)

Principle	$\mathcal{H}_{\text{gem}}$ Axiom
Unit	$\mathsf{A}\langle\leq\rangle\mathbf{1}$
Bottom	$\mathsf{A}\langle\geq\rangle\mathbf{0}$
Bound	$\xi(i, j) \to \mathsf{E}(\langle\geq\rangle i \wedge \langle\geq\rangle j)$
Supremum existence	$\xi(i, j) \to \mathsf{E}\langle+\rangle(i, j)$
Fusion existence a	$\xi(i, j) \to \mathsf{E}\langle+_a\rangle(i, j)$
Fusion existence b	$\xi(i, j) \to \mathsf{E}\langle+_b\rangle(i, j)$
Product existence	$\xi(i, j) \to \mathsf{E}\langle\times\rangle(i, j)$
Strong ψ-bound	$\mathsf{E}\phi \wedge \mathsf{A}(\phi \to \psi) \to \mathsf{E}[\overline{\geq}]\phi$
Strong ψ-supremum	$\mathsf{E}\phi \wedge \mathsf{A}(\phi \to \psi) \to \mathsf{E}\langle\bigoplus\rangle\phi$
Strong ψ-fusion a	$\mathsf{E}\phi \wedge \mathsf{A}(\phi \to \psi) \to \mathsf{E}\langle\bigoplus_a\rangle\phi$
Strong ψ-fusion b	$\mathsf{E}\phi \wedge \mathsf{A}(\phi \to \psi) \to \mathsf{E}\langle\bigoplus_b\rangle\phi$
Strong ψ-product	$\mathsf{E}\phi \wedge \mathsf{A}(\phi \to \psi) \to \mathsf{E}\langle\bigotimes\rangle\phi$
Unrestricted fusion a	$\mathsf{E}\phi \to \mathsf{E}\langle\bigoplus_a\rangle\phi$
Unrestricted fusion b	$\mathsf{E}\phi \to \mathsf{E}\langle\bigoplus_b\rangle\phi$

4.1 Definability of Classes of BAs

It is well known that the FO formulas in Table 4 define the class of BAs. By the standard translation, the definability of the operators in Proposition 2, and Lemma 1, it is easy to show that each $\mathcal{H}_{\text{m}}$-formula in Table 4 defines the corresponding condition. We call the set containing the eight $\mathcal{H}_{\text{m}}$-formulas in the table BA. We say ψ is a *pure instance* of ϕ if ψ is obtained from ϕ by uniformly substituting nominals for nominals.

Proposition 9 *Let $\mathcal{M} = (\mathcal{F}, V)$ be a named model. Suppose that for all pure instances ψ of each formula ϕ in* BA*, $\mathcal{M} \models \psi$. Then, $\mathcal{F}$ is a BA.*

Proof As each formula in BA is pure and defines the required property, then by Lemma 1 the desired result is immediate. □

A BA $\mathcal{F}$ is *atomic* if for all $x \in \mathcal{F}$, there is an atom y such that $y \leq x$. $\mathcal{F}$ is *atomless* if $\mathcal{F}$ contains no atoms. Observe that in this case the atomic and atomless frames can, again, be defined as $\mathsf{A}\langle\geq\rangle\alpha$ and $\mathsf{A}\neg\alpha$, respectively. And therefore it follows that the modified $\mathcal{H}_{\text{ba}}$-semantics for α again yields a language without the finite model property; for consider the formula $\mathsf{A}(\langle\geq\rangle\top \wedge \neg\mathbf{0} \wedge \neg\alpha)$.

4.2 Second-order Frame Definability in $\mathcal{H}_{\text{o}}$ and ESOPHL

Let us return to languages for GEMS structures. The SO unrestricted fusion condition is definable in $\mathcal{H}_{\text{o}}$ by $\mathsf{E}p \to \mathsf{E}([\overline{\geq}]p \wedge [\geq]\langle\mathsf{O}\rangle p)$. Specifically, the formula defines

Table 4 $\mathcal{H}_{\mathsf{m}}$ formulas defining the class of BAs

	Principle	Formula
FO-Formulation	Identity	$\forall x(x \sqcap 1 = x)$
		$\forall x(x \sqcup 0 = x)$
	Complement	$\forall x(\complement x \sqcup x = 1)$
		$\forall x(\complement x \sqcap x = 0)$
	Associativity	$\forall x \forall y \forall z((x \sqcap (y \sqcup z)) = ((x \sqcap y) \sqcup (y \sqcap z)))$
		$\forall x \forall y \forall z((x \sqcup (y \sqcap z)) = ((x \sqcup y) \sqcap (y \sqcup z)))$
	Commutativity	$\forall x \forall y(x \sqcup y = y \sqcup x)$
		$\forall x \forall y(x \sqcap y = y \sqcap x)$
$\mathcal{H}_{\mathsf{m}}$-Formulation	Identity	$@_i \langle \sqcap \rangle (i, \mathbf{1})$
		$@_i \langle \sqcup \rangle (i, \mathbf{0})$
	Complement	$@_{\mathbf{0}} \langle \sqcap \rangle (\langle \complement \rangle i, i)$
		$@_{\mathbf{1}} \langle \sqcup \rangle (\langle \complement \rangle i, i)$
	Associativity	$\langle \sqcup \rangle (i, \langle \sqcap \rangle (j, k)) \leftrightarrow \langle \sqcup \rangle (\langle \sqcap \rangle (i, j), \langle \sqcap \rangle (i, k))$
		$\langle \sqcap \rangle (i, \langle \sqcup \rangle (j.k)) \leftrightarrow \langle \sqcap \rangle (\langle \sqcup \rangle (i, j), \langle \sqcup \rangle (i, k))$
	Commutativity	$\langle \sqcap \rangle (i, j) \leftrightarrow \langle \sqcap \rangle (j, i)$
		$\langle \sqcup \rangle (i, j) \leftrightarrow \langle \sqcup \rangle (j, i)$

the condition $\forall P(\exists x\,Px \rightarrow \exists z(\forall w(Pw \rightarrow w \leq z) \wedge \forall w(w \leq z \rightarrow \exists v(vOw \wedge Pv))))$. In other words, for any set of objects P, there is an extensional fusion. And the condition is expressible in ESOPHL: $\forall p.(\mathsf{E}p \rightarrow \mathsf{E}([\overline{\geq}]p \wedge [\geq]\langle \mathsf{O} \rangle p))$. There is also a Boolean formulation of this principle definable in $\mathcal{H}_{\mathsf{bo}}$. Boolean completeness is the property that every set of objects in a BA has a supremum: $\mathsf{E}p \rightarrow \mathsf{E}([\overline{\geq}]p \wedge [\geq](\neg \mathbf{0} \rightarrow \langle \mathsf{V} \rangle p))$. Is there is a FO set of formulas defining these properties? Consider the following definitions due to Koppelberg [17]. The *cofinality* $cf(A)$ of a BA A is the least limit ordinal κ such that A is the union of an increasing chain of length κ of proper subalgebras of A, provided such a chain exists.

Proposition 10 (Koppelberg [17]) *Each infinite complete BA has cofinality $\aleph_1$.*

Clearly the cofinality of an infinite BA B is an infinite regular cardinal bounded by the size of B. If C is an infinite quotient of B then $cf(B) \leq cf(C)$. Koppelberg showed that $Pow(\omega)$, and in fact every infinite complete BA, has cofinality $\aleph_1$. And the same obviously holds for complete 0/BAs. A class $\mathcal{K}$ of structures is EC if there is a FO sentence σ such that the set of models of σ Mod $\sigma = \mathcal{K}$. $\mathcal{K}$ is an elementary class in the wider sense (EC_Δ) iff $\mathcal{K} = $ Mod Σ for some set Σ of FO sentences.

Theorem 2 *Boolean completeness is neither EC nor EC_Δ.*

Proof Suppose there were a set of FO-sentences Σ defining Boolean completeness. By the downward Löwenheim Skolem theorem, Σ has a countable model. Thus Mod Σ contains Boolean incomplete models by the preceding proposition, a contradiction. □

Despite whether or not the unrestricted fusion axiom is true, one might think that it's *expression* is an essential ingredient. If this is true, any suitable language for mereology must be a SO logic. This obviously would be "out of bounds" for a nominalistic mereology. We note a recent result by A. Kuusisto [19] that ESOPHL is expressively equivalent to a fragment of monadic SO logic (MSOL).

Corollary 1 (Kuusisto [19]) *ESOPHL is expressively equivalent to MSOL in vocabulary containing a single binary relation and countably many constants.*

Nonetheless, recall that in Tarski's *Foundations of the Geometry of Solids* he shows that there is a universal second order theory of regular open sets of the Euclidean space which is ω_1 categorical. Thus we note in passing that in ESOPHL we have the same potential. And it is easy to check that there is an axiomatization of infinite atomic complete Boolean algebras as well in ESOPHL.

5 Hybrid Mereologics

Let **R** be the set $\{\leq, \geq, <, >, \overline{\leq}, \overline{\geq}, \overline{O}, \sim\}$ of relation symbols. If $R \in \mathbf{R}$, let Dual R denote the $\mathcal{H}_{\text{gem}}$-formula $\langle R \rangle p \leftrightarrow \neg[R]\neg p$. Let Σ be a set of $\mathcal{H}_{\text{gem}}$-formulas. For each $\phi \in \Sigma$, if for each θ, where θ is obtained by ϕ by uniformly replacing proposition letters by arbitrary formulas and nominals by nominals, we have $\theta \in \Sigma$, we say Σ *is closed under sorted substitution*.

Definition 4 ($\mathbf{K}_{\text{gem}}$) $\mathbf{K}_{\text{gem}}$ is the set of $\mathcal{H}_{\text{gem}}$-formulas whose axioms are the tautologies, each Dual R for $R \in \mathbf{R}$, and the following:

(**K** $\leq$)	$[\leq](p \to q) \to ([\leq]p \to [\leq]q)$	(back $\geq$)	$\langle \geq \rangle @_i p \to @_i p$
(**K** $\geq$)	$[\geq](p \to q) \to ([\geq]p \to [\geq]q)$	(back $\leq$)	$\langle \leq \rangle @_i p \to @_i p$
($\langle\leq\rangle$-$\langle\geq\rangle$)	$@_i\langle\leq\rangle j \leftrightarrow @_j\langle\geq\rangle i$	($\mathbf{K}_@$)	$@_i(p \to q) \to$
			$(@_i p \to @_i q)$
($\langle<\rangle$-$\langle\leq\rangle$)	$@_i\langle<\rangle p \leftrightarrow @_i\langle\leq\rangle(\neg i \land p)$	(Self Dual)	$@_i p \leftrightarrow \neg@_i\neg p$
($\langle>\rangle$-$\langle\geq\rangle$)	$@_i\langle>\rangle p \leftrightarrow @_i\langle\geq\rangle(\neg i \land p)$	(E)	$\phi \to \mathsf{E}\phi$
($[\overline{\geq}]$-$\langle\leq\rangle$)	$@_i[\overline{\geq}]p \land @_j p \to @_j\langle\leq\rangle i$	(Intro)	$i \land p \to @_i p$
($[\overline{O}]$-$\langle O\rangle$)	$@_i[\overline{O}]p \land @_j p \to @_j\langle\geq\rangle\langle\leq\rangle i$	(ref)	$@_i i$
($\langle\overline{\leq}\rangle$-$\langle\geq\rangle$)	$@_i\langle\overline{\leq}\rangle p \leftrightarrow \mathsf{E}(\neg\langle\geq\rangle i \land \neg p)$	(sym)	$@_i j \leftrightarrow @_j i$
($\langle\overline{\geq}\rangle$-$\langle\leq\rangle$)	$@_i\langle\overline{\geq}\rangle p \leftrightarrow \mathsf{E}(\neg\langle\leq\rangle i \land \neg p)$	(nom)	$@_i j \land @_j p \to @_i p$
($\langle\overline{O}\rangle$-$\langle O\rangle$)	$@_i\langle\overline{O}\rangle p \leftrightarrow \mathsf{E}(\neg\langle\geq\rangle\langle\leq\rangle i \land \neg p)$	(agree)	$@_j @_i p \leftrightarrow @_i p$
($\langle\sim\rangle$-$\langle D\rangle$)	$@_i\langle\sim\rangle p \leftrightarrow \mathsf{E}([\overline{\geq}]\langle D\rangle i \land [\overline{\geq}]\langle D\rangle i \land p)$		
(α-$[\geq]$)	$@_i\alpha \leftrightarrow @_i[\geq]i$		
(Clip)	$@_i\alpha \land @_j\langle\leq\rangle i \to @_i j$		

closed under modus ponens, sorted substitution, and the rules below closed under uniform substitution of formulas ξ, and θ:

(Gen $\leq$)	If $\vdash \xi$, then $\vdash [\leq]\xi$
(Gen $\geq$)	If $\vdash \xi$, then $\vdash [\geq]\xi$
(Gen@)	If $\vdash \xi$, then $\vdash @_i\xi$ for any $i \in \Omega$
(NAME)	If $\vdash j \rightarrow \theta$, then $\vdash \theta$
(PASTE)	If $\vdash @_i\langle\leq\rangle j \wedge @_j\xi \rightarrow \theta$, then $\vdash @_i\langle\leq\rangle\xi \rightarrow \theta$
(SPLIT)	If $\vdash @_i\langle\geq\rangle j \wedge @_j\xi \rightarrow \theta$, then $\vdash @_i\langle\geq\rangle\xi \rightarrow \theta$
(UP)	If $\vdash @_i\langle\leq\rangle j \wedge @_j(\neg i \wedge \xi) \rightarrow \theta$, then $\vdash @_i\langle<\rangle\xi \rightarrow \theta$
(DOWN)	If $\vdash @_i\langle\geq\rangle j \wedge @_j(\neg i \wedge \xi) \rightarrow \theta$, then $\vdash @_i\langle>\rangle\xi \rightarrow \theta$
(INV1)	If $\vdash @_i\neg\langle\leq\rangle j \wedge @_j\neg\xi \rightarrow \theta$, then $\vdash @_i\langle\overline{\leq}\rangle\xi \rightarrow \theta$
(INV2)	If $\vdash @_i\neg\langle\geq\rangle j \wedge @_j\neg\xi \rightarrow \theta$, then $\vdash @_i\langle\overline{\geq}\rangle\xi \rightarrow \theta$
(INV3)	If $\vdash @_i\neg\langle\geq\rangle\langle\leq\rangle j \wedge @_j\neg\xi \rightarrow \theta$, then $\vdash @_i\langle\overline{O}\rangle\xi \rightarrow \theta$
(SIM)	If $\vdash @_i([\geq]\langle D\rangle j \wedge [\overline{\geq}]\langle D\rangle j) \wedge @_j\xi \rightarrow \theta$, then $\vdash @_i\langle\sim\rangle\xi \rightarrow \theta$

In the final nine rules, j is a nominal distinct from i that does not occur in ξ or θ. The final nine rules are called the *decomposition rules*.

The eleven axioms directly below the axiom (**K** $\geq$) are intuitively valid interaction principles. The axioms in the second column contain the naming validities, well known to hybrid logic, and the E axiom. The decomposition rules allow us to expand any set of formulas to a maximally consistent set with the required number of *named witnesses*. It is well known that if we replace ϕ by $\neg\phi$ in the Intro axiom, contrapose, and make use of Self Dual, we obtain $(i \wedge @_i\phi) \rightarrow \phi$, the Elim formula. The transitivity of naming follows from nom. For example, by substituting the nominal k for ϕ yields $@_i j \wedge @_j k \rightarrow @_i k$. The Back axioms express how the @-operator interacts with $\langle\leq\rangle$ and $\langle\geq\rangle$. We can derive $\langle\leq\rangle i \wedge @_i\phi \rightarrow \langle\leq\rangle\phi$ and $\langle\geq\rangle i \wedge @_i\phi \rightarrow \langle\geq\rangle\phi$ called Bridge $\leq$ and Bridge $\geq$, respectively.

Lemma 3 [3, p.435] *Both* Bridge $\leq$ *and* Bridge $\geq$ *are provable in* $\mathbf{K_{gem}}$.

Proof We do a sketch for Bridge $\leq$. By Elim, $(i \wedge @_i\phi) \rightarrow \phi$ or tautologously $(@_i\phi \wedge i) \rightarrow \phi$. Thus as $\mathbf{K_{gem}}$ is a normal modal logic, we can prove $\langle\leq\rangle(@_i\phi \wedge i) \rightarrow \langle\leq\rangle\phi$. As in standard modal logic, $([\leq]\phi \wedge \langle\leq\rangle\psi) \rightarrow \langle\leq\rangle(\phi \wedge \psi)$ is $\mathbf{K_{gem}}$-theorem for any $\mathcal{H}_{gem}$-formulas ϕ and ψ. By tautologous reasoning $([\leq]@_i\phi \wedge \langle\leq\rangle i) \rightarrow \langle\leq\rangle\phi$ and $[\leq]@_i\phi \rightarrow (\langle\leq\rangle i \rightarrow \langle\leq\rangle\phi)$. By substituting $\neg\phi$ for back $\leq$ we can prove that $@_i\phi \rightarrow [\leq]@_i\phi$. By tautologous reasoning we have $@_i\phi \rightarrow (\langle\leq\rangle i \rightarrow \langle\leq\rangle\phi)$ which implies $\langle\leq\rangle i \wedge @_i\phi \rightarrow \langle\leq\rangle\phi$. $\square$

It can be checked easily that the axioms above are sound. So we now will set out to show that there is a completeness result. A $\mathbf{K_{gem}}$-maximally consistent set (henceforth $\mathbf{K_{gem}}$-MCS) is *named* if and only if it contains a nominal, and call any nominal belonging to a $\mathbf{K_{gem}}$ a name for that MCS. If Γ is a $\mathbf{K_{gem}}$-MCS and i is a nominal, then we will call $\{\phi \mid @_i\phi \in \Gamma\}$ *the set named i yielded by* Γ and denote this set by Δ_i.

Lemma 4 *Let Γ be a $\mathbf{K}_{\mathsf{gem}}$-MCS. Then:*

(i) *For all nominals i, Δ_i is a $\mathbf{K}_{\mathsf{gem}}$-MCS that contains i.*
(ii) *For all nominals i and j, $i \in \Delta_j \Longrightarrow \Delta_j = \Delta_i$.*
(iii) *For all nominals i and j, $@_i\phi \in \Delta_j \Longleftrightarrow @_i\phi \in \Gamma$ [Agreement Property].*
(iv) *If k is a name for Γ, then $\Gamma = \Delta_k$.*

Proof A well-known result. See [3, p.439 Lemma 7.24]. □

A $\mathbf{K}_{\mathsf{gem}}$-MCS Γ is:

– $\leq$-*pasted if* $@_i\langle\leq\rangle\phi \in \Gamma \Longrightarrow$ for some nominal j, $@_i\langle\leq\rangle j \wedge @_j\phi \in \Gamma$,
– $\geq$-*pasted if* $@_i\langle\geq\rangle\phi \in \Gamma \Longrightarrow$ for some nominal j, $@_i\langle\geq\rangle \wedge @_j\phi \in \Gamma$,
– $<$-*pasted if* $@_i\langle<\rangle\phi \in \Gamma \Longrightarrow$ for some nominal j, $@_i\langle\leq\rangle j \wedge @_j(\neg i \wedge \phi) \in \Gamma$,
– $>$-*pasted if* $@_i\langle>\rangle\phi \in \Gamma \Longrightarrow$ for some nominal j, $@_i\langle\geq\rangle j \wedge @_j(\neg i \wedge \phi) \in \Gamma$,
– $\overline{\leq}$-*pasted if* $@_i\langle\overline{\leq}\rangle\phi \in \Gamma \Longrightarrow$ for some nominal j, $@_i\neg\langle\geq\rangle j \wedge @_j\neg\phi \in \Gamma$,
– $\overline{\geq}$-*pasted if* $@_i\langle\overline{\geq}\rangle\phi \in \Gamma \Longrightarrow$ for some nominal j, $@_i\neg\langle\leq\rangle j \wedge @_j\neg\phi \in \Gamma$,
– $\overline{\mathsf{O}}$-*pasted if* $@_i\langle\overline{\mathsf{O}}\rangle\phi \in \Gamma \Longrightarrow$ for some nominal j, $@_i\neg\langle\geq\rangle\langle\leq\rangle j \wedge @_j\neg\phi \in \Gamma$,
– $\sim$-*pasted if* $@_i\langle\sim\rangle\phi \in \Gamma \Longrightarrow$ for some nominal j, $@_i([\geq]\langle\mathsf{D}\rangle j \wedge [\overline{\geq}]\langle\mathsf{D}\rangle j) \wedge @_j\phi \in \Gamma$.

If a $\mathbf{K}_{\mathsf{gem}}$-MCS has every property listed above, we say that it is *decomposed*.

Lemma 5 (Lindenbaum Lemma) *Let Ω' be a countably infinite set of nominals disjoint from Ω. Suppose $\mathcal{L}'$ is the language obtained by adding all these new nominals to $\mathcal{H}_{\mathsf{gem}}$. Then every $\mathbf{K}_{\mathsf{gem}}$-consistent set of formulas in language $\mathcal{H}_{\mathsf{gem}}$ can be extended to a decomposed $\mathbf{K}_{\mathsf{gem}}$-MCS in language $\mathcal{L}'$.*

Proof Enumerate Ω'. Given a consistent set of $\mathcal{H}_{\mathsf{gem}}$-formulas Σ, define Σ_k to be $\Sigma \cup \{k\}$, where k is the first new nominal in Ω'. Toward contradiction suppose that Σ_k is inconsistent. Then for some conjunction of formulas θ from $\Sigma, \vdash k \to \neg\theta$. But as k is a new nominal, it does not occur in θ; hence, by the **NAME** rule, $\vdash \neg\theta$. But this contradicts the consistency of Σ, so Σ_k must be consistent. Next we enumerate all the formulas of $\mathcal{L}'$, define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. let ϕ_{m+1} be the $(m+1)$-th formula in the enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows. If $\Sigma^{m+1} \cup \{\phi_{m+1}\}$ is inconsistent, let $\Sigma^{m+1} = \Sigma^m$. Otherwise let:

1. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\}$ if ϕ_{m+1} is in none of the following forms $@_i\langle\leq\rangle\phi$, $@_i\langle\geq\rangle\phi$, $@_i\langle<\rangle\phi$, $@_i\langle>\rangle\phi$, $@_i\langle\overline{\leq}\rangle\phi$, $@_i\langle\overline{\geq}\rangle\phi$, $@_i\langle\overline{\mathsf{O}}\rangle\phi$, or $@_i\langle\sim\rangle\phi$.
2. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\langle\leq\rangle j \wedge @_j\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\leq\rangle\phi$.
3. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\langle\geq\rangle j \wedge @_j\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\geq\rangle\phi$.
4. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\langle\leq\rangle j \wedge @_j(\neg i \wedge \phi)\}$, if ϕ_{m+1} is of the form $@_i\langle<\rangle\phi$.
5. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\langle\geq\rangle j \wedge @_j(\neg i \wedge \phi)\}$, if ϕ_{m+1} is of the form $@_i\langle>\rangle\phi$.
6. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\neg\langle\geq\rangle j \wedge @_j\neg\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\overline{\leq}\rangle\phi$.
7. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\neg\langle\leq\rangle j \wedge @_j\neg\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\overline{\geq}\rangle\phi$.
8. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\neg\langle\geq\rangle\langle\leq\rangle j \wedge @_j\neg\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\overline{\mathsf{O}}\rangle\phi$.

9. $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i([\geq]\langle D\rangle j \wedge [\overline{\geq}]\langle D\rangle j) \wedge @_j\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\sim\rangle\phi$.

In steps 2–9, j is the next nominal in the enumeration of nominals in Ω' and thus occurring in neither Σ^m nor ϕ_{m+1}. Let $\Sigma^+ = \bigcup_{n\geq 0}\Sigma^n$. Clearly Σ^+ is decomposed. And it is also consistent, since the consistency of sets obtained by steps 2–9 is what the decomposition rules guarantee. $\square$

Definition 5 Let Γ be a decomposed $\mathbf{K}_{\mathsf{gem}}$-MCS. The *named model yielded by* Γ, is $\mathcal{M}^\Gamma = (W^\Gamma, \leq^\Gamma, 1^\Gamma, V^\Gamma)$, where $W^\Gamma = \{\Delta_i \mid i \in \Omega\}$, and $\leq$ is the restriction to W^Γ of the canonical relation between MCSs: $u \leq^\Gamma v \iff \forall\phi \in \mathcal{H}_{\mathsf{gem}} (\phi \in v \Rightarrow \langle\leq\rangle\phi \in u)$. And $1^\Gamma = \Delta_\mathbf{1}$. And finally $V^\Gamma = \{(i, \{\Delta_i\}) \mid i \in \Omega\}$.

Lemma 6 *Let Γ be a decomposed $\mathbf{K}_{\mathsf{gem}}$-MCS. And let $\mathcal{M} = (W, \leq, 1, V)$ be the named model yielded by Γ. Then $u \leq v \iff \forall\phi \in \mathcal{H}_{\mathsf{gem}}(\phi \in u \Rightarrow \langle\geq\rangle\phi \in v)$.*

Proof ($\Rightarrow$) Let $u \leq v$ and $\psi \in u$. By Definition 5, $\forall\phi \in \mathcal{H}_{\mathsf{gem}} (\phi \in v \Rightarrow \langle\leq\rangle\phi \in u)$. As $\mathcal{M}$ is named, there are nominals i and j such that $i \in \Delta_i = v$ and $j \in \Delta_j = u$. Then $\langle\leq\rangle i \in u$, $@_j\langle\leq\rangle i \in u$ and $@_j\langle\leq\rangle i \in \Gamma$ by agreement. By axiom $(\langle\leq\rangle$-$\langle\geq\rangle)$, $@_i\langle\geq\rangle j \in \Gamma$. Then $\langle\geq\rangle j \in v$. As $\psi \in u$ we have $@_j\psi \in u$. Hence by agreement $@_j\psi \in \Gamma$ and $@_j\psi \in v$. By (**Bridge** $\geq$), $\langle\geq\rangle\psi \in v$. ($\Leftarrow$) Suppose that $\forall\phi \in \mathcal{H}_{\mathsf{gem}}(\phi \in u \Rightarrow \langle\geq\rangle\phi \in v)$. Let $\psi \in v$. There are nominals i, j such that $i \in v$ and $j \in u$. Thus $\langle\geq\rangle j \in v$ and $@_i\langle\geq\rangle j \in \Gamma$. By axiom $\langle\leq\rangle$-$\langle\geq\rangle$, $@_j\langle\leq\rangle i \in \Gamma$. So $\langle\leq\rangle i \in u$. Again by agreement, $@_i\psi \in u$. Hence by (**Bridge** $\leq$), $\langle\leq\rangle\psi \in u$. We conclude therefore that $u \leq v$. $\square$

Lemma 7 (Existence Lemma) *Let Γ be a decomposed $\mathbf{K}_{\mathsf{gem}}$-MCS, and let $\mathcal{M} = (W, \leq, 1, V)$ be the named model yielded by Γ. Suppose $u \in W$. (i) $\langle\leq\rangle\phi \in u$ implies $\exists v \in W$ such that $u \leq v$ and $\phi \in v$. (ii) $\langle\geq\rangle\phi \in u$ implies $\exists v \in W$ such that $v \leq u$ and $\phi \in v$. (iii) $\langle<\rangle\phi \in u$ implies $\exists v \in W$ such that $u < v$ and $\phi \in v$. (iv) $\langle>\rangle\phi \in u$ implies $\exists v \in W$ such that $v < u$ and $\phi \in v$. (v) $\langle\overline{\leq}\rangle\phi \in u$ implies $\exists v \in W$ such that $u \not\leq v$ and $\neg\phi \in v$. (vi) $\langle\overline{\geq}\rangle\phi \in u$ implies $\exists v \in W$ such that $v \not\leq u$ and $\neg\phi \in v$. (vii) $[\overline{\geq}]\phi \in u$ implies that if $\phi \in v$, then $v \leq u$. (viii) $\langle\overline{O}\rangle\phi \in u$ implies $\exists v \in W$ such that $\neg u O v$ and $\neg\phi \in v$. (ix) $\langle\sim\rangle\phi \in u$ implies $\exists v \in W$ such that $\sim u = v$ and $\phi \in v$.*

Proof (i) Let $\langle\leq\rangle\phi \in u$. For some nominal i, $u = \Delta_i$. Thus $@_i\langle\leq\rangle\phi \in \Gamma$. Since Γ is pasted, for some nominal j, $@_i\langle\leq\rangle j \wedge @_j\phi \in \Gamma$. So $\langle\leq\rangle j \in \Delta_i$ and $\phi \in \Delta_j$. It suffices to show that $\Delta_i \leq \Delta_j$. So let $\psi \in \Delta_j$. Then $@_j\psi \in \Delta_j$. By agreement $@_j\psi \in \Gamma$ and $@_j\psi \in \Delta_i$. Since $\langle\leq\rangle j \in \Delta_i$, by (**Bridge** $\leq$), $\langle\leq\rangle\psi \in \Delta_i$. Thus by definition $\Delta_i \leq \Delta_j$. (ii) Analogous to (i). (iii) Let $\langle<\rangle\phi \in u$. For some nominal i, $u = \Delta_i$. Thus $@_i\langle<\rangle\phi \in \Gamma$. Since Γ is $<$-pasted, for some nominal j, $@_i\langle\leq\rangle j \wedge @_j(\neg i \wedge \phi) \in \Gamma$. So $\langle\leq\rangle j \in \Delta_i$ and $\neg i \wedge \phi \in \Delta_j$. By (i) it suffices to show $@_j\neg i$ which is virtually immediate. (iv) Analogous to (iii). (v) Let $\langle\overline{\leq}\rangle\phi \in u$. For some nominal i, $u = \Delta_i$. So $@_i\langle\overline{\leq}\rangle\phi \in \Gamma$. Since Γ is $\overline{\leq}$-pasted, for some nominal j, $@_i\neg\langle\geq\rangle j \wedge @_j\neg\phi \in \Gamma$. So $\neg\langle\geq\rangle j \in \Delta_i$ and $\neg\phi \in \Delta_j$.

– *Claim:* $\Delta_j \not\leq \Delta_i$. *Proof.* It suffices to show that there is a formula ψ such that $\psi \in \Delta_i$ but $\langle\leq\rangle\psi \notin \Delta_j$. By consistency and maximality, $@_i\neg\langle\geq\rangle j \in \Gamma$. By (Self Dual), we have $\neg@_i\langle\geq\rangle j \in \Gamma$. It follows by $(\langle\leq\rangle\text{-}\langle\geq\rangle)$ that $\neg@_j\langle\leq\rangle i \in \Gamma$. Again by (Self Dual) we have $@_j\neg\langle\leq\rangle i \in \Gamma$. So $\neg\langle\leq\rangle i \in \Delta_j$, and by consistency $\langle\leq\rangle i \notin \Delta_j$. Finally note that $i \in \Delta_i$ and $\langle\leq\rangle i \notin \Delta_j$.

(vi) Analogous to (v). (vii) Let $[\geq]\phi \in u$. Suppose $\phi \in v$. We must show $v \leq u$. This is virtually immediate by part (i) and the axiom $([\geq]\text{-}\langle\leq\rangle)$. (viii) Let $\langle\overline{O}\rangle\phi \in u$. For some nominal i, $u = \Delta_i$. As $\langle\overline{O}\rangle\phi \in u$, by agreement $@_i\langle\overline{O}\rangle\phi \in \Gamma$. As Γ is $\overline{O}$-pasted, for some nominal j, $@_i\neg\langle\geq\rangle\langle\leq\rangle j \land @_j\neg\phi \in \Gamma$. So $\neg\langle\geq\rangle\langle\leq\rangle j \in \Delta_i$ and $\neg\phi \in \Delta_j$. If we could show that $\forall y(\Delta_y \leq \Delta_j \to \Delta_y \not\leq \Delta_i)$, then Δ_j would be disjoint from Δ_i as required. So suppose that $\Delta_k \leq \Delta_j$. To prove $\Delta_k \not\leq \Delta_i$, it suffices to show by Lemma 6 that there is a formula $\psi \in \Delta_k$ such that $\langle\geq\rangle\psi \notin \Delta_i$. As $\Delta_k \leq \Delta_j$, $\forall\phi \in \mathcal{H}_{\mathsf{gem}}$, $\phi \in \Delta_j$ implies $\langle\leq\rangle\phi \in \Delta_k$. As $j \in \Delta_j$, $\langle\leq\rangle j \in \Delta_k$. As $\neg\langle\geq\rangle\langle\leq\rangle j \in \Delta_i$, by consistency of Δ_i, $\langle\geq\rangle\langle\leq\rangle j \notin \Delta_i$, and we are done. (ix) Suppose $\langle\sim\rangle\phi \in u$. We will show $\exists v \in W \; \forall w(w \leq u \leftrightarrow wDv)$ and $\phi \in v$. As $u \in W$, for some nominal i we have $u = \Delta_i$. By agreement $@_i\langle\sim\rangle\phi \in \Gamma$. Since Γ is $\sim$-pasted, for some nominal j, $@_i([\geq]\langle D\rangle j \land [\overline{\geq}]\langle D\rangle j) \in \Gamma$ and $@_j\phi \in \Gamma$. By maximality and $\mathbf{K}_@$, $@_i[\geq]\langle D\rangle j \in \Gamma$ and $@_i[\overline{\geq}]\langle D\rangle j \in \Gamma$. Thus $[\geq]\langle D\rangle j \in \Delta_i$. If we could show that $\forall x(\Delta_x \leq \Delta_i \Leftrightarrow \Delta_x D \Delta_j)$, then Δ_j would be the complement of Δ_i, as required. We do this in the following two claims.

– *Claim 1.* $\forall x(\Delta_x \leq \Delta_i \Rightarrow \Delta_x D \Delta_j)$. *Proof.* Let $\Delta_k \leq \Delta_i$. Observe that Lemma 6 implies that: $\Delta_k \leq \Delta_i \iff \forall\varphi \in \mathcal{H}_{\mathsf{gem}}([\geq]\varphi \in \Delta_i \implies \varphi \in \Delta_k)$. Thus as $[\geq]\langle D\rangle j \in \Delta_i$, $\langle D\rangle j \in \Delta_k$. By definition of $\langle D\rangle$, we have $\neg[\overline{O}]j \in \Delta_k$. By (Dual $\overline{O}$), $\langle\overline{O}\rangle\neg j \in \Delta_k$. By the previous case it suffices to show $\langle\overline{O}\rangle\neg j \in \Delta_k$. So we are done.
– *Claim 2. Proof.* $\forall x(\Delta_x D \Delta_j \Rightarrow \Delta_x \leq \Delta_i)$. Let $\Delta_k D \Delta_j$.

  – *Subclaim: Proof.* $\langle D\rangle j \in \Delta_k$. Suppose toward contradiction that $\langle D\rangle j \notin \Delta_k$. By maximality, $\neg\langle D\rangle j \in \Delta_k$. By definition of $\langle D\rangle$, $\neg\neg[\overline{O}]j \in \Delta_k$. By tautologous reasoning, we have $[\overline{O}]j \in \Delta_k$. By agreement, $@_k[\overline{O}]j \in \Gamma$. By (ref), $@_j j \in \Gamma$. So by axiom $([\overline{O}]\text{-}\langle O\rangle)$, we have that $@_k\langle\geq\rangle\langle\leq\rangle j \in \Gamma$. By parts (i) and (ii) of this lemma, this implies that $\Delta_k O \Delta_j$, a contradiction. Thus $\langle D\rangle j \in \Delta_k$.

  By agreement $@_k\langle D\rangle j \in \Gamma$. As $@_i[\overline{\geq}]\langle D\rangle j \in \Gamma$ by $([\overline{\geq}]\text{-}\langle\leq\rangle)$ we have $@_k\langle\leq\rangle i \in \Gamma$. By case (i) we are done. $\square$

Lemma 8 (Atom Lemma) *Let Γ be a decomposed $\mathbf{K}_{\mathsf{gem}}$-MCS, and let $\mathcal{M}$ be the named model yielded by Γ. Then $\alpha \in u \in W \iff \mathcal{M}, u \models \alpha$.*

Proof $(\Rightarrow)$ Let $\alpha \in u \in W$. For some nominal i, $u = \Delta_i$. As $\alpha \in u$, $@_i\alpha \in \Gamma$. Let $v \leq \Delta_i$. For some nominal j, $v = \Delta_j$. It suffices to show $\Delta_i = \Delta_j$. So it suffices to show $i \in \Delta_j$, by Lemma 4. We have $\Delta_j \leq \Delta_i$. Thus for all formulas ϕ, $\phi \in \Delta_j$ implies $\langle\leq\rangle\phi \in \Delta_i$. Since $i \in \Delta_i$, $\langle\leq\rangle i \in \Delta_j$. By agreement $@_j\langle\leq\rangle i \in \Gamma$. So $@_i\alpha \land @_j\langle\leq\rangle i \in \Gamma$ by consistency and maximality. By the (Clip) axiom, $@_j i \in \Gamma$.

Hence $i \in \Delta_j$ as required. ($\Leftarrow$) Let $\mathcal{M}, u \models \alpha$. So $\forall y \in W(y \leq u \Rightarrow y = u)$. As $\mathcal{M}$ is named, there is a nominal i such that $V(i) = \{u\}$. So $\forall y \in W(y \leq u \Rightarrow i \in y)$. Toward contradiction assume $@_i[\geq]i \notin \Gamma$. Then $\neg@_i[\geq]i \in \Gamma$ by maximality. By (Self Dual) $@_i\neg[\geq]i \in \Gamma$. So $\neg[\geq]i \in \Delta_i$ and by (Dual $\geq$), $\langle\geq\rangle\neg i \in \Delta_i$. By part (ii) of the existence lemma, there is a $v \in \mathcal{M}$ such that $v \leq u$ and $\neg i \in v$—a contradiction. Thus $@_i[\geq]i \in \Gamma$. By axiom (α-$[\geq]$), $@_i\alpha$. Hence $\alpha \in \Delta_i$. $\qquad\square$

Lemma 9 (Truth Lemma) *Let $\mathcal{M} = (W, \leq, 1, V)$ be the named model yielded by a decomposed $\mathbf{K}_{\mathsf{gem}}$-MCS Γ, and let $u \in W$. Then $\forall\phi \in \mathcal{H}_{\mathsf{gem}}$, $\mathcal{M}, w \models \phi \Longleftrightarrow \phi \in w$.*

Proof By induction on the complexity of ϕ. *Base Case:* The α case follows by the atom lemma. Cases for atomic formulas are well known. *Inductive Step:* The case for the Boolean connectives and $\langle\leq\rangle$, $\langle\geq\rangle$ are well-known in tense logic [5]. We show cases for $\langle\overline{\leq}\rangle$ and $\langle\sim\rangle$.

- $\langle\overline{\leq}\rangle$: $\mathcal{M}, w \models \langle\overline{\leq}\rangle\phi \Longleftrightarrow \exists v \in W(w \not\leq v \wedge \mathcal{M}, v \models \neg\phi) \Longleftrightarrow w \not\leq v \wedge \neg\phi \in v$ by the inductive hypothesis. It suffices to show $\langle\overline{\leq}\rangle\phi \in w$, since the converse implication: $\langle\overline{\leq}\rangle\phi \in w \Longrightarrow \exists v \in W$ $(w \not\leq v \wedge \neg\phi \in v)$ follows by part (v) of the existence lemma.

  - *Claim: $\exists v \in W$ and a nominal $i \in w$ such that $\neg\langle\geq\rangle i \in v$. Proof.* As $\mathcal{M}$ is named there is a nominal $i \in w$. Suppose toward contradiction that $\neg\langle\geq\rangle i \notin v$. By consistency and maximality of v, $\langle\geq\rangle i \in v$. By the existence lemma, $\exists u \in W$ such that $u \leq v$ and $i \in u$. By Lemma 4 part (ii), $u = \Delta_i = w$, thus $w \leq v$—a contradiction.

  So $\neg\langle\geq\rangle i \in v$. As $\neg\phi \in v$, $\neg\langle\geq\rangle i \wedge \neg\phi \in v$. By axiom (E), $\mathsf{E}(\neg\langle\geq\rangle i \wedge \neg\phi) \in v$. Then by axiom ($\langle\overline{\leq}\rangle$-$\langle\geq\rangle$), $@_i\langle\overline{\leq}\rangle\phi \in v$. Hence $@_i\langle\overline{\leq}\rangle\phi \in \Gamma$ by agreement. So $\langle\overline{\leq}\rangle\phi \in w$.

- $\langle\sim\rangle$: $\mathcal{M}, w \models \langle\sim\rangle\phi \Longleftrightarrow \exists v \in W(\sim w = v \wedge \mathcal{M}, v \models \phi) \Longleftrightarrow \sim w = v \wedge \phi \in v$ by the inductive hypothesis. It suffices to show $\langle\sim\rangle\phi \in w$, since the converse implication: $\langle\sim\rangle\phi \in w \Longrightarrow \exists v \in W$ $(\sim w = v \wedge \phi \in v)$ follows by part (ix) of the existence lemma.

  - *Claim: There is a nominal $i \in w$ such that $[\geq]\langle\mathsf{D}\rangle i \wedge [\overline{\geq}]\langle\mathsf{D}\rangle i \in v$. Proof.* Let $i \in w$.

    - *Subclaim 1: $[\geq]\langle\mathsf{D}\rangle i \in v$. Proof.* Toward contradiction let $[\geq]\langle\mathsf{D}\rangle i \notin v$. By maximality, definition of $\langle\mathsf{D}\rangle$, and Dual $\geq$, $\langle\geq\rangle[\overline{\mathsf{O}}]i \in v$. By part (viii) of the existence lemma, $\exists u \in W$ such that $u \leq v$ and $[\overline{\mathsf{O}}]i \in u$. Now u has some name j. So $@_j[\overline{\mathsf{O}}]i \in u$. By agreement $@_j[\overline{\mathsf{O}}]i \in \Gamma$. By ref, $@_i i \in \Gamma$. By $[\overline{\mathsf{O}}]$-$\langle\mathsf{O}\rangle$, $@_i\langle\geq\rangle\langle\leq\rangle j \in \Gamma$. Thus $\langle\geq\rangle\langle\leq\rangle j \in w$ and by (i) and (ii) of the existence lemma $\exists z(z \leq w \wedge z \leq u \wedge u \leq v)$ contradicting $\sim w = v$.

    - *Subclaim 2: $[\overline{\geq}]\langle\mathsf{D}\rangle i \in v$. Proof.* Toward contradiction let $[\overline{\geq}]\langle\mathsf{D}\rangle i \notin v$. By Dual$\overline{\geq}$, $\langle\overline{\geq}\rangle[\overline{\mathsf{O}}]i \in v$. By the existence lemma part (ii) $\exists x \in W$

such that $x \not\leq v$ and $\neg[\overline{O}]i \in x$. By the existence lemma part (viii) we have $\exists y \neg x O y$ and $i \in y$. Hence $y = \Delta_i = w$ and therefore $\exists x (x D w \wedge x \not\leq v)$ contradicting $\sim w = v$.

Thus $[\geq]\langle D \rangle i \wedge [\overline{\geq}]\langle D \rangle i \in v$ and $[\geq]\langle D \rangle i \wedge [\overline{\geq}]\langle D \rangle i \wedge \phi \in v$. By the existence axiom (E), $\mathsf{E}([\geq]\langle D \rangle i \wedge [\overline{\geq}]\langle D \rangle i \wedge \phi) \in v$. By axiom $(\langle \sim \rangle\text{-}\langle D \rangle)$, $@_i \langle \sim \rangle \phi \in v$. By agreement, $@_i \langle \sim \rangle \phi \in \Gamma$. Hence $\langle \sim \rangle \phi \in w$. $\square$

Theorem 3 (General Completeness) *Every $\mathbf{K_{gem}}$-consistent set of $\mathcal{H}_{gem}$-formulas is satisfied on a countable named hybrid model.*

Proof Given a $\mathbf{K_{gem}}$-consistent set of formulas Σ, use the Lindenbaum Lemma to expand it to a decomposed set Σ^+ in a countable language $\mathcal{L}'$. Let $\mathcal{M} = (W, \leq, 1, V)$ be the named model yielded by Σ^+. As Σ^+ is named, by item (iv) of Lemma 4, $\Sigma^+ \in W$. By the truth lemma, $\mathcal{M}, \Sigma^+ \models \Sigma$. The model is countable because each state is named by some $\mathcal{L}'$ nominal, and there are only countably many of these. $\square$

There are analogous $\mathcal{H}_m$, $\mathcal{H}_o$, $\mathcal{H}_{ba}$, $\mathcal{H}_{bo}$, and $\mathcal{H}_{bem}$ axiom systems complete w.r.t the class of structures. We must only change Definition 4 slightly. Logics $\mathbf{K_{ba}}$, $\mathbf{K_{bo}}$, and $\mathbf{K_{bem}}$ for the languages $\mathcal{H}_{ba}$, $\mathcal{H}_{bo}$, and $\mathcal{H}_{bem}$, respectively, require the removal of all axioms containing operators outside the language. And the α-interaction axioms (Clip) and $(\alpha\text{-}[\leq])$ are switched for

$$(@_i \alpha \wedge @_j (\langle \leq \rangle i \wedge \neg \mathbf{0})) \rightarrow @_i j$$

$$@_i \alpha \leftrightarrow @_i (\neg \mathbf{0} \wedge \neg \langle \geq \rangle (\neg \mathbf{0} \wedge \neg i)).$$

And in addition for $\mathcal{H}_{bem}$, we change the axioms $(\langle \overline{O} \rangle\text{-}\langle O \rangle)$, $([\overline{O}]\text{-}\langle O \rangle)$, $(\langle \sim \rangle\text{-}\langle D \rangle)$ and rules (INV3) and (SIM) in an obvious way to ones for the operator $\langle V \rangle$. General logics $\mathbf{K_{hm}}$, $\mathbf{K_{ba}}$, $\mathbf{K_o}$, $\mathbf{K_{bo}}$, for the languages $\mathcal{H}_{hm}$, $\mathcal{H}_{ba}$, $\mathcal{H}_o$, $\mathcal{H}_{bo}$, respectively, are obtained by removing the required axioms. And as $\mathbf{K_{hm}}$ and $\mathbf{K_{ba}}$ are pure logics containing no formulas with proposition symbols, for every occurrence of p and q in Definition 4, we substitute formula ϕ and ψ respectively. Thus in $\mathbf{K_{hm}}$ and $\mathbf{K_{ba}}$, each such axiom will be an axiom *schema*. In all cases the lemmas leading up to the completeness result go through analogously. The following is then immediate:

Corollary 2 (General Completeness) *(i) Any $\mathbf{K_{hm}}$/$\mathbf{K_{ba}}$-consistent set of $\mathcal{H}_m$/$\mathcal{H}_{ba}$-formulas is satisfied on a countable named pure hybrid model. (ii) Every $\mathbf{K_o}$/$\mathbf{K_{bo}}$/$\mathbf{K_{bem}}$-consistent set of $\mathcal{H}_o$/$\mathcal{H}_{bo}$/$\mathcal{H}_{bem}$-formulas is satisfied on a countable named hybrid model.*

$\mathbf{K_{gem}}$+GEM GEM is the infinite set of $\mathcal{H}_{gem}$-formulas consisting of the axioms for partial orders, extensionality, $\mathsf{A}\langle \leq \rangle \mathbf{1}$, and finally the axiom schema $\mathsf{E}\phi \rightarrow \mathsf{E}\langle \bigoplus_a \rangle \phi$, where ϕ is any *pure* formula of $\mathcal{H}_{gem}$. Where $\mathbf{1}$ appears, no other nominals will be substituted for it.

Let F be a class of frames. A normal hybrid logic Λ is sound w.r.t F if for all formulas ϕ and all frames $\mathcal{F} \in \mathsf{F}, \vdash_\Lambda \phi \Rightarrow \mathcal{F} \models \phi$.

Proposition 11 *The axiom system* $\mathbf{K}_{\mathsf{gem}} + \mathsf{GEM}$ *is sound w.r.t the class of GEMSs.*

Proof Let $\mathcal{F}$ be a GEMS. *Claim*: $\mathbf{K}$ is sound. Let $\mathcal{F} \models [\leq](\phi \rightarrow \psi) \wedge [\leq]\phi$. For any hybrid valuation V, $(\mathcal{F}, V) \models [\leq](\phi \rightarrow \psi) \wedge [\leq]\phi$. Therefore we have $\forall w \in \mathcal{F}, (\mathcal{F}, V), w \models [\leq](\phi \rightarrow \psi) \wedge [\leq]\phi$. So $(\mathcal{F}, V), w \models [\leq](\phi \rightarrow \psi)$ and $(\mathcal{F}, V), w \models [\leq]\phi$. Then $\forall v \in \mathcal{F}$, $w \leq v$ implies $(\mathcal{F}, V), v \models \phi \rightarrow \psi$ and $(\mathcal{F}, V), v \models \phi$. By modus ponens $(\mathcal{F}, V), v \models \psi$. So $(\mathcal{F}, V), v \models [\leq]\psi$ as required. *Claim*: The dual axioms are sound. we do just the case for $\leq$. The others are analogous. Let $\mathcal{F} \models \langle\geq\rangle\phi$. For any hybrid valuation V, $(\mathcal{F}, V) \models \langle\geq\rangle\phi$. Thus $\forall w \in \mathcal{F}, (\mathcal{F}, V), w \models \langle\geq\rangle\phi$. Then $\exists v \in \mathcal{F}$ such that $w \leq v$ and $(\mathcal{F}, V), v \models \phi$. Thus it is not the case that $\forall v \in \mathcal{F}$ $w \leq v$ implies $(\mathcal{F}, V), v \models \neg\phi$. Hence $(\mathcal{F}, V), v \models \neg[\leq]\neg\phi$ as required; and similarly for the other direction.

That the interaction axioms are all sound is trivial since they are semantic equivalences which are valid with respect to the dominance relation $\leq$. And it is equally clear that the decomposition rules are all sound. The final nine modal naming rules are trivially sound. It therefore remains to check that the GEM axioms are sound. Observe that by Proposition 2 the definitions of the operators is sufficient. Thus each represents an instance of the FO GEMS axioms and is therefore sound. $\square$

Corollary 3 $\mathbf{K}_{\mathsf{gem}} + \mathsf{GEM}$ *is sound and complete w.r.t the class of GEMS frames.*

Proof Soundness follows by the previous proposition. By general completeness, every consistent set of $\mathbf{K}_{\mathsf{gem}}$ formulas is satisfied on a countable named model $\mathcal{M} = (\mathcal{F}, V)$ of mereological type. As the GEM axioms are pure, by Proposition 2, $\mathcal{F}$ is a GEMS. $\square$

5.1 Philosophical Benefits of the Modal Approach to Mereology

Indexicality in $\mathcal{H}_\mathsf{m}$ $\mathbf{K}_{\mathsf{hm}}$ can also serve as a nominalized mereo-egocentric logic in the sense of Prior [31]. Let's consider the following sentences:

(1) I have an arm and my arm has an elbow.
(2) Tom has an arm and his arm has an elbow.

Suppose Tom utters (1) and we wish to evaluate various mereological facts from his first-personal standpoint. By thinking of him, his arm, and is elbow as individuals, (1) and (2) are translated into $\mathcal{H}_\mathsf{m}$ as:

(3) $i_{\text{Tom}} \wedge \langle>\rangle(i_{\text{arm}} \wedge \langle>\rangle i_{\text{elbow}})$
(4) $@_{i_{\text{Tom}}}\langle>\rangle(i_{\text{arm}} \wedge \langle>\rangle i_{\text{elbow}})$

(3) is an indexical statement supplying content about the utterer, Tom. In (1) Tom states that he has an arm, and by using "I" supplies information that he is referring to himself. Next consider a series of inferences that can be made in $\mathbf{K}_{\mathsf{hm}} + \mathcal{MM}$ where

$\mathcal{MM}$ is the set of $\mathcal{H}_m$-axioms for partial orders and supplementation. Formula (5) below is an instance of (Intro).

(5) $\quad \vdash_{\mathbf{K}_{hm}+\mathcal{MM}} \quad i_{Tom} \wedge \langle > \rangle(i_{arm} \wedge \langle > \rangle i_{elbow}) \rightarrow @_{i_{Tom}}\langle > \rangle(i_{arm} \wedge \langle > \rangle i_{elbow})$

(6) $\quad \vdash_{\mathbf{K}_{hm}+\mathcal{MM}} \quad i_{Tom} \wedge \langle > \rangle(i_{arm} \wedge \langle > \rangle i_{elbow}) \rightarrow @_{i_{Tom}}\langle > \rangle i_{elbow}$

(7) $\quad \vdash_{\mathbf{K}_{hm}+\mathcal{MM}} \quad i_{Tom} \wedge \langle > \rangle(i_{arm} \wedge \langle > \rangle i_{elbow}) \rightarrow \mathsf{E}(\langle < \rangle i_{Tom} \wedge \langle \mathsf{D} \rangle i_{elbow})$

(8) $\quad \vdash_{\mathbf{K}_{hm}+\mathcal{MM}} \quad @_{i_{Ana}}((i_{Tom} \wedge \langle > \rangle(i_{arm} \wedge \langle > \rangle i_{elbow})) \rightarrow \mathsf{E}(\langle < \rangle i_{Tom} \wedge \langle \mathsf{D} \rangle i_{elbow}))$

The transitivity of proper extension $>$, is valid on any frame of $\mathcal{MM}$. Thus by the completeness of $\mathbf{K}_{hm}+\mathcal{MM}$ with respect to $\mathcal{MM}$, (6) can be derived from (5). We pass from (6) to (7) by supplementation, and from (7) to (8) by @-generalization. So what is interesting with the present hybrid approach is that there is both indexical expressivity and the ability to "lift" the interpretation to a global, third-personal perspective. We moved from a first-personal perspective *at* John. But then the same mereological facts will hold also at Ana.

One obvious deficiency in translating (1) and (2) into $\mathcal{H}_m$ is that there is no way to characterize particular things as *types*. In $\mathcal{H}_o$ there is no such problem:

(10) $\quad i_{Tom} \wedge \langle > \rangle(i_{arm} \wedge p_{arm} \wedge \langle > \rangle(i_{elbow} \wedge p_{elbow}))$

(11) $\quad @_{i_{Tom}}\langle > \rangle(i_{arm} \wedge p_{arm} \wedge \langle > \rangle(i_{elbow} \wedge p_{elbow}))$

And therefore we can derive analogous inferences as those in (5) through (8) above with the properties characterizing the various individuals. The addition of the proposition symbols will be nominalistically acceptable only if we view these as, in some sense, non-abstract. And this is implied by the modal semantics: properties hold *at* locations. For an interesting example of this way of understanding properties, see Armstrong [1].

Mereological changes plus indexicality in $\mathcal{H}_o$ Indexical expressivity is one consideration in favor of the modal approach, but it will only get us so far. In $\mathcal{H}_o$ propositional variables can be understood as representing the presence of an object that exists in various locations. For example, consider the famous case of Tibbles the cat [7]. Times on this view are parts: maximal fusions of all three-dimensional objects located within the three dimensions of a feline body. The propositional variable represents the existence of Tibbles at various times. At particular time i (where 'i' is a nominal), Tibbles is involved in an accident and tragically loses his tail. Suppose that k is before i. Tibbles' change can then be expressed in one sentence:

(12) $\quad @_k(\langle > \rangle(j_{Body1} \wedge p_{Tibbles} \wedge \langle > \rangle i_{tail})) \wedge @_i(\langle > \rangle(j_{Body2} \wedge p_{Tibbles} \wedge \neg\langle > \rangle i_{tail}))$

That is, we think of the bodies of Tibbles as ordered sequentially, and each such body "tibblizes". So a passage of egocentric facts is then representable by sequences of open formulae. Let $j_{Body1}...j_{Body\lambda}$ be names for a sequence of λ bodies before and after his accident.

(13) $\quad \langle(j_{Body1} \wedge p_{Tibbles} \wedge \langle > \rangle i_{tail}), ..., (j_{Body\lambda} \wedge p_{Tibbles} \wedge \neg\langle > \rangle i_{tail})\rangle$

Obviously this approach will only be fully realized within the context of a temporal language and logic with temporal operators. Perhaps a mereological tense language

or one for products can be employed. We note these possible extensions to the language in passing as other avenues of research.

6 Invariance

Definition 6 (Mereobisimulation) Let $\mathcal{M} = (W, \leq, V)$ and $\mathcal{M}' = (W', \leq', V')$ be two models. A nonempty binary relation $Z \subseteq W \times W'$ is called a *mereobisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ (notation: $Z : \mathcal{M} \triangleq \mathcal{M}'$) if the following conditions are satisfied:

1. $wZw' \Longrightarrow (V^{-1}(\{w\}) = V'^{-1}(\{w'\}))$.
2. $(V(i) = \{w\}$ and $V'(i) = \{w'\}$ for some $i \in \Omega) \Longrightarrow wZw'$.
3. $wZw' \Longrightarrow (At(w) \Leftrightarrow At(w'))$.
4. $(wZw'$ and $w \leq v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' \leq' v')$ (*Back*).
5. $(wZw'$ and $w' \leq' v') \Longrightarrow \exists v \in W(vZv'$ and $w \leq v)$ (*Forth*).
6. $(wZw'$ and $w \geq v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' \geq' v')$ (*Back*).
7. $(wZw'$ and $w' \geq' v') \Longrightarrow \exists v \in W(vZv'$ and $w \geq v)$ (*Forth*).
8. $(wZw'$ and $w \nleq v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' \nleq' v')$ (*Back*).
9. $(wZw'$ and $w' \nleq' v') \Longrightarrow \exists v \in W(vZv'$ and $w \nleq v)$ (*Forth*).
10. $(wZw'$ and $w \ngeq v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' \ngeq' v')$ (*Back*).
11. $(wZw'$ and $w' \ngeq' v') \Longrightarrow \exists v \in W(vZv'$ and $w \ngeq v)$ (*Forth*).

If, in addition to the conditions above, $Z \subseteq W \times W'$ satisfies the following condition:

14. If wZw', then w and w' satisfy the same proposition letters.

then we say that Z is a *ontobisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ (notation: $Z : \mathcal{M} \stackrel{\circ}{=} \mathcal{M}'$). If in addition to conditions 1-14 we have:

15. $(wZw'$ and $w < v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' <' v')$ (*Back*).
16. $(wZw'$ and $w' <' v') \Longrightarrow \exists v \in W(vZv'$ and $w < v)$ (*Forth*).
17. $(wZw'$ and $w > v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' >' v')$ (*Back*).
18. $(wZw'$ and $w' >' v') \Longrightarrow \exists v \in W(vZv'$ and $w > v)$ (*Forth*).
19. $(wZw'$ and $w\mathsf{D}v) \Longrightarrow \exists v' \in W'(vZv'$ and $w'\mathsf{D}v')$ (*Back*).
20. $(wZw'$ and $w'\mathsf{D}v') \Longrightarrow \exists v \in W(vZv'$ and $w\mathsf{D}v)$ (*Forth*).
21. $(wZw'$ and $w =\sim v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' =\sim v')$ (*Back*).
22. $(wZw'$ and $w' =\sim v') \Longrightarrow \exists v \in W(vZv'$ and $w =\sim v)$ (*Forth*).

then we say that Z is a *gem-bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ (notation: $Z : \mathcal{M} \simeq \mathcal{M}'$). We write $w \triangleq w'$ if those states are mereobisimilar, $w \stackrel{\circ}{=} w'$ if they are onto-bisimilar, and $w \simeq w'$ if they are gem-bisimilar. And we let $w \leftrightsquigarrow_{\mathcal{L}} w'$ denote that those states are indistinguishable by $\mathcal{L}$-formulas, for some language $\mathcal{L}$.

Theorem 4 *Let $\mathcal{M}, \mathcal{M}'$ be two models. (a) Then for every $w \in W$ and $w' \in W'$, $w \triangleq w'$ implies that $w \leftrightsquigarrow_{\mathcal{H}_m} w'$. (b) $w \in W$ and $w' \in W'$, $w \stackrel{\circ}{=} w'$ implies that $w \leftrightsquigarrow_{\mathcal{H}_o} w'$. (c) $w \in W$ and $w' \in W'$, $w \simeq w'$ implies that $w \leftrightsquigarrow_{\mathcal{H}_{gem}} w'$.*

The converses of the above are easy to prove for a restricted case. We say that $\mathcal{M}$ is *image finite* if for each state $u \in \mathcal{M}$, the set $\{(w, v) \mid w \le v\}$ is finite. We name this the Mereo-Hennessy Milner Theorem given its similarity to that seminal result. The proof is entirely analogous to the original.

Theorem 5 (Mereo-Hennessy-Milner Theorem) *Let $\mathcal{M}$ and $\mathcal{M}'$ be two image-finite models. Then, for every $w \in W$ and $w' \in W'$, $w \simeq w'$ iff $w \leftrightsquigarrow_{\mathcal{H}_{\mathsf{gem}}} w'$.*

By $\mathcal{H}(O_1, ..., O_n, \alpha)$ we denote the recursively defined hybrid language obtained by a countable set of nominals Ω, the atom constant α, closed under the Boolean operations, and operators $O_1, ..., O_n$. It follows from Theorem 4 that if $\{O_1, ..., O_n\}$ is a subset of the operators in $\mathcal{H}_{\mathsf{gem}}$, then a corresponding invariance result for $\mathcal{H}(O_1, ..., O_n, \alpha)$ follows. We now show that that our language is well motivated in terms of its selection of operators. That is, by properly restricting the set of operators, we shall not be able to define important Boolean and mereological operators.
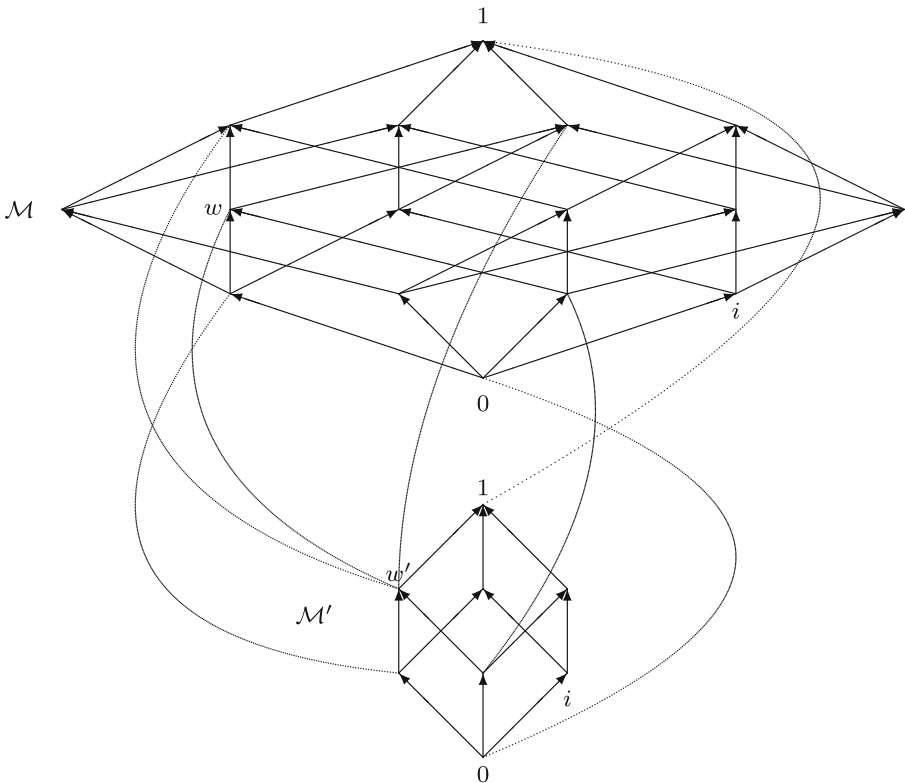
**Fig. 1** Models for Proposition 13. Transitive edges and reflexive loops are omitted

Proposition 12 *Over BAs (GEMS frames), no formula in $\mathcal{H}([\leq], [\geq], \alpha)$ expresses $\langle\mathsf{C}\rangle i$ ($\langle\sim\rangle i$) with respect to an arbitrary nominal i.*

Proof Consider the two BA-models in Fig. 1. $\mathcal{M}$ is the atomic BA-model with exactly 4 atoms with just one named element except for 1 and 0. All nominals name this element in $\mathcal{M}$. $\mathcal{M}'$ is the atomic BA-model with 3 atoms again with just one named element. All nominals name this element in $\mathcal{M}'$. The denotation of i in both models is neither the top nor the bottom. In both models there is a single point that is named i. Now $\mathsf{C}\sum V(i) = w'$ but $w \neq \mathsf{C}\sum V'(i)$. However $w \leftrightsquigarrow_{\mathcal{H}([\leq],[\geq],\alpha)} w'$. The restriction of the models and the bisimulation to those elements properly above 0 demonstrates the corresponding result over GEMS frames for $\langle\mathsf{C}\rangle i$. $\qquad\square$

Proposition 13 *Over the general class of structures, no formula in $\mathcal{H}([\leq], [\geq], [\overline{\geq}], \alpha)$ expresses $\langle\sqcup\rangle(i, j)$ with respect to arbitrary nominals i and j.*

Proof Consider the two models in Fig. 2. $\mathcal{M}$ is a GEMS-model with two atoms. $\mathcal{M}'$ is an atomic model with two atoms. All nominals besides j and i name the denotation of j in $\mathcal{M}$ and the denotation of j in $\mathcal{M}'$. Thus $w = \sqcup(\sum V(i), \sum V(j)) \in \mathcal{M}$, and $w' \neq \sqcup(\sum V'(i), \sum V'(j)) \in \mathcal{M}'$. However, $w \leftrightsquigarrow_{\mathcal{H}([\leq],[\geq],[\overline{\geq}],\alpha)} w'$ by the relation indicated in the figure. $\qquad\square$

As is only suitable for a nominalistic language, $\mathcal{H}_\mathsf{m}$ has virtually no pure arithmetical expressive capabilities. This is observed in the following proposition.
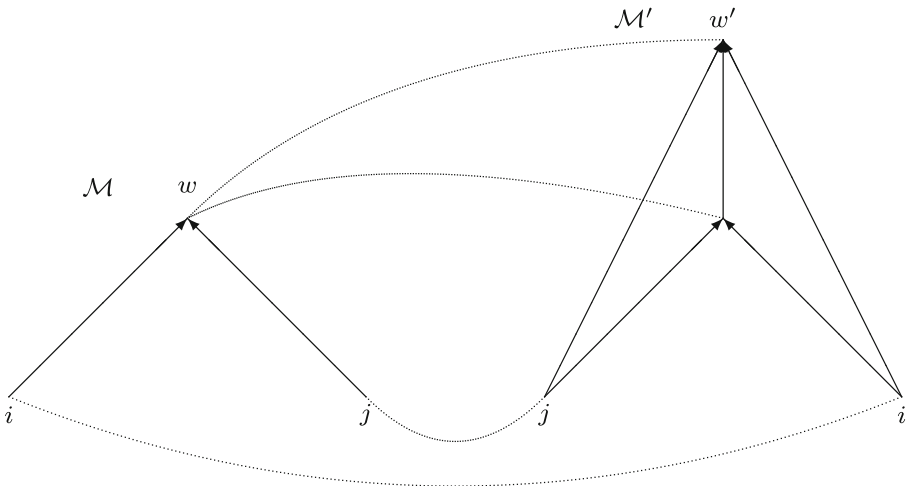
Fig. 2 Models for Proposition 14. Reflexive loops have been omitted

Proposition 14 *No formula in $\mathcal{H}_m$ expresses that there are exactly $n > 2$ objects.*

Proof Take two hybrid models $(W, \{\}, V)$, $(W', \{\}, V')$ such that $|W| = 2$ and $|W'| = n$ such that $n > 2$. The elements of both models are atoms. Assume that all nominals name one and only one point in each model. It is easy to see that there exists a mereobisimulation between them. □

Arithmetical properties over BAs ultimately boil down to arithmetical statements about atoms. A natural question is whether over BAs and GEMSs there are $\mathcal{H}_m$-expressions like "there are at least n-atoms." Now $\alpha \wedge \langle \leq \rangle (\neg \alpha \wedge \langle \not\leq \rangle \alpha)$ implies that there are at least two atoms. And observe that $\bigwedge_{1 \leq k \leq n} @_{i_k} \alpha \wedge \bigwedge_{1 \leq k \neq l \leq n} @_{i_k} \neg i_l$ implies there are n *named* atomic states $i_1, ..., i_n$. But in general there is no such formula.

Proposition 15 *Over BAs and GEMSs, there exists no formula in $\mathcal{H}_m$ expressing that there are at least n atoms for $n > 2$.*

Proof Consider the BA-models in Fig. 3 which both are unnamed except for the top and bottom. Let each nominal that is neither **0** nor **1** have as a denotation the top. One is the BA with 3 atoms and the other is the BA with 4. Let Z be the following relation: $Z = \{(x, x') \in (\mathcal{M}/\{0, 1\}) \times (\mathcal{M}'/\{0, 1\}) \mid At(x) \Leftrightarrow At(x')\} \cup \{(1, 1), (0, 0)\}$. The result over GEMS structures follows by removing the bottoms from the models. □

7 Characterization

We now show which formulas are equivalent to the standard translation of an $\mathcal{H}_{gem}$-formula. This is done in a fashion totally analogous to Johan van Benthem's original characterization of modal logic [6]. The $\mathcal{H}_o$ and $\mathcal{H}_m$ cases are immediate consequences.

Definition 7 (Hennessy-Milner Classes) We say a class K of models of mereological type have the Hennessy-Milner Property if for every two models of mereological type $\mathcal{M}, \mathcal{M}' \in \mathsf{K}$ and any two states w, w' of $\mathcal{M}$ and $\mathcal{M}'$, respectively, $w \leftrightsquigarrow_{\mathcal{H}_{gem}} w$ implies $\mathcal{M}, w \simeq \mathcal{M}', w'$.

We now introduce a notion of modal completeness. To explain informally, suppose that we are working over a model $\mathcal{M}$ of mereological type with unit and $w \in \mathcal{M}$ where w has successors $v_0, v_1, v_2, ...$ and, respectively, $\phi_0, \phi_0 \wedge \phi_1, \phi_0 \wedge \phi_1 \wedge \phi_2, ...$ hold. If there is no successor v of w where all formulas from Σ hold *at the same time*, then the model is in some sense *modally incomplete*. To formalize the corresponding notion of completeness observe the following definition.
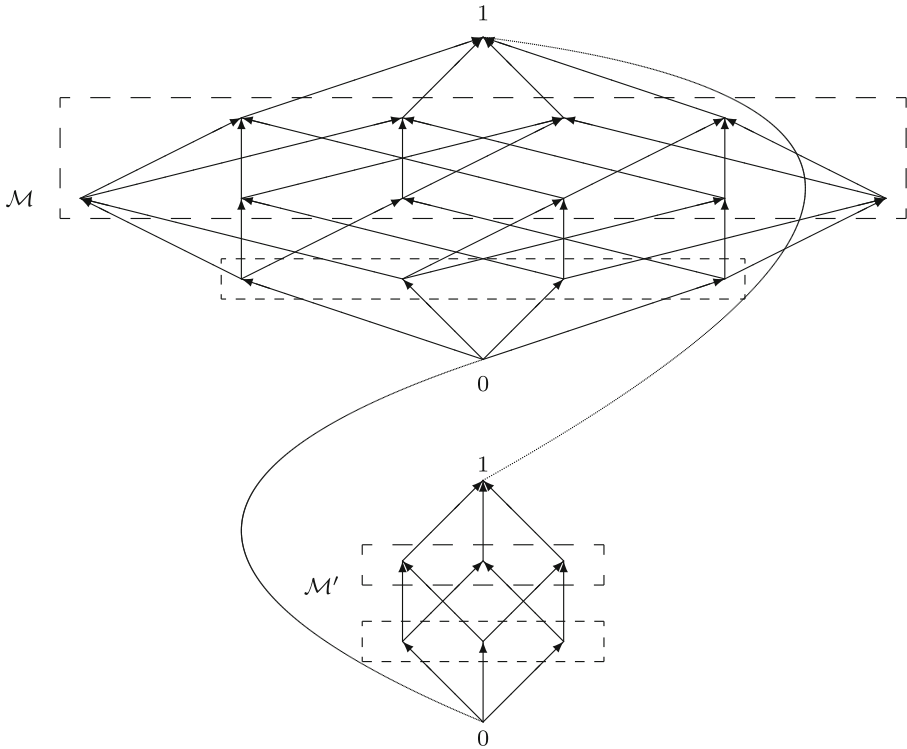
Fig. 3 Models for Proposition 16. Transitive edges and reflexive loops are omitted

Definition 8 (Modal Saturation) Assume $\mathcal{M}$ be a model of mereological type, $X \subseteq W$, and Σ a set of $\mathcal{H}_{\mathsf{gem}}$-formulas. Σ is *satisfiable* in the set X if there is a state $x \in X$ such that $\mathcal{M}, x \models \phi$ for all ϕ in Σ; Σ is *finitely satisfiable* in X if every finite subset of Σ is satisfiable in X. The model $\mathcal{M}$ is called *modally saturated* or *m-saturated*, for short, if it satisfies the following conditions for every state $w \in W$ and every set Σ of $\mathcal{H}_{\mathsf{gem}}$-formulas:

– *If Σ is finitely satisfiable in the set of $\leq$-successors of w, then Σ is satisfiable in the set of $\leq$-successors of w.*
– *If Σ is finitely satisfiable in the set of $\geq$-successors of w, then Σ is satisfiable in the set of $\geq$-successors of w.*
– *If Σ is finitely satisfiable in the set of $<$-successors of w, then Σ is satisfiable in the set of $<$-successors of w.*
– *If Σ is finitely satisfiable in the set of $>$-successors of w, then Σ is satisfiable in the set of $>$-successors of w.*
– *If Σ is finitely satisfiable in the set of $\nleq$-successors of w, then Σ is satisfiable in the set of $\nleq$-successors of w.*
– *If Σ is finitely satisfiable in the set of $\ngeq$-successors of w, then Σ is satisfiable in the set of $\ngeq$-successors of w.*

- *If Σ is finitely satisfiable in the set of D-successors of w, then Σ is satisfiable in the set of D-successors of w.*

Proposition 16 *Let K be the class of models of mereological type $(W, \leq, 1, V)$. Then the class $\mathsf{K}' \subseteq \mathsf{K}$ of m-saturated models of K has the Hennessy-Milner Property.*

Proof It suffices to prove that the relation $\longleftrightarrow_{\mathcal{H}_{\mathsf{gem}}}$ between states in $\mathcal{M}$ and states in $\mathcal{M}'$ (where $\mathcal{M}, \mathcal{M}'$ are any members of K') is an ontobisimulation. The conditions concerning the nominals, proposition symbols are trivially satisfied, as is the case for the atom constant. The forth and back conditions are analogously proved and are virtually immediate by the definition of m-saturation. We do just the forward case for $\leq$.

Let $\mathcal{M} = (W, \leq, 1, V)$ and $\mathcal{M}' = (W', \leq', 1', V')$ be models of mereological type. Assume that $w, v \in W$ and $w' \in W'$ are such that $w \leq v$ and $w \longleftrightarrow_{\mathcal{H}_{\mathsf{gem}}} w'$. Let Σ be the set of formulas true at v. It is clear that for every finite subset Δ of Σ we have $\mathcal{M}, v \models \bigwedge \Delta$. Hence $\mathcal{M}, w \models \langle \leq \rangle \bigwedge \Delta$. As $w \longleftrightarrow_{\mathcal{H}_{\mathsf{gem}}} w'$, $\mathcal{M}', w' \models \langle \leq \rangle \bigwedge \Delta$, so w' has an $\leq'$-successor v_Δ such that $\mathcal{M}', v_\Delta \models \bigwedge \Delta$. In other words, Σ is finitely satisfiable in the set of successors of w'; but then by m-saturation, Σ itself is satisfiable in a successor v' of w'. Thus $v \longleftrightarrow_{\mathcal{H}_{\mathsf{gem}}} v'$. $\square$

Definition 9 (Filters and Ultrafilters) Let W be a non-empty set. A *filter F* over W is a set $F \subseteq \mathcal{P}(W)$ such that (i) $W \in F$, (ii) If $X, Y \in F$, then $X \cap Y \in F$, and (iii) $X \in F$ and $X \subseteq Z \subseteq W$ implies $Z \in F$. A filter is called *proper* if it is distinct from $\mathcal{P}(W)$. An *ultrafilter over W* is a proper filter U such that for all $X \in \mathcal{P}(W)$, $X \in U$ if and only if $(W/X) \notin U$.

Suppose that $I \neq \varnothing$, U is an ultrafilter over I, and for each $x \in I$, W_x is a non-empty set. Let $C = \Pi_{x \in I} W_x$ be the cartesian product of those sets. That is: C is the set of all functions f with domain I such that for each $x \in I$, $f(x) \in W_x$. For two functions $f, g \in C$ we say that f and g are *U-equivalent* (notation $f \sim_U g$) if $\{x \in I \mid f(x) = g(x)\} \in U$. It is easy to check that $\sim_U$ is an equivalence relation on C.

Definition 10 (Ultraproducts of Sets) Let f_U be the equivalence class of f modulo $\sim_U$, that is $f_U = \{g \in C \mid g \sim_U f\}$. The *ultraproduct of W_x modulo U*, denoted as $\Pi_U W_x$, is the set of all equivalence classes of $\sim_U$. So $\Pi_U W_x = \{f_U \mid f \in \Pi_{x \in I} W_x\}$. If every W_x is identical (i.e. if $W_x = W$ for all $x \in I$), the ultraproduct is called the *ultrapower of W modulo U*, and written $\Pi_U W$.

Definition 11 (Ultraproduct of Hybrid Models of Mereological Type) Let $\mathcal{M}_x (x \in I)$ be a set of models of mereological type. The *ultraproduct $\Pi_U \mathcal{M}_x$ of $\mathcal{M}_x$ modulo U* is the model described as follows:

(i) The universe W_U of $\Pi_U \mathcal{M}_x$ is the set $\Pi_U W_x$, where W_x is the universe of $\mathcal{M}_x$.

(ii)   Let V_x be the hybrid valuation of $\mathcal{M}_x$. Then the hybrid valuation V_U and distinguished elements 1_U and 0_U of $\Pi_U \mathcal{M}_x$ are defined by

$$f_U \in V_U(p) \iff \{x \in I \mid f(x) \in V_x(p)\} \in U \quad for\ p \in \Phi$$

$$\{f_U\} = V_U(i) \iff \{x \in I \mid \{f(x)\} = V_x(i)\} \in U \quad for\ i \in \Omega$$

$$1_U \quad = \quad \{(x, 1_x) \mid x \in I\}_U$$

(iii)   Let $\leq_x$ be the dominance relation in the model $\mathcal{M}_x$. The relation $\leq_U$ in $\Pi_U \mathcal{M}_x$ is given by $f_U \leq_U g_U \iff \{x \in I \mid f(x) \leq_x g(x)\} \in U$.

(iv)   Finally we have some definitions. Let $At(f_U) \iff \{x \in I \mid At_x(f(x))\} \in U$, where $At_x(y)$ indicates that in model $\mathcal{M}_x$ we have $At(y)$.

Proposition 17 *Let $\Pi_U \mathcal{M}$ be an ultrapower of $\mathcal{M}$ where $\mathcal{M}$ is a model of mereological type $(W, \leq, 1)$. Then $\forall \phi \in \mathcal{H}_{\mathsf{gem}} : \mathcal{M}, w \models \phi \iff \Pi_U \mathcal{M}, (f_w)_U \models \phi$, where f_w is the constant function such that $f_w(x) = w$, for all $x \in I$.*

Proof   Proof by induction on ϕ.

Base Case $\mathcal{M}, w \models i \iff \{w\} = V(i) \iff \{x \in I \mid \{w\} = V_x(i)\} \in U \iff$
$\{x \in I \mid \{f_w(x)\} = V_x(i)\} \in U \iff \{(f_w)_U\} = V_U(i) \iff \Pi_U \mathcal{M}, (f_w)_U \models i$.
$\quad \mathcal{M}, w \models \alpha \iff At(w) \iff \{x \in I \mid At_x(w)\} \in U \iff \{x \in I \mid At_x(f_w(x))\} \in U \iff At((f_w)_U) \iff \Pi_U \mathcal{M}, (f_w)_U \models \alpha$. The case for proposition letters is standard.

Inductive Step   To prove closure under negation requires that U is an ultrafilter and in particular $X \in U$ if and only if $(W/X) \notin U$. The other Boolean cases are easy. We do only the case for the inverse operator $\langle \overline{\leq} \rangle \phi$.

$$\mathcal{M}, w \models \langle \overline{\leq} \rangle \phi \iff \exists v \in W\ w \not\leq v \text{ and } \mathcal{M}, v \not\models \phi$$
$$\iff \{x \in I \mid f_w(x) \not\leq_x f_v(x)\} \in U \text{ and}$$
$$\Pi_U \mathcal{M}, (f_v)_U \not\models \phi \text{ [by IH]}$$
$$\iff \{x \in I \mid f_w(x) \leq_x f_v(x)\} \notin U \text{ and } \Pi_U \mathcal{M}, (f_v)_U \not\models \phi$$
$$\text{[as U is an ultrafilter]}$$
$$\implies \exists (f_v)_U \in \Pi_U \mathcal{M}\ ((f_w)_U \not\leq (f_v)_U \text{ and } \Pi_U \mathcal{M}, (f_v)_U \not\models \phi)$$
$$\implies \Pi_U \mathcal{M}, (f_w)_U \models \langle \overline{\leq} \rangle \phi$$

$\square$

Let $\Gamma(x)$ be a set of FO formulas in which a single individual variable x may occur free. We call $\Gamma(x)$ a *type*. We say that a FO model $\mathcal{M}$ *realizes* a type $\Gamma(x)$ if there is an element $w \in \mathcal{M}$ such that for all $\gamma \in \Gamma(x)$, $\mathcal{M} \models \gamma[w]$.

Assume that $\mathcal{M}$ is a model for a given FO language $\mathcal{L}^1$ with domain W. For a subset $A \subseteq W$, $\mathcal{L}^1[A]$ is the language obtained by extending $\mathcal{L}^1$ with new constants $\underline{a}$ for all elements $a \in A$. $\mathcal{M}_A$ is the expansion of $\mathcal{M}$ to a structure for $\mathcal{L}^1[A]$ in which each $\underline{a}$ is interpreted as a. We now recall the notion of κ-saturated models.

Definition 12 (κ-saturated Models) Let κ be a natural number or ω. A model $\mathcal{M}$ is κ-*saturated* if for every subset $A \subseteq W$ of size less than κ, the expansion $\mathcal{M}_A$ realizes every set $\Gamma(x)$ of $\mathcal{L}^1[A]$-formulas (with only x occurring free) that is consistent with the FO theory of $\mathcal{M}_A$. An ω-saturated model is called *countably saturated*.

Lemma 10 (Hennessy-Milner property) *Let $\mathcal{M}$ be an model of mereological type with unit. If $\mathcal{M}$ is countably saturated, then it is m-saturated. It follows that the class of countably saturated models of mereological type has the Hennessy-Milner property.*

Proof Assume that $\mathcal{M}$ is of mereological type and, viewed as a FO model, is countably saturated. We do only the case for the $\leq$-relation. The others are similar. Let a be a state in W, and consider a set of Σ of $\mathcal{H}_{\mathsf{gem}}$-formulas which is finitely satisfiable in the $\leq$-successor set. Define Σ' to be $\Sigma' = \{\underline{a} \leq x\} \cup ST_x(\Sigma)$, where $ST_x(\Sigma)$ is the set $\{ST_x(\phi) \mid \phi \in \Sigma\}$ of standard translations of formulas in Σ. Clearly, Σ' is consistent with the FO theory of $\mathcal{M}_a$: $\mathcal{M}_a$ realizes every finite subset of Σ', namely in some successor of a. So, by the countable saturation of $\mathcal{M}$, Σ' is realized in some state b. By $\mathcal{M}_a \models \underline{a} \leq x[b]$ it follows that b is a successor of a. By Proposition 6 and $\mathcal{M}_a \models ST_x(\phi)[b]$ for all $\phi \in \Sigma$, it follows that $\mathcal{M}, b \models \Sigma$. So Σ is satisfiable in a successor of $\underline{a}$. $\square$

An ultrafilter is *countably incomplete* if it is not closed under countable intersections (but just closed under finite intersections). For example, consider an ultrafilter over $\mathbb{N}$ which does not contain any singletons $\{n\}$. Then, for any n, $(\mathbb{N}/\{n\}) \notin U$. But $\varnothing = \bigcap_{n \in \mathbb{N}}(\mathbb{N}/\{n\}) \notin \mathbb{N}$. Thus U is countably incomplete. The following is a standard result. See [8, Theorem 6.1.1].

Lemma 11 *Let $\mathcal{L}$ be a countable FO language, U a countably incomplete ultrafilter over a non-empty set I, and $\mathcal{M}$ an $\mathcal{L}$-model. The ultrapower $\Pi_U \mathcal{M}$ is countably saturated.*

Lemma 12 (Detour Lemma) *Let $\mathcal{M}$ and $\mathcal{N}$ be models of mereological type and w and v states in $\mathcal{M}$ and $\mathcal{N}$, respectively. Then the following are equivalent:*

(i) *For all $\mathcal{H}_{\mathsf{gem}}$-formulas ϕ: $\mathcal{M}, w \models \phi \iff \mathcal{N}, v \models \phi$.*
(ii) *There exist ultrapowers $\Pi_U \mathcal{M}$ and $\Pi_U \mathcal{N}$ and a bismulation Z: $\Pi_U \mathcal{M}, (f_w)_U \simeq \Pi_U \mathcal{N}, (f_v)_U$ linking $(f_w)_U$ and $(f_v)_U$, where $f_w(f_v)$ is the constant function mapping every index to $w(v)$.*

Proof (ii) $\Rightarrow$ (i). By Proposition 17 $\mathcal{M}, w \models \phi$ iff $\Pi_U \mathcal{M}, (f_w)_U \models \phi$. By assumption this is equivalent to $\Pi_U \mathcal{N}, (f_v)_U \models \phi$ and the latter is equivalent to $\mathcal{N}, v \models \phi$. (i) $\Rightarrow$ (ii). Assume that for all $\mathcal{H}_{\mathsf{gem}}$-formulas ϕ we have $\mathcal{M}, w \models \phi$ iff $\mathcal{N}, v \models \phi$. We need to create bisimilar ultrapowers. Take the set of natural numbers $\mathbb{N}$ as the index set and let U be a countably incomplete ultrafilter (as in the example above). By Lemma 11, the ultrapowers $\Pi_U \mathcal{M}, (f_w)_U$ and $\Pi_U \mathcal{N}, (f_v)_U$ are countably saturated. Now $(f_w)_U$ and $(f_v)_U$ are $\mathcal{H}_{\mathsf{gem}}$-equivalent: for all $\mathcal{H}_{\mathsf{gem}}$-formulas ϕ we have $\Pi_U \mathcal{M}, (f_w)_U \models \phi$ iff $\Pi_U \mathcal{N}, (f_v)_U \models \phi$. This follows from the assumption that

w and v are $\mathcal{H}_{\mathsf{gem}}$-equivalent together with Proposition 17. Next use Lemma 10: as $(f_w)_U$ and $(f_v)_U$ are $\mathcal{H}_{\mathsf{gem}}$-equivalent and $\Pi_U \mathcal{M}$ and $\Pi_U \mathcal{N}$ are countably saturated, there is the required ontobisimulation Z. $\qquad\square$

Definition 13 A FO formula $\phi(x)$ in $\mathcal{L}^1$ in the signature of mereological type is *invariant for gem-bisimulations* if for all models $\mathcal{M}$ and $\mathcal{N}$ and all states w in $\mathcal{M}$, $v \in \mathcal{N}$, and all gem-bisimulations Z between $\mathcal{M}$ and $\mathcal{N}$ such that wZv, we have $\mathcal{M} \models \phi(x)[w]$ iff $\mathcal{N} \models \phi(x)[v]$.

Theorem 6 (Characterization Theorem) *Let $\phi(x)$ be a FO formula in $\mathcal{L}^1$, where the latter is in the signature of mereological types, Then $\phi(x)$ is invariant for gem-bisimulations if and only if it is equivalent to the standard translation of a $\mathcal{H}_{\mathsf{gem}}$-formula.*

Proof ($\Longleftarrow$) follows from Theorem 4. ($\Rightarrow$) With the detour lemma, this direction is proven analogously to that found in [3, p. 103]. $\qquad\square$

8 Conclusion

It is in $\mathcal{H}_{\mathsf{m}}$ that we arrived at a streamlined language for nominalistic mereology. Extending $\mathbf{K}_{\mathsf{hm}}$ with any of the pure formulas defining the mereological classes of frames in Table 3, gave rise to $\mathcal{H}_{\mathsf{m}}$-mereologics complete with respect to any traditional class of mereological structures. Although I have suggested that $\mathbf{K}_{\mathsf{hm}}$ is a maximally nominalistic alternative to FO extensional mereology, it will be too weak to define the property of unrestricted fusion and Boolean completeness. Even FOL shared this limitation. A benefit of $\mathcal{H}_{\mathsf{o}}$ is that it has the capability to define these principles. We did not delve deeply into $\mathcal{H}_{\mathsf{o}}$. This language and the corresponding logic $\mathbf{K}_{\mathsf{o}}$ usher in at least two important questions. Is there an axiomatization of unrestrictedly fused structures or complete Boolean algebras in $\mathcal{H}_{\mathsf{o}}$? And are the general logics we outlined decidable?

Note as well that there is a sense in which both ESOPHL and FOL are too strong. By Proposition 5, $\mathcal{H}_{\mathsf{m}}$ will be sufficient to capture any finite named frame up-to-isomorphism. So in the movement from finite models to infinite ones, mereological reasoning over unrestrictedly fused structures "skips a beat." Finite named structures are unrestrictedly fused and captured by a $\mathcal{H}_{\mathsf{m}}$-formula. But we must jump up to a fragment of SOL to capture reasoning over infinite classes of these structures. For by Kuusisto's [19] result we saw that ESOPHL is expressively equivalent to MSOL (in the relevant signature).

So to arrive at our main philosophical argument, suppose that the purpose of formal mereology is to model reasoning over the entire decompositional structure of concrete objects. If reality is infinitely articulated and closed under unrestricted fusion, an optimal system of extensional mereology must, by the downward Löwenheim Skolem theorem, be provided in a second-order system. And thus if impure sets are abstract, the goal of nominalistic mereology will be unachievable.

If, however, there are only finitely many material objects, there will be a maximally nominalistic system for mereology obtainable in $\mathcal{H}_m$.

There were also other benefits we observed concerning the modal approach to mereology. Indexical statements are expressible in $\mathcal{H}_m$ which admit of novel patterns in mereological reasoning. And mereological changes in which objects are multiply located were shown to be expressible in $\mathcal{H}_o$. Traditional FO mereologies lack these features. This indicates an important sense in which extensions like $\mathcal{H}_o$ of $\mathcal{H}_m$ with propositional variables may be anti-nominalistic. According to the robust version of nominalism I sketched in the introduction motivated by Leśniewski, propositional variables in $\mathcal{H}_o$ will denote "general objects" or unary properties. A nominalistic conception of reality and reasoning over mereological structures is therefore best represented by $\mathcal{H}_m$-logics. Consequently, situations in which objects like lifeforms change or engage in intentional movement and even those in which inanimate objects are in motion will not be representable. The vision of reality that emerges from $\mathcal{H}_m$ is thoroughly static and unmovable.

It is therefore intriguing to examine $\mathcal{H}_m$ in isolation. In particular, is there a sound and complete logic of infinite complete BAs in $\mathcal{H}_m$? And finally, this is related to an earlier intimation: there may be interesting uses for $\mathcal{H}_o$ in the context of mereological spatiotemporal logics over models for reasoning about changes in objects over time. Questions such as these connect mereology with contemporary spatial logic. And spatial reasoning over traditional models of space is of particular interest in nominalistic inquiry, since the criterion of concrete objects is framed in terms of spatiotemporality.

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