THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

CONTINUUM-MANY BOOLEAN ALGEBRAS OF THE FORM $\mathcal{P}(\omega)/\mathcal{I}$, $\mathcal{I}$ BOREL

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Abstract. We examine the question of how many Boolean algebras, distinct up to isomorphism, that are quotients of the powerset of the naturals by Borel ideals, can be proved to exist in ZFC alone. The maximum possible value is easily seen to be the cardinality of the continuum $2^{\aleph_0}$; earlier work by Ilijas Farah had shown that this was the value in models of Martin's Maximum or some similar forcing axiom, but it was open whether there could be fewer in models of the Continuum Hypothesis.

We develop and apply a new technique for constructing many ideals whose quotients must be nonisomorphic in any model of ZFC. The technique depends on isolating a kind of ideal, called *shallow*, that can be distinguished from the ideal of all finite sets even after any isomorphic embedding, and then piecing together various copies of the ideal of all finite sets using distinct shallow ideals. In this way we are able to demonstrate that there are continuum-many distinct quotients by Borel ideals, indeed by analytic P-ideals, and in fact that there is in an appropriate sense a Borel embedding of the Vitali equivalence relation into the equivalence relation of isomorphism of quotients by analytic P-ideals. We also show that there is an uncountable definable wellordered collection of Borel ideals with distinct quotients.

Typeset April 2, 2004.

0022-4812/00/0000-0000/$00.00

§1. Introduction and nomenclature.

1.1. Background. In [Far02], Farah asks the question "How Many Boolean Algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ Are There?", with the understanding that there is some definability criterion being imposed on $\mathcal{I}$, since if no such criterion is imposed then every Boolean algebra of cardinality $2^{\aleph_0}$ is isomorphic to one of the form $\mathcal{P}(\omega)/\mathcal{I}$ and there were known to be $2^{2^{\aleph_0}}$ such (pairwise nonisomorphic) Boolean algebras. In this work we shall address only the case where $\mathcal{I}$ is Borel.

On the face of it the answer might appear likely to be independent of ZFC; certainly it is possible for different models of ZFC to answer differently the question of whether two given Borel ideals have isomorphic quotients, even when the models are both wellfounded and have the same reals (so that the quotients being compared are identical across the models). For example, it follows from CH that $\mathcal{P}(\omega)/\text{Fin} \cong \mathcal{P}(\omega)/(\text{Fin} \times \emptyset)$, because both $\mathcal{P}(\omega)/\text{Fin}$ and $\mathcal{P}(\omega)/(\text{Fin} \times \emptyset)$ are $\aleph_1$-saturated in the model-theoretic sense (see [Far02, Proposition 6.1]). However it follows from OCA+MA that $\mathcal{P}(\omega)/\text{Fin} \ncong \mathcal{P}(\omega)/(\text{Fin}\times\emptyset)$ (see [Far00b, Corollary 3.4.5]). Here OCA stands for the Open Coloring Axiom; see [Far00b, Chapter 2] for definitions.

Thus Farah has addressed the question from two sides: In [Far00b] he looks at set-theoretic propositions consistent with ZFC, such as Martin's Maximum, that tend to minimize the opportunity for given definable ideals to have isomorphic quotients. On the other hand, in [Far02] he examines the question of what quotients must be isomorphic if CH holds, which tends to maximize the opportunity to find isomorphisms between definable structures of cardinality $2^{\aleph_0}$, and therefore (potentially) to minimize the number of isomorphism types. In this latter case he found many partial classification results, showing for example (given CH) that there are exactly two quotients, up to isomorphism, by dense density ideals, but leaving open the question of whether there are $2^{\aleph_0}$ (or indeed even infinitely many) distinct quotients by Borel (or even $\utilde{\mathbf{\Sigma}}^1_1$) ideals.

Steprāns, in [Step03], uses a variation on Sacks Forcing to show that there is a family of $2^{\aleph_0}$ distinct $\utilde{\mathbf{\Pi}}^0_3$ ideals on a certain *Polish lattice* (that is, a lattice ordering on a Polish space that is closed as a subset of the Cartesian product of the space with itself; an ideal on such a lattice is a subset closed downward and under join) that have pairwise nonisomorphic quotients. The method also works to give ideals on the natural numbers, but apparently at the cost of increasing the complexity to $\utilde{\mathbf{\Pi}}^1_1$. At this writing it is not clear whether the method can be refined to give $\utilde{\mathbf{\Pi}}^0_3$ ideals on the natural numbers; if so, it would provide an alternative proof of much of the content of Theorem 3.4.

It should be noted that Steprāns' method provides information that the present work does not; namely, he shows that two lattices (or two Boolean algebras, as the case may be) are nonisomorphic by showing that neither can be completely embedded into the other (indeed, that there is no complete embedding from the regular open algebra of one to the regular open algebra of the other).

In this work we provide an answer, in some sense maximal, to Farah's question. We show that there are $2^{\aleph_0}$ Borel ideals with pairwise-nonisomorphic quotients, and that these may be chosen to be analytic P-ideals, in particular $\utilde{\mathbf{\Pi}}^0_3$. It is not possible (in ZFC alone) to reduce this complexity to $\utilde{\mathbf{\Sigma}}^0_2$, because, by [JK84], CH implies that all $\underset{\sim}{\mathbf{\Sigma}}^0_2$ ideals have isomorphic quotients. We also give information about the *definable* cardinality of isomorphism types of Borel quotients; our results here do not appear to be maximal and leave room for further inquiry.

1.2. Basic definitions and nomenclature. By an *ideal* we shall always mean a collection $\mathcal{I}$ of subsets of a countably infinite index set I (usually $I = \omega$ or $I = \omega \times \omega$) such that

i) if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$
ii) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$
iii) if $A \subseteq I$ and A is finite, then $A \in \mathcal{I}$

Condition (iii) is not part of the standard definition of an ideal, but we include it to avoid trivialities.

Elements of $\mathcal{I}$ are said to be *$\mathcal{I}$-null*; subsets of I that are not $\mathcal{I}$-null are called *$\mathcal{I}$-positive*, and we write $\mathcal{I}^+$ for the collection of all $\mathcal{I}$-positive sets.

$\mathcal{I}$ induces an equivalence relation $\approx_{\mathcal{I}}$ on $\mathcal{P}(I)$ by

$$X \approx_{\mathcal{I}} Y \iff X \bigtriangleup Y \in \mathcal{I}$$

If $X \subseteq I$ we write $[X]_{\mathcal{I}}$ for the $\approx_{\mathcal{I}}$-equivalence class of X. We write $\mathcal{P}(I)/\mathcal{I}$ for the Boolean algebra whose underlying set is the collection of all $\approx_{\mathcal{I}}$-equivalence classes, and whose $\wedge$ and $\vee$ are induced by $\cap$ and $\cup$ respectively.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on index sets I and J respectively, we define

$$\mathcal{I} \times \mathcal{J} \triangleq \{A \subseteq I \times J | \{m \in I | \{n| < m, n > \in A\} \in \mathcal{J}^+\} \in \mathcal{I}\}$$

That is, the $\mathcal{I} \times \mathcal{J}$-positive subsets of $I \times J$ are the ones with $\mathcal{I}$-positively many $\mathcal{J}$-positive vertical sections.

Note that while we do not officially consider $\{\emptyset\}$ to be an ideal, we do define $\mathcal{I} \times \emptyset$ and $\emptyset \times \mathcal{I}$ as though it were (leaving off the braces around $\emptyset$). That is, a subset of $I \times \omega$ is $\mathcal{I} \times \emptyset$-positive if and only if it has $\mathcal{I}$-positively many nonempty vertical sections, whereas a subset of $\omega \times I$ is $\emptyset \times \mathcal{I}$-positive just in case it has no $\mathcal{I}$-positive vertical sections.

Ideals $\mathcal{I}$ and $\mathcal{J}$ are *Rudin–Keisler isomorphic*, $\mathcal{I} \approx_{RK} \mathcal{J}$, if (modulo null sets) there is a bijection between the underlying sets I and J that respects the ideals—that is, there are $A \in \mathcal{I}$, $B \in \mathcal{J}$, and a bijection $h : I \setminus A \to J \setminus B$ such that, for any $X \subseteq I$, $X \in \mathcal{I} \iff h"X \in \mathcal{J}$. (By $h"X$ we mean $\{h(n) | n \in X\}$.) If $\mathcal{I} \approx_{RK} \mathcal{J}$ then easily $\mathcal{P}(I)/\mathcal{I} \cong \mathcal{P}(J)/\mathcal{J}$ as Boolean algebras.

If A is $\mathcal{I}$-positive, we write $\mathcal{I} \restriction A$ for $\{X \subseteq I | X \cap A \in \mathcal{I}\}$. If $\mathcal{B}$ is a Boolean algebra and $x \in \mathcal{B}$, $x \neq \emptyset_{\mathcal{B}}$, we write $\mathcal{B} \restriction x$ for the Boolean algebra $\{y | y \leq_{\mathcal{B}} x\}$ with the Boolean operations inherited from $\mathcal{B}$. Clearly

$$\mathcal{P}(A)/\mathcal{I} \cong \mathcal{P}(I)/\left(\mathcal{I} \restriction A\right) \cong \left(\mathcal{P}(I)/\mathcal{I}\right) \restriction [A]_{\mathcal{I}}$$

via canonical isomorphisms.

Ideals in our context can never be closed under countable unions; the entire underlying set is the union of countably many singletons, and singletons are null. In some sense the closest we can get is the notion of a *P-ideal.* $\mathcal{I}$ is a P-ideal if, for any countable collection of $\mathcal{I}$-null sets $\{B_k | k \in \omega\}$ there's a $\mathcal{I}$-null set B that misses only finitely much of each B_k (that is, $(\forall k) B_k \setminus B \in \text{Fin}$).

An ideal on ω is a subset of $\mathcal{P}(\omega)$; on the latter we take the product topology. Any reference to the descriptive-set-theoretic complexity of an ideal (say, "Borel ideal", "analytic ideal", "$\underset{\sim}{\mathbf{\Pi}}^0_3$ ideal") should be understood in terms of that topology, as should any reference to Wadge reducibility between ideals.

Some specific ideals to which we shall make frequent reference:

Fin	the ideal of all finite subsets of ω
$\mathcal{Z}_0$	the ideal of all subsets of ω with asymptotic zero density (i.e. $X \in \mathcal{Z}_0 \iff \lim_{n\to\infty} \frac{\lvert X\cap n\rvert}{n} = 0$)
$\emptyset \times \mathrm{Fin}$	all subsets of $\omega \times \omega$ with no infinite vertical sections
$\mathrm{Fin} \times \emptyset$	all subsets of $\omega \times \omega$ with only finitely many nonempty vertical sections

§2. Construction of the ideals.

2.1. Motivation. The construction is an idea of Hjorth, who noted that it was possible to "integrate" ideals with respect to a partition of the natural numbers in such a way that the original ideal could be recovered from the order-theoretic properties of the quotient algebra corresponding to the "integral".

2.2. Formal definition.

Definition 2.1. *Given $\mathcal{I}$ an ideal on ω containing* Fin *and $\vec{A} = (A_n)_{n\in\omega}$ a sequence of disjoint subsets of ω (some possibly empty), we write:*

$$\mathcal{I}(\vec{A}) = \{X \subseteq \omega \mid \forall n\ X \cap A_n \in \mathrm{Fin} \wedge \{n \mid X \cap A_n \neq \emptyset\} \in \mathcal{I}\}$$

Definition 2.2. *Given $\vec{A} = (A_n)_{n\in\omega}$ a sequence of disjoint subsets of ω (some possibly empty), we write:*

$$P_{\vec{A}} = \{n \in \omega \mid A_n \text{ is infinite}\}$$

It may be easier to think of $\mathcal{I}(\vec{A})$ in terms of the positive sets: A set is $\mathcal{I}(\vec{A})$-positive just in case it either has infinite intersection with some A_n, or meets $\mathcal{I}$-positively many A_n. For example, any infinite A_n is itself an $\mathcal{I}(\vec{A})$-positive set, below which $\mathcal{P}(\omega)/\mathcal{I}(\vec{A})$ is isomorphic to $\mathcal{P}(\omega)/\mathrm{Fin}$. On the other hand, given any $\mathcal{I}$-positive set C, we can choose one element (say, the least) from A_n for each $n \in C$; the set of these is now a $\mathcal{I}(\vec{A})$-positive set below which $\mathcal{P}(\omega)/\mathcal{I}(\vec{A})$ is isomorphic to $\mathcal{P}(\omega)/\mathcal{I}$ restricted to C. By choosing ideals $\mathcal{I}$ with a structural property distinguishing them from Fin (see Section 3.2.1 below) we can rule out isomorphisms of certain types between quotient algebras.

We should note as well that we can consider our ideals as living on any countably infinite set, say $\omega \times \omega$, and that if every A_n is infinite (that is, if $P_{\vec{A}} = \omega$) then $\mathcal{I}(\vec{A})$ is Rudin–Keisler isomorphic to the ideal $(\emptyset \times \mathrm{Fin}) \cap (\mathcal{I} \times \emptyset)$ on $\omega \times \omega$.

The following simple facts will come in handy:

Lemma 2.1. *Let $\mathcal{I}$ be an ideal and $\vec{A} = (A_n)_{n\in\omega}$ a sequence of disjoint subsets of ω.*

i) *For any $X \subseteq \omega$, if we write X_n for $X \cap A_n$ and $\vec{X}$ for the sequence of X_n, then*

$$\mathcal{I}(\vec{X}) = \mathcal{I}(\vec{A}) \upharpoonright X$$

ii) *For any $X \subseteq \omega$, if we let*

$$A'_n \triangleq \begin{cases} A_n & \text{if } n \in X \\ \emptyset & \text{otherwise} \end{cases}$$

and write $\vec{A} \restriction X$ for the sequence of A'_n, then

$$(\mathcal{I} \restriction X)(\vec{A} \restriction X) = \mathcal{I}(\vec{A} \restriction X)$$

iii) *If every A_n is either empty or a singleton and $\{n | A_n \neq \emptyset\} \in \mathcal{I}^+$, then $\mathcal{I}(\vec{A})$ is RK-isomorphic to the restriction of $\mathcal{I}$ to a positive set.*

$\dashv$

§3. Non-isomorphism results on the quotients by the ideals $\mathcal{I}(\vec{A})$.

3.1. Connection between "input" ideals and structure of Boolean algebra.

Lemma 3.1. *Let $\mathcal{I}$ be an ideal on ω containing* Fin, *let $\vec{A} = (A_n)_{n\in\omega}$ be a sequence of disjoint subsets of ω, and let $\mathcal{I}(\vec{A})$ be as defined above. Then for every $C \subseteq P_{\vec{A}}$, $C \in \mathcal{I}$ if and only if $\{A_n | n \in C\}$ has a least upper bound with respect to $\mathcal{I}(\vec{A})$ (that is, $\{[A_n]_{\mathcal{I}(\vec{A})} | n \in C\}$ has a least upper bound in $\mathcal{P}(\omega)/\mathcal{I}(\vec{A})$).*

Proof.
$\Rightarrow$: The least upper bound will be represented by $X \triangleq \bigcup_{n\in C} A_n$. Clearly this is an upper bound. Suppose that Y is also a representative for an upper bound. Then for each $n \in C$, $(X \setminus Y) \cap A_n = A_n \setminus Y$ must be finite, as otherwise Y would not be above A_n with respect to $\mathcal{I}(\vec{A})$. But these are the only n for which $(X \setminus Y) \cap A_n$ is nonempty, and C is $\mathcal{I}$-null. Therefore $X \setminus Y$ is $\mathcal{I}(\vec{A})$-null.

$\Leftarrow$: Suppose X (no longer defined as above) is a representative for the least upper bound. For each $n \in C$, $A_n \setminus X$ must be finite, so $X \cap A_n$ must be infinite, and in particular nonempty (this uses $n \in P_{\vec{A}}$). Form Y by removing from X one element of $X \cap A_n$ for each $n \in C$. Then Y must still be an upper bound, but if C were $\mathcal{I}$-positive we would have $Y <_{\mathcal{I}(\vec{A})} X$, contradicting the assumption that X is a least upper bound. Therefore C is $\mathcal{I}$-null. $\dashv$

3.2. Wadge reduction. The goal of this section, culminating in Theorem 3.1, is to show that, given ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ with a certain property (called *shallowness*), and given $\vec{A}$ a partition of ω into countably many infinite sets, if $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{J}_2(\vec{A})$, then $\mathcal{J}_1 \leq_W \mathcal{J}_2$.

3.2.1. *ω-partitions.* In [Far02], Proposition 6.1, Farah gives a list of equivalent conditions for an atomless ideal on ω to have a quotient that is not $\aleph_1$-saturated in the model–theoretic sense. This derives from a result of Just and Mijajlović; see [JM87]. We shall not make use of the model theory in this paper, and in the interest of independent readability we instead isolate the following equivalent:

Definition 3.1. *Given a Boolean algebra $\mathcal{B}$ and $x \in \mathcal{B}$, $x \neq 0_{\mathcal{B}}$, we say $(x_n)_{n\in\omega}$ is an ω-*partition *of x if:*

i) $\forall n$ ($x_n \leq x$ *and* $x_n \neq 0_{\mathcal{B}}$),

ii) $\forall n \neq m$ $x_n \wedge x_m = 0_{\mathcal{B}}$, *and*

iii) $\forall y \leq x[(\forall n\ y \wedge x_n = 0_{\mathcal{B}}) \Rightarrow y = 0_{\mathcal{B}}]$

DEFINITION 3.2. *Given an ideal $\mathcal{I}$ on ω and a subset X of ω, X $\mathcal{I}$-positive, we say $(X_n)_{n\in\omega}$ is an* ω-partition of X with respect to $\mathcal{I}$ *if:*

i) $\forall n$ *($X_n \leq_{\mathcal{I}} X$ and X_n is $\mathcal{I}$-positive),*
ii) $\forall n \neq m$ *($X_n \cap X_m$ is $\mathcal{I}$-null), and*
iii) $\forall Y \leq_{\mathcal{I}} X[(\forall n\ Y \cap X_n$ *is $\mathcal{I}$-null* $) \Rightarrow Y$ *is $\mathcal{I}$-null* $]$

Though not necessary for our current purposes, it is useful to note that the *nonexistence* of an ω-partition is equivalent, at least in our context, to what Farah calls countable saturation and Chang and Keisler (see [CK90], p. 256) call $\aleph_1$-saturation, as this provides an important constraint on nonisomorphism results from ZFC alone. If CH holds, then any two countably saturated Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ are isomorphic.

Also note that (iii) of Definition 3.2 implies that X is a least upper bound for the X_n in the order given by $\mathcal{I}$, and mutatis mutandis for Definition 3.1.

DEFINITION 3.3. *An ideal $\mathcal{I}$ is* shallow *if $\mathcal{I} \neq \mathcal{P}(\omega)$ and every $\mathcal{I}$-positive set has an ω-partition with respect to $\mathcal{I}$.*

We call these ideals "shallow" because they are the ones with respect to which there are no "deep" sets in the sense of [Far02]. It should be noted that shallowness is *not* a "smallness" or "simplicity" condition—there are shallow ideals of arbitrarily high complexity, and $\mathcal{P}(\omega)/\text{Fin}$ has smaller Borel cardinality than $\mathcal{P}(\omega)/\mathcal{Z}_0$, though Fin is not shallow (in fact *every* set is deep with respect to Fin) and $\mathcal{Z}_0$ is shallow.

It is easy to verify that $\mathcal{P}(\omega)/\text{Fin}$ has no ω-partition below any point. On the other hand we can get large collections of shallow ideals, whose quotients have ω-partitions below *every* point (see Section 4 below). Thus our plan for establishing nonisomorphism results between various quotients: We shall construct ideals with shallow restrictions and restrictions RK-isomorphic to Fin; an isomorphism cannot send anything below the former sort of point to anything below the latter sort.

3.2.2. *How an isomorphism must behave on the A_n.* Given sequences $\vec{A} = (A_n)_{n\in\omega}$ and $\vec{B} = (B_i)_{i\in\omega}$ of subsets of ω, each sequence pairwise disjoint, and shallow ideals $\mathcal{J}_1$ and $\mathcal{J}_2$, write $\mathcal{B}_1$ for $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A})$ and $\mathcal{B}_2$ for $\mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, and suppose

$$\phi : \mathcal{B}_1 \to \mathcal{B}_2$$

is an isomorphism.

Let us overload the symbol ϕ by choosing once and for all an arbitrary lift of ϕ to a function from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$, which we shall also call ϕ.

Now let $S \subseteq \omega \times \omega$ be the relation defined by

$$\begin{aligned} nSi &\iff \phi(A_n) \cap B_i \text{ is infinite} \\ &\iff [\phi(A_n) \cap B_i]_{\mathcal{J}_2(\vec{B})} > 0 \\ &\iff [\phi(A_n)]_{\mathcal{J}_2(\vec{B})} \wedge [B_i]_{\mathcal{J}_2(\vec{B})} > 0 \\ &\iff \phi\left([A_n]_{\mathcal{J}_1(\vec{A})}\right) \wedge [B_i]_{\mathcal{J}_2(\vec{B})} > 0 \end{aligned}$$

Note by the last equivalent that S does not depend on how we lifted ϕ.

LEMMA 3.2. *$\phi(A_n)$ is essentially the union of its infinite B_i pieces. That is, for each n, $\phi(A_n) \approx_{\mathcal{J}_2(\vec{B})} \bigcup_{i|nSi}[\phi(A_n) \cap B_i]$.*

PROOF. Clearly [1]

$$\begin{aligned} \phi(A_n) = &\bigcup_{i|nSi} [\phi(A_n) \cap B_i] \\ &\cup \bigcup_{i|\neg nSi} [\phi(A_n) \cap B_i] \\ &\cup \left[\phi(A_n) \setminus \bigcup_{i\in\omega} B_i\right] \end{aligned}$$

The final summand does not meet any B_i and so is $\mathcal{J}_2(\vec{B})$-null.

Thus if the claim fails, $\bigcup_{i|\neg nSi}[\phi(A_n) \cap B_i]$ is $\mathcal{J}_2(\vec{B})$-positive (call this set T). All the $T \cap B_i$ are finite, by the definition of S, so there must be a $\mathcal{J}_2$-positive set U of indices i on which $T \cap B_i$ is nonempty. For each $i \in U$, choose one element (say, the least) of $T \cap B_i$ and let D be the set of all these.

By Lemma 2.1(iii), the subsets of D modulo $\mathcal{J}_2(\vec{B})$ are an isomorphic copy of the restriction of $\mathcal{P}(\omega)/\mathcal{J}_2$ to some nonzero point. Therefore there is an ω-partition of D with respect to $\mathcal{J}_2(\vec{B})$. However, below the $\mathcal{J}_1(\vec{A})$ equivalence class of A_n, $\mathcal{B}_1$ is isomorphic to $\mathcal{P}(\omega)/\mathrm{Fin}$, and therefore there is no nonzero point below $[A_n]$ in $\mathcal{B}_1$ having an ω-partition. As ϕ is an isomorphism, this is a contradiction. ⊣

LEMMA 3.3. *There are only finitely many pieces as in Lemma 3.2. That is, for each n, $\{i|nSi\}$ is finite.*

PROOF. First observe that for any n, the set of i such that $\phi(A_n)$ meets B_i is $\mathcal{J}_2$-null. Otherwise, let D be the set of all least elements of nonempty sets of the form $\phi(A_n) \cap B_i$. Now $0 <_{\mathcal{J}_2(\vec{B})} D \leq_{\mathcal{J}_2(\vec{B})} \phi(A_n)$, so there is some $\mathcal{J}_1(\vec{A})$-positive $D' \leq_{\mathcal{J}_1(\vec{A})} A_n$ such that $\phi(D') \approx_{\mathcal{J}_2(\vec{B})} D$. But $\mathcal{B}_1$ restricted to D' is isomorphic to $\mathcal{P}(\omega)/\mathrm{Fin}$ and therefore there is no ω-partition below D' in $\mathcal{B}_1$, whereas $\mathcal{B}_2$ restricted to D is isomorphic to $\mathcal{P}(\omega)/\mathcal{J}_2$ restricted to the set of all i such that $\phi(A_n)$ meets B_i, so there *is* an ω-partition below D with respect to $\mathcal{J}_2(\vec{B})$. This is a contradiction.

Now suppose for some n, $\{i|nSi\}$ is infinite (but by the above argument, necessarily $\mathcal{J}_2$-null). Then $\phi(A_n)$ is above (by Lemma 3.2, actually equivalent to) $\bigcup_{i|nSi}[\phi(A_n) \cap B_i]$ in the order given by $\mathcal{J}_2(\vec{B})$. But a subset of $\bigcup_{i|nSi}[\phi(A_n) \cap B_i]$ is $\mathcal{J}_2(\vec{B})$-positive just in case it is infinite on at least one of the B_i (because the set of indices i being considered is $\mathcal{J}_2$-null). That means that $\mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$ restricted to $\bigcup_{i|nSi}[\phi(A_n) \cap B_i]$ is isomorphic to $\mathcal{P}(\omega \times \omega)/(\emptyset \times \mathrm{Fin})$. But it is easily seen that $\omega \times \omega$ has an ω-partition with respect to $\emptyset \times \mathrm{Fin}$, whereas there is no ω-partition below A_n with respect to $\mathcal{J}_1(\vec{A})$. This again is a contradiction. ⊣

[1] The notation $\bigcup_{i|nSi}$ is to be read "the union over all i such that nSi holds".

Lemma 3.4. *$\phi(A_n)$ is the least upper bound (mod $\mathcal{J}_2(\vec{B})$) of its infinite B_i pieces:*

$$[\phi(A_n)]_{\mathcal{J}_2(\vec{B})} = \bigvee_{i|nSi} [\phi(A_n) \cap B_i]_{\mathcal{J}_2(\vec{B})}$$

Proof. Immediate from Lemmata 3.2 and 3.3. ⊣

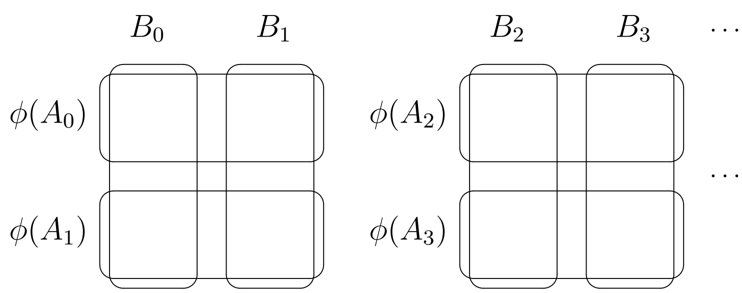
Lemma 3.5. *For each i, $B_i \approx_{\mathcal{J}_2(\vec{B})} \bigcup_{n|nSi}[\phi(A_n) \cap B_i]$, and moreover there are only finitely many n such that nSi. Therefore*

$$[B_i]_{\mathcal{J}_2(\vec{B})} = \bigvee_{n|nSi} \left[\phi\left([A_n]_{\mathcal{J}_1(\vec{A})}\right) \wedge [B_i]_{\mathcal{J}_2(\vec{B})}\right]$$

Proof. We apply Lemmata 3.2, 3.3 and 3.4 to a lift of ϕ^{-1}, noticing that the S relation for ϕ^{-1} is precisely the inverse of S.

⊣

Figure 1. Example isomorphism



3.2.3. *Example.* Suppose for example that for each n we have

$$2n\ S\ 2n$$
$$2n\ S\ 2n+1$$

and that no other pairs of natural numbers bear the S relation. That is, $\phi(A_{2n})$ is essentially the union of infinite pieces of B_{2n} and B_{2n+1}, and $\phi(A_{2n+1})$ occupies the "other half" of B_{2n} and B_{2n+1}. This situation is illustrated in Figure 1.

Then for a given set of natural numbers C, we know by Lemma 3.1 that $C \in \mathcal{J}_1$ if and only if $\{A_n | n \in C\}$ has a least upper bound with respect to $\mathcal{J}_1(\vec{A})$, which happens just in case $\{\phi(A_n) | n \in C\}$ has a least upper bound with respect to $\mathcal{J}_2(\vec{B})$; that is, if $D_0 \cup D_1$ has such a least upper bound, where

$$D_0 \triangleq \{\phi(A_{2n}) | 2n \in C\}$$
$$D_1 \triangleq \{\phi(A_{2n+1}) | 2n+1 \in C\}$$

which we can rewrite

$$D_0 = \{(\phi(A_{2n}) \cap B_{2n}) \cup (\phi(A_{2n}) \cap B_{2n+1}) | 2n \in C\}$$
$$D_1 = \{(\phi(A_{2n+1}) \cap B_{2n+1}) \cup (\phi(A_{2n+1}) \cap B_{2n+1}) | 2n+1 \in C\}$$

Now given any $X \subseteq \omega$, it is easy to see that X is an upper bound for $D_0 \cup D_1$ if and only if X is an upper bound for $D_{00} \cup D_{01} \cup D_{10} \cup D_{11}$, where

$$D_{00} \triangleq \{\phi(A_{2n}) \cap B_{2n} | 2n \in C\}$$
$$D_{01} \triangleq \{\phi(A_{2n}) \cap B_{2n+1} | 2n \in C\}$$
$$D_{10} \triangleq \{\phi(A_{2n+1}) \cap B_{2n} | 2n+1 \in C\}$$
$$D_{11} \triangleq \{\phi(A_{2n+1}) \cap B_{2n+1} | 2n+1 \in C\}$$

Therefore $C \in \mathcal{J}_1$ if and only if $D_{00} \cup D_{01} \cup D_{10} \cup D_{11}$ has a least upper bound with respect to $\mathcal{J}_2(\vec{B})$. The method used in Lemma 3.1 shows that such a least upper bound exists just in case the set of all indices represented in the D_{ij}, namely $\{2n, 2n+1 | 2n \in C \vee 2n+1 \in C\}$, is $\mathcal{J}_2$-null.

3.2.4. *Formal reduction.* The example in Section 3.2.3 suggests the following claim: for each $C \subseteq P_{\vec{A}}$,

$$C \in \mathcal{J}_1 \iff \{i | \exists n (nSi \wedge n \in C)\} \in \mathcal{J}_2$$

In the case where $P_{\vec{A}} = \omega$, this equivalence gives us a Wadge reduction demonstrating $\mathcal{J}_1 \leq_W \mathcal{J}_2$. To see this, we must check that the function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ given by $f(C) = \{i | \exists n(nSi \wedge n \in C)\}$ is continuous. Choose a basic open set U in the topology on $\mathcal{P}(\omega)$; say, let $\{a_0, \dots, a_k\}$ and $\{b_0, \dots, b_\ell\}$ be given disjoint finite sets of natural numbers and let

$$U = \{X \subseteq \omega | (\forall i \leq k)(a_i \in X) \wedge (\forall j \leq \ell)(b_j \notin X)\}$$

What we need is for $f^{-1}[U]$ to be open. Choose an element C of $f^{-1}[U]$. As $f(C) \in U$, there must be $n_0 \dots n_k$ taken from C such that $(\forall i \leq k)(n_k S a_k)$. Moreover no b_j is an element of $f(C)$, so for each $j \leq \ell$ and any n such that nSb_j, we have $n \notin C$; by Lemma 3.5 there are only finitely many such n.

Now given any C' such that $(\forall i \leq k)(n_i \in C')$ and such that C' does not contain any of the finitely many n bearing the S relation to any b_j, $j \leq \ell$, we have that $f(C') \in U$, so $C' \in f^{-1}[U]$. The collection of all such C' is an open neighborhood of C included in $f^{-1}[U]$, so $f^{-1}[U]$ is open.

3.2.5. *Proof of reduction.*

Theorem 3.1. *for each $C \subseteq P_{\vec{A}}$,*

$$C \in \mathcal{J}_1 \iff \{i | \exists n(nSi \wedge n \in C)\} \in \mathcal{J}_2$$

Proof. The proof is a generalization of the argument in the example in 3.2.3. We will argue that the following are equivalent:

i) $C \in \mathcal{J}_1$
ii) $\{A_n | n \in C\}$ has a least upper bound with respect to $\mathcal{J}_1(\vec{A})$
iii) $\{\phi(A_n) | n \in C\}$ has a least upper bound with respect to $\mathcal{J}_2(\vec{B})$
iv) $\{\phi(A_n) \cap B_i | nSi \wedge n \in C\}$ has a least upper bound with respect to $\mathcal{J}_2(\vec{B})$
v) $\{i | \exists n(nSi \wedge n \in C)\} \in \mathcal{J}_2$

The equivalence of (i) and (ii) is immediate from Lemma 3.1. (ii) is equivalent to (iii) because ϕ is an isomorphism.

To see that (iii) is equivalent to (iv), note that $D \triangleq \{\phi(A_n) | n \in C\}$ and $E \triangleq \{\phi(A_n) \cap B_i | nSi \wedge n \in C\}$ have the same collection of upper bounds with respect to $\mathcal{J}_2(\vec{B})$ by Lemma 3.4. The equivalence of (iv) and (v) follows by the method of proof of Lemma 3.1. ⊣

3.3. $\aleph_1$ distinct quotients.

Theorem 3.2. *There is an uncountable collection of Borel ideals on ω such that if $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals from the collection, then $\mathcal{P}(\omega)/\mathcal{I}_1$ is not isomorphic to $\mathcal{P}(\omega)/\mathcal{I}_2$. Moreover, there is a* definable *embedding from ω_1 into the isomorphism types of quotients by Borel ideals, in the sense that there is a Borel map $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, for $X, Y \subseteq \omega$ coding countable ordinals, $f(X)$ and $f(Y)$ are Borel codes for ideals, and their quotients are isomorphic if and only if X and Y code the same ordinal.*

Proof. In Section 4.1 below, we show that there is a (Borel in the codes) map $\alpha \mapsto \mathcal{J}_\alpha$ such that, for α a countable ordinal, $\mathcal{J}_\alpha$ is a shallow ideal, and such that if $\alpha < \beta$, then $\mathcal{J}_\alpha <_W \mathcal{J}_\beta$.

Now let $\vec{A} = (A_n)_{n\in\omega}$ be a partition of ω into infinite sets. Then if $X \subseteq \omega$ codes a countable ordinal α, we will let $f(X)$ be a code for $\mathcal{J}_\alpha(\vec{A})$. (If X does not code a countable ordinal, we do not care what $f(X)$ is.) The fact that there is a Borel such f follows from the Suslin–Kleene theorem.

To see that it works, note that if $\mathcal{P}(\omega)/\mathcal{J}_\alpha(\vec{A})$ were isomorphic to $\mathcal{P}(\omega)/\mathcal{J}_\beta(\vec{A})$, $\beta < \alpha$, then by Theorem 3.1 we would have $\mathcal{J}_\alpha$ Wadge-reducible to $\mathcal{J}_\beta$, contrary to the construction. ⊣

3.4. Continuum many distinct quotients by analytic P-ideals. The goal of this section is to show that there is a collection of $2^{\aleph_0}$ analytic P-ideals (therefore necessarily $\boldsymbol{\Pi}^0_3$) with pairwise nonisomorphic quotients. This is not simply a strengthening of Theorem 3.2 because that theorem establishes a "definable" injection from ω_1 into the isomorphism types of quotients by Borel ideals, which the result of this section will not.

3.4.1. *Preservation of analytic P-property.* Thanks to Farah for pointing out that this next result follows directly from the definition, making it unnecessary to appeal to Solecki's result that analytic P-ideals are precisely the exhaustions of lower semicontinuous submeasures (see [Sol99]). (It is by that result of Solecki that we know, as mentioned above, that all analytic P-ideals are $\boldsymbol{\Pi}^0_3$.)

Lemma 3.6. *If $\mathcal{I}$ is an analytic P-ideal, then so is $\mathcal{I}(\vec{A})$.*

Proof. Let $B_0, B_1, \ldots$ be a sequence of $\mathcal{I}(\vec{A})$-null sets of naturals. For each k let $C_k \triangleq \{n | A_n \cap B_k \neq \emptyset\}$; then C_k is $\mathcal{I}$-null.

As $\mathcal{I}$ is a P-ideal, there is some $\mathcal{I}$-null set C such that $C_k \setminus C$ is finite for every k. Now define

$$B \triangleq \{m | \exists k[m \in B_k \wedge \forall \ell(m \in A_\ell \Longrightarrow (\ell \geq k \wedge \ell \in C))]\}$$

Now $\{\ell | B \cap A_\ell \neq \emptyset\} \subseteq C \in \mathcal{I}$, and the intersection of B with a given A_ℓ is contained in the union of the intersections of finitely many B_k with A_ℓ (namely those with $k \leq \ell$); each of those intersections is finite (since $B_k \in \mathcal{I}(\vec{A})$), so $B \cap A_\ell$ is finite. Thus $B \in \mathcal{I}(\vec{A})$.

For each k, the elements of $B_k \setminus B$ are either in A_ℓ for $\ell < k$ (there can be only finitely many of these), or in A_ℓ for some $\ell \notin C$. However in the latter case we have $\ell \in C_k \setminus C$, and $C_k \setminus C$ is finite. Since each $B_k \cap A_\ell$ is finite, we get that $B_k \setminus B$ is finite. Thus B is the required witness demonstrating that $\mathcal{I}(\vec{A})$ is a P-ideal. (That $\mathcal{I}(\vec{A})$ is analytic is trivial quantifier-counting.) ⊣

(Actually the method gives that if $\mathcal{I}$ is a P-ideal, analytic or not, then so is $\mathcal{I}(\vec{A})$, but we will not make use of this.)

Remark. An alternative proof of Lemma 3.6 applies the result of Solecki to which we have made reference: If $\mathcal{I} = \mathrm{Exh}(\varphi)$ for a lower semicontinuous submeasure φ, then $\mathcal{I}(\vec{A}) = \mathrm{Exh}(\psi)$, where

$$\begin{aligned}\psi(X) &= \varphi(\{n | X \cap A_n \neq \emptyset\}) \\ &\quad + \frac{1}{m+1}, \text{ where } m \text{ is least such that } X \cap A_m \neq \emptyset\end{aligned}$$

See [Sol99] for definitions.

3.4.2. *Attempt using Borel reducibility.* It was originally hoped that the method of Section 3.2 would show, given that $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{J}_2(\vec{A})$, not merely that $\mathcal{J}_1 \leq_W \mathcal{J}_2$, but that $\mathcal{J}_1 \leq_B \mathcal{J}_2$ as equivalence relations. As Louveau points out, Borel reducibility of equivalence relations is in some sense the dimension-2 analogue of Wadge reducibility.

Had this held, then we could have used the ideals given by Louveau and Veličkovič in [LV94] to establish the result immediately (they give a collection of $2^{\aleph_0}$ analytic P-ideals such that none is Borel reducible to another as equivalence relations).

However there does not seem to be any direct way to establish this implication. The natural thing to try would be to establish, for $X, Y \subseteq \omega$, that

$$X \bigtriangleup Y \in \mathcal{J}_1 \iff f(X) \bigtriangleup f(Y) \in \mathcal{J}_2$$

where

$$f(X) \triangleq \{m \in \omega | (\exists n \in X) m S n\}$$

But that's false. Look again at the example in Section 3.2.3, illustrated in Figure 1, and take X to be the set of even natural numbers and Y to be the set of odd natural numbers. Then $X \bigtriangleup Y$ is ω, but $f(X) \bigtriangleup f(Y)$ is $\emptyset$.

Note that what goes wrong has to do with the fact that the S relation in this example is neither a function nor one-one. If we knew that S were a bijection, then the proposed reduction would not be merely a Borel reduction of equivalence relations but a Rudin–Keisler isomorphism between the ideals. I am indebted to Farah for the idea that we can make S do what we want by paring down the underlying set.

3.4.3. *More Technical Lemmata.* In this section we prove some easy, yet notationally messy, facts about the possible structure of ideals $\mathcal{I}(\vec{A})$. The reader may wish to skip ahead to Section 3.4.4 and refer back to this section as necessary.

LEMMA 3.7. *If $\mathcal{J}_1$ and $\mathcal{J}_2$ are shallow ideals and $\vec{A}$ and $\vec{B}$ are sequences of pairwise disjoint subsets of ω such that $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, and if $P_{\vec{A}}$ is $\mathcal{J}_1$-positive, then $P_{\vec{B}}$ is $\mathcal{J}_2$-positive.*

PROOF. This lemma will be proved below after two preliminary results. ⊣

Definition 3.4. *A set $X \subseteq \omega$ is* shallow with respect to an ideal $\mathcal{I}$ *if X is $\mathcal{I}$-positive and $\mathcal{I} \restriction X$ is shallow.*

(Note that it is stronger to say that a set is shallow than to say it is not deep —a shallow set is not only not deep; it has no deep sets below it.)

Lemma 3.8. *If $\mathcal{I}$ is shallow and $P_{\vec{A}}$ is $\mathcal{I}$-null, then one of the following six cases holds:*

i) $\mathcal{I}(\vec{A}) = \mathcal{P}(\omega)$
ii) $\mathcal{I}(\vec{A}) \approx_{RK} \mathrm{Fin}$
iii) $\mathcal{I}(\vec{A}) \approx_{RK} \emptyset \times \mathrm{Fin}$
iv) $\mathcal{I}(\vec{A})$ *is shallow*
v) $[\omega]_{\mathcal{I}(\vec{A})} = [X]_{\mathcal{I}(\vec{A})} \dot{\vee} [Y]_{\mathcal{I}(\vec{A})}$, *where $\mathcal{I}(\vec{A}) \restriction X \approx_{RK} \mathrm{Fin}$ and Y is $\mathcal{I}(\vec{A})$-shallow.* [2]
vi) $[\omega]_{\mathcal{I}(\vec{A})} = [X]_{\mathcal{I}(\vec{A})} \dot{\vee} [Y]_{\mathcal{I}(\vec{A})}$, *where $\mathcal{I}(\vec{A}) \restriction X \approx_{RK} \emptyset \times \mathrm{Fin}$ and Y is $\mathcal{I}(\vec{A})$-shallow.*

Proof. Let $Q_{\vec{A}} \triangleq \{n | A_n \in \mathrm{Fin} \wedge A_n \neq \emptyset\}$. Then $P_{\vec{A}}$ is either empty, finite nonempty, or infinite, and $Q_{\vec{A}}$ is either $\mathcal{I}$-null or $\mathcal{I}$-positive. The cases break down as follows:

	$P_{\vec{A}} = \emptyset$	$P_{\vec{A}}$ finite nonempty	$P_{\vec{A}}$ infinite
$Q_{\vec{A}} \in \mathcal{I}$	(i)	(ii)	(iii)
$Q_{\vec{A}} \in \mathcal{I}^+$	(iv)	(v)	(vi)

⊣

Lemma 3.9. *Suppose $\mathcal{I}$ is shallow and $P_{\vec{A}}$ is $\mathcal{I}$-positive. Then there is an $\mathcal{I}(\vec{A})$-shallow set, and moreover, given any $X \subseteq \omega$ such that X is $\mathcal{I}(\vec{A})$-shallow, there is $Y \subseteq \omega$ disjoint from X such that Y is also $\mathcal{I}(\vec{A})$-shallow.*

Proof. Let Y equal $\{k | (\exists n \in P_{\vec{A}})(k \text{ is the least element of } A_n)\}$. Then Y is $\mathcal{I}(\vec{A})$-positive, and $\mathcal{I}(\vec{A}) \restriction Y$ is Rudin–Keisler isomorphic to $\mathcal{I}$, which is shallow by hypothesis. Thus Y is $\mathcal{I}(\vec{A})$-shallow.

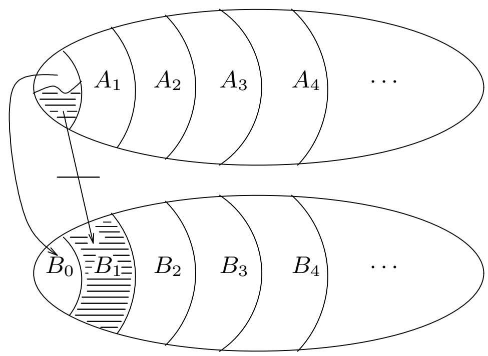
Given an $\mathcal{I}(\vec{A})$-shallow set X, for every n, $X \cap A_n$ must be finite, because otherwise $X \cap A_n$ would be an $(\mathcal{I}(\vec{A}) \restriction X)$-positive set without an ω-partition. So for each $n \in P_{\vec{A}}$ let $A'_n \triangleq A_n \setminus X$; each such A'_n is infinite and in particular nonempty. Now, much as before, let

$$Y \triangleq \{k | (\exists n \in P_{\vec{A}})(k \text{ is the least element of } A'_n)\}$$

then Y is disjoint from X and $\mathcal{I}(\vec{A})$-shallow. ⊣

Proof of Lemma 3.7. Suppose to the contrary that $P_{\vec{B}}$ is $\mathcal{J}_2$-null. Then $\mathcal{J}_2(\vec{B})$ falls into one of the six cases of Lemma 3.8. But cases (i), (ii) and (iii) are ruled out because they imply there is no $\mathcal{J}_2(\vec{B})$-shallow set, when there must be a $\mathcal{J}_1(\vec{A})$-shallow set because $P_{\vec{A}}$ is $\mathcal{J}_1$-positive. Cases (iv), (v) and (vi), pulled back via the isomorphism to $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A})$, would contradict the "moreover" clause of Lemma 3.9. ⊣

FIGURE 2. Paring an isomorphism



3.4.4. *Paring technique.* First we explain how, given an arbitrary isomorphism from $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A})$ onto $\mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, to cut down the underlying sets to get an isomorphism between pieces of these Boolean algebras, whose corresponding S relation is now a function. The technique will then be run in the other direction to make S into a bijection.

Each A_n is sent by ϕ to some subset of the union of all B_i with nSi (modulo a $\mathcal{J}_2(\vec{B})$-null difference). We take i_n to be the first such i, and we pare away, from the underlying set of $\mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, the image of A_n restricted to each of the *other* i with nSi. Then we must also pare away from the underlying set of $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A})$ the part of A_n that is sent to those other i. See Figure 2.

This now leaves an isomorphism between two Boolean algebras that are restrictions of the previous ones, such that in the new S relation i_n is the unique natural number satisfying nSi_n. Moreover the ideal formed by restricting $\mathcal{J}_1(\vec{A})$ is in fact Rudin–Keisler isomorphic to the original, because we had to leave an infinite piece of each A_n.

This description is not quite precise because we have not shown that ϕ sends the remaining part of the first algebra to the remaining part of the second, and in fact it may not, exactly. But if we define

$$\begin{aligned} i_n &\triangleq \text{least } i \text{ such that } nSi \text{ (for each } n\text{)} \\ D_n &\triangleq \phi(A_n) \cap B_{i_n} \\ A'_n &\triangleq A_n \cap \phi^{-1}(D_n) \\ X &\triangleq \bigcup_{n\in\omega} A'_n \\ B'_i &\triangleq \phi(X) \cap B_i \text{ (for each } i\text{)} \end{aligned}$$

then certainly ϕ restricts to an isomorphism

$$\phi : \mathcal{B}_1 \restriction [X]_{\mathcal{J}_1(\vec{A})} \cong \mathcal{B}_2 \restriction [\phi(X)]_{\mathcal{J}_2(\vec{B})}$$

[2]By $a \dot\vee b$, where a and b are elements of a Boolean algebra, we mean $a \vee b$, but imply as well that $a \wedge b = 0$.

Roughly, X is the part left after paring away the top half of Figure 2, and B'_i is what should be left of B_i. The technicality here is that we do not know (at least by the methods so far developed) that B'_i is empty in the case that B_i has been entirely pared away. However we do have:

Claim 3.1. *If $i = i_n$ for some n then B'_i is infinite, otherwise not.*

Proof. Suppose $i = i_n$. Then $[D_n]_{\mathcal{J}_2(\vec{B})}$ is positive, so

$$\begin{aligned}[B'_i]_{\mathcal{J}_2(\vec{B})} &= [\phi(X)]_{\mathcal{J}_2(\vec{B})} \wedge [B_i]_{\mathcal{J}_2(\vec{B})} \\ &\geq [\phi(A'_n)]_{\mathcal{J}_2(\vec{B})} \wedge [B_i]_{\mathcal{J}_2(\vec{B})} \\ &= [\phi(A_n)]_{\mathcal{J}_2(\vec{B})} \wedge [D_n]_{\mathcal{J}_2(\vec{B})} \wedge [B_i]_{\mathcal{J}_2(\vec{B})} \\ &= [D_n]_{\mathcal{J}_2(\vec{B})} \wedge [D_n]_{\mathcal{J}_2(\vec{B})} \\ &> 0\end{aligned}$$

Otherwise, for each of the finitely many n such that nSi, we have $[A'_n]_{\mathcal{J}_1(\vec{A})} \wedge [\phi^{-1}(B_i)]_{\mathcal{J}_1(\vec{A})} = 0_{\mathcal{B}_1}$, so $[B'_i]_{\mathcal{J}_2(\vec{B})} = 0_{\mathcal{B}_2}$. ⊣

So we know that the S relation of the pared-down isomorphism is a function (it sends n to i_n) and that it is a subset of the original S.

We can now establish

Theorem 3.3. *Given shallow ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ and sequences $\vec{A}$ and $\vec{B}$ such that $P_{\vec{A}}$ is $\mathcal{J}_1$-positive, and given that $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, there is an injective partial function $f : \omega \rightharpoonup \omega$ such that the domain of f is $\mathcal{J}_1$-positive, and such that for $X \subseteq \mathrm{dom}(f)$, $X \in \mathcal{J}_1 \iff f``X \in \mathcal{J}_2$.*

Proof. Given an isomorphism ϕ and working as above, writing $\vec{A}'$ and $\vec{B}'$ for the sequences $(A'_n)_{n\in\omega}$ and $(B'_i)_{i\in\omega}$ respectively, we obtain a restricted isomorphism

$$\phi : \mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}') \to \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B}')$$

such that the new S relation, call it $S^{(1)}$, given by

$$nS^{(1)}i \iff \phi\left([A'_n]_{\mathcal{J}_1(\vec{A}')}\right) \wedge [B'_i]_{\mathcal{J}_2(\vec{B}')} > 0$$

is a function.

Moreover by Lemma 3.7 we know that $P_{\vec{B}}$ is $\mathcal{J}_2$-positive; therefore the inverse isomorphism

$$\phi^{-1} : \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B}') \to \mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}')$$

can similarly be pared down to

$$\phi^{-1} : \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B}'') \to \mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}'')$$

and a new inverse S relation, $S^{(2)}$, given by

$$iS^{(2)}n \iff \phi^{-1}\left([B''_i]_{\mathcal{J}_2(\vec{B}'')}\right) \wedge [A''_n]_{\mathcal{J}_1(\vec{A}'')} > 0$$

Now the desired f is simply the inverse of $S^{(2)}$; we have $\mathrm{dom}(f) = P_{\vec{A}''}$, which again by Lemma 3.7 must be $\mathcal{J}_1$-positive.

⊣

The following is the main result of this Section 3.4:

THEOREM 3.4. *There are at least E_0-many distinct quotients by analytic P-ideals; that is, there is a Borel function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, for any $X, Y \subseteq \omega$, $f(X)$ and $f(Y)$ are Borel codes for analytic P-ideals, and the quotients of $\mathcal{P}(\omega)$ by said ideals are isomorphic as Boolean algebras if and only if $X \triangle Y$ is finite.*

PROOF. In Section 4.2 below, we describe a shallow ideal $\mathcal{Z}_{\vec{a}}$ and an injective algebra-of-sets homomorphism $\widehat{\ }: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, $C \mapsto \widehat{C}$, such that $\widehat{C}$ is $\mathcal{Z}_{\vec{a}}$-positive if and only if C is infinite, such that if $C \triangle D$ is finite then so is $\widehat{C} \triangle \widehat{D}$, and such that if C and D are almost disjoint, then there is no RK-reduction from any nontrivial piece of $\mathcal{Z}_{\vec{a}} \restriction \widehat{C}$ to $\mathcal{Z}_{\vec{a}} \restriction \widehat{D}$. That is, there is no injective partial function $h : \omega \rightharpoonup \omega$ such that $\mathrm{dom}(h) \in (\mathcal{Z}_{\vec{a}} \restriction \widehat{C})^+$ with

$$\forall Y \subseteq \mathrm{dom}(h) \qquad Y \in \mathcal{Z}_{\vec{a}} \restriction \widehat{C} \iff h"Y \in \mathcal{Z}_{\vec{a}} \restriction \widehat{D}$$

Fix a partition $\vec{A} = (A_n)_{n\in\omega}$ of ω into infinite sets. For $C \subseteq \omega$, write $\mathcal{I}_C$ for $\mathcal{Z}_{\vec{a}} \restriction \widehat{C}$, and let let $f(C)$ be a Borel code for the ideal $\mathcal{I}_C(\vec{A})$. (That we can find a Borel—indeed, recursive—function f that accomplishes this is a consequence of the Suslin–Kleene theorem; see the discussion in Section 4.1.3.)

If $C \triangle D$ is finite, then $\mathcal{I}_C$ and $\mathcal{I}_D$ are literally the same ideal, so trivially $\mathcal{P}(\omega)/\mathcal{I}_C(\vec{A})$ and $\mathcal{P}(\omega)/\mathcal{I}_D(\vec{A})$ are isomorphic.

If on the other hand $C \triangle D$ is infinite, then without loss of generality suppose $C \setminus D$ is infinite, and assume there is an isomorphism $\phi : \mathcal{P}(\omega)/\mathcal{I}_C(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{I}_D(\vec{A})$. Note $\mathcal{I}_{C\setminus D} = \mathcal{Z}_{\vec{a}} \restriction \widehat{C \setminus D} = \mathcal{Z}_{\vec{a}} \restriction (\widehat{C} \setminus \widehat{D}) = (\mathcal{Z}_{\vec{a}} \restriction \widehat{C}) \restriction (\widehat{C} \setminus \widehat{D}) = \mathcal{I}_C \restriction (\widehat{C} \setminus \widehat{D})$. By Lemma 2.1 (and using its notation) we have $\mathcal{I}_{C\setminus D}(\vec{A} \restriction \widehat{C \setminus D}) = \mathcal{I}_C(\vec{A} \restriction \widehat{C \setminus D}) = \mathcal{I}_C(\vec{A}) \restriction X$, where $X \triangleq \bigcup_{n\in\widehat{C\setminus D}} A_n$ is clearly $\mathcal{I}_C(\vec{A})$-positive. Then ϕ restricts to an isomorphism

$$\phi : \mathcal{P}(\omega)/\mathcal{I}_{C\setminus D}\left(\vec{A} \restriction \widehat{C \setminus D}\right) \cong \mathcal{P}(\omega)/\left(\mathcal{I}_D(\vec{A}) \restriction \phi(X)\right) = \mathcal{P}(\omega)/\mathcal{I}_D(\vec{B})$$

(writing $\phi(X)$ for some representative of ϕ of the equivalence class of X, and $\vec{B} = (B_n)_{n\in\omega}$ where $B_n \triangleq A_n \cap \phi(X)$). But now by Theorem 3.3 there is an injective partial $h : \omega \rightharpoonup \omega$, $\mathrm{dom}(h) \in \left(\mathcal{I}_{C\setminus D}\right)^+$, that RK-reduces a positive piece of $\mathcal{I}_{C\setminus D}$ to $\mathcal{I}_D$. This is a contradiction.

⊣

3.5. Remarks on the use of the Axiom of Choice. In Sections 3.2 and 3.4 we have used the Axiom of Choice to conclude that any isomorphism $\phi : \mathcal{P}(\omega)/\mathcal{I} \cong \mathcal{P}(\omega)/\mathcal{J}$, for ideals $\mathcal{I}$ and $\mathcal{J}$, must lift to a map $\phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$. This application of AC is mostly for notational and expository convenience. For example the S relation defined in Section 3.2.2 does not require the lifting at all. Also for Lemmata 3.2, 3.3, 3.4 and 3.5 we do not need any Choice at all.

In the proof of Theorem 3.1, we are finally making use of a fragment of AC in a way that does not seem to be eliminable; to get the equivalence between (iv) and (v) we appear to need representatives for all the $\phi\left([A_n]_{\mathcal{J}_1(\vec{A})}\right)$ at once. However we do not need full AC; countable AC for reals is enough. This same fragment also suffices for all the results of Section 3.

§4. Existence of collections of "input" ideals with desired properties.

4.1. $\aleph_1$ "input" ideals. In order to get the result of Section 3.3, we need to find a "definable" collection of $\aleph_1$ shallow ideals, all of distinct Wadge rank.

4.1.1. *If $\mathcal{I}$ is shallow, so is $\mathcal{I} \times \mathcal{J}$.*

LEMMA 4.1. *Let $\mathcal{I}$ be a shallow ideal on ω, and let $\mathcal{J}$ be an arbitrary ideal on ω. Then $\mathcal{I} \times \mathcal{J}$ is shallow.*

PROOF. Let X be an $\mathcal{I} \times \mathcal{J}$-positive subset of $\omega \times \omega$; we wish to find an ω-partition of X with respect to $\mathcal{I} \times \mathcal{J}$.

The intuition is simple: $\mathcal{I} \times \mathcal{J}$ is the ideal whose positive sets are the ones that have $\mathcal{I}$-positively many $\mathcal{J}$-positive vertical sections. So we will project everything down to the horizontal axis, the one corresponding to $\mathcal{I}$ and work with the properties of $\mathcal{I}$. Nothing interesting happens in the vertical sections; $\mathcal{J}$ is treated as a black box.

Let $\check{X} \subseteq \omega$ be the projection, to the first coordinate, of all $\mathcal{J}$-positive vertical sections of X:

$$\check{X} \triangleq \{n \in \omega | \{m | \langle n, m\rangle \in X\} \text{ is } \mathcal{J}\text{-positive}\}$$

Now $\check{X}$ is an $\mathcal{I}$-positive subset of ω (by the definition of $\mathcal{I} \times \mathcal{J}$), so there is an ω-partition of $\check{X}$ with respect to $\mathcal{I}$, call it $(\check{X}_n)_{n\in\omega}$.

Now we can recover an ω-partition of the original set by taking the vertical sections above the $\check{X}_n$: Let

$$X_n \triangleq \{\langle k, \ell\rangle \in X | k \in \check{X}_n\}$$

We must now check that this $(X_n)_{n\in\omega}$ is an ω-partition of X with respect to $\mathcal{I} \times \mathcal{J}$. Clauses (i) and (ii) of Definition 3.2 are easy. We shall verify the contrapositive of clause (iii): Let $Y \subseteq X$ be $\mathcal{I} \times \mathcal{J}$-positive; we must check that $Y \cap X_n$ is $\mathcal{I} \times \mathcal{J}$-positive for some n. Project the $\mathcal{J}$-positive vertical sections of Y to the first coordinate as we did for X:

$$\check{Y} \triangleq \{n \in \omega | \{m | \langle n, m\rangle \in Y\} \text{ is } \mathcal{J}\text{-positive}\}$$

Now $\check{Y}$ is $\mathcal{I}$-positive, so for some n, $\check{Y} \cap \check{X}_n$ is $\mathcal{I}$-positive. But then easily $Y \cap X_n$ is $\mathcal{I} \times \mathcal{J}$-positive. ⊣

4.1.2. *The ideal of density $\mathcal{Z}_0$ is shallow.*

LEMMA 4.2 (Folklore?). *$\mathcal{Z}_0$ is shallow.*

PROOF. See for example [Far02, Prop. 3.3(5)], or work directly as an exercise: Define an ω-partition of ω with respect to $\mathcal{Z}_0$ by

$$X_n \triangleq \{i \in \omega | 2^n \text{ divides } i \text{ but } 2^{n+1} \text{ does not}\}$$

(For completeness, take 0 to be an element of X_0.) Now generalize to get an ω-partition of any $\mathcal{Z}_0$-positive set. ⊣

4.1.3. *There are $\mathcal{J}$ of arbitrarily high Borel rank.* We apply an elementary argument found in [Kec95], Exercise 23.4 on page 180, with the hint on page 362. Here it is shown that for any set of reals A there is an ideal $\mathcal{I}_A$ such that A is Wadge–reducible to $\mathcal{I}_A$. We reproduce the argument: For $x \in 2^\omega$, let $C_x \subseteq 2^{<\omega}$ be $\{ x \upharpoonright n \mid n \in \omega\}$, and given $A \subseteq 2^\omega$, let $\mathcal{I}_A$ be the ideal on $2^{<\omega}$ generated by all sets C_x for $x \in A$, and all finite subsets of $2^{<\omega}$. That is, for $B \subseteq 2^{<\omega}$, $B \in \mathcal{I}_A \iff (\exists\{x_0, x_1, \ldots, x_{n-1}\} \subseteq A)(B \setminus \cup_{i<n} C_{x_i}$ is finite).

If $x \neq y$, then $C_x \cap C_y$ is finite. The map $x \mapsto C_x$ is a continuous function from 2^ω to $\mathcal{P}(2^{<\omega})$ reducing A to $\mathcal{I}_A$.

A finer analysis shows that if $\xi \geq 3$ is a countable ordinal and A is $\mathbf{\Sigma}^0_\xi$ then $\mathcal{I}_A$ is also $\mathbf{\Sigma}^0_\xi$; therefore, if A is $\mathbf{\Sigma}^0_\xi$-complete, then so is $\mathcal{I}_A$. (This calculation, reported by Kechris in a personal communication, may be found in [Oli03].) This finishes the part of the argument taken from Kechris' book.

Now for a given countable ordinal β, let $\mathrm{WO}_{<\beta}$ be the set of all elements of 2^ω that code an ordinal less than β. By [Ster78], $\mathrm{WO}_{<\omega^\alpha}$ is $\mathbf{\Sigma}^0_{2\cdot\alpha}$-complete. For each α let $A_\alpha \triangleq \mathrm{WO}_{<\omega^{\alpha\cdot\omega}}$ and let $\mathcal{I}_\alpha \triangleq \mathcal{I}_{A_\alpha}$ in the sense of the Kechris argument above. Then $\mathcal{I}_\alpha$ is $\mathbf{\Sigma}^0_{\alpha\cdot\omega}$-complete.

(See also Zafrany ([Zaf89]), who proves the complexity claims about $\mathrm{WO}_{<\omega^\alpha}$ by first defining ideals of unbounded Borel complexity; such ideals are exactly what we need here. However Zafrany does not define these ideals in a uniform way on ω, but rather on different countable sets as α increases; it is not clear to me whether any slight modification of his work would provide the ideals we want directly, without going through the Kechris argument.)

Now let $\mathcal{J}_\alpha \triangleq \mathcal{Z}_0 \times \mathcal{I}_\alpha$. Then $\mathcal{J}_\alpha$ is a shallow ideal and is $\mathbf{\Sigma}^0_{\alpha\cdot\omega+n}$ for some n, but is also $\mathbf{\Sigma}^0_{\alpha\cdot\omega}$-hard. Therefore, for any countable α, β with $\alpha < \beta$ we have that $\mathcal{J}_\alpha <_W \mathcal{J}_\beta$.

It remains to check that there is a Borel function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, if X is a code for a countable ordinal α, then $f(X)$ is a Borel code for $\mathcal{J}_\alpha$. For this we appeal to the Suslin–Kleene theorem; see [Mos80, 7B], and observe that we can find recursive functions that, given a code for a countable α, return $\mathbf{\Sigma}^1_1$-codes for $\mathcal{J}_\alpha$ and its complement.

4.2. Mutually RK-irreducible analytic P-ideals. In this section, using a construction reminiscent of (but much simpler than) that of Louveau and Veličkovič in [LV94], we define continuum-many analytic P-ideals with the property that every positive set has an ω-partition, such that if $\mathcal{J}_1$, $\mathcal{J}_2$ are distinct ideals from the collection and $X \subseteq \omega$ is $\mathcal{J}_1$-positive, there is no injection $h : X \to \omega$ such that

$$(\forall Y \subseteq X)\ Y \in \mathcal{J}_1 \iff h\text{"}Y \in \mathcal{J}_2$$

Let $\vec{a} = (a_i)_{i\in\omega}$ be a sequence increasing fast enough that the ratio $a_{i+1}/(\sum_{k=0}^{i} a_k)$ goes to infinity as i goes to infinity. Let $n_i = \sum_{k=0}^{i-1} a_i$, and let $I_i = [n_i, n_{i+1})$, so that $|I_i| = a_i$. Write $\mathcal{Z}_{\vec{a}}$ for the ideal

$$X \in \mathcal{Z}_{\vec{a}} \iff \lim_{i\to\infty} \frac{|X \cap I_i|}{a_i} = 0$$

Note $\mathcal{Z}_{\vec{a}}$ is shallow.

Now we can define the algebra-of-sets homomorphism $\widehat{\ }: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ promised in the proof of Theorem 3.4:

$$\widehat{C} \triangleq \bigcup_{i \in C} I_i$$

For any $A \subseteq \omega$, let $\mathcal{I}_A$ be the ideal of sets whose density on I_i goes to zero as i goes to infinity for $i \in A$:

$$X \in \mathcal{I}_A \iff \lim_{\substack{i \to \infty \\ i \in A}} \frac{|X \cap I_i|}{a_i} = 0$$

In other words, $\mathcal{I}_A = \mathcal{Z}_{\vec{a}} \restriction \widehat{A}$.

Claim 4.1. *If A and B are almost disjoint, then there is no $\mathcal{I}_A$-positive set X such that $\mathcal{I}_A \restriction X$ is Rudin–Keisler isomorphic to any piece of $\mathcal{I}_B$.*

Proof. Suppose to the contrary that X is $\mathcal{I}_A$-positive and $h : X \to \omega$ is an injection such that for all $Y \subseteq X$,

$$Y \in \mathcal{I}_A \iff h"Y \in \mathcal{I}_B$$

Now since X is positive, there is $\epsilon > 0$ such that for infinitely many $i \in A$, $\mu_i(X) > \epsilon$, where $\mu_i(X)$ is the density of X on I_i.

Choose M_0 such that for $i > M_0$, $a_i/(\sum_{k=0}^{i-1} a_k) > 3/\epsilon$. Define:

$$\begin{aligned} I_i^- &= \{n \in I_i | h(n) \in I_j, j < i\} \\ I_i^+ &= \{n \in I_i | h(n) \in I_j, j > i\} \\ I_i^= &= \{n \in I_i | h(n) \in I_i\} \end{aligned}$$

$$\begin{aligned} X^- &= \bigcup_i (X \cap I_i^-) \\ X^+ &= \bigcup_i (X \cap I_i^+) \\ X^= &= \bigcup_i (X \cap I_i^=) \end{aligned}$$

Now for $i > M_0$ we have

$$|I_i^-| \leq \sum_{k=0}^{i-1} a_k \leq (\epsilon/3) a_i$$

so $\mu_i(X^-) \leq \mu_i(I_i^-) \leq \epsilon/3$.

Now there must be $M_1 > M_0$ such that for all $i > M_0$, $i \in A$, we have $\mu_i(X^=) \leq \epsilon/3$, because if for infinitely many $i \in A$ this inequality failed, we could take Y to be $X^=$ on those infinitely many intervals, and then $\mu_i(Y)$ would not approach 0 as $i \to \infty$ in A, but $\mu_i(h"Y)$ *would* equal zero for all but finitely many $i \in B$. (A and B are almost disjoint, and h sends elements of each $Y \cap I_i$ into the same I_i.)

So now for infinitely many $i > M_1$ we have that $\mu_i(X) > \epsilon$, but $\mu_i(X^-) \leq \epsilon/3$ and $\mu_i(X^=) \leq \epsilon/3$. So for such i we have $\mu_i(X^+) > \epsilon/3$.

Now let Y be the union of $X^+ \cap I_i$ for the infinitely many i mentioned in the previous paragraph. Then Y is *not* in $\mathcal{I}_A$, because $\mu_i(Y)$ is infinitely often greater than $\epsilon/3$. However $h"Y$ *is* in $\mathcal{I}_B$, because everything in Y gets sent by h to a higher interval, and the condition $a_{i+1}/(\sum_{k=0}^{i} a_k) \to \infty$ now guarantees that $\mu_i(h"Y) \to 0$ as $i \to \infty$ (in fact we need not even restrict to $i \in B$). ⊣

Each $\mathcal{I}_A$ (for A infinite) is shallow because it is the restriction of $\mathcal{Z}_{\vec{a}}$ to a positive set, and is an analytic P-ideal because it is the exhaustion of a lower semicontinuous submeasure φ given by

$$\varphi(X) \triangleq \sup_{i \in A} \frac{|X \cap I_i|}{a_i}$$

(See [Sol99] for the result that the analytic P-ideals are precisely the exhaustions of lower semicontinuous submeasures.)

REFERENCES

[CK90] C. C. CHANG and H. JEROME KEISLER, ***Model theory***, third ed., North–Holland, 1990.

[Far00a] ILIJAS FARAH, ***Analytic quotients: Theory of liftings for quotients over analytic ideals on the integers***, Memoirs of the AMS, no. 702, American Mathematical Society, November 2000, Volume 148.

[Far00b] ———, *Rigidity conjectures*, ***Proceedings of Logic Colloquium 2000***, (2000).

[Far02] ———, *How many Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?*, ***Illinois Journal of Mathematics***, vol. 46 (2002), no. 4, pp. 999–1033.

[JK84] WINFRIED JUST and ADAM KRAWCZYK, *On certain Boolean algebras $\mathcal{P}\omega/I$*, ***Transactions of the American Mathematical Society***, vol. 285 (1984), no. 1, pp. 411–429.

[JM87] WINFRIED JUST and ŽARKO MIJAJLOVIĆ, *Separation properties of ideals over ω*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 33 (1987), no. 3, pp. 267–276.

[Kec95] ALEXANDER S. KECHRIS, ***Classical descriptive set theory***, Springer–Verlag, 1995.

[LV94] ALAIN LOUVEAU and BOBAN VELIČKOVIČ, *A note on Borel equivalence relations*, ***Proceedings of the American Mathematical Society***, vol. 120 (1994), no. 1, pp. 255–259.

[Mos80] YIANNIS MOSCHOVAKIS, ***Descriptive set theory***, North–Holland, 1980.

[Oli03] MICHAEL RAY OLIVER, *An inquiry into the number of isomorphism classes of Boolean algebras and the Borel cardinality of certain Borel equivalence relations*, ***Ph.D. thesis***, UCLA, April 2003.

[Sol99] SŁAWOMIR SOLECKI, *Analytic ideals and their applications*, ***Annals of Pure and Applied Logic***, vol. 99 (1999), no. 1–3, pp. 51–72.

[Step03] JURIS STEPRĀNS, *Many quotient algebras of the integers modulo co-analytic ideals*, preprint, York University, 2003.

[Ster78] JACQUES STERN, *Évaluation du rang de Borel de certains ensembles*, ***Comptes Rendus Hebdomaires des Séances de l'Académie des Sciences***, vol. 286 (1978), no. 20, pp. A855–857, Série A–B.

[Zaf89] SAMY ZAFRANY, *Borel ideals vs. Borel sets of countable relations and trees*, ***Annals of Pure and Applied Logic***, vol. 43 (1989), no. 2, pp. 161–195.

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