Conservativity for Logics of Justified Belief: Two Approaches

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Abstract

In [1], Fitting showed that the standard hierarchy of logics of justified knowledge is conservative (e.g. a logic with positive introspection operator ! is conservative over the logic without !). We do the same with most logics of justified belief, showing both conservation of sequent proofs and extensibility of models. A brief example shows that conservativity does not hold for logics of justified consistent belief.

Key words: justification logic, epistemic logic, logic of belief, proof theory

1. Introduction

In [1], Fitting showed conservativity of logics of justified knowledge, including **JT**, **JT4** (also known as **LP**), and **JT45** as well as many weaker logics. His proof showed something stronger than simple conservation of validity; he showed that simple omission of symbols missing in the smaller language, carefully done, leads to a line-by-line translation of all Hilbert-style proofs in the stronger logic into proofs in the weaker one. He observed that his method did not extend to logics of justified belief (such as **J** and **J4**) and left the question of conservativity in these logics open.

As Fitting noted, approaches to conservativity in logics of justified belief rooted in Hilbert-style deduction do not work out easily. However, an approach based on cut-free sequent proofs works almost effortlessly. One can also take a semantic approach, extending models of the smaller logic to those of the larger. In what follows, I will outline the basic definitions of logics of justified belief and a simple semantics for these. I will then present two arguments in detail for the conservativity of **J4** over **J**, one syntactic and one semantic, then outline others in broad strokes. After an example showing the lack of conservativity of **JD4** over **JD**, I will close with a few comments on open problems.

2. Preliminaries

Modal logic has long been a way of attempting to formalize the idea of knowledge and belief (among other concepts). What separates knowledge from

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"mere" belief is the truth of what is known, reflected in the modal axiom scheme **T** ($\Box F \to F$). I will assume the reader is familiar with the modal logics of belief (the basic normal modal logic **K** with or without the assumption of positive introspection $\Box F \to \Box\Box F$). I will deal later with the deontic axiom **D** and its explicit counterpart, which insist on the consistency of belief.

In a series of papers ([2], [3], [4] and others), Sergei Artemov defined the Logic of Proofs (**LP**), partially as a solution to an open problem dating back to Gödel ([5]) regarding the proper interpretation of the **S4** modality as arithmetic proof. However, **LP** proved useful and interesting well beyond answering Gödel's question, as a general way to make reasoning about knowledge explicit. Many variations of Artemov's original **LP** have appeared over the last decade or so, and have come under the common heading of *justification logics*.

Why "justification" logics? Because in each of these systems, we augment propositional logic with *justification terms* which are intended to make explicit the reasons for knowing/believing a particular proposition. In a formula $t:F$, the justification term t makes explicit the reasons for asserting that formula F (which may itself contain nested justification terms) is known/believed.

I will limit myself to fairly standard justification logics without truth axioms (the analog of the modal axiom **T**), and in the next section will briefly define languages and axiom systems for these. In the two sections that follow, I will show that both a syntactic approach (based on sequent calculi) and a semantic approach will serve to show the conservativity results we are pursuing. The final section will be a very brief exploration of logics of consistent belief, with a simple example showing that conservativity does not hold in these cases.

2.1. Languages

As mentioned several times above, we will be examining several logics of justified belief, which will differ in the richness of their language. All logics of justified belief contain *justification terms*, which include *justification variables* $x_1, x_2, \ldots$ and *justification constants* $c_1, c_2, \ldots$. In addition, a particular language may contain one or more of the following symbols for operations on justification terms:

- $\cdot$ (binary)
- $+$ (binary)
- ! (unary)

The intended meanings of these symbols are as follows:

- $\cdot$ is known as "application," the idea being that if s is a justification for believing $F \to G$ and t is a justification for believing F, then $s \cdot t$ is a justification for believing G;
- $+$ is known as "sum," sort of a concatenation of justifications, the idea being that $s + t$ is justification for believing anything justified either by s or by t;

- ! is used to represent positive introspection,[1] so that if t is the justification for believing F, then $!t$ is the justification for believing that t is justification for believing F.

We will limit ourselves to $\cdot$, $+$, and !, since the negative introspection operator ? is a recent addition to the literature and the history of results involving the other operators is substantially richer; in addition, negative introspection presents difficulties for reasons mentioned in the concluding section of this paper.

We will define *formulas* as being built from atomic propositions $P_1, P_2, \ldots$, the propositional constant $\perp$ and the implication operator $\rightarrow$ in the usual way; in addition, given any formula F and justification term t, we will allow the formula $t\!:\!F$.

Following Fitting ([1]), as I will for much of the following section, I will adopt the notation $L(S)$ (where S is a subset of $\{\cdot, +, !\}$) to indicate the language where only justification operations from S are permitted, and $B(S)$ to indicate the logic of justified belief based on $L(S)$.

2.2. Logics, Axiomatically

Of course languages with different sets of justification operators will have different collections of axioms to govern the behavior of those operators.

- All justification logics include a propositionally complete set of classical axiom schemes.
- If $\cdot \in S$, include the axiom scheme $s : (F \rightarrow G) \rightarrow (s : F \rightarrow (s \cdot t) : G)$ in $B(S)$.
- If $+ \in S$, include the axiom schemes $s\!:\!F \rightarrow (s\!+\!t)\!:\!F$ and $t\!:\!F \rightarrow (s\!+\!t)\!:\!F$ in $B(S)$.
- If $! \in S$, include the axiom scheme $t\!:\!F \rightarrow !t\!:\!t\!:\!F$ in $B(S)$.

All justification logics share the rule *modus ponens* (from $F \rightarrow G$ and F conclude G), but they differ in their treatment of constants. Quoting Fitting, "Constant symbols are intended to serve as justification of truths we cannot further analyze, but our ability to analyze is dependent on available machinery." Thus, variations in the rules governing constants.

- *The Axiom Necessitation Rule*: If A is an axiom an c is a constant, then $c\!:\!A$ is a theorem.
- *The Iterated Axiom Necessitation Rule*: If A is an axiom and $c_1, c_2, \ldots, c_n$ are constants, then $c_n : c_{n-1} : \cdots : c_2 : c_1 : A$ is a theorem.

[1] Also known as "proof checker" in the context of the Logic of Proofs.

- *The Theorem Necessitation Rule*: If X is a theorem and c is a constant, then $c{:}X$ is a theorem.

If both $\cdot$ and $!$ are in S, then $B(S)$ needs only the Axiom Necessitation Rule. If S has $\cdot$ but lacks $!$, then $B(S)$ requires the Iterated Axiom Necessitation Rule, and if S lacks $\cdot$, then $B(S)$ requires the Theorem Necessitation Rule.

Since $\{\cdot, +, !\}$ has eight subsets, we have just defined eight logics of justified belief, but as far as I know, those lacking $\cdot$ have never been studied or found any application. $B(\{\cdot, +\})$ is also known as **J**, the basic logic of justification and $B\{\cdot, +, !\}$ is known as **J4**; both were defined by Brezhnev ([6]). Their counterparts without $+$ are known as $\mathbf{J}^-$ and $\mathbf{J4}^-$ and were defined by Fitting in [7]. I will focus primarily on **J** and **J4** for the rest of the paper, but the techniques dealing with the presence/absence/addition of ! apply just as well to $+$ and $\cdot$.

In what was just described above, each constant may serve as justification for any axiom. One may also restrict the roles of various constants by means of a *constant specification*, associating individual constants with sets of individual instances of axioms. (In the case of the Iterated Axiom Necessitation rule, finite sequences of constants are associated with sets of axiom instances, and in the case of the Theorem Necessitation Rule constants are associated with sets of theorems, of course.) When each constant is associated with all axioms, the constant specification is called *full*.

For the rest of the paper, I will assume that my constant specifications are *schematic*. In the context of Iterated Constant Necessitation, that means that if a particular axiom A is in $\mathcal{C}(\langle c_n, c_{n-1}, \ldots, c_2, c_1\rangle)$ for some sequence c_n, c_{n-1}, …, c_2, c_1 of justification constants, then so are all other instances of the schema of which A is an instance. In other words, entire schemas are specified by a particular sequence of justification constants.

Secondly, I will assume that any constants in **J4** used to justify instances of positive introspection ($t{:}F \rightarrow !t{:}t{:}F$) will be constants not acting as justification for any other axiom schemes. This would be a corollary of having a schematically injective constant specification.[2] For the sake of simplicity, I will assume that there is a single such constant and refer to it as $c_!$.

2.3. Several Examples

Let us look at three proofs of the same proposition: $y{:}(Q \rightarrow R) \rightarrow (P \rightarrow (P \rightarrow P))$. This is clearly a tautology, since it is of the form $\beta \rightarrow \varphi$ and φ is itself a tautology. I am going to write out three proofs in some detail because examining these proofs, and variants of these proofs, will help guide and illustrate what is going on in the next section. To this end, I will specify that I am considering the propositional schemes $\alpha \rightarrow (\beta \rightarrow \alpha)$ and $(\alpha \rightarrow$

[2]In a *schematically injective* constant specification, each constant corresponds to either no axioms at all or all instances of a single axiom schema.

$\beta) \to (\beta \to \gamma) \to (\alpha \to \gamma)$ as axiomatic.[3] I will also abbreviate the instance $P \to (P \to P)$ of the first schema as φ for brevity in what follows.

Proof #1, which is straightforward:

1.	$P \to (P \to P) = \varphi$	Axiom 1
2.	$\varphi \to (y\!:\!(Q \to R) \to \varphi)$	Axiom 1
3.	$y\!:\!(Q \to R) \to \varphi$	MP 1, 2

Proof #2, which is roundabout and makes essential use of positive introspection:

1.	$P \to (P \to P) = \varphi$	Axiom 1
2.	$\varphi \to (!y\!:\!y\!:\!(Q \to R) \to \varphi)$	Axiom 1
3.	$(!y\!:\!y\!:\!(Q \to R)) \to \varphi$	MP 1, 2
4.	$y\!:\!(Q \to R) \to !y\!:\!y\!:\!(Q \to R)$	Positive Introspection
5.	$(y\!:\!(Q \to R) \to !y\!:\!y\!:\!(Q \to R))$ $\to ((!y\!:\!y\!:\!(Q \to R)) \to \varphi) \to (y\!:\!(Q \to R) \to \varphi)$	Axiom 2
6.	$((!y\!:\!y\!:\!(Q \to R)) \to \varphi) \to (y\!:\!(Q \to R) \to \varphi)$	MP 4, 5
7.	$y\!:\!(Q \to R) \to \varphi$	MP 3, 6

Proof #3, which is roundabout makes inessential use of the ! operator:

1.	$P \to (P \to P) = \varphi$	Axiom 1
2.	$\varphi \to ((!z\!:\!Q \to (y\cdot !z)\!:\!R) \to \varphi)$	Axiom 1
3.	$(!z\!:\!Q \to (y\cdot !z)\!:\!R) \to \varphi$	MP 1, 2
4.	$y\!:\!(Q \to R) \to (!z\!:\!Q \to (y\cdot !z)\!:\!R)$	Application
5.	$y\!:\!(Q \to R) \to (!z\!:\!Q \to (y\cdot !z)\!:\!R)$ $\to ((!z\!:\!Q \to (y\cdot !z)\!:\!R) \to \varphi) \to (y\!:\!(Q \to R) \to \varphi)$	Axiom 2
6.	$((!z\!:\!Q \to (y\cdot !z)\!:\!R) \to \varphi) \to (y\!:\!(Q \to R) \to \varphi)$	MP 4, 5
7.	$y\!:\!(Q \to R) \to \varphi$	MP 3, 6

The difference between logics of justified belief and logics of justified knowledge is that the latter contain a factivity or "truth" axiom $(t\!:\!F) \to F$. That is, if we have a justified belief in F, then F must be true. This notion of justified true belief as a definition for knowledge is ancient. For a wide-ranging discussion related to Justification Logics, see [8] and many items cited in its references.

For our purposes, I would simply like to point out that there seems no straightforward and uniform way to eliminate occurrences of ! in both axioms like $t\!:\!F \to !t\,!\,t\!:\!F$ and $s\!:\!(F \to G) \to (!t\!:\!F \to (s\cdot !t)\!:\!G)$ in a logic of justified belief, but if we are allowed truth axioms it is easy. If we simply eliminate any justification term containing !, then the first item becomes the propositional tautology $t\!:\!F \to t\!:\!F$ and the second becomes an instance of the truth axiom $s\!:\!(F \to G) \to (F \to G)$. This is the essence of Fitting's approach in [1].

One can also "lift" these proofs and encode them into ground proof terms:

Proof #1 lifted:

[3] This is not standard but it makes for brief and illustrative examples.

1. $c_1:(P \to (P \to P)) = c_1:\varphi$
2. $c_1:(\varphi \to (y:(Q \to R) \to \varphi))$
3. $(c_1 \cdot c_1):(y:(Q \to R) \to \varphi)$

Proof #2 lifted:

1. $c_1:(P \to (P \to P)) = c_1:\varphi$
2. $c_1:(\varphi \to (!y:y:(Q \to R) \to \varphi))$
3. $(c_1 \cdot c_1):((!y:y:(Q \to R)) \to \varphi)$
4. $c_!:(y:(Q \to R) \to !y:y:(Q \to R))$
5. $c_2:((y:(Q \to R) \to !y:y:(Q \to R)))$
   $\quad \to ((!y:y:(Q \to R)) \to \varphi) \to (y:(Q \to R) \to \varphi))$
6. $(c_2 \cdot c_!)(((!y:y:(Q \to R)) \to \varphi) \to (y:(Q \to R) \to \varphi))$
7. $((c_2 \cdot c_!) \cdot (c_1 \cdot c_1))(y:(Q \to R) \to \varphi)$

Proof #3 lifted:

1. $c_1:(P \to (P \to P)) = c_1:\varphi$
2. $c_1:(\varphi \to ((!z:Q \to (y{\cdot}!z):R) \to \varphi))$
3. $(c_1 \cdot c_1):((!z:Q \to (y{\cdot}!z):R) \to \varphi)$
4. $c_{\cdot}:(y:(Q \to R) \to (!z:Q \to (y{\cdot}!z):R))$
5. $c_2:(y:(Q \to R) \to (!z:Q \to (y{\cdot}!z):R)$
   $\quad \to ((!z:Q \to (y{\cdot}!z):R) \to \varphi) \to (y:(Q \to R) \to \varphi))$
6. $(c_2 \cdot c_{\cdot}):(((!z:Q \to (y{\cdot}!z):R) \to \varphi) \to (y:(Q \to R) \to \varphi))$
7. $((c_2 \cdot c_{\cdot}) \cdot (c_1 \cdot c_1)):(y:(Q \to R) \to \varphi)$

Note that we are now proving three different propositions, as each is a lifting of an essentially different proof. Note also the presence of $c_!$ and $c_{\cdot}$ in the lifted versions proof #2 and #3 to indicate the essential uses of the axioms for positive introspection and application in the original versions of proof #2 and #3.

We will revisit all six of these proofs in the next section.

3. Sequent Proofs

I will draw on [4], essentially word-for-word, for a sequent formulation of **J4**. The formulations of $\mathbf{J4}^-$, **J** and other weaker logics are obtained simply by omission of the rules pertinent to the missing justification operators.

By *sequent* we mean a pair $\Gamma \Longrightarrow \Delta$ where Γ and Δ are finite multisets of **J4** formulas. By Γ, F we will mean $\Gamma \cup \{F\}$. We will assume a boolean basis $\bot, \to$ and consider the other connectives to be defined in terms of these.

Axioms of $\mathbf{J4G}_0$ are sequents of the form $\Gamma, F \Longrightarrow F, \Delta$ and $\Gamma, \bot \Longrightarrow \Delta$. Along with the usual Gentzen rules $L \to$, $R \to$ and contraction (see **G2c** in [9]), the system $\mathbf{J4G}_0$ contains

$$\frac{\Gamma \Longrightarrow \Delta, s:(F \to G) \qquad \Gamma \Longrightarrow \Delta, t:F}{\Gamma \Longrightarrow \Delta, (s \cdot t):G}$$

$$\frac{\Gamma \Longrightarrow \Delta, s:F}{\Gamma \Longrightarrow \Delta, (s+t):F} \qquad \frac{\Gamma \Longrightarrow \Delta, t:F}{\Gamma \Longrightarrow \Delta, (s+t):F}$$

$$\frac{\Gamma \Longrightarrow \Delta, t{:}F}{\Gamma \Longrightarrow, \Delta, !t{:}t{:}F}$$

Of course $\mathbf{JG}_0$ omits the last of these and $\mathbf{J4G}_0^-$ omits the middle pair.

You may have noted that I did not mention cut among the "usual" Gentzen rules. Although cut is certainly admissible, the cut-free calculus is sound and complete ([4]) and in an unfortunate confluence of notation, the same superscript $-$ is used to denote both justification logics missing the $+$ operator and cut-free sequent calculi. I will try to avoid any possible confusion by using only cut-free sequent calculi throughout the paper and reserving the superscript $-$ for justification logics without $+$.

Given a constant specification $\mathcal{C}$, we may also add axioms of the form

$$\Gamma \Longrightarrow \Delta, c_n{:}\cdots{:}c_1{:}A$$

where $c_n{:}\cdots{:}c_1{:}A \in \mathcal{C}$ to form the calculus $\mathbf{J4G}_{\mathcal{C}}$ (or $\mathbf{JG}_{\mathcal{C}}$ etc.).[4]

3.1. Conservation of Sequent Proofs

It turns out that any $\mathbf{J4G}_{\mathcal{C}}$ proof of a formula in the language of $\mathbf{J}$ (that is, !- and $c_!$-free) is readily, almost trivially, convertible to a $\mathbf{JG}_{\mathcal{C}}$ proof. That the conversion is nearly trivial depends on an observation, which in turn depends on a definition.

We will define the *:-depth* of a particular occurrence of a justification term t by induction:

- If t occurs in a formula of the form $s : F$
  - If t occurs as a subterm of s then that occurrence of t has :-depth 0.
  - If t occurs with :-depth n in the formula F, then the corresponding occurrence of t within $s{:}F$ is considered to be of depth $n + 1$.
- If t occurs in a formula of the form $F \to G$, then that occurrence of t has the same :-depth in $F \to G$ as it had in F or in G.

With the notion of :-depth in mind, we can observe:

Proposition 3.1. *In a $\mathbf{J4G}_{\mathcal{C}}$ cut-free sequent proof, if a justification term t ever appears at :-depth 0, then t will be a subterm of a :-depth 0 justification term in all sequents below the first appearance of that t.*

[4]In Artemov's formulation [4] the constant specification is expressed as rules of the form $\frac{\Gamma \Longrightarrow \Delta, A}{\Gamma \Longrightarrow \Delta, c_n{:}\cdots{:}c_1{:}A}$. This is much closer in spirit to the necessitation rule from modal logic of which this is the analogue, but it will simplify our work to treat these steps of the proof as axioms. This difference is not essential but aesthetic in both cases. It could, of course, affect complexity, etc.

This is immediate from inspection of the rules set forth earlier in this section. Immediate in turn is:

Corollary 3.2. *If a* !*- and* $c_!$*-free sequent has a* $\boldsymbol{J4G}_{\mathcal{C}}$ *proof, that proof made no use of the* !*-introduction rule or of an axiom of the form* $\Gamma \Longrightarrow \Delta, c_! : A$*.*

Given a formula F, let $F^\ddagger$ be obtained from F by replacing all justification terms $!t$ and $c_!$ with the justification variable x.

A couple more statements of what I hope are obvious truths:

- If A in an instance of any **J4** axiom other than positive introspection ($t{:}F \to !t{:}t{:}F$), then it is an instance of a **J** axiom, though maybe not in the language of **J**.
- If A is an instance of a **J** axiom containing the symbols ! or $c_!$, then $A^\ddagger$ is an instance of that same axiom.

Convincing oneself of the second item might require looking back at subsection 2.2 for a moment, but I do not believe it requires explicit proof.

So if we have a $\mathbf{J4G}_{\mathcal{C}}$ proof of a !- and $c_!$-free sequent $\Gamma \Longrightarrow \Delta$, then that same proof is also a $\mathbf{JG}_{\mathcal{C}}$ proof, though it may make inessential use of ! and $c_!$. Also, $\mathcal{C}$ is a **J4**-constant specification rather than a **J** one.

However, based on the above observations we can see that if we eliminate all occurrences of ! and $c_!$ in the most simple-minded way possible, by replacing every formula F occurring anywhere in the proof with $F^\ddagger$, the **JG** sequent proof will remain a **JG** sequent proof. Furthermore, if $\mathcal{C}$ was schematic, then we can also take the restriction of $\mathcal{C}$ to the language of **J** in a similarly simple-minded way, eliminating all instances of axiom schema which contain either ! or $c_!$. If we call this constant specification $\mathcal{C}^\ddagger$, then our "sanitized" **JG** sequent proof is now a $\mathbf{JG}_{\mathcal{C}^\ddagger}$ proof, fully in the language of **J**.

Since formulas containing no occurrences of ! and $c_!$ are unaffected by the ‡ operation, we have argued for the following:

Theorem 3.3. *If the* !*- and* $c_!$*-free sequent* $\Gamma \Longrightarrow \Delta$ *has a* $\boldsymbol{J4G}_{\mathcal{C}}$ *deduction, then that same deduction, modified by replacing each formula* F *occurring in the deduction with* $F^\ddagger$*, is a* $\boldsymbol{JG}_{\mathcal{C}^\ddagger}$ *deduction of* $\Gamma \Longrightarrow \Delta$*.*

An explicit proof of this would proceed by induction, but each step is obvious from inspection of the sequent rules (and the fact that the !-introduction rule was not used in the deduction). We will look at examples of this process in the next subsection, and that might also convince the reader that everything that I assert is clear really is so.

The theorem leads us to our first proof of the following corollary. (The second will come in the next section.)

Corollary 3.4. *If* F *is a* !*- and* $c_!$*-free formula valid in* ***J4*** *with constant specification* $\mathcal{C}$*, then* F *is also valid in* ***J*** *with constant specification* $\mathcal{C}^\ddagger$*.*

Finally, a note about constant specifications. While Iterated Constant Necessitation is not necessary in logics with positive introspection (!), it does no harm to include it. So if we want $\mathcal{C}^{\ddagger}$ to be appropriate for **J**, we would want to include iterated constant necessitation in the original $\mathcal{C}$, even though it was not necessary in a **J4** specification.

3.2. Examples Revisited

Let us revisit the three Hilbert-styles proofs of the proposition $y:(Q \to R) \to (P \to (P \to P))$ from subsection 2.3 and their lifted versions.

It turns out that if we want a cut-free sequent proof of $y:(Q \to R) \to (P \to (P \to P))$, we are forced into the straightforward propositional one, the analog of Proof #1. Without cut, we cannot take a roundabout approach. However, things become more interesting when we look at the three propositions which arose from the lifted versions of the three proofs:

- $(c_1 \cdot c_1):(y:(Q \to R) \to \varphi)$
- $((c_2 \cdot c_!) \cdot (c_1 \cdot c_1))(y:(Q \to R) \to \varphi)$
- $((c_2 \cdot c_.) \cdot (c_1 \cdot c_1)):(y:(Q \to R) \to \varphi)$

We would need to find deductions of the sequents:

- $\Longrightarrow (c_1 \cdot c_1):(y:(Q \to R) \to \varphi)$
- $\Longrightarrow ((c_2 \cdot c_!) \cdot (c_1 \cdot c_1))(y:(Q \to R) \to \varphi)$
- $\Longrightarrow ((c_2 \cdot c_.) \cdot (c_1 \cdot c_1)):(y:(Q \to R) \to \varphi)$

Ignoring the structural rule of contraction[5] we find that each of these must have resulted from the sequent rule governing the behavior of $\cdot$.

In the first instance, the sequent proof is straightforward:

$$\frac{\Longrightarrow c_1:(F \to (y:(Q \to R) \to \varphi)) \qquad \Longrightarrow c_1:F}{(c_1 \cdot c_1):(y:(Q \to R) \to \varphi)}$$

This necessitates finding a formula F such that F itself and $F \to (y:(Q \to R) \to \varphi)$ are both instances of the axiom scheme $\alpha \to (\beta \to \alpha)$. Fortunately (and not coincidentally) φ itself fits the bill exactly and is our only option in this exceedingly simple unification problem.

In the second instance, things get significantly more complicated. Our sequent proof must look like:

$$\frac{\dfrac{\Longrightarrow c_2:(G \to F \to y:(Q \to R) \to \varphi) \qquad \Longrightarrow c_!:G}{\Longrightarrow (c_2 \cdot c_!):(F \to (y:(Q \to R) \to \varphi)} \qquad \dfrac{\Longrightarrow c_1:(H \to F) \qquad \Longrightarrow c_1:H}{\Longrightarrow (c_1 \cdot c_1):F}}{\Longrightarrow ((c_2 \cdot c_!) \cdot (c_1 \cdot c_1)):(y:(Q \to R) \to \varphi)}$$

[5] Which, in our case merely postpones and/or duplicates our work.

That is, we need to find F, G, and H so that $G \to F \to (y\!:\!(Q \to R) \to \varphi)$ is an instance of the scheme $(\alpha \to \beta) \to (\beta \to \gamma) \to (\alpha \to \gamma)$, G is of the form $t\!:\!\alpha \to !t\!:\!t\!:\!\alpha$, and $H \to F$ and H are instances of the scheme $\alpha \to (\beta \to \alpha)$.

We are forced to make G be $y : (Q \to R) \to (!y : y : (Q \to R))$, forcing F to be $!y\!:\!y\!:\!(Q \to R) \to \varphi$, forcing H to be φ. Again, this all works out nicely and deterministically.

Two related observations about this case: Because of the presence and placement of the $c_!$ constant in the final sequent, this is not the sort of proof which our theorem asserts can be converted to a $\mathbf{JG}_{\mathcal{C}^\ddagger}$ proof. Also, if we had not cast Axiom Necessitation as a sequent axiom but as a sequent rule (as is more standard), above each sequent of the form (for example)

$$\Longrightarrow c_!\!:\!(y\!:\!(Q \to R) \to (!y\!:\!y\!:\!(Q \to R)))$$

we would have the sequent

$$\Longrightarrow y\!:\!(Q \to R) \to (!y\!:\!y\!:\!(Q \to R))$$

which could only be deduced by use of the !-introduction sequent rule from **J4G**. This illustrates more emphatically why such proofs cannot be converted to **JG** proofs.

In the third instance, things are nearly identical. The only difference is that now G must be an instance of $s : (\alpha \to \beta) \to (t : \alpha) \to (s \cdot t) : \beta)$. We are forced to let unify $y : (Q \to R)$ with $s : (\alpha \to \beta)$, and all else about F and H follows. However, we are left with the choice of an arbitrary t. We must choose some value for t, but it is entirely immaterial to the rest of the proof. If we happen to choose a value for t which contains the ! operator, we will have what is essentially a **JG** deduction, but in the language of **J4G**. This is exactly the circumstance in which converting G to $G^\ddagger$, converting whatever subterms of the form $!u$ occurred in t into the variable x, yields a pure **JG** proof.

4. A Semantic Approach

The original intended semantics for **LP** was arithmetic proofs, but a more adaptable semantics was defined by Fitting in [7], generalizing the idea of Kripke models for modal logics. We will not have need of the full strength of Fitting models here, though, and will revert to an older semantics due to Mkrtychev [10]. Mkrtychev models are essentially one-world Fitting models.

The first notion we will need is that of an *evidence function*, which is simply any function $\mathcal{E}$ from justification terms in $L(S)$ to sets of $L(S)$ formulas. We may impose additional conditions on $\mathcal{E}$:

- If $\cdot$ is in S, we will insist that whenever $F \to G \in \mathcal{E}(s)$ and $F \in \mathcal{E}(t)$ it is also the case that $G \in \mathcal{E}(s \cdot t)$.
- If $+$ is in S, we will insist that $\mathcal{E}(s) \cup \mathcal{E}(t) \subseteq \mathcal{E}(s + t)$.

- If ! is in S, we will insist that whenever $F \in \mathcal{E}(t)$, it is also the case that $t{:}F \in \mathcal{E}(!t)$.

Finally, for an evidence function to be appropriate for a language $L(S)$ and a particular constant specification $\mathcal{C}$ in that language, it must behave properly on constants.

- If $\cdot$ and ! are both in S, it must be that $A \in \mathcal{E}(c)$ for each axiom $A \in \mathcal{C}(c)$ where c is a justification constant.
- If $\cdot$ is in S but ! is not in S, it must be that $c_2 : \cdots : c_n : A \in \mathcal{E}(c_1)$, $c_3{:}\cdots{:}c_n{:}A \in \mathcal{E}(c_2)$, $\ldots c_n{:}A \in \mathcal{E}(c_{n-1})$, and $A \in \mathcal{E}(c_n)$ for each axiom $A \in \mathcal{C}(\langle c_1, c_2 \ldots, c_n\rangle)$ where c_1, c_2, $\ldots$, c_n are justification constants. ($n \geq 1$.)
- If S lacks $\cdot$, it must be that $F \in \mathcal{E}(c)$ for each theorem $F \in \mathcal{C}(c)$ where c is a justification constant.

A *(Mkrtychev) structure* $\mathcal{M}$ for a language $L(S)$ is a pair $\langle \mathcal{E}, \mathcal{V}\rangle$ where $\mathcal{E}$ is an evidence function appropriate to $L(S)$ and $\mathcal{V}$ is a propositional valuation.

We will define satisfaction of a formula F in a structure $\mathcal{M}$ (written $\mathcal{M} \Vdash F$) as follows:

- $\mathcal{M} \Vdash P$ for propositional variable P if and only if $\mathcal{V}(P)$ is true.
- $\mathcal{M} \nVdash \bot$.
- $\mathcal{M} \Vdash F \to G$ if and only if either $\mathcal{M} \nVdash F$ or $\mathcal{M} \Vdash G$.
- $\mathcal{M} \Vdash t{:}F$ if and only if $F \in \mathcal{E}(t)$.

Kuznets ([11]) proved the soundness and completeness of Mkrtychev models for $B(\{\cdot, +\})$ and $B(\{\cdot, +, !\})$ (that is, **J** and **J4**). Essentially the same proof goes through for $\cdot$-free and $+$-free logics of justified belief.

4.1. A Fuller Semantics

I will treat this only briefly, but Fitting in [7] combined Mkrtychev's notion of evidence function with Kripke frames to obtain a much more expressive semantics. See [8] for a fuller treatment and an explanation of the relative merits of Mkrtychev and Kripke-Fitting models. I will draw on Artemov's exposition but adapt the notation a little.

The basic idea is that we begin with a Kripke frame, a set W of possible worlds and an accessibility relation R on these, and have an evidence function $\mathcal{E}$ (as defined above) for each world. So instead of talking about $F \in \mathcal{E}(t)$, we will have $F \in \mathcal{E}(t, w)$ for possible world w. Mkrtychev models would constitute one-world Kripke-Fitting models with an empty accessibility relation.

The same closure conditions on evidence functions would pertain, but to say that $t{:}F$ holds at world w we would insist not only that $F \in \mathcal{E}(t, w)$ as before, but also that F holds at world v for all worlds v accessible from w (that is, with wRv).

In addition to the closure under introspection (if $F \in \mathcal{E}(t,w)$ then $t:F \in \mathcal{E}(!t,w)$) for logics with !, in such cases we would also require that R be transitive (a condition familiar from Kripke models) and satisfy a monotonicity condition: If $F \in \mathcal{E}(t,w)$ and wRv then $F \in \mathcal{E}(t,v)$.

4.2. Extension of Models

Another approach to showing the conservativity of **J4** over **J**, and the one taken in the preliminary version of this paper presented at LFCS 2009, is to work with Mkrtychev models, showing that we can extend **J** models into **J4** models. This is more work, but may well have application beyond the present result.

Theorem 4.1. *Let $\mathcal{C}$ be a constant specification in $L(\{\cdot,+\})$. Any Mkrtychev model $\mathcal{M}$ for **J** satisfying constant specification $\mathcal{C}$ can be extended to a **J4** model $\mathcal{M}'$ satisfying $\mathcal{C}$ with $\mathcal{M} \subseteq \mathcal{M}'$. Furthermore, exactly the same !-free formulas are true in $\mathcal{M}'$ as in $\mathcal{M}$.*

Proof. We begin with a Mkrtychev model $\mathcal{M}$ for **J** satisfying $\mathcal{C}$. Recall that this means that we have a propositional valuation $\mathcal{V}$ and an evidence function $\mathcal{E}$ with the properties that $\mathcal{E}(s) \cup \mathcal{E}(t) \subseteq \mathcal{E}(s+t)$ and whenever $F \to G \in \mathcal{E}(s)$ and $F \in \mathcal{E}(t)$ it is also the case that $G \in \mathcal{E}(s \cdot t)$. The handy thing about a semantic approach logics of belief is that beliefs need not have anything to do with the "real world" so we do not have to worry any further about our propositional valuation $\mathcal{V}$.

We will extend $\mathcal{E}$ to a **J4**-appropriate evidence function $\mathcal{E}'$ in stages. Because we are leaving the constant specification alone, we need only one additional property: that whenever $F \in \mathcal{E}'(t)$ we also have $t:F \in \mathcal{E}'(!t)$. (We essentially taking the transitive closure of $\mathcal{E}$.)

We will define $\mathcal{E}_n$ recursively, taking the closure under operations at successive stages. We can then set $\mathcal{E}' = \bigcup_{n=0}^{\infty} \mathcal{E}_n$.

We begin by setting $\mathcal{E}_0 = \mathcal{E}$. Now we can define $\mathcal{E}_{n+1}$.

- If c is a justification constant, $\mathcal{E}_{n+1}(c) = \mathcal{E}_n(c)$.
- If x is a justification variable, $\mathcal{E}_{n+1}(x) = \mathcal{E}_n(x)$.
- $\mathcal{E}_{n+1}(s+t) = \mathcal{E}_n(s+t) \cup \mathcal{E}_n(s) \cup \mathcal{E}_n(t)$.
- $\mathcal{E}_{n+1}(s \cdot t) = \mathcal{E}_n(s \cdot t) \cup \{G | F \to G \in \mathcal{E}_n(s) \text{ and } F \in \mathcal{E}_n(t)\}$.
- $\mathcal{E}_{n+1}(!t) = \mathcal{E}_n(!t) \cup \{t:F | F \in \mathcal{E}_n(t)\}$.

The only evidence for formulas containing the ! operator will be justification terms which themselves contain !.

Lemma 1. *If t is !-free and $F \in \mathcal{E}_n(t)$, then F is !-free as well.*

Proof. We will prove this by induction, and the base ($n = 0$) case is trivial, since $\mathcal{E}_0(t) = \mathcal{E}(t)$ and $\mathcal{E}$ was an evidence function in a !-free language.

For the inductive step of the proof, let us assume that $F \in \mathcal{E}_{k+1}(t)$. If $F \in \mathcal{E}_k(t)$ as well, we are finished by our inductive hypothesis. So let us assume that $F \notin \mathcal{E}_k(t)$. It is impossible by the definition of $\mathcal{E}_{k+1}$ that t is a justification constant or justification variable, so t is either $u \cdot v$ or $u + v$. (Recall that t was !-free.)

- If $t = u \cdot v$, then there is some G with $G \to F \in \mathcal{E}_k(u)$ and $G \in \mathcal{E}_k(v)$. By our inductive hypothesis, $G \to F$ is !-free, so F will be as well.

- If $t = u + v$, then either $F \in \mathcal{E}_k(u)$ or $F \in \mathcal{E}_k(v)$. In either case, we know by our inductive hypothesis that F is !-free.

This ends the inductive argument and the proof of the first lemma. □

Now we can get almost all of the way home with a second lemma.

Lemma 2. *Let t and F be !-free. $F \in \mathcal{E}'(t)$ if and only if $F \in \mathcal{E}(t)$.*

Proof. That $F \in \mathcal{E}(t)$ implies $F \in \mathcal{E}'(t)$ is immediate from the construction of $\mathcal{E}'$.

To show the converse, we will prove that if $F \in \mathcal{E}_n(t)$ then $F \in \mathcal{E}(t)$ by induction on n. Again, the base case is trivial.

For the induction, we may assume that if $G \in \mathcal{E}_k(s)$ then $G \in \mathcal{E}(s)$ for all !-free pairs s and G. We wish to show that if F and t are !-free and $F \in \mathcal{E}_{k+1}(t)$ then $F \in \mathcal{E}(t)$. If $F \in \mathcal{E}_k(t)$, we are done immediately by our inductive hypothesis. So let us examine the other possible cases:

- If $t = u \cdot v$ and there is $G \in \mathcal{E}_k(v)$ with $G \to F \in \mathcal{E}_k(u)$. Because $u \cdot v$ is !-free, we know by our earlier lemma that it is also the case that $G \to F$ and G are !-free. By our inductive hypothesis, $G \to F \in \mathcal{E}(v)$ an $G \in \mathcal{E}(u)$. Since $\mathcal{E}$ was an evidence function, it must be that $F \in \mathcal{E}(u \cdot v)$.

- If $t = u + v$ and $F \in \mathcal{E}_k(u)$, then by our inductive hypothesis, $F \in \mathcal{E}(u)$. Because $\mathcal{E}$ was an evidence function, we know that $\mathcal{E}(u) \subseteq \mathcal{E}(u + v)$, so $F \in \mathcal{E}(u + v)$. The case for $F \in \mathcal{E}_k(v)$ is identical.

This completes the induction and the proof of our second lemma. □

It is clear from the definition of $\mathcal{E}'$ that it is an evidence function appropriate to the logic **J4**, so we can define a **J4** Mkrtychev model $\mathcal{M}' = \langle \mathcal{E}', \mathcal{V} \rangle$ where $\mathcal{V}$ is the propositional valuation from our original **J**-model $\mathcal{M}$. What remains to be shown is that if F is a !-free formula, then $\mathcal{M} \Vdash F$ if and only if $\mathcal{M}' \Vdash F$. We can prove this by a very easy induction on the construction of F.

Because both $\mathcal{M}$ and $\mathcal{M}'$ are built from the same propositional valuation $\mathcal{V}$ the case for propositional variables is immediate, as is the case for $\perp$. That

$\mathcal{M} \Vdash t\!:\!G$ if and only if $\mathcal{M}' \Vdash t\!:\!G$ is immediate from the definition of $\Vdash$ and the second lemma above. The argument for $F = G_1 \rightarrow G_2$ is standard and straightforward. $\square$

Corollary 4.2. *If $F \in L(\{\cdot, +\})$ is provable in **J4** under constant specification $\mathcal{C}$, then F is provable in **J** under constant specification $\mathcal{C}$.*

Proof. By the completeness of Mkrtychev models for **J**, if F is not provable in **J**, then there will be Mkrtychev model for **J** making F false. This will be extended to a **J4** model in which F is also false, showing that F was not provable in **J4** by the soundness of Mkrtychev models. $\square$

Of course we do not want to hamstring ourselves with constant specifications for **J4** which are entirely !-free. In particular, it would be good to have conservativity hold for axiomatically appropriate constant specifications. With a few reasonable conditions, we can generalize Theorem 4.1 to a broader class of constant specifications.

Let $\mathcal{C}$ be a schematic constant specification for **J4**, and let $\mathcal{C}^\ddagger$ be as defined in section 3.1.

Note that while in section 3.1 we defined $\mathcal{C}^\ddagger$ as a reduction of $\mathcal{C}$, it is also straightforward given any appropriate schematic constant specification for **J** to extend it to an appropriate constant specification $\mathcal{C}$ for **J4** such that $\mathcal{C}^\ddagger$ is the original **J** specification. Also note that if we start with a $\mathcal{C}^\ddagger$ which is axiomatically appropriate and extend to $\mathcal{C}$, $\mathcal{C}$ will be axiomatically appropriate as well, though with unnecessary instances of iterated justification constants.

Theorem 4.3. *Let $\mathcal{C}$ be a schematic constant specification for **J4**, and $\mathcal{C}^\ddagger$ its corresponding **J** specification. Any Mkrtychev model $\mathcal{M}$ for **J** satisfying constant specification $\mathcal{C}^\ddagger$ can be extended to a **J4** model $\mathcal{M}'$ satisfying $\mathcal{C}$ with $\mathcal{M} \subseteq \mathcal{M}'$. Furthermore, exactly the same !- and $c_!$-free formulas are true in $\mathcal{M}'$ as in $\mathcal{M}$.*

Proof. The proof will be nearly identical in its outline to that of Theorem 4.1, but there will be one small change to the construction of the extension $\mathcal{E}'$ of the evidence relation $\mathcal{E}$, entailing some extra work in the lemmas.

As above, we begin with a Mkrtychev model $\mathcal{M}$ for **J** which respects the constant specification $\mathcal{C}^\ddagger$, consisting of a propositional valuation $\mathcal{V}$ and an evidence function $\mathcal{E}$.

This time when we extend $\mathcal{E}$ to a **J4**-appropriate evidence function $\mathcal{E}'$ we will need two additional properties: First, we need to extend the behavior of constants in $\mathcal{E}$ to include axioms from $L(\{\cdot, +, !\})$, and as before we need it to be the case that whenever $F \in \mathcal{E}'(t)$ we also have $t\!:\!F \in \mathcal{E}'(!t)$.

We will again define $\mathcal{E}_n$ recursively, treating constants at the initial stage and closure under operations at successive stages. We can still set $\mathcal{E}' = \bigcup_{n=0}^{\infty} \mathcal{E}_n$.

We begin with $\mathcal{E}_0$.

- If t is not a justification constant, let $\mathcal{E}_0(t) = \mathcal{E}(t)$.

- If c is a justification constant, let $\mathcal{E}_0(c) = \mathcal{E}(c) \cup \mathcal{C}(c)$

Now we can define $\mathcal{E}_{n+1}$ exactly as in Theorem 4.1.

- If c is a justification constant, $\mathcal{E}_{n+1}(c) = \mathcal{E}_n(c)$.
- If x is a justification variable, $\mathcal{E}_{n+1}(x) = \mathcal{E}_n(x)$.
- $\mathcal{E}_{n+1}(s+t) = \mathcal{E}_n(s+t) \cup \mathcal{E}_n(s) \cup \mathcal{E}_n(t)$.
- $\mathcal{E}_{n+1}(s \cdot t) = \mathcal{E}_n(s \cdot t) \cup \{G | F \to G \in \mathcal{E}_n(s) \text{ and } F \in \mathcal{E}_n(t)\}$.
- $\mathcal{E}_{n+1}(!t) = \mathcal{E}_n(!t) \cup \{t{:}F | F \in \mathcal{E}_n(t)\}$.

At this point, we will need a somewhat more complicated version of Lemma 1.

Revised Lemma 1. *If t is !- and $c_!$-free, then if $F \in \mathcal{E}_n(t)$, we have $(F)^\ddagger \in \mathcal{E}_n(t)$ as well.*

Proof. We will prove this, of course, by induction. First, for $n = 0$. If t is not a justification constant and $F \in \mathcal{E}_0(t)$, then $F \in \mathcal{E}(t)$. Because $\mathcal{E}$ was the evidence function for a **J**-model in a language without ! or $c_!$, we know that F is !- and $c_!$-free, meaning that $(F)^\ddagger$ is identical to F. A similar argument works in the case that t is a justification constant and $F \in \mathcal{E}(c)$.

To complete the base case of the induction, we need to show that if $F \in \mathcal{C}(c)$, $(F)^\ddagger \in \mathcal{C}(c)$ as well. That is, we need to show that if F is an instance of a **J4** axiom schema other than positive introspection (recall that $c \neq c_!$) then $(F)^\ddagger$ is also an instance of that same schema. However, this is immediate from the fact that $\ddagger$ leaves intact all instances of the justification operators $\cdot$ and $+$ and all propositional connectives. (For example, $(s+t{:}G)^\ddagger = (s)^\ddagger + (t)^\ddagger{:}(G)^\ddagger$.)

For the inductive step of the proof, let us assume that $F \in \mathcal{E}_{k+1}(t)$. If $F \in \mathcal{E}_k(t)$ as well, we are finished by our inductive hypothesis. So let us assume that $F \notin \mathcal{E}_k(t)$. It is impossible by the definition of $\mathcal{E}_{k+1}$ that t is a justification constant or justification variable, so t is either $u \cdot v$ or $u + v$. (Recall that t was !- and $c_!$-free.)

- If $t = u \cdot v$, then there is some G with $G \to F \in \mathcal{E}_k(u)$ and $G \in \mathcal{E}_k(v)$. By our inductive hypothesis, $(G \to F)^\ddagger \in \mathcal{E}_k(u)$ and $(G)^\ddagger \in \mathcal{E}_k(v)$. Since $\ddagger$ is not concerned with propositional connectives, $(G \to F)^\ddagger = (G)^\ddagger \to (F)^\ddagger$. Since we have $(G)^\ddagger \in \mathcal{E}_k(v)$ and $(G)^\ddagger \to (F)^\ddagger \in \mathcal{E}_k(u)$, the definition of $\mathcal{E}_{k+1}$ tells us that $(F)^\ddagger \in \mathcal{E}_{k+1}(u \cdot v)$.
- If $t = u + v$, then either $F \in \mathcal{E}_k(u)$ or $F \in \mathcal{E}_k(v)$. By our inductive hypothesis, $(F)^\ddagger \in \mathcal{E}_k(u)$ or $(F)^\ddagger \in \mathcal{E}_k(v)$. By the definition of $\mathcal{E}_{k+1}$, $(F)^\ddagger \in \mathcal{E}_{k+1}(u+v)$.

This ends the inductive argument and the proof of the first lemma. $\square$

The complexities of the first lemma lead to a few changes in the proof of second lemma as well.

Revised Lemma 2. *Let t and F be !- and $c_!$-free. $F \in \mathcal{E}'(t)$ if and only if $F \in \mathcal{E}(t)$.*

Proof. That $F \in \mathcal{E}(t)$ implies $F \in \mathcal{E}'(t)$ is again immediate from the construction of $\mathcal{E}'$.

To show the converse, we will as usual prove that if $F \in \mathcal{E}_n(t)$ then $F \in \mathcal{E}(t)$ by induction on n.

If t is not a justification constant, then $\mathcal{E}_0(t) = \mathcal{E}(t)$. If t is a justification constant $c \neq c_!$, $\mathcal{C}^\ddagger$ and $\mathcal{C}$ agree on F by the definition of $\mathcal{C}^\ddagger$. (Recall that F was !- and $c_!$-free.) Thus, if $F \in \mathcal{E}_0(t)$ then $F \in \mathcal{E}(t)$.

Now we may assume that if $G \in \mathcal{E}_k(s)$ then $G \in \mathcal{E}(s)$ for all !- and $c_!$-free pairs s and G. We wish to show that if F and t are !- and $c_!$-free and $F \in \mathcal{E}_{k+1}(t)$ then $F \in \mathcal{E}(t)$. If $F \in \mathcal{E}_k(t)$, we are done immediately by our inductive hypothesis. So let us examine the other possible cases:

- If $t = u \cdot v$ and there is $G \in \mathcal{E}_k(v)$ with $G \to F \in \mathcal{E}_k(u)$. Because $u \cdot v$ is !- and $c_!$-free, we know by our earlier lemma that it is also the case that $(G)^\ddagger \in \mathcal{E}_k(v)$ and $(G \to F)^\ddagger \in \mathcal{E}_k(u)$. Because we can move $\ddagger$ past propositional connectives, and because F is !- and $c_!$-free, we know that $(G \to F)^\ddagger = (G)\ddagger \to (F)^\ddagger = (G)^\ddagger \to F$. Thus, we have $(G)^\ddagger \in \mathcal{E}_k(v)$ and $(G)^\ddagger \to F \in \mathcal{E}_k(u)$. By our inductive hypothesis, $(G)^\ddagger \in \mathcal{E}(v)$ and $(G)^\ddagger \to F \in \mathcal{E}(u)$. Because $\mathcal{E}$ was an evidence function for a Mkrtychev model for **J**, it must be the case that $F \in \mathcal{E}(u \cdot v)$.

- If $t = u + v$ and $F \in \mathcal{E}_k(u)$, then by our inductive hypothesis, $F \in \mathcal{E}(u)$. Because $\mathcal{E}$ was an evidence function, we know that $\mathcal{E}(u) \subseteq \mathcal{E}(u + v)$, so $F \in \mathcal{E}(u+v)$. The case for $F \in \mathcal{E}_k(v)$ is identical. (This case is unchanged from the original version of the lemma.)

This completes the induction and the proof of our second lemma. $\square$

The remainder of the proof of the current theorem is both standard and identical with the end of the proof of Theorem 4.1. $\square$

We again get a corollary nearly for free:

Corollary 4.4. *If $F \in L(\{\cdot, +\})$ and containing no instances of $c_!$ is provable in **J4** under constant specification $\mathcal{C}$, then F is provable in **J** under constant specification $\mathcal{C}^\ddagger$.*

Similar proofs work to show the conservativity of, say, **J4** over **J4**$^-$ (the +-free fragment of **J4**). (In fact, the proofs of conservativity over +-free fragments have much less need for equivocation about constant specifications.) The only substantial differences would come in the inductive steps of the lemmas.

For example, in the (revised) first lemma, we would need the following argument (for **J4** over **J4**$^-$, assuming that $(F)^-$ was defined analogously with $(F)^\ddagger$):

- If $t = !s$, then $F = s:G$ and $G \in \mathcal{E}_k(s)$. By our inductive hypothesis[6] we know that $(G)^- \in \mathcal{E}_k(s)$ as well. Because t was $+$-free, so is s, so $(s:G)^- = s:(G)^-$. Because $(G)^- \in \mathcal{E}_k(s)$, $s:(G)^- \in \mathcal{E}_{k+1}(!s)$, and so $(s:G)^- \in \mathcal{E}_{k+1}(!s)$.

And in the second:

- If $t = !s$ and $F = s:G$ for some $G \in \mathcal{E}_k(s)$, then we know by our inductive hypothesis[7] that $G \in \mathcal{E}(s)$. Because $\mathcal{E}$ was a $\mathbf{J4}^-$-appropriate evidence function, it must be that $s:G \in \mathcal{E}(!s)$.

The interested reader can work out details for other cases.

5. Consistent Belief

What we have been examining so far could be called the logic of "pure normal belief" (with or without positive introspection). From a modal standpoint, the only axioms are $\Box(F \to G) \to \Box F \to \Box G$ and possibly $\Box F \to \Box\Box F$. No other constraints are placed on what is believed. In logics of knowledge, consistency of belief is automatic because things believed/known are also true, and no inconsistency can be true. However, one can fairly simply mandate consistency of beliefs without requiring that all which is believed be true. The modal axiom **D** (from the word deontic[8]) $((\Box\bot) \to \bot)$ accomplishes this. This can also be introduced in logics of justified belief as the axiom scheme $(t:\bot) \to \bot$.

In a way, insisting on consistent belief can be seen as a middle ground between unconstrained (normal) belief and knowledge. This makes it surprising that while **J4** is conservative over **J** and **JT4** over **JT**, it is not the case that **JD4** is conservative over **JD**. In particular, introspection can introduce inconsistencies in otherwise consistent belief systems.

The potential that introspection has for havoc can be seen from a simple example. I might both believe that the sky is blue and believe that I do not believe that the sky is blue. (This could be expressed as $x:P$ and $y:((x:P) \to \bot)$.) Absent positive introspection, I can hold both these beliefs. But in the presence of positive introspection (and application), an inconsistent belief appears. (From $x:P$ deduce $!x:x:P$, and by application $(y\cdot!x):\bot$.)

The same example works in the absence of the operator $+$. I have not explored the conservativity of **JD** over $\mathbf{JD}^-$ or of **JD4** over $\mathbf{JD4}^-$.

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[6]Having, of course, to do with $+$-free justification terms and formulas $(G)^-$ which have had the $+$-terms stripped out.

[7]Again, this would be a different hypothesis than in our original version of the lemma.

[8]The word *deontic* denotes a connection to duty or obligation. The more common form of the axiom **D** is $\Box F \to \Diamond F$, which could be interpreted as "What is mandatory is also permitted." In normal modal logics, this scheme is equivalent to our formulation.

6. Conclusion and Acknowledgments

In Fitting's proof of conservativity for logics of knowledge [1], he showed that not only conclusions but entire proofs could be preserved by careful elimination of justification operators. However, his reduction relied heavily on the presence of the truth axiom ($t{:}F \to F$) and its variants. For example, if we were trying to eliminate occurrences of ! from a proof, and the justification term t contained ! (while u did not), the axiom $u{:}F \to (t+u){:}F$ would become $u{:}F \to F$, an instance of the truth axiom.

Thus, one problem clearly still open in this area is the existence or impossibility of uniform direct translations of Hilbert-style proofs in a stronger logic of belief to those in a weaker logic of belief.

We will also note here with the briefest of examples that the naïve world-by-world application of our extension of evidence functions does not work in general for multi-world Kripke-Fitting models as defined in Section 4.1. If W consists of two worlds u and v, and the accessibility relation $R = \{\langle u, v\rangle\}$, if $P \in \mathcal{E}(t, u)$ and $P \notin \mathcal{E}(t, v)$ then we run into trouble with whether $!t : t : P$ should be true at u. Our definition of $\mathcal{E}'$ would put $t{:}P$ into $\mathcal{E}'(!t, u)$, but for $!t{:}t{:}P$ to be true at u we should also have $t{:}P$ true at v since v is accessible from u. It may be that a subtler definition of $\mathcal{E}'$ would work, but that has not been explored, to the best of my knowledge. [9]

Also still open is conservativity of logics of belief with the negative introspection (?) operator. Because the semantic arguments in the present paper relied heavily on the monotonicity of the construction of the extension of evidence relations, they would seem incompatible with negative introspection. I am not aware of a straightforward cut-free sequent calculus for logics of justified belief incorporating negative introspection.

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