# HOMEOMORPHISMS AND FINITE SOLVABILITY OF THEIR PERTURBATIONS FOR FREDHOLM MAPS OF INDEX ZERO WITH APPLICATIONS [1] [2]

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**Abstract.** We prove a number of homeomorphism results for nonlinear Fredholm maps of index zero and their perturbations. Moreover, we show that k-ball and k-set perturbations of these homeomorphisms are again homeomorphisms or that the corresponding equations are finitely solvable. Various generalized first Fredholm theorems are given and finite solvability of general ( odd ) Fredholm maps is also studied. We apply these results to finite solvability of quasilinear elliptic equations on $R^N$ as well as on bounded domains. The basic tool used are the recent degree theories for nonlinear $C^1$-Fredholm maps of index zero and their perturbations.

## 1. Intoduction

In the early nineties, a rather complete degree theory for nonlinear $C^2$-Fredholm maps of index zero has been developed by Fitzpatrick-Pejsachowisz-Rabier [6]. Subsequently, their degree has been extended to $C^1$ Fredholm maps of index zero and to their (non) compact perturbations by Pejsachowicz, Rabier, Salter, Benevieri, Calamai and Furi ([12], [15], [2] and [1]). They have also given applications to global bifurcation problems for equations envoling these maps and to quasilinear elliptic equations. Part of this paper is devoted to proving various homeomorphism and finite solvability results for these maps using these degrees and applying them to quasilinear elliptic equations. New types of generalized first Fredholm theorems are proven. We also study the stability of these homeomorphisms under k-ball and k-set perturbations.

Let us describe our main results in more detail. Throughout the paper, we assume that $X$ and $Y$ are infinite dimensional Banach spaces. In Section 2, we establish a number of homeomorphism results for nonlinear Fredholm maps of index zero $T : X \to Y$ and their perturbations. Using the recent open mapping theorem for such maps of Calamai [4] and Rabier-Salter [15], we establish first a number of homeomorphism results for $T$ and its suitable perturbations $C$

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assuming that $T + C$ is locally injective, satisfies condition $(+)$ (i.e., $\{x_n\}$ is bounded whenever $\{(T + C)x_n\}$ converges), and $\alpha(C) < \beta(T)$, where ([6]), using the set measure of noncompactness $\alpha$,

$$\alpha(T) = \sup\{\alpha(T(A))/\alpha(A) \mid A \subset X \text{ bounded}, \ \alpha(A) > 0\}$$
$$\beta(T) = \inf\{\alpha(T(A))/\alpha(A) \mid A \subset X \text{ bounded}, \ \alpha(A) > 0\}.$$

$\alpha(T)$ and $\beta(T)$ are related to the properties of compactness and properness of the map T, respectively. In particular, we prove such results when $T + C$ is asymptotically close to a map that is positively homogeneous outside some ball. In the latter case, these results can be considered as generalizations of the first Fredholm theorem for linear compact perturbations of the identity. Next, we also show that these homeomorphism results are stable under k-ball or k-set contractive perturbations. They are based on a new invariance of domain theorem for such maps. An alternative result for equations involving $k$ set or ball contractive perturbations of homeomorphisms is also given.

Section 3 is devoted to studying equations of the form $Tx + Cx + Dx = f$ with T Fredholm of index zero, $\alpha(D) < \beta(T) - \alpha(C)$ and $T + C$ a homeomorphism. We show that they are either uniquely solvable or are finitely solvable for almost all right hand sides and that the cardinality of the solution set is constant on certain connected components in Y. Several generalized first Fredholm theorems are proved when T+C is asymptotically close to a map that is positively homogeneous outside some ball as well as some Borsuk type results and a general finite solvability result are given. These results are proved using the recent degree theories of nonlinear perturbations for Fredholm maps of index zero as defined by Fitzpatrick, Pejsachowicz-Rabier ([6],[11]), Benevieri-Furi [2], Rabier-Salter [15] and Benevieri-Calamai-Furi [1].

In Section 4, we apply some of our results to the unique and finite solvability of quasilinear elliptic equations on $R^N$ of the form

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, u, \nabla u)\partial_{\alpha\beta}^2 u + b(x, u, \nabla u) + c(x, u, \nabla u, D^2 u) = f(x)$$

in $W^{2,p}(R^N)$ and $f \in L_p(R^N)$. The Fredholm and properness properties of these equations have been established by Rabier-Stuart [16]. We refer to [17] for bifurcation problems for these equations. Combining a result of Rabier [13,14] with our results, we also obtain a Fredholm like result for these equations involving asymptotic limits of $a_{\alpha\beta}(x, \xi)$ and $b(x, \xi)$.

Finaly, in Section 5, some of our results are applied to the finite solvability of quasilinear elliptic equations on a bounded domain. The Fredholm part is a $C^1$ map of type $(S_+)$ that is asymptotically close to a k-homogeneous map and the perturbation is a $k_1$-set contraction.

## 2. Homeomorphism results for perturbed Fredholm maps of index zero

In this section we shall first prove various homeomorphism results for perturbations of nonlinear Fredholm maps of index zero. Then we shall look at their stability under k-$\phi$-contractive perturbations.

We begin by recalling some definitions. We say that a map $T : X \to Y$ satisfies condition (+) if whenever $Tx_n \to f$ in $Y$ then $\{x_n\}$ is bounded in $X$. $T$ is locally injective at $x_0 \in X$ if there is a neighborhood $U(x_0)$ of $x_0$ such that $T$ is injective on $U(x_0)$. $T$ is locally injective on $X$ if it is locally injective at each point $x_0 \in X$. A continuous map $T : X \to Y$ is said to be locally invertible at $x_0 \in X$ if there are a neighborhood $U(x_0)$ and a neighborhood $U(T(x_0))$ of $T(x_0)$ such that $T$ is a homeomorphism of $U(x_0)$ onto $U(T(x_0))$. It is locally invertible on $X$ if it is locally invertible at each point $x_0 \in X$.

**Theorem 2.1.** *Let $T : X \to Y$ be a continuous, locally injective and open map that is closed (in particular, proper) on bounded and closed subsets of $X$. Then $T$ is a homeomorphism if and only if $T$ satisfies condition (+).*

**Proof.** Let T be a homeomorphism and $Tx_n \to f$. Then $x_n = T^{-1}(Tx_n) \to T^{-1}f$ and therefore $T$ satisfies condition (+). Conversely, let $T$ satisfy condition(+). Let T be closed on bounded closed subsets of X. We shall show that T is closed on X. Let $D \subset X$ be closed and $x_n \in D$ such that $Tx_n \to y$ in Y. Then $\{x_n\}$ is bounded by condition (+) and $\{x_n\} \subset D \cap \bar{B}(0,r)$ for some $r > 0$. Since $T(D \cap \bar{B}(0,r))$ is closed, it follows that $y \in T(D \cap \bar{B}(0,r)) \subset T(D)$. Hence, $T(D)$ is closed and $T$ is a homeomorphism by Browder's theorem [3].

Next, let T be proper on bounded closed subsets of X. It is closed on such subsets and T is a homeomorphism by the first part. Let us prove this also independently by proving that T is proper on X. Namely, we shall prove that a proper on bounded closed subset map that satisfies condition (+) is proper. This fact will be used througout the paper. Let $K$ be a compact subset of Y. Then, $T^{-1}(K)$ is a bounded set in X. Indeed, if we had $\{x_n\} \subset T^{-1}(K)$ with $\|x_n\| \to \infty$, then $Tx_n = y_n$ for some $y_n \in K$. We may assume that $y_n \to y \in K$ and so $Tx_n \to y \in K$ with $\{x_n\}$ unbounded, in contradiction to condition (+). Hence, $T^{-1}(K)$ is bounded. Since T is continuous, $T^{-1}(K)$ is closed. Thus, $T^{-1}(K)$ is bounded and closed and therefore compact since T is proper when restricted to bounded and closed subsets of X. Hence, T is proper and therefore a homeomorphism by the global inversion theorem. ■

To look at $\phi$-condensing maps, we recall that the *set measure of noncompactness of a bounded set $D \subset X$ is defined as $\alpha(D) = \inf\{d > 0 \mid D$ has a finite covering

by sets of diameter less than d }. The *ball-measure of noncompactness* of $D$ is defined as $\chi(D) = \inf\{r > 0 | D \subset \cup_{i=1}^{n} B(x_i, r), \ x_i \in X, \ n \in N\}$. Let $\phi$ denote either the set or the ball-measure of noncompactness. Then a map $T : D \subset X \to Y$ is said to be $k - \phi$-*contractive* ($\phi$-*condensing*) if $\phi(T(Q)) \leq k\phi(Q)$ (respectively $\phi(T(Q)) < \phi(Q)$) whenever $Q \subset D$ (with $\phi(Q) \neq 0$). We note that T is completely continuous if and only if it is $\alpha$-0-contractive. Moreover, if T is Lipschitz continuous with constant k, then it is $\alpha$-k-contractive.

**Corollary 2.1.** *Let $F : X \to X$ be continuous, locally $\phi$-condensing and $I - F$ be locally injective. Then $I - F$ is a homeomorphism if and only if it satisfies condition (+).*

**Proof.** Let $T = I - F$. It was shown by Nussbaum [11] that $T$ is proper on bounded and closed subsets of $X$ and that it is an open map (cf. also [5]). Hence, the conclusions follow from Theorem 2.1. ∎

Theorem 2.1 applies also to maps $T : X \to X^*$ of type $(S_+)$, i.e., if $x_n \rightharpoonup x$ and $\limsup(Tx_n, x_n - x) \leq 0$ then $x_n \to x$.

**Corollary 2.2.** *Let $X$ be reflexive and $T : X \to X^*$ be continuous, bounded, of type $(S_+)$ and be locally injective. Then $T$ is a homeomorphism if and only if it satisfies condition (+).*

**Proof.** T is an open map ( see [18] ). We claim that it is proper on the bounded closed subsets of X. Let $D \subset X$ be closed and bounded and $K \subset X^*$ be compact. Let $x_n \in T^{-1}(K) \cap D$. Then $Tx_n = y_n \in K$ and we may assume that $x_n \rightharpoonup x$ and $y_n \to y \in K$. Then $\limsup(Tx_n, x_n - x) = 0$ and therefore $x_n \to x \in D$. Hence, $Tx_n \to Tx = y \in K$ and $x \in T^{-1}(K)$, which proves the claim. Thus, Theorem 2.1 applies. ∎

Next, we shall extend Corollary 2.1 to condensing perturbations of Fredholm maps of index zero. Let X,Y be infinite dimensional Banach spaces, U be an open subset of X and $T : U \to Y$ be as above. We recall the following properties ( see [7]) of $\alpha(T)$ and $\beta(T)$ defined in the introduction. First, we note that $\alpha(T)$ is related to the property of compactness of the map T and the number $\beta(T)$ is related to the properness of T.

(1) $\alpha(\lambda T) = |\lambda|\alpha(T)$ and $\beta(\lambda T) = |\lambda|\beta(T)$ for each $\lambda \in R$.

(2) $\alpha(T + C) \leq \alpha(T) + \alpha(C)$.

(3) $\beta(T)\beta(C) \leq \beta(ToC) \leq \alpha(T)\beta(C)$ (when defined)

(4) If $\beta(T) > 0$, then T is proper on bounded closed sets.

(5) $\beta(T) - \alpha(C) \leq \beta(T + C) \leq \beta(T) + \alpha(C)$.

(6) If T is a homeomorphism and $\beta(T) > 0$, then $\alpha(T^{-1})\beta(T) = 1$.

If $T : X \to Y$ is a homeomorphism, then (3) implies $1 = \beta(I) = \beta(T^{-1}oT) \leq$

$\alpha(T^{-1})\beta(T)$. Hence, $\beta(T) > 0$.

If $L : X \to Y$ is a bounded linear operator, then $\beta(L) > 0$ if and only if Im L is closed and dim $Ker L < \infty$ and $\alpha(L) \leq \|L\|$. Moreover, one can prove that L is Fredholm if and only if $\beta(L) > 0$ and $\beta(L^*) > 0$, where $L^*$ is the adjoint of L.

Let $T : U \to Y$ be, as before, a map from an open subset U of a Banach space X into a Banach space Y, and let $p \in U$ be fixed. Let $B_r(p)$ be the open ball in X centered at p with radius r. Suppose that $B_r(p) \subset U$ and set

$$\alpha(T|_{B_r(p)}) = \sup\{\alpha T(A)/\alpha(A) \mid A \subset B_r(p) \text{ bounded}, \ \alpha(A) > 0\}.$$

This is non-decreasing as a function of r, and clearly $\alpha(T|_{B_r(p)}) \leq \alpha(T)$. Hence, the following definition makes sense:

$$\alpha_p(T) = \lim_{r \to 0} \alpha(T|_{B_r(p)}).$$

Similarly, we define $\beta_p(T)$. We have $\alpha_p(T) \leq \alpha(T)$ and $\beta_p(T) \geq \beta(T)$ for any p. If $T$ is of class $C^1$, then $\alpha_p(T) = \alpha(T'(p))$ and $\beta_p(T) = \beta(T'(p))$ for any p ([4]). Note that for a Fredholm map $T : X \to Y$, $\beta_p(T) > 0$ for all $p \in X$.

Next, we shall apply Theorem 2.1 to certain perturbations of nonlinear Fredholm maps of index zero.

**Theorem 2.2.** *Let $T : X \to Y$ be a Fredholm map of index zero and $C : X \to Y$ be a continous map with $\alpha_p(C) < \beta_p(T)(= \beta(T'(p))$ for each $p \in X$. Suppose that $T + C$ is locally injective and closed ( in particular, proper) on bounded and closed subsets of $X$. Then $T + C$ is a homeomorphism if and only if it satisfies condition (+).*

**Proof.** We have that $T+C$ is an open map by the invariance of domain theorem of Calamai [4] ( see Rabier-Salter [15] when $C$ is compact). Hence, $T + C$ is a local homeomorphism. Since $T + C$ is closed ( proper ) on closed bounded subsets of $X$, the conclusions follow from Theorem 2.1. ∎

**Corollary 2.3.** *Let $T : X \to Y$ be a Fredholm map of index zero and $C : X \to Y$ be a continous map with $\alpha(C) < \beta(T)$. Suppose that $T + C$ is locally injective. Then $T + C$ is a homeomorphism if and only if it satisfies condition (+).*

**Proof.** Let $p \in X$ be fixed. Then $\alpha_p(C) \leq \alpha(C) < \beta(T) \leq \beta_p(T)(= \beta(T'(p))$. Since $\beta(T + C) \geq \beta(T) - \alpha(C) > 0$, then $T + C$ is proper on closed bounded subsets of $X$. Hence, the conclusions follow from Theorem 2.2. ∎

Next, the following simple lemma gives some conditions on $T$ and $C$ that imply condition (+) for $T+C$ in Theorem 2.2. Recall that $C : X \to Y$ is quasibounded if, for some $k > 0$ the quasinorm

$$|C| = \limsup_{\|x\| \to \infty} \|Cx\|/\|x\|^k < \infty.$$

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**Lemma 2.1.** *Suppose that $T, C : X \to Y$ and either one of the following conditions holds*

*(i) $\|Cx\| \leq a\|Tx\| + b$ for some constants $a \in [0, 1)$ and $b > 0$ and all $\|x\|$ large and $\|Tx\| \to \infty$ as $\|x\| \to \infty$.*

*(ii) There exist constants $c, c_0, k > 0$ and $R > R_0$ such that*

$$\|Tx\| \geq c\|x\|^k - c_0 \text{ for all } \|x\| \geq R.$$

*and $C$ is quasibounded with the quasinorm $|C| < c$.*

*Then $T + tC$ satisfies condition $(+)$, $t \in [0, 1]$.*

**Proof.** It suffices to observe that, in either case, $\|Tx + tCx\| \to \infty$ as $\|x\| \to \infty$ uniformly in t. ∎

Corollary 2.3 and Lemma 2.1 imply the following result.

**Corollary 2.4.** *Let $T : X \to Y$ be a Fredholm map of index zero and $C : X \to Y$ be a continous map with $\alpha(C) < \beta(T)$. Suppose that $T + C$ is locally injective and either one of the conditions (i)-(ii) of Lemma 2.1 holds. Then $T + C$ is a homeomorphism.*

A map T is positive k-homogeneous outside some ball if $T(\lambda x) = \lambda^k T(x)$ for some $k \geq 1$, all $\|x\| \geq R$ and all $\lambda \geq 1$. We say that T is asymptotically close to a positive k-homogeneous map $A$ if

$$|T - A| = \limsup_{\|x\| \to \infty} \|Tx - Ax\|/\|x\|^k < \infty.$$

We note that $T$ is asymptotically close to a positively k-homogeneous map $A$ if there is a functional $c : X \to [0, a]$ such that

$$\limsup_{t \to \infty} \|T(tx)/t^k - Ax\| = c(t)\|x\|^k,$$

In this case, $|T - A| \leq a$.

Next, we shall prove a nonlinear extension of the Fredholm alternative for linear compact vector fields involving asymptotically homogeneous maps. To that end, we need the following result.

**Theorem 2.3.** *Let $T : X \to Y$ be a locally injective, continuous and open map that is closed ( in particular, proper ) on bounded and closed subsets of $X$. Assume that $T$ is asymptotically close to a continuous, closed (proper, in particular) on bounded and closed subsets of $X$ positive k-homogeneous map $A$ outside some ball in $X$, i.e., there are $k \geq 1$ and $R_0 > 0$ such that*

$$A(\lambda x) = \lambda^k Ax$$

*for all* $\|x\| \geq R_0$ *and either for all* $\lambda > 0$, *or for all* $\lambda \geq 1$ *and* $(A)^{-1}(0)$ *bounded. Then* $T$ *is a homeomorphism.*

For the proof of the theorem, we need the following lemma, which gives other conditions that imply condition $(+)$.

**Lemma 2.2.** *a) Let* $A : X \to Y$ *be continuous, closed (in particular, proper) on bounded and closed subsets of* $X$ *and for some* $R_0 \geq 0$

$$A(\lambda x) = \lambda^k(Ax) \tag{2.1}$$

*for all* $\|x\| \geq R_0$, $\lambda \geq 1$ *and some* $k \geq 1$. *Suppose that either one of the following conditions holds*

*(i) There is a constant* $M > 0$ *such that if* $Ax = 0$, *then* $\|x\| \leq M$

*(ii) A is injective*

*(iii) A is locally injective and (2.1) holds for all* $\lambda > 0$.

*Then there exist constants* $c > 0$ *and* $R > R_0$ *such that*

$$\|Ax\| \geq c\|x\|^k \text{ for all } \|x\| \geq R \tag{2.2}$$

*and, in addition,* $A^{-1}$ *is bounded when (ii) holds. Moreover, if* $A$ *is positively* $k$-*homogeneous, then* $Ax = 0$ *has only the trivial solution if and only if (2.2) holds.*

*b) If* $T : X \to Y$ *is asymptotically close to* $A$ *with* $|T - A|$ *sufficiently small, then* $T$ *also satisfies (2.2) with* $c$ *replaced by* $c - |T - A|$.

**Proof.** a) We may assume that $R_0 \geq M$. Let (i) hold and $R > R_0$ be fixed. We shall show that condition (2.2) holds for some $c > 0$. If not, then there would exist a sequence $\{x_n\}$ in $X$ such that $\|x_n\| \geq R$ and

$$\|Ax_n\| \leq 1/n\|x_n\|^k \text{ for all } n \geq 1.$$

Set $v_n = Rx_n/\|x_n\|$. Then $\|v_n\| = R$, $x_n = R^{-1}\|x_n\| \, v_n$, and

$$Ax_n = A(R^{-1}\|x_n\| \, v_n) = R^{-k}\|x_n\|^k A v_n$$

since $R^{-1}\|x_n\| \geq 1$. Hence,

$$\|Ax_n\| = R^{-k}\|x_n\|^k \|Av_n\| \leq 1/n\|x_n\|^k.$$

Thus

$$\|Av_n\| \leq R^k/n \to 0 \text{ as } n \to \infty.$$

If $A$ is proper on bounded closed subsets, it maps each bounded closed subset of $X$ on a closed subset. Hence, $A(\partial B(0, R))$ is closed in either case and contains

0, and therefore there is a $v \in \partial B(0, R)$ such that $Av = 0$. This contradicts our choice of $R$ since $R_0 \geq M$. Thus, inequality (2.2) holds.

Next, we shall show that (ii) implies that $A^{-1}$ is bounded, and therefore (i) holds. Select $R > R_0$ such that $\|x\| < R$ if $Ax = 0$. If $A^{-1}$ is not bounded, then there would exist $\{x_n\}$ such that $\{Ax_n\}$ is bounded and $\|x_n\| \to \infty$. We may assume that $\|x_n\| \geq R$ for all n. As above, setting $v_n = Rx_n/\|x_n\|$, we get that

$$Av_n = R^k Ax_n/\|x_n\| \to 0 \text{ as } n \to \infty.$$

As above, we get a contradiction, and therefore $A^{-1}$ is bounded.

Finally, we shall show that (iii) implies (i) with M=0. Clearly, (2.1) implies that it holds for all x and all $\lambda > 0$. Hence, T is positively homogeneous. If $Ax = 0$ with $x \neq 0$, set $v = r/\|x\|x$ with $r > 0$ such that $A$ is injective on $\bar{B}(0, r)$. Then $Av = 0$ in contradiction to our choice of $r$. Hence, $x = 0$ and (i) holds with $M = 0$. Finally, if (2.1) holds for each x and all $\lambda > 0$, then it is clear that (2.2) implies that $Ax = 0$ has only the triavial solution.

b) Let $\epsilon > 0$ be such that $\epsilon + |T - A| < c$ for $c > 0$ in part a). Then there is an $R > R_0$ such that $\|Tx - Ax\| \leq (\epsilon + |T - A|)\|x\|^k$ for all $\|x\| \geq R$. This implies that T satisfies condition (2.2) with $c_1 = c - |T - A|$. ∎

**Proof of Theorem 2.3.** By Lemma 2.2, $T$ satisfies condition (+). Hence, the conclusion follows by Theorem 2.1. ∎

When $X$ is reflexive, the k-homogeneity in Lemma 2.2 can be relaxed. Recall that ([10]) a map $T : X \to Y$ is called $k$-quasihomogeneous relative to $T_0$, $k > 0$, if there is a positive k-homogeneous map $T_0$ such that if $x_n \rightharpoonup x$, $0 < t_n \to 0$ and $t_n^k T(x_n/t_n) \to y$, then $T_0 x = y$. It is strongly k-quasihomogeneous if $x_n \rightharpoonup x$ and $0 < t_n \to 0$ imply that $t_n^k T(x_n/t_n) \to T_0 x$. In the latter case, $|T - T_0| = 0$ and $T_0$ is completely continuous if $X$ is reflexive.

**Lemma 2.3** *Let $X$ be a reflexive Banach space, $T : X \to Y$ be asymptotically close to a k-quasihomogeneous map $F$ at 0 relative to $F_0$. Let $\|Fx\| \geq l\|x\|^k$ for some $l > 0$ and $A : X \to Y$ be asymptotically close to a strongly k-quasihomogeneous map $G$ relative to $G_0$. Assume that $|T - F|$ and $|A - G|$ are sufficiently small and $x = 0$ if $F_0 x + G_0 x = 0$. Then there are $c > 0$ and $R > 0$ such that*
$$\|Tx + Ax\| \geq c\|x\|^k \text{ for all } \|x\| \geq R.$$

**Proof.** We claim that this inequality holds for some $c_1 > 0$ for $F + G$. If not, then there are $\|x_n\| \to \infty$ such that for each n

$$\|Fx_n + Gx_n\| \leq 1/n\|x_n\|^k.$$

Set $v_n = x_n/\|x_n\|$ and we may assume that $v_n \rightharpoonup v$. Then

$$F(\|x_n\|v_n)/\|x_n\|^k + G(\|x_n\|v_n)/\|x_n\|^k = (Fx_n + Gx_n)/\|x_n\|^k \to 0$$

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as $n \to \infty$. The strong k-quasihomogeneity of $G$ implies that $G(\|x_n\|v_n)/\|x_n\|^k \to G_0 v$. Hence, $F(\|x_n\|v_n)/\|x_n\|^k \to -G_0 v$ and $F_0 v = -G_0 v$ by the k-quasihomogeneity of $F$. But,

$$\|F(\|x_n\|v_n)\|/\|x_n\|^k \geq l(\|x_n\|v_n)^k/\|x_n\|^k = l > 0$$

and therefore $\|F_0 v\| = \|G_0 v\| \geq l$. Since $G_0$ is k-homogeneous, we get that $v \neq 0$, in contradiction to our assumption on $F_0 + G_0$. Hence, the above inequality holds for some $c_1 > 0$. This and the smallness of $|T - F|$ and $|A - G|$ imply the conclusion with $c = c_0 - |T - F| - |A - G|$. ∎

**Remark 2.1** In view of Lemma 2.3, the k-homogeneity in the results of this paper can be weaken to k-quasihomogeneity when X is reflexive.

**Theorem 2.4.** (Generalized First Fredholm Theorem) *Let $T : X \to Y$ be a Fredholm map of index zero and $C : X \to Y$ be a continuous map with $\alpha(C) < \beta(T)$. Suppose that $T + C$ is locally injective and is asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of X positive k-homogeneous map A, outside some ball in X, i.e., there are $k \geq 1$ and $R_0 > 0$ such that*

$$A(\lambda x) = \lambda^k A x$$

*for all $\|x\| \geq R_0$ and either for all $\lambda > 0$, or for all $\lambda \geq 1$ and $(A)^{-1}(0)$ bounded. Then $T + C$ is a homeomorphism.*

**Proof.** The map $T + C$ satisfies condition (+) by Lemma 2.2. Thus, the conclusion follows by Corollary 2.3. ∎

**Remark 2.2.** In view of Theorem 2.1, the condition $\alpha(C) < \beta(T)$ in Theorem 2.4 can be weaken to $T + C$ is closed ( in particular, proper) on bounded and closed subsets of X and $\alpha_p(C) < \beta_p(T)$ for each $p \in X$.

The following proposition provides a large class of nonlinear Fredholm maps of index zero.

**Proposition 2.1.** *Let $T = A + N : X \to Y$ be such that A is a linear Fredholm map of index zero and $N \in C^1(X, Y)$ such that $A + tN'(x) : X \to Y$ is proper when restricted to bounded closed subsets of X for all $x \in X$ and $t \in [0, 1]$. Then T is a Fredholm map of index zero.*

**Proof.** Since for all $x \in X$ and $t \in [0, 1]$, $A + tN'(x) : X \to Y$ is proper when restricted to bounded closed subsets of X, it is semi-Fredholm. Hence, $\text{ind}(A + N'(x)) = \text{ind}A$ for each $x \in X$ and therefore $T$ is Fredholm of index zero (see [5]) . ∎

Next, we shall show that k-$\phi$-contractive perturbations of some ( e.g., expansive ) homeomorphisms are still homeomorphisms. We need the following open mapping stability result for condensing perturbations of these homeomorphisms.

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**Theorem 2.5.** *Let $T : X \to Y$ be a homeomorphism and $C : X \to Y$ be continuous. Then $T + C$ is an open map if either*

*(i) $T$ is expansive, i.e., $\|Tx - Ty\| \geq c\|x - y\|$ for all $x, y \in X$ and some $c > 0$, $C$ is $k$-$\phi$-contraction with $k < c$ and $T + C$ is locally injective, or*

*(ii) $\alpha(C) < \beta(T)$ and $T + C$ is injective.*

**Proof.** The equation $Tx + Cx = f$ is equivalent to $y + CT^{-1}y = f$, $y = Tx$. Let (i) hold. Then the map $CT^{-1}$ is $k/c$-$\phi$-contactive with $k/c < 1$. If (ii) holds, then the map $CT^{-1}$ is $\alpha(CT^{-1})$-contractive since

$$\alpha(CT^{-1}) \leq \alpha(C)\alpha(T^{-1}) = \alpha(C)/\beta(T) < 1.$$

Moreover, $I + CT^{-1}$ is ( locally ) injective since such is $T + C$. Hence, $I + CT^{-1}$ is an open map ([11], see also [5]). Thus, $T + C$ is an open map since $T + C = (I + CT^{-1})T$. ∎

To discuss various special cases, we need the following lemma.

**Lemma 2.4.** *Let $T : X \to Y$ be a homeomorphism and $C : X \to Y$.*

*(i) If $I + CT^{-1} : Y \to Y$ is proper on bounded closed subsets of $Y$ and if either $T$ or $C$ is bounded, then $T + C$ is proper on bounded closed subsets of $X$.*

*(ii) If $T^{-1}$ is bounded and $I + CT^{-1} : Y \to Y$ saisfies condition (+), then $T + C$ satisfies condition (+). Conversely, if either $T$ or $C$ is bounded, and $T + tC$, $t \in [0, 1]$, satisfies condition (+), or $C$ has a linear growth and $T^{-1}$ is quasibounded with a sufficiently small quasinorm, then $I + tCT^{-1} : Y \to Y$ satisfies condition (+), $t \in [0, 1]$.*

**Proof.** (i) Let $B \subset X$ be bounded and closed and $K$ be a compact subset of $Y$. Suppose that $x_n \in (T + C)^{-1}(K) \cap B$ and $y_n \in K$ such that $x_n \in (T + C)^{-1}(y_n)$. Then $Tx_n + Cx_n = y_n$ and $z_n + CT^{-1}z_n = y_n$ if we set $z_n = Tx_n$. Thus, $\{z_n\}$ is bounded if $T$ is bounded. If $C$ is bounded, then $\{Cx_n\}$ is bounded and so is $\{z_n\}$ since $z_n = y_n - Cx_n$. Let $D$ be a bounded closed subset of $Y$ with $\{z_n\} \subset D$. Then $z_n \in (I + CT^{-1})^{-1}(K) \cap D$ and therefore $\{z_n\}$ is compact by our assumption on $I + CT^{-1}$. Hence, $\{x_n = T^{-1}z_n\}$ is compact by the continuity of $T^{-1}$ and a subsequence $x_{n_k} \to x \in B$. Thus, $T + C$ is proper on bounded closed subsets of $X$.

(ii) Let $I + CT^{-1}$ satisfy condition (+), $T^{-1}$ be bounded and $Tx_n + Cx_n \to f$. Then $y_n + CT^{-1}y_n \to f$ in Y, where $x_n = T^{-1}y_n$. Since $\{y_n\}$ is bounded, so is $\{x_n\}$ by the boundedness of $T^{-1}$. Hence, $T + C$ satisfies condition (+). Conversely, let $T + tC$ satisfy condition (+) and $y_n + t_nCT^{-1}y_n \to f$ in Y, $t_n \in [0, 1]$. Then $Tx_n + t_nCx_n \to f$, where $y_n = Tx_n$. Hence, $\{x_n\}$ is bounded and therefore such is $\{y_n\}$ by the boundedness of either $T$ or $C$ as in (i). Thus, $I + tCT^{-1}$ satisfies condition (+). Next, let $T^{-1}$ be quasibounded and C have

a linear growth. Then $CT^{-1}$ is quasibounded with the quasinorm $|CT^{-1}| \leq |T^{-1}||C| < 1$. Thus, $I + tCT^{-1}$ satisfies condition $(+)$. ∎

**Theorem 2.6.** *Let $T : X \to Y$ be a homeomorphism and $C : X \to Y$ be continuous. Suppose that either one of the following conditions holds*

*(i) $\|Tx - Ty\| \geq c\|x - y\|$ for all $x, y \in X$ and some $c > 0$, $C$ is $k$-$\phi$-contraction with $k < c$ and $T + C$ is locally injective, or*

*(ii) $\alpha(C) < \beta(T)$ and $T + C$ is injective.*

*Then $T + C$ is a homeomorphism if and only if it satisfies condition $(+)$.*

**Proof.** Note that $C$ is bounded since it is $k - \phi$-contractive. The map $T + C$ is open by Theorem 2.5. Moreover, it is proper on bounded closed subsets of X by Lemma 2.4(i) when (i) holds. The same is true if (ii) holds since then $\beta(T + C) \geq \beta(T) - \alpha(C) > 0$. Thus, $T + C$ is a homeomorphism if and only if it satisfies condition $(+)$ by Theorem 2.1. ∎

**Corollary 2.5.** *Let $T : X \to Y$ be a homeomorphism and $C : X \to Y$ be continuous. Suppose that either one of the following conditions holds*

*(i) $\|Tx - Ty\| \geq c\|x - y\|$ for all $x, y \in X$ and some $c > 0$, $C$ is $k$-$\phi$-contraction with $k < c$ and $T + C$ is locally injective, or*

*(ii) $\alpha(C) < \beta(T)$ and $T + C$ is injective.*

*Then $T + C$ is a homeomorphism if either one of the following conditions holds*

*(a) (i) holds and $C$ is quasibounded with the quasinorm $|C| < c$.*

*(b) $C$ has a linear growth and $T^{-1}$ is quasibounded with a sufficiently small quasinorm.*

**Proof.** It suffices to show that $T + C$ satisfies condition $(+)$. This follows from Lemma 2.1-(ii) and Lemma 2.4-(ii) in either case since $T^{-1}$ is bounded being $1/c$ and $1/\beta(T)$-contractive, respectively. ∎

If we are interested only in the bijectivity of $T + C$, then we have the following result.

**Theorem 2.7.** *Let $T : X \to Y$ be a bijection and $C : X \to Y$ be continuous. Suppose that either one of the following conditions holds*

*(i) $\|Tx - Ty\| \geq c\|x - y\|$ for all $x, y \in X$ and some $c > 0$, $C$ is $k$-$\phi$-contraction with $k < c$ and is quasibounded with the quasinorm $|C| < c$, or*

*(ii) $CT^{-1}$ is $k_1 - \phi$-contractive, $k_1 < 1$, $T^{-1}$ is quasibounded with sufficiently small quasinorm and $C$ has a linear growth.*

*Then $T + C$ is surjective and is a bijection if it is injective.*

**Proof.** The equation $Tx + Cx = f$ is equivalent to $y + CT^{-1}y = f$, where

$y = Tx$. The map $CT^{-1}$ is $k_1 - \phi$-contractive with $k_1 < 1$ in either case. If (i) holds, then $T + tC$ satisfies condition $(+)$ by Lemma 2.1-(ii) and, since C is bounded, $I + tCT^{-1}$ satisfies condition $(+)$, $t \in [0,1]$, by Lemma 2.4-(ii). If (ii) holds, then again $H_t = I + tCT^{-1}$ satisfies condition $(+)$ by Lemma 2.4-(ii). Thus, the equation $y + CT^{-1}y = f$ is solvable for each $f \in Y$ ( just use the degree in [11]). Hence, $T + C$ is surjective. ∎

Now, using Theorem 2.7, we shall establish a surjectivity and a homeomorphism result for $k-\phi$-perturbations of strongly accretive maps. ( A similar result holds for such perturbations of strongly monotone maps. ) If $J : X \to 2^{X^*}$ is the duality map, define $(x, y)_+ = \max\{y(x) \mid y \in Jx\}$ for each $x, y \in X$. Then $(x, y)_+ \geq (x, z)$ for each $z \in Jx$.

**Corollary 2.6.** *Let $T : X \to X$ be c-strongly accretive, i.e., $(Tx - Ty, z)_+ \geq c\|x - y\|^2$ for all $x, y \in X$, all $z \in J(x - y)$ and some $c > 0$. Suppose that $C$ is $k$-$\phi$-contraction with $k < c$. Then*

*(i) $T + C$ is surjective if $C$ is quasibounded with the quasinorm $|C| < c$ and $T$ is either continuous or demicontinuous and $X^*$ is uniformly conxex,*

*(ii) if $T$ is continuous and $T + C$ is locally injective, then $T + C$ is a homeomorphism if and only if it satisfies condition $(+)$.*

**Proof.** (i) It is known that $T$ is a bijection ( see [5] ) . Since $(x, y)_+ \geq (x, z)$ and $\|z\| = \|x\|$ for each $z \in Jx$, we get that $\|Tx - Ty\| \geq c\|x - y\|$ for each $x, y \in X$. The conclusion follows from Theorem 2.7.

(ii) Since T is a homeomorphism and $k < c$, T+C is a homeomorphism if and only if it satisfies condition $(+)$ by Theorem 2.6-(i). ∎

We have the following variant of Theorem 2.4

**Theorem 2.8.** ( Generalized First Fredholm Theorem) *Let $T : X \to Y$ be a Fredholm map of index zero and $C : X \to Y$ be a continuous map with $\alpha(C) < \beta(T)$. Suppose that $T$ is locally injective, $T + C$ is injective, the quasinorm $|C|$ of $C$ is sufficiently small, where*

$$|C| = \limsup_{\|x\| \to \infty} \|Cx\|/\|x\|^k,$$

*and $T$ is asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of X positive k-homogeneous map A, outside some ball in X,i.e., there are $k \geq 1$ and $R_0 > 0$ such that*

$$A(\lambda x) = \lambda^k A x$$

*for all $\|x\| \geq R_0$ and either for all $\lambda > 0$, or for all $\lambda \geq 1$ and $(A)^{-1}(0)$ bounded. Then $T + C$ is a homeomorphism.*

**Proof.** T is a homeomorphism by Theorem 2.4 ( with C=0 in this theorem ). Moreover, $T + C$ is proper on bounded closed subsets of X since $\beta(T + C) \geq$

$\beta(T) - \alpha(C) > 0$. Since $\|Tx\| \geq c\|x\|^k$ by Lemma 2.2-b) and $|C|$ is sufficiently small, $T + C$ satisfies condition $(+)$. Thus, $T + C$ is a homeomorphism by Theorem 2.6-(ii). ∎

Finally, we shall now prove an alternative result for perturbations of general homeomorphisms by k-set-contractive maps. ( This result is also valid for $k$-ball contractive perturbations and it will be given elsewhere. ) For a continuous map $F : X \to Y$, let $\Sigma$ be the set of all points $x \in X$ where $F$ is not locally invertible and let $\text{card}F^{-1}(\{f\})$ be the cardinal number of the set $F^{-1}(\{f\})$. We need the following result.

**Theorem 2.9.** (Ambrosetti) *Let $F \in C(X, Y)$ be a proper map. Then the cardinal number $\text{card}F^{-1}(\{f\})$ is constant, finite (it may be even 0) on each connected component of the set $Y \setminus F(\Sigma)$.*

**Theorem 2.10** (Nonlinear Alternative) *Let $T : X \to Y$ be a homeomorphism, $C : X \to Y$ be such that $T + C$ satisfies condition $(+)$ and $\alpha(C) < \beta(T)$ ( $\|Tx - Ty\| \geq c\|x - y\|$ for all $x, y \in X$ and some $c > 0$ and $C$ is k-$\phi$-contraction with $k < c$, respectively ). Then either*

*(i) $T + C$ is injective ( locally injective, respectively ), in which case it is a homeomorphism, or*

*(ii) $T + C$ is not injective ( not locally injective, respectively ), in which case, assuming additionally that $T + tC$ satisfies condition $(+)$, the equation $Tx + Cx = f$ is solvable for each $f \in Y$ with $(T + C)^{-1}(f)$ compact and the cardinal number $\text{card}(T + C)^{-1}(f)$ is positive, constant and finite on each connected component of the set $Y \setminus (T + C)(\Sigma)$.*

**Proof.** (i) If $T + C$ is injective ( locally injective, respectively ), then it is a homeomorphism by Theorem 2.6.

(ii) Suppose that $T + C$ is not injective ( locally injective, respectively ). As above, the equation $Tx + Cx = f$ is equivalent to the equation $y + CT^{-1}y = f$ in $Y$, with $y = Tx$. The map $CT^{-1}$ is $\alpha(CT^{-1})$-contractive in the first case since

$$\alpha(CT^{-1}) \leq \alpha(C)\alpha(T^{-1}) = \alpha(C)/\beta(T) < 1.$$

It is $k/c$-$\phi$-contractive in the second case. Moreover, $I + tCT^{-1}$ satisfies condition $(+)$ by Lemma 2.4-(ii) since $C$ is bounded. Hence, using the homotopy $H(t, x) = x + tCT^{-1}x$ and the degree theory for condensing maps [11], we get that $I + CT^{-1}$ is surjective. Therefore, $T + C$ is surjective.

Since $\beta(T + C) \geq \beta(T) - \alpha(C) > 0$, it follows that $T + C$ is proper on closed bounded subsets of $X$. Hence, $T + C$ is proper on X since it satisfies condition $(+)$. Thus, the cardinal number $\text{card}(T + C)^{-1}(f)$ is positive, constant and finite on each connected component of the set $Y \setminus (T + C)(\Sigma)$ by Theorem 2.9. ∎

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**Theorem 2.11** ( Generalized First Fredholm Theorem) *Let* $T : X \to Y$ *be a homeomorphism and* $C, D : X \to Y$ *be continuous maps such that* $\alpha(D)$, $\alpha(C)$ *and* $|D|$ *are sufficiently small, where*

$$|D| = \limsup_{\|x\| \to \infty} \|Dx\|/\|x\|^k < \infty.$$

*Assume that* $T + C$ *is injective and either* $\|Tx + Cx\| \geq c\|x\|^k - c_0$ *for all* $\|x\| \geq R$ *for some* $R$, $c$ *and* $c_0$, *or is asymptotically close to a continuous, closed (proper, in particular) on bounded and closed subsets of* $X$ *positive* $k$-*homogeneous map* $A$, *outside some ball in* $X$, *i.e., there are* $k \geq 1$ *and* $R_0 > 0$ *such that*

$$A(\lambda x) = \lambda^k A x$$

*for all* $\|x\| \geq R_0$ *and all* $\lambda \geq 1$. *Then either*

*(i)* $T + C + D$ *is injective, in which case* $T + C + D$ *is a homeomorphism, or*

*(ii)* $T + C + D$ *is not injective, in which case the solution set* $(T + C + D)^{-1}(\{f\})$ *is nonempty and compact for each* $f \in Y$ *and the cardinal number* $\mathrm{card}(T + C + D)^{-1}(\{f\})$ *is constant, finite and positive on each connected component of the set* $Y \setminus (T + C + D)(\Sigma)$.

**Proof.** Since T is a homeomorphism, it is proper and $\beta(T) > 0$. Thus, $\alpha(C) < \beta(T)$ and $T + C$ satisfies condition $(+)$ by Lemma 2.2-(ii). It follows that $T + C$ is a homeomorphism by Theorem 2.6. If (i) holds, then $T + C + D$ is a homeomorphism by Theorem 2.6-(ii) and Lemmas 2.1-(ii) and 2.2-(ii)-b) since $D$ has a sufficiently small quasinorm and $\alpha(D) < \beta(T) - \alpha(C) \leq \beta(T + C)$. Let (ii) hold. We claim that $T + C + tD$ satisfies condition $(+)$.

Let $y_n = (T + C + t_n D)x_n \to y$ as $n \to \infty$ with $t_n \in [0, 1]$, and suppose that $\|x_n\| \to \infty$. Then, by Lemma 2.2 with $c_1 = c - |T + C|$

$$c_1 \|x_n\|^k - c_0 \leq \|(T + C)x_n\| \leq \|y_n\| + (|D| + \epsilon)\|x_n\|^k$$

for all $n$ large and any $\epsilon > 0$ fixed. Dividing by $\|x_n\|^k$ and letting $n \to \infty$, we get that $c \leq |D|$. This contradicts our assumption that $|D|$ is sufficiently small and therefore condition $(+)$ holds. Hence, the equation $Tx + Cx + Dx = f$ is solvable by Theorem 2.10. Since $\alpha(C)$ and $\alpha(D)$ are sufficiently small, $\beta(T + C + D) \geq \beta(T) - \alpha(C + D) \geq \beta(T) - \alpha(C) - \alpha(D) > 0$. Hence, the map $T + C + D$ is proper on closed bounded sets. Since $T + C + D$ satisfies condition $(+)$, it is a proper map and the other conclusions follow from Theorem 2.9. ■

## 3. Finite solvability for perturbations of Fredholm maps of index zero

In this section, we shall study equations of the form $Tx + Cx + Dx = f$ with T Fredholm of index zero, $\alpha(D) < \beta(T) - \alpha(C)$ and $T + C$ a homeomorphism.

We shall show that they are either uniquely solvable or are finitely solvable for almost all right hand sides and that the cardinality of the solution set is constant on certain connected components in Y. Some generalized first Fredholm theorems will be proved when T+C is positively homogeneous outside some ball as well as some Borsuk type results and a general finite solvability result will be given. They are based on the recent degree theory of nonlinear perturbations for $C^1$-Fredholm maps of index zero as defined in Rabier-Salter [15], Benevieri-Furi [2] and Benevieri-Calamai-Furi [1] (see also Fitzpatrick-Pejsachowisz-Rabier [6], [12]).

We begin with the following extension of the first Fredholm theorem to perturbed nonlinear Fredholm maps of index zero.

**Theorem 3.1.** ( Generalized First Fredholm Theorem) *Let $T : X \to Y$ be a Fredholm map of index zero and $C, D : X \to Y$ be continuous maps such that $\alpha(D) < \beta(T) - \alpha(C)$ with $|D|$ sufficiently small, where*

$$|D| = \limsup_{\|x\| \to \infty} \|Dx\|/\|x\|^k < \infty.$$

*Assume that $T + C$ is locally injective and either $\|Tx + Cx\| \geq c\|x\|^k - c_0$ for all $\|x\| \geq R$ for some $R$, $c$ and $c_0$, or is asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of $X$ positive $k$-homogeneous map $A$, outside some ball in $X$, i.e., there are $k \geq 1$ and $R_0 > 0$ such that*

$$A(\lambda x) = \lambda^k Ax$$

*for all $\|x\| \geq R_0$ and either for all $\lambda > 0$, or for all $\lambda \geq 1$ and $(A)^{-1}(0)$ bounded. Then either*

*(i) $T+C+D$ is injective, in which case $T+C+D$ is a homeomorphism, or*

*(ii) $T+C+D$ is not injective, in which case the solution set $(T+C+D)^{-1}(\{f\})$ is nonempty and compact for each $f \in Y$ and the cardinal number $\text{card}(T+C+D)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (T + C + D)(\Sigma)$.*

**Proof.** Since $\alpha(C) < \beta(T)$, it follows that $T + C$ is a homeomorphism by Corollary 2.4 and Theorem 2.4, respectively. Since $\alpha(D) < \beta(T) - \alpha(C) \leq \beta(T+C)$, by Theorem 2.10 if suffices to show that $T+C+tD$ satisfies condition (+). But this follows as in the proof of Theorem 2.11. Hence, the equation $Tx + Cx + Dx = f$ is solvable for each $f \in Y$.

Since $\beta(T + C + D) \geq \beta(T) - \alpha(C + D) \geq \beta(T) - \alpha(C) - \alpha(D) > 0$, the map $T + C + D$ is proper on closed bounded sets. Since $T + C + D$ satisfies condition (+), it is a proper map and the other conclusions of the theorem follow from Theorem 2.9. ∎

**Corollary 3.1.** *Let $T, C : X \rightarrow Y$ and $T + C$ be Fredholm maps of index zero such that $\alpha(C) < \beta(T)$ and $|C|$ sufficiently small, where*

$$|C| = \limsup_{\|x\| \rightarrow \infty} \|Cx\|/\|x\|^k < \infty.$$

*Assume that $T$ is locally injective and either $\|Tx\| \geq c\|x\|^k - c_0$ for all $\|x\| \geq R$ for some $R$, $c$ and $c_0$, or $T$ is asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of $X$ positive k-homogeneous map $A$, outside some ball in $X$, i.e., there are $k \geq 1$ and $R_0 > 0$ such that*

$$A(\lambda x) = \lambda^k A x$$

*for all $\|x\| \geq R_0$ and either for all $\lambda > 0$, or for all $\lambda \geq 1$ and $(A)^{-1}(0)$ bounded. Then either*

*(i) $T+C$ is injective, in which case $T+C$ is a homeomorphism, or*

*(ii) $T+C$ is not injective, in which case the solution set $(T + C)^{-1}(\{f\})$ is nonempty and compact for each $f \in Y$ and the cardinal number $card(T + C)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the open and dense set of regular values $R_{T+C} = Y \setminus (T + C)(S)$ of $Y$, where $S$ is the set of singular points of $T + C$.*

**Proof.** Part (i) and the surjectivity of $T + C$ follow from Theorem 3.1 ( with C replaced by D ). As before, we have that $T + C$ is proper on bounded and closed set and satisfies condition (+). Hence, it is proper and the other conclusions follow from the general theorem on nonlinear Fredholm maps of index zero ( see [19]). ∎

Denote by $\deg_{BCF}$ the degree of Benevieri-Calamai-Furi. When $T + C$ is not locally injective, we have the following extension of Theorem 3.1.

**Theorem 3.2.** ( Generalized First Fredholm Theorem) *Let $T : X \rightarrow Y$ be a Fredholm map of index zero and $C, D : X \rightarrow Y$ be continuous maps such that $\alpha(D) < \beta(T) - \alpha(C)$ with $|D|$ sufficiently small, where*

$$|D| = \limsup_{\|x\| \rightarrow \infty} \|Dx\|/\|x\|^k < \infty.$$

*Assume that $T+C$ is asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of $X$ positive k-homogeneous map $A$, outside some ball in $X$ for all $\lambda \geq 1$ and some $k \geq 1$, $\|x\| \leq M < \infty$ if $Ax = 0$ and $\deg_{BCF}(T + C, B(0, r), 0) \neq 0$ for all large $r$. Then the equation $Tx + Cx + Dx = f$ is solvable for each $f \in Y$ with $(T+C+D)^{-1}(\{f\})$ compact and the cardinal number $card(T+C+D)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (T + C + D)(\Sigma)$.*

**Proof.** Since $\beta(T + C) \geq \beta(T) - \alpha(C) > 0$, we have that $T + C$ is proper on closed bounded sets and satisfies condition (2.2) holds for some $R > 0$ by Lemma

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2.2 with $c_= c - |T - C|$. Moreover, $\alpha(D+C) \leq \alpha(D) + \alpha(C) < \beta(T)$. Let $f \in Y$ be fixed and $\epsilon > 0$ and $R_1 > R$ such that $|D| + \epsilon + \|f\|/R_1^k < c$ and $\|Dx\| \leq (|D|+\epsilon)\|x\|$ for all $\|x\| \geq R_1$. Define the homotopy $H(t,x) = Tx + Cx + tDx - tf$ for $t \in [0,1]$. Then, $H(t,x) \neq 0$ for $(t,x) \in [0,1] \times (X \setminus B(0,R_1))$. If not, then there is a $t \in [0,1]$ and x with $\|x\| \geq R_1$ such that $H(t,x) = 0$. Then

$$c_1\|x_n\|^k \leq \|Tx_n + Cx_n\| = t_n\|Dx_n - f\| \leq (|D| + \epsilon)\|x_n\|^k + \|f\|.$$

Hence, $c_1 < |D| + \epsilon + \|f\|/R_1^k$, in contradiction to our choice of $\epsilon$ and $R_1$. Similarly, arguing by cotradiction, we get that $H(t,x)$ satisfies condition $(+)$.

Next, we will show that $H(t,x)$ is an admissible oriented homotopy on $[0,1] \times B(0,R_2)$ for some $R_2 \geq R_1$. Set F(t,x)=tDx. Then for any subset $A \subset B(0,R_2)$, $\alpha(F([0,1] \times A)) = \alpha(\{tDx \mid t \in [0,1], x \in A\}) = \alpha(\{ty \mid t \in [0,1], y \in D(A)\}) = \alpha(D(A)) \leq \alpha(D)\alpha(A)$. Hence, $\alpha(F) \leq \alpha(D)$. Moreover, $\alpha(C + F) \leq \alpha(C) + \alpha(F) \leq \alpha(C) + \alpha(D) < \beta(T)$. Thus, we get that $\beta(H) \geq \beta(T) - \alpha(C + F) > 0$. This implies that $H$ is proper on bounded closed subsets of $[0,1] \times X$ with the norm $\|(t,x)\| = \max\{|t|, \|x\|\}$ for $(t,x) \in R \times X$. Since $H$ satisfies condition $(+)$, it is proper on $[0,1] \times X$. Thus, $H^{-1}(0)$ is compact and contained in $[0,1] \times B(0,R_2)$ for some $R_2 \geq R_1$. Since $B(0,R_2)$ is simply connected, $H(0,.) = T + C : B(0,R_2) \to Y$ is oriented by Proposition 3.7 in [2]. Hence, $H$ is oriented by Proposition 3.5 in [2] and the homotopy Theorem 6.1 in [1] implies that

$$\deg_{BCF}(H_1, B(R_1,0),0) = \deg_{BCF}(T + C + D - f, B(R_1,0),0) =$$

$$\deg_{BCF}(T + C, B(R_1,0),0) \neq 0.$$

Thus, the equation $Tx + Cx + Dx = f$ is solvable. The other conclusions follow from Theorem 2.9 since $H(0,.) = T + C + D$ satisfies condition $(+)$ and is therefore proper. ∎

Let us now look at a special case when $T + C$ is odd. Now, we will use the Fitzpatrick-Pejsachowisz-Rabier-Salter degree.

**Theorem 3.3.** ( Generalized First Fredholm Theorem) *Let $T : X \to Y$ be a Fredholm map of index zero that is proper on bounded and closed subsets of $X$ and $C, D : X \to Y$ be compact maps with $|D|$ sufficiently small, where*

$$|D| = \limsup_{\|x\|\to\infty} \|Dx\|/\|x\|^k < \infty.$$

*Assume that $T + C$ is odd, asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of $X$ positive k-homogeneous map A, outside some ball in $X$, i.e., there exists $R_0 > 0$ such that*

$$A(\lambda x) = \lambda^k Ax$$

*for all $\|x\| \geq R_0$, for all $\lambda \geq 1$ and some $k \geq 1$, and $\|x\| \leq M < \infty$ if $Ax = 0$. Then the equation $Tx + Cx + Dx = f$ is solvable for each $f \in Y$ with*

$(T + C + D)^{-1}(\{f\})$ *compact and the cardinal number* $card(T + C + D)^{-1}(\{f\})$ *is constant, finite and positive on each connected component of the set* $Y \setminus (T + C + D)(\Sigma)$.

**Proof.** Step 1. Let $p$ be a base point of $T$. By Lemma 2.2, there is an $R > 0$ such that condition (2.2) holds for $T + C$. Define the homotopy $H(t,x) = Tx + Cx + tDx - tf$ for $t \in [0,1]$. We have shown in the proof of Theorem 3.2 that $T + C + D$ satisfies condition (+) and $H(t,x) \neq 0$ for $(t,x) \in [0,1] \times \partial B(0,R_1)$ for some $R_1 \geq R$.

Next, we note that $H(t,x)$ is a compact perturbation tD of the odd mapping $T + C$ with $T$ Fredholm of index zero. Then, by the homototy theorem ( Corollary 7.2 in [15] ) and the Borsuk's theorem for such maps of Rabier-Salter [15], we get that the Fitzpatrick-Pejsachowisz-Rabier-Salter degree

$$\deg_{T,p}(H_1, B(R_1, 0), 0) = \deg_{T,p}(T + C + D - f, B(R_1, 0), 0) =$$

$$\nu \deg_{T,p}(T + C, B(R_1, 0), 0) \neq 0$$

where $\nu$ is 1 or -1.

Step 2. $T$ has no base point. Then pick a point $q \in X$ and let $A : X \to Y$ be a continuous linear map with finite dimensional range such that $T'(q) + A$ is invertible. Then $T + A$ is Fredholm of index zero, proper on $\bar{B}(0,r)$ and $q$ is a base point of $T + A$. Moreover, $T + C + D = (T + A) + (C - A) + D$. We have reduced the problem to the case when there is a base point for $T + A$ and the maps $T + A$, $C - A$ and $D$ satisfy the same conditions of the theorem as the maps $T$, $C$ and $D$. Then, as in Step 1,

$$\deg_{T+A,q}(H_1, B(R_1, 0), 0) = \deg_{T+A,q}((T+A) + (C-A) + D - f, B(R_1, 0), 0) =$$

$$\nu \deg_{T+A,q}(T + C, B(R_1, 0), 0) \neq 0$$

where $\nu$ is 1 or -1.

Hence, by the existence theorem of this degree, we have that $Tx + Cx + Dx = f$ is solvable in either case. Next, since $T + C + D$ is continous and satisfies condition (+), it is proper since it is proper on bounded closed sets as a compact perturbation of such a map. Hence, the second part of the theorem follows from Theorem 2.9. ∎

**Remark 3.1.** Theorems 3.2 and 3.3 are valid if the k-positive homogeneity of $T + C$ is replaced by $\|Tx + Cx\| \geq c\|x\|^k$ for all x outside some ball ( see Lemma 2.2).

**Remark 3.2.** Earlier generalizations of the first Fredholm theorem to condensing vector fields, maps of type $(S_+)$, monotone like maps and ( pseudo ) A-proper maps assumed the homogeneity of $T$ with $Tx = 0$ only if $x = 0$ ( see [7],[8], [19] and the references therein).

Next, we provide some generalizations of the Borsuk-Ulam principle for odd compact perturbations of the identity. The first result generalizes Theorem 3.3 when $D = 0$.

**Theorem 3.4.** *Let $T : X \to Y$ be a Fredholm map of index zero that is proper on closed bounded subsets of $X$ and $C : X \to Y$ be compact such that $T + C$ is odd outside some ball $B(0, R)$. Suppose that $T + C$ satisfies condition (+). Then $Tx + Cx = f$ is solvable, $(T + C)^{-1}(f)$ is compact for each $f \in Y$ and the cardinal number $\mathrm{card}(T + C)^{-1}(f)$ is positive and constant on each connected component of $Y \setminus (T + C)(\Sigma)$.*

**Proof.** Condition (+) implies that for each $f \in Y$ there is an $r = r_f > R$ and $\gamma > 0$ such that

$$\|Tx + Cx - tf\| \geq \gamma \text{ for all } t \in [0, 1], \|x\| = r.$$

The homotopy $H(t, x) = Tx + Cx - tf$ is admissible for the Rabier-Salter degree and $H(t, x) \neq 0$ on $[0, 1] \times \partial B(0, r)$. Hence, by the homotopy Corollary 7.2 in [15], if $p \in X$ is a base point of $T$, then

$$\deg_{T,p}(T + C - f, B(0, r), 0) = \nu \deg_{T,p}(T + C, B(0, r), 0) \neq 0$$

since $\nu$ is plus or minus one and the second degree is odd by the generalized Borsuk theorem in [15]. If $T$ has no base point, then proceed as in Step 2 of the proof of Theorem 3.3. Hence, the equation $Tx + Cx = f$ is solvable in either case. The second part follows from Theorem 2.9 since $T + C$ is proper on bounded closed subsets and satisfies condition (+). ∎

Next, we shall prove a more general version of Theorem 3.4.

**Theorem 3.5.** *Let $T : X \to Y$ be a Fredholm map of index zero that is proper on closed bounded subsets of $X$ and $C_1, C_2 : X \to Y$ be compact such that $T + C_1$ is odd outside some ball $B(0, R)$. Suppose that $H(t, x) = Tx + C_1 x + tC_2 x - tf$ satisfies condition (+). Then $Tx + C_1 x + C_2 x = f$ is solvable for each $f \in Y$ with $(T + C_1 + C_2)^{-1}(f)$ compact and the cardinal number $\mathrm{card}(T + C_1 + C_2)^{-1}(f)$ is positive and constant on each connected component of $Y \setminus (T + C_1 + C_2)(\Sigma)$.*

**Proof.** Condition (+) implies that for each $f \in Y$ there is an $r = r_f > R$ $0 \notin H([0, 1] \times \partial B(0, r))$. If $p$ is a base points of $T$, then by Theorem 7.1 in [15]

$$\deg_{T,p}(H(1, .), B(0, r), 0) = \nu \deg_{T,p}(T_1 + C_1, B(0, r), 0) \neq 0$$

where $\nu \in \{-1, 1\}$.

Next, if $T$ has no a base point, then pick $q \in X$ and let $A$ be a continuous linear map from $X$ to $Y$ with finite dimensional ranges such that $T'(q) + A$ is invertible. Then we can rewrite $H$ as $H(t, x) = (T + A)x + (C_1 - A)x + tC_2 x - tf$, where $T + A$, $C_1 - A$ and $C_2$ satisfy all the conditions of the theorem and $T + A$ has a base point. As in the first case, we get that

$$\deg_{T+A,q}(H(1, .), B(0, r), 0) =$$

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$$\deg_{T+A,q}((T+A) + (C_1 - A) + C_2 - f, B(0,r), 0) =$$
$$\nu\deg_{T+A,q}((T+A) + (C_1 - A), B(0,r), 0) \neq 0$$

where $\nu \in \{-1, 1\}$. Hence, the equation $Tx + C_1x + C_2x = f$ is solvable in either case. The other conclusions follow from Theorem 2.9. ∎

Next, we prove a general surjectivity result.

**Theorem 3.6.** ( Global Leray-Schauder condition ) *Let $T, G : X \to Y$ be Fredholm maps of index zero that are proper on closed bounded subsets of $X$, $F(t,x) = tTx + (1-t)Gx$ be Fredholm of index 1 from $[0,1] \times X \to Y$ and proper on bounded closed subsets, and $C, M : X \to Y$ be compact maps. Suppose that $T + C$ satisfies condition $(+)$, and $G + M : X \to Y$ is odd and such that*

*(i) $(G + M)x \neq 0$ for $x$ with large norm.*

*(ii) $Tx + Cx \neq \gamma(G + M)(x)$ for all $x \in X \setminus B(0, R)$ for all $\gamma < 0$ and some $R > 0$.*

*Then $Tx + Cx = f$ is solvable for each $f \in Y$ with $(T+C)^{-1}(f)$ compact and the cardinal number $card(T + C)^{-1}(f)$ is positive and constant on each connected component of $Y \setminus (T + C)(\Sigma)$.*

**Proof.** Let $f \in Y$ be fixed and suppose that $p_1$ and $p_2$ are based points for T and G. Then condition $(+)$ implies that there are $r \geq R$ and $c > 0$ such that

$$\|Tx + Cx - tf\| > c \text{ for } x \in \partial B(0,r), t \in [0,1]. \qquad (3.1)$$

Hence, the homotopy $F(t,x) = Tx + Cx - tf$ is admissible and the Fitzpatrick-Pejsachowisz-Rabier-Salter degree $\deg_{T,p_1}(T + C - f, B(0,r), 0) = \deg_{T,p_1}(T + C, B(0,r), 0)$. Thus, it suffices to show that the last degree is nonzero. Consider the homotopy

$$H(t,x) = tTx + tCx + (1 - t)(G + M)x \text{ for } t \in [0,1], x \in \bar{B}(0,r).$$

$H$ is continuous and can be written as $H = H_\Phi + H_k$ with $H_\Phi(t,x) = tTx + (1-t)Gx$ and $H_k = tCx + (1-t)Mx$. Since $H_\Phi = G + t(T - G) \in \Phi_1 C^1([0,1] \times X, Y)$, is proper on $[0,1] \times \bar{B}(0,r)$, and $H_k = tCx + (1-t)Mx$ is continous and compact, the homotopy $H(t,x)$ is admissible. We claim that $H(t,x) \neq 0$ for $x \in \partial B(0,r)$ and $t \in [0,1]$. If not, then there would exist $x \in \partial B(0,r)$ and $t \in [0,1]$ such that

$$H(t,x) = tTx + tCx + (1 - t)(G + M)x = 0.$$

Then $t \neq 0$ by (i) and $t \neq 1$ by (3.1). Hence, $t \in (0,1)$ and setting $\gamma = (t-1)/t < 0$ we get that $Tx + Cx = \gamma(G + M)x$, in contradiction to (ii). Hence, $H(t,x) \neq 0$ on $[0,1] \times \partial B(0,r)$ and $H_t$ is proper on closed bounded subsets of X and is the sum of a Fredholm map of index zero and a compact map. Hence, by the homotopy theorem [15],

$$\deg_{T,p_1}(T + C - f, B(0,r), 0) = \nu \deg G, p_2(G + M, B(0,r), 0) \neq 0$$

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since $G + M$ is odd.

Next, suppose that $T$ and $G$ have no base points. Pick $q_1, q_2 \in X$ and linear maps $A_1, A_2 : X \to Y$ with finite dimensional ranges and such that $T'(q_1) + A_1$ and $G'(q_2) + A_2$ are invertible. Then $T + A_1$ and $G + A_2$ have base points. Moreover, the maps $T + A_1$, $C - A_1$, $G + A_2$ and $M - A_2$ satisfify the same conditions as $T, C, G$ and $M$ and $H(t, x) = t(T + A_1)x + t(C - A_1)x + (1 - t)(G + A_2)x + (1 - t)(M - A_2)x$. Hence, by the fist case, we get again that

$$\deg_{T+A_1, q_1}((T + A_1) + (C - A_1) - f, B(0, r), 0) =$$

$$\nu \deg_{G+A_2, q_2}((G + A_2) + (M - A_2), B(0, r), 0) \neq 0$$

since $G + M$ is odd. Therefore, the equation $Tx + Cx = f$ is solvable in either case for each $f \in X$. The second part follows from Theorem 2.9 since $T + C$ is proper. ∎

**Remark 3.3.** The oddness of $G + M$ in Theorem 3.6 can be replaced by deg $(G + M, B(0, r), 0) \neq 0$ (mod 2) for each large $r$. Using the degree of Benevieri, Calamai, Furi [2], we see that Theorem 3.6 is valid if $T, G : X \to Y$ are Fredholm maps of index zero and $C, M : X \to Y$ are continuous maps with $\alpha(C) < \beta(T)$ and $\alpha(M) < \beta(G)$ with deg $_{BCF}(G + M, B(0, r), 0) \neq 0$ for each large $r$.

## 4. Unique and finite solvability of quasilinear elliptic equations on $R^N$

Consider the quasilinear elliptic equation

$$-\sum_{\alpha, \beta=1}^{N} a_{\alpha\beta}(x, u, \nabla u)\partial^2_{\alpha\beta}u + b(x, u, \nabla u) + c(x, u, \nabla u, D^2 u) = f(x) \qquad (4.1)$$

for $x \in R^n$ and the function $u$ is required to satisfy the condition

$$\lim_{|x| \to \infty} u(x) = 0.$$

This condition is satisfied for all elements $u \in W^{2,p}(R^N)$ with $N < p < \infty$. For this reason, we define, for $u \in W^{2,p}(R^N)$, the maps

$$Tu = -\sum_{\alpha, \beta=1}^{N} a_{\alpha\beta}(x, u, \nabla u)\partial^2_{\alpha\beta}u + b(x, u, \nabla u)$$

$$Cu = c(x, u, \nabla u, D^2 u).$$

Then Eq. (4.1) can be written in the operator form as

$$Tu + Cu = f \qquad (4.2)$$

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where $T + C : X_p = W^{2,p}(R^N) \to Y_p = L^p(R^N)$ and no growth restrictions are needed for the functions $a_{\alpha\beta}(x, \xi)$ and $b(x, \xi)$ as $|x| \to \infty$.

We need to impose conditions on the functions involved so that $T$ is Fredholm of index zero and is proper on bounded closed subsets of $X_p$ and $C$ is $\alpha$-k-contractive with $k$ sufficiently small. The Fredholm and properness properties of $T$ have been studied in detail by Rabier and Stuart [16,17]. We impose the following assumptions on the functions $a_{\alpha\beta} : R^N \times (R \times R^N) \to R$ and $b : R^N \times (R \times R^N) \to R$ and $c : R^{s_2} \to R$, where $R^{s_2}$ is the vector space whose elements are $\xi = \{\xi_\alpha : |\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)| \leq 2\}$. Set $x = (x_1, ..., x_N)$ and $\xi = (\xi_0, ..., \xi_N)$. By viewing $R^N \times (R \times R^N)$ as a bundle over $R^N$, a function $f(= f(x, \xi)) : R^N \times (R \times R^N) \to R$ can be identified with the bundle map

$$(x, \xi) \in R^N \times (R \times R^N) \to (x, f(x, \xi)) \in R^N \times R.$$

Next, we recall the definition of equicontinuous bundles.

**Definition 4.1** We say that $f$ is an equicontinuous $C^0$ bundle map if $f$ is continouous and the collection $(f(x,.))_{x \in R^N}$ is equicontinuous at every point of $R \times R^N$. If $k \geq 0$ is an integer, we say that $f$ is an equicontinuous $C_\xi^k$ bundle map if the partial derivatives $D_\xi^\alpha f$, $|\alpha| \leq k$, exist and are equicontinuous $C^0$ bundle maps.

Define the space

$$C_d^1(R^N) = \{u \in C^1(R^N) : \lim_{|x| \to \infty} |u(x)| = \lim_{|x| \to \infty} |\nabla u(x)| = 0\}$$

where "d" stands for "decay". It is a Banach space for the norm

$$\|u\|_{C_d^1(R^N)} = \max_{x \in R^N} |u(x)| + \max_{x \in R^N} |\nabla u(x)|.$$

We impose the following assumptions on the functions in (4.1), used in [16,17].

(A1) $a_{\alpha\beta}$ is an equicontinuous $C_\xi^1$ bundle map, $1 \leq \alpha, \beta \leq N$, $0 \leq i \leq N$,

(A2) $a_{\alpha\beta}(., 0) \in L^\infty(R^N)$, $\partial_{\xi_i} a_{\alpha\beta}(., 0) \in L^\infty(R^N)$, $1 \leq \alpha, \beta \leq N$, $0 \leq i \leq N$,

$$(A3) \qquad \sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, \xi)\eta_\alpha\eta_\beta \geq \gamma(x, \xi)|\eta|^2$$

for all $\eta = (\eta_1, ..., \eta_N) \in R^N$, for all $(x, \xi) \in R^N \times (R \times R^N)$, where $\gamma : R^N \times (R \times R^N) \to (0, \infty)$ is bounded from below by a positive constant $\gamma_{\tilde{K}}$ on every compact subset $\tilde{K}$ of $R^N \times (R \times R^N)$ ( for example, $\gamma$ is lower semicontinuous),

(B1) $b : R^N \times R^{N+1} \to R$ is an equicontinuous $C_\xi^1$ bundle map

(B2) $b(., 0) \in L^p(R^n)$, $\partial_{\xi_i} b(., 0) \in L^\infty(R^N)$, $0 \leq i \leq N$

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(C) $c(.,0) \in L^\infty(R^N)$ and there is a function $c(x) \in L^\infty(R^N)$ and a constant $k > 0$ sufficiently small such that

$$|c(x,\xi) - c(x,\xi')| \le kc(x) \sum_{|\alpha| \le 2} |\xi_\alpha - \xi'_\alpha|^{1/(p-1)}$$

for all $x \in R^N$, all $\xi, \xi' \in R^{s_2}$, where $p' = p/(p-1)$.

(F1) There is an element $u_0 \in W^{2,p}(R^N)$ such that the Frechet derivative $T'(u_0) : X_p \to Y_p$ is Fredholm of index zero.

(F2) Every bounded sequence $\{u_n\} \subset W^{2,p}(R^N)$ such that $\{Tu_n\}$ converges in $L^p(R^N)$ has a subsequence $\{u_{n_k}\}$ converging in $C_d^1(R^N)$.

**Theorem 4.1.** *Let assumptions (A1)-(A2), (B1)-(B2), (C) and (F1)-(F2) hold. Suppose that $T + C$ is locally injective and satisfies condition (+) or, in particular, $\|Tu\| \to \infty$ as $\|u\| \to \infty$ and $\|Cu\| \le a\|Tu\| + b$ for some $a \in [0,1)$, $b > 0$ and all $\|u\|$ large. Then $T + C : X_p \to Y_p$ is a homeomorphism, and, in particular, Eq. (4.1) is uniquely solvable in $X$ for each $f \in Y_p$.*

**Proof.** It has been shown by Rabier and Stuart [16] that conditions (A1)-(A2), (B1)-(B2), and (F1)-(F2) imply that $T : X_p \to Y_p$ is a proper on bounded closed subsets Fredholm map of index zero. Note that if (A1)-(A2) and (B1)-(B2) hold, then (F1) is equivalent to $T'(u)$ being Fredholm of index zero for each $u \in X_p$ ([16]). Condition (F2), together with (A1)-(A2) and (B1)-(B2), is used to show the properness of $T$ on closed bounded subsets. Moreover, it is easy to see that condition (C1) implies that $C : X_p \to Y_p$ is a $k$-contraction and therefore it is $\alpha$-k- contractive. Hence, $\beta(T) > 0$ and $\beta(T) > \alpha(C)$. Thus, the conclusions follow from Corollary 2.4. ∎

**Example 4.1** If $b(x,\xi) = b_1(x)b_2(\xi)$, then $b$ is an equicontinuous $C^0$ bundle map if and only if $b_1$ is bounded and continuous and $b_2$ is nonconstant and continuous. Moreover, $b$ is an equicontinuous $C_\xi^k$ bundle map if $b_2$ is also $C^k$. If $a_{\alpha\beta}$ have this form also, then they are $C_\xi^k$ bundle maps. Moreover, if $c(x,\xi)$ is bounded with bounded partial derivatives $\partial c(x,\xi)/\partial\xi_\alpha$, then (C) holds and $T + C$ satisfies condition (+) if $\|Tu\| \to \infty$ as $\|u\| \to \infty$.

**Theorem 4.2.** *Let assumptions (A1)-(A2), (B1)-(B2), (C) and (F1)-(F2) hold and $H_t = T + tC$ satisfy condition (+), and, in particular, $\|Tu\| \to \infty$ as $\|u\| \to \infty$ and $\|Cu\| \le a\|Tu\| + b$ for some $a \in [0,1)$, $b > 0$ and all $\|u\|$ large. Suppose that $T$ is locally injective. Then the solution set of Eq. (4.1) is a nonempty compact set for each $f \in Y_p$. Moreover, either Eq.(4.1) is uniquely solvable for each $f \in L_p$, or the cardinal number $card(T + C)^{-1}(f)$ is constant, finite and positive on each connected component of the set $Y_p \backslash (T+C)(\Sigma)$, where $\Sigma$ is the set of all $u \in X_p$ where $T + C$ is not locally invertible.*

**Proof.** As above, we have that $T : X_p \to Y_p$ is a proper on bounded closed subsets Fredholm map of index zero and therefore $\beta(T) > 0$. Since T is locally

injective, it is an open map by [4,15] and satisfies condition (+). Hence, T is a homeomorphism by Theorem 2.1. Moreover, $C : X_p \to Y_p$ is $\alpha$-k- contractive with the quasinorm $|C|$ sufficiently small and $\beta(T) > \alpha(C)$. Hence, the conclusions follow from Theorem 2.10, and Lemma 2.1 in the particular case.∎

**Theorem 4.3.** *Let assumptions (A1)-(A2), (B1)-(B2), (C) and (F1)-(F2) hold. Suppose that there are constants $c, R > 0$ such that $\|Tu\| \geq c\|u\|$ for all $\|u\| \geq R$ and $deg_{BCF}(T, B(0, r), 0) \neq 0$ for all large $r$. Then the solution set of Eq. (4.1) is a nonempty compact set for each $f \in Y$. Moreover, the cardinal number $card(T + C)^{-1}(f)$ is constant, finite and positive on each connected component of the set $Y \setminus (T + C)(\Sigma)$, where $\Sigma$ is the set of all $u \in X_p$ where $T + C$ is not locally invertible.*

**Proof.** We have seen above that $T : X_p \to Y_p$ is a proper on bounded closed subsets Fredholm map of index zero and therefore $\beta(T) > 0$. Then the conclusions follow from Theorem 3.2 and Remark 3.1 with $D = C$ and $T + C$ replaced by T. ∎

Next, following [16,17], we shall look at some more explicit conditions on $a_{\alpha\beta}$ and $b$ that imply the properness of T on closed bounded subsets of $X_p$. Namely, we assume that they are asymptotically $N$-periodic. First, we need to recall some definitions.

Let $P = (P_1, ..., P_N) \in R^N$ with $P_i > 0$, $1 \leq i \leq N$. A function $f : R^N \to R$ is said to be $N$-periodic with period $P$ if $f(x_1, ..., x_i + P_i, ..., x_n) = f(x_1, ..., x_i, ..., x_n)$ for each $x \in R^N$ and every $1 \leq i \leq N$.

(D) There exist equicontinuous $C^0$-bundle maps $a_{\alpha\beta}^\infty = a_{\beta\alpha}^\infty : R^N \times R^{N+1} \to R$ for $\alpha, \beta = 1, ..., N$, and an equicontinuous $C_\eta^1$-bundle $b^\infty : R^N \times R^{N+1} \to R$ such that $b^\infty(x, 0) \equiv 0$ and

$$\lim_{|x| \to \infty} [a_{\alpha\beta}(x, \xi) - a_{\alpha\beta}^\infty(x, \xi)] = \lim_{|x| \to \infty} [\partial_{\xi_i} b(x, \xi) - \partial_{\xi_i} b^\infty(x, \xi)] = 0$$

uniformly for $\xi$ in bounded subsets of $R^{N+1}$, where $1 \leq \alpha, \beta \leq N$ and $i = 0, 1, ..., N$. Moreover, $a_{\alpha\beta}^\infty(., \xi)$ and $b^\infty(., \xi) : R^N \to R$ are $N$-periodic in $x$.

Define the asymptotic limit map $T^\infty : X_p \to Y_p$ by

$$T^\infty(u) = -\sum_{\alpha, \beta=1}^N a_{\alpha\beta}^\infty(x, u, \nabla u)\partial_{\alpha\beta}^2 u(x) + b^\infty(x, u(x), \nabla u(x)) \qquad (4.3)$$

Assumptions (A1)-(A2), (B1)-(B2) and (D) imply that $T^\infty : X_p \to Y_p$ and $T^\infty(0) = 0$.

(F3) The equation $T^\infty u = 0$ has no nonzero solutions in $X_p$.

**Theorem 4.4.** *Let assumptions (A1)-(A2), (B1)-(B2), (C), (D), (F1) and (F3) hold. Suppose that $T + C$ is locally injective and satisfies condition (+),*

*and, in particular, $\|Tu\| \to \infty$ as $\|u\| \to \infty$ and $\|Cu\| \leq a\|Tu\| + b$ for some $a \in [0,1)$, $b > 0$ and all $\|u\|$ large. Then $T + C : X_p \to Y_p$ is a homeomorphism, and, in particular, Eq. (4.1) is uniquely solvable in $X$ for each $f \in Y_p$.*

**Proof.** As shown by Rabier and Stuart [15], the map $T : X_p \to Y_p$ is proper on bounded closed subsets Fredholm map of index zero. Since $C : X_p \to Y_p$ is a $k$-contraction and therefore $\alpha$-k- contractive, we get that $\beta(T) > \alpha(C)$. Thus, the conclusions follow from Corollary 2.4. ∎

**Theorem 4.5.** *Let assumptions (A1)-(A2), (B1)-(B2), (C), (D), (F1) and (F3) hold and $H_t = T + tC$ satisfy condition (+), and, in particular, $\|Tu\| \to \infty$ as $\|u\| \to \infty$ and $\|Cu\| \leq a\|Tu\| + b$ for some $a \in [0,1)$, $b > 0$ and all $\|u\|$ large. Suppose that $T$ is locally injective. Then the solution set of Eq. (4.1) is a nonempty compact set for each $f \in Y_p$. Moreover, either Eq.(4.1) is uniquely solvable for each $f \in L_p$, or the cardinal number $card(T + C)^{-1}(f)$ is constant, finite and positive on each connected component of the set $Y_p \backslash (T + C)(\Sigma)$, where $\Sigma$ is the set of all $u \in X_p$ where $T + C$ is not locally invertible.*

**Proof.** As in Theorem 4.4, we have that $T : X_p \to Y_p$ is a proper on bounded closed subsets Fredholm map of index zero, and therefore $\beta(T) > 0$. Since T is locally injective, it is an open map by [4,15] and satisfies condition (+). Hence, T is a homeomorphism by Theorem 2.1. Moreover, $C : X_p \to Y_p$ is $\alpha$-k-contractive with the quasinorm $|C|$ sufficiently small and $\beta(T) > \alpha(C)$. Hence, the conclusions follow from Theorem 2.10, and Lemma 2.1 in the particular case. ∎

**Theorem 4.6.** *Let assumptions (A1)-(A2), (B1)-(B2), (C), (D), (F1) and (F3) hold. Suppose that there are constants $c, R > 0$ such that $\|Tu\| \geq c\|u\|$ for all $\|u\| \geq R$ and $deg_{BCF}(T, B(0,r), 0) \neq 0$ for all large $r$. Then the solution set of Eq. (4.1) is a nonempty compact set for each $f \in Y$. Moreover, the cardinal number $card(T + C)^{-1}(f)$ is constant, finite and positive on each connected component of the set $Y \backslash (T + C)(\Sigma)$, where $\Sigma$ is the set of all $u \in X_p$ where $T + C$ is not locally invertible.*

**Proof.** We have seen above that $T : X_p \to Y_p$ is a proper on bounded closed subsets Fredholm map of index zero and therefore $\beta(T) > 0$. Then the conclusions follow from Theorem 3.2 and Remark 3.1 with $D = C$ and $T + C$ replaced by T. ∎

**Remark 4.1.** We note that $deg(T, B(0,r), 0) \neq 0$ for all large $r$ if $T$ is an odd map outside a ball by Borsuk's theorem (see [14]). Moreover, if $T^{-1}(0) = 0$ and $T'(0)$ is invertible then $|deg(T, B(0,r), 0)| = |deg(T'(0), B(0,r), 0)| = 1$ for all large $r$.

**Remark 4.2.** Rabier and Stuart have shown in [16] that if the conditions (A1)-(A2), (B1)-(B2) are satisfied, then $T : X_p \to Y_p$ is Fredholm of index zero and proper on the closed bounded subsets of $X_p$ if and only if (F1) and (F3) hold.

**Remark 4.3.** If the map T does not depend on $\nabla u$, then all the results in this section remain valid for $p \in (\max(1, N/2), \infty)$.

If (F3) does not hold, then $T^{-1}(f)$ is not compact for almost all $f \in T(X)$ in the sense of the Baire category relative to $T(X)$. More precisely, the following result is valid.

**Theorem 4.7.** (Rabier [14]) *Let assumptions (A1)-(A2), (B1)-(B2), (D) and (F1) hold. Suppose also that the equation $T^\infty u = 0$ has a nonzero solution in $X_p$ and that the set of critical points of $T$ has empty interior in $X_p$. Then $T(X_p)$ is a Baire space and the set of the regular values $f \in Y_p$ of $T$ such that $T^{-1}(f)$ is an infinite sequence of distinct isolated points is residual in $T(X_p)$.*

This result does not say anything about the solvability of $Tx = f$ with $f \neq 0$. It is of qualitative nature for $T^{-1}(f)$ when $f \in T(X)$. Combining Theorems 4.6-4.7, we have the following Fredholm like result.

**Theorem 4.8.** ( Fredholm alternative) *Let assumptions (A1)-(A2), (B1)-(B2), (D) and (F1) hold. Then, either*

*(i) The equation $T^\infty u = 0$ has no nonzero solution in $X_p$, in which case, assuming additionaly that $\|Tu\| \geq c\|u\|$ for some $c > 0$ and all $\|u\|$ large and the Fitzpatrick-Pejsachowisz-Rabier degree $deg(T, B(0, r), 0) \neq 0$ for all large $r$, the solution set $(T)^{-1}(f)$ of Eq.(4.1) ( with $c \equiv 0$) is a nonempty compact set for each $f \in Y$. Moreover, the cardinal number $card(T)^{-1}(f)$ is constant, finite and positive on each connected component of the set $Y \setminus (T)(\Sigma)$, where $\Sigma$ is the set of all $u \in X_p$ where $T$ is not locally invertible, or*

*(ii) The equation $T^\infty u = 0$ has a nonzero solution in $X_p$, in which case, assuming additionaly that the set of critical points of $T$ has empty interior in $X_p$, then $T(X_p)$ is a Baire space and the set of the regular values $f \in Y_p$ of $T$ such that $T^{-1}(f)$ is an infinite sequence of distinct isolated points is residual in $T(X_p)$.*

If $Tu = -\Delta u + b(x, u)$, conditions on $b$ are given in Rabier [14] that imply that the set of critical points of $T$ has empty interior in $X_p$ with $p \in (\max(1, N/2), \infty)$.

## 5. Solvability of quasilinear elliptic BVP's with asymptotic positive homogeneous nonlinearities

Let $Q \subset R^n$ be a bounded domain with the smooth boundary and consider the boundary value problem in a divergent form

$$\Sigma_{|\alpha| \leq m}(-1)^{|\alpha|} D^\alpha A_\alpha(x, u, ..., D^m) + k\Sigma_{|\alpha| \leq m}(-1)^{|\alpha|} D^\alpha B_\alpha(x, u, ..., D^m) = f \tag{5.1}$$

Let $X$ be the closed subspace of $W_p^m(Q)$ corresponding to the Dirichlet condi-

tions. Define the maps $T, D : X \to X^*$ by

$$(Tu, v) = \Sigma_{|\alpha| \leq m} \int_Q A_\alpha(x, u, ..., D^m u) D^\alpha v \, dx,$$

$$(Du, v) = \Sigma_{|\alpha| \leq m} \int_Q B_\alpha(x, u, ..., D^m u) D^\alpha v \, dx.$$

Then weak solutions of (5.1) are solutions of the operator equation

$$Tu + Du = f, \ u \in X. \tag{5.2}$$

We impose the following conditions on $A_\alpha$.

(A1) For each $\alpha$, let $A_\alpha : Q \times R^{s_m} \to R$ be a Caratheodory function, i.e., $A_\alpha(x, \xi)$ is measurable in x and for each fixed $\xi$, and has continuous derivatives in $\xi$ for a.e. x.

(A2) Assume that for $p > 2$, $x \in Q$, $\xi \in R^{s_m}$, $\eta \in R^{s_m - s_{m-1}}$, $|\alpha|, |\beta| \leq m$ the $A_\alpha$'s satisfy

$|A_\alpha(x, \xi)| \leq g_0(|\xi_0|)(h(x) + \Sigma_{m-n/p \leq \gamma| \leq m} |\xi_\gamma|^{p-1})$

(A3) $|A_{\alpha,\beta}(x, \xi)| \leq g_1(|\xi_0|)(b(x) + \Sigma_{m-n/p \leq \gamma| \leq m} |\xi_\gamma|^{p-2})$,

(A4) $\Sigma_{|\alpha|=|\beta|=m} A_{\alpha,\beta}(x, \xi) \eta_\alpha \eta_\beta \geq g_2(|\xi_0|)(1 + \Sigma_{|\gamma|=m} |\xi_\gamma|)^{p-2} \Sigma_{|\alpha|=m} \eta_\alpha^2$,

where $A_{\alpha,\beta}(x, \xi) = \partial / \partial \xi_\beta A_\beta(x, \xi)$, $h, b \in L_q(Q)$, $g_0, g_1, g_2$ are continuous positive nondecreasing and nonincreasing functions, respectively, and $\xi_0 = \{\xi_\alpha \mid |\alpha| < m - n/p\}$.

For $|\alpha| \leq m$, there are Caratheodory functions $a_\alpha$ such that

(a1) $a_\alpha(x, t\xi) = t^{p-1} a_\alpha(x, \xi)$ for all $t > 0, \xi \in R^{s_m}$

(a2) $|1/t^{p-1} A_\alpha(x, t\xi) - a_\alpha(x, \xi)| \leq c(t)(1 + |\xi|)^{p-1}$

for each $t > 0$, $x \in Q$, and $\xi \in R^{s_m}$, where $0 < \lim c(t)$ is sufficiently small as $t \to \infty$.

**Proposition 5.1** *Let conditions* $(A1) - (A4)$ *hold. Then* $T : X \to X^*$ *is Fredholm of index zero and is proper on bounded closed subsets of X.*

**Proof.** The map $T : X \to X^*$ is continuous and of type $(S_+)$ ([18]) and, as shown before, it is proper on bounded closed subsets of X. By Lemma 3.1 in [18], $T$ has a Frechet derivative $T'(u)$ for each $u \in X$ and is given by

$$(T'(u)v, w) = \Sigma_{|\alpha|,|\beta| \leq m} \int_Q A_{\alpha,\beta}(x, u, ..., D^m u) D^\beta v D^\alpha w \, dx \tag{5.3}$$

Next, we shall show that $T'(u)$ is Fredholm of index zero for each $u \in X$. First, we shall show that $T'(u)$ satisfies condition $(S_+)$. We can write it in the form

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$T'(u) = T_1'(u) + T_2'(u)$, where

$$(T_1'(u)v, w) = \Sigma_{|\alpha|=|\beta|=m} \int_Q A_{\alpha,\beta}(x, u, ..., D^m u) D^\beta v D^\alpha w dx \qquad (5.4)$$

$$(T_2'(u)v, w) = \Sigma_{\substack{|\alpha|,|\beta|\leq m \\ |\alpha+\beta|<2m}} \int_Q A_{\alpha,\beta}(x, u, ..., D^m u) D^\beta v D^\alpha w dx. \qquad (5.5)$$

It is easy to see that $|(T_2'(u)v, w)| \leq c\|v\|_{m,p} \|w\|_{m-1,p}$ for some constant $c > 0$. Hence, $T_2'(u) : X \to X^*$ is compact, since the embedding of $W_p^m$ into $W_q^k$ is compact for $0 \leq k \leq m - 1$ if $1/q > 1/p - (m - k)/n > 0$, or if $q < \infty$ and $1/p = (m - k)/n$.

Next, we shall show that $T_1'(u) : X \to X^*$ is of type $(S_+)$. Let $v_n \in X$ be such that $v_n \rightharpoonup v$ in X and

$$\limsup_{n\to\infty}(T_1'(u)v_n, v_n - v) \leq 0.$$

It follows that $D^\alpha v_n \to D^\alpha v$ in $L_p$ for each $|\alpha| < m$ by the Sobolev embedding theorem. Next, we shall show that $D^\alpha v_n \to D^\alpha v$ in $L_p$ for each $|\alpha| = m$. Since X is separable, there are finite dimensional subspaces $\{X_n\}$ in X whose union is dense in it. Since dist$(v, X_n) \to 0$ for each $v \in X$, there is a $w_n \in X_n$ such that $w_n \to v$ in X as $n \to \infty$. Then

$$\limsup_n(T_1'(u)v_n - T_1'(u)w_n, v_n - w_n) \leq \limsup_n(T_1'(u)v_n, v_n - v - (w_n - v)) -$$

$$\liminf_n(T_1'(u)w_n, v_n - w_n) \leq \limsup_n(T_1'(u)v_n, v_n - v) + \lim_n(T_1'(u)v_n, w_n - v) -$$

$$\liminf_n(T_1'(u)w_n, v_n - w_n) \leq 0. \qquad (5.6)$$

But,

$$(T_1'(u)(v_n - w_n), v_n - w_n) =$$

$$\Sigma_{|\alpha|=|\beta|=m} \int_Q A_{\alpha\beta}(x, u, ...D^m u) D^\alpha(v_n - w_n) D^\beta(v_n - w_n) dx \geq$$

$$\int_Q (g_2(|D^{\alpha_0} u|) \Sigma_{|\alpha|=m} D^\alpha(v_n - w_n)^2 dx \geq c \int_Q \Sigma_{|\alpha|=m} D^\alpha(v_n - w_n)^2 dx =$$

$$c\Sigma_{|\alpha|=m}\|D^\alpha(v_n - w_n)\|^2 \qquad (5.7)$$

where we may assume $g_2 = c > 0$. This and (5.6) imply that $D^\alpha(v_n - w_n) \to 0$ in $L_p$ for each $|\alpha| = m$. Hence, $D^\alpha v_n \to D^\alpha v$ in $L_p$ for each $|\alpha| = m$, and therefore $v_n \to v$ in X. This shows that $T_1'(u)$ is continuous and of type $(S_+)$ as is $T'(u) = T_1'(u) + T_2'(u)$. Hence, as shown before, $T'(u)$ is proper on bounded closed subsets of X. By Yood's criterion, the index of $T'(u)$ =dim$N(T'(u))$−codim$R(T'(u)) \geq 0$. Moreover, $T'(u)^* = T_1'(u)^* + T_2'(u)^*$ with $T_2'(u)^*$ compact. Hence, using (A4),

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as above, we get that $T_1'(u)^*$ is continuous and of type $(S_+)$. Thus, the index $i(T_1'(u)^*) \geq 0$ and $i(T_1'(u)^*) = -i(T_1'(u)) \leq 0$. It follows that $i(T_1'(u)) = 0$. It is left to show that $T'(u)$ is a continuous map in u. Let $u_n \to u$. Then

$$(T'(u_n)v - T'(u)v, w) =$$

$$\Sigma_{|\alpha|,|\beta|\leq m} \int_Q [A_{\alpha\beta}(x, u_n, ..., D^m u_n) - A_{\alpha\beta}(x, u, ..., D^m u) D^\beta v D^\alpha w dx.$$

The Nemytski map $Nu = A_{\alpha\beta}(x, u, ..., D^m u)$ is continuous from $X$ to $L_{p'}(Q)$, $1/p + 1/p' = 1$. Hence, $T'(u_n) \to T'(u)$ using also the Sobolev imbedding theorem. This completes the proof that $T$ is a Fredholm map of index zero that is proper on bounded closed subsets of X. ∎

**Remark 5.1** We can put $T_2$ together with $D$ and require the differentiability (Fredholmness) of only $T_1$.

We assume that the $B_\alpha's$ satisfy

(B1) For each $|\alpha| \leq m$, $B_\alpha(x, \xi)$ is a Caratheodory function and, for $p > 2$ there exist a constant $c > 0$ and $h_\alpha(x) \in L_q(Q)$, $1/p + 1/q = 1$, such that

$$|B_\alpha(x, \xi)| \leq c(h_\alpha(x) + |\xi|^{p-1})$$

(B2) There is a sufficiently small $k_1 > 0$ such that

$$\Sigma_{|\alpha|=m}|B_\alpha(x, \eta, \xi_\alpha) - B_\alpha(x, \eta, \xi_\alpha')| \leq k_1 \Sigma_{|\alpha|=m}|\xi_\alpha - \xi_{\alpha'}|$$

for each a.e. $x \in Q$, $\eta \in R^?$ and $\xi_\alpha, \xi_{\alpha'} \in R^?$.

We note that if the $B_\alpha$'s are differentiable for $|\alpha| = m$ and $B_{\alpha\alpha}(x, \xi) = \partial/\partial\xi_\alpha B_\alpha(x, \xi)$ are sufficiently small, then $(B2)$ holds.

In view of Proposition 5.1, the results in the form of Theorems $4.1 - 4.3$ are valid for Eq. (5.1) as well as the corresponding ones involving maps that are asymptotically close to positively k-homogeneous maps. A sample of such a theorem is given next.

Consider also the equation

$$\Sigma_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha a_\alpha(x, u, ..., D^m) = f \tag{5.8}$$

in X. Define the map $A : X \to X^*$ by

$$(Au, v) = \Sigma_{|\alpha|\leq m} \int_Q a_\alpha(x, u, ..., D^m u)D^\alpha v dx. \tag{5.9}$$

Then weak solutions of (5.8) are solutions of the operator equation

$$Au = f, \ u \in X. \tag{5.10}$$

**Theorem 5.1** *Let conditions* $(A1) - (A4)$, *(a1)-(a2) and (B1)-(B2) hold and* $deg_{BCF}(T, B(0,r), 0) \neq 0$ *for all large* $r$. *Suppose that* $A$ *is of type* $(S_+)$ *and* $Au = 0$ *has no a nontrivial solution. Then Eq. (5.1) is solvable for each* $f \in X^*$, *has a compact set of solutions whose cardinal number is constant, finite and positive on each connected component of the set* $X^* \setminus (T + D)(\Sigma)$, *where* $\Sigma = \{u \in X \mid T + D \text{ is not locally invertible at } u\}$.

**Proof.** In view of our discussion above, $T$ is a Fredholm map of index zero that is proper on bounded closed subsets of X. It is proper on X since it satisfies condition $(+)$, and therefore $\beta(T) > 0$. We need to swow that $\alpha(D) < \beta(T)$. We note that the boundedness and continuity of D follows from $(A1) - (A2)$, the Sobolev embedding theorem and the continuity of the Nemitski maps in $L_p$ spaces. We can write $D = D_1 + D_2$, where

$$(D_1 u, v) = \Sigma_{|\alpha|=m} \int_Q B_\alpha(x, u, ..., D^m u) D^\alpha v dx.$$

$$(D_2 u, v) = \Sigma_{|\alpha|<m} \int_Q B_\alpha(x, u, ..., D^m u) D^\alpha v dx,$$

The map $D_1 : X \to X^*$ is $k_1$-set contractive by (B2), while $D_2 : X \to X^*$ is compact by the Sobolev embedding theorem. Hence, $\alpha(D) = \alpha(D_1) \leq k_1 < \beta(T)$ since $k_1$ is sufficiently small. Finally, condition (a2) implies that T is asymptotically close to the $(p-1)-$ positive homogeneous map $A$ given by (5.9). Hence, Theorem 3.2 applies with $C = 0$. ∎

**Example 5.1** Let $s > 0$, k be sufficiently small, and look at

$$-\Delta u - \mu u \frac{|u|^s}{1 + |u|^s} + kF(x, u, \nabla u) = f \tag{5.11}$$

and

$$-\Delta u - \mu u = 0 \tag{5.12}$$

Let $A_0(x, \xi_0, \xi_1, ..., \xi_n) = \xi_0 \frac{|\xi_0|^s}{1+|\xi_0|^s}$ and $A_i(x, \xi_0, \xi_1, ..., \xi_n) = a_i(x, \xi_0, \xi_1, ..., \xi_n) = \xi_i$ and $a_0(x, \xi_0, \xi_1, ..., \xi_n) = \xi_0$. Then $A_0, A_i, a_0$ and $a_i$ satisfy (a1)-(a2). Let F satisfy (B1). Then Eq. (5.11) has a solution $u \in W_2^1(Q)$, $u = 0$ on $\partial Q$, for each $f \in L_2(Q)$ if $\mu$ is not an eigenvalue of Eq. (5.12).

**Example 5.2** Let $p > 2$, $k$ be sufficiently small, and look at

$$-\Sigma_{i=1}^n \frac{\partial}{\partial x_i}[(1 + \Sigma_{j=1}^n |\frac{\partial u}{\partial x_j}|^2)^{p/2-1} \frac{\partial u}{\partial x_i}] + \mu(1 + |u|^2)^{p/2-1}u + kDu = f \tag{5.13}$$

and

$$-\Sigma_{i=1}^n \frac{\partial}{\partial x_i}(\Sigma_{j=1}^n |\frac{\partial u}{\partial x_j}|^2)^{p/2-1} \frac{\partial u}{\partial x_i}) + \mu(|u|^2)^{p/2-1}u = 0. \tag{5.14}$$

where $D : W_p^1 \to W_p^1$ is k-set contractive, e.g., $Du = F(x, u, \nabla u)$ in which case it is compact, or $Du = \Sigma_{i=1}^n \partial/\partial x_i c_i(x, u, \nabla u)$ with the $c_i$ $k_i$-contractive in $\nabla u$ with $k_i$ small. Let $A_0(x, \xi_0, \xi_1, ...\xi_n) = (1 + \xi_0^2)^{p/2-1}\xi_0$, $A_i(x, \xi_0, \xi_1, ..., \xi_n) = (1 + \Sigma_{j=1}^n \xi_j^2)^{p/2-1}\xi_i$, $a_0(x, \xi_0, \xi_1, ...\xi_n) = (\xi_0^2)^{p/2-1}\xi_0$ and $a_i(x, \xi_0, \xi_1, ..., \xi_n) = \Sigma_{j=1}^n(\xi_j^2)^{p/2-1}\xi_i$. Then the matrix $(A_{ij}(x, \xi_0, \xi_1, ...\xi_n))$ is symmetric. Let $n = 2$ for simplicity. Then the eigenvalues of the matrix are $\lambda_1 = (p/2 - 1)(1 + \xi_1^2 + \xi_2^2)^{p/2-1}$ and $\lambda_2 = \lambda_1 + (p-2)(1 + \xi_1^2 + \xi_2^2)^{p/2-2}(\xi_1^2 + \xi_2^2)$. Hence, $A_0, A_i, a_0$ and $a_i$ satisfy conditions $(A1) - (A4)$ and $(a1) - (a2)$, respectively. Then Eq. (5.13) has a compact set of solution $u \in W_p^1(Q)$, $u = 0$ on $\partial Q$, for each $f \in X^*$, if $\mu$ is not an eigenvalue of Eq. (5.14) and $n = 2$. The solution set is finite for all f as in Theorem 5.1.

# References

[1] P. Benevieri, A. Calamai, M. Furi, A degree theory for a class of perturbed Fredholm maps, Fixed Point Theory Appl., 2 (2005) 185-206.

[2] P. Benevieri, M. Furi, (2006) A degree theory for locally compact perturbations of nonlinear Fredholm maps of index zero, Abstr. Appl. Anal., 1-20.

[3] F.E. Browder, Covering spaces, fiber spaces and local homeomorphisms, Duke Math. J., 21 (1954), 329-336.

[4] A. Calamai, The invariance of domain theorem for compact perturbations of nonlinear Fredholm maps of index zero, Nonlinear Funct. Anal. Appl., 9 (2004) 185-194.

[5] K. Deimling, Nonlinear Functional Analysis, Springer, Berlin, 1985.

[6] P.M. Fitzpatrick, J. Pejsachowisz, P.J. Rabier, The degree of proper $C^2$ Fredholm mappings I, J. Reine Angew. Math. 247 (1992) 1-33.

[7] M. Furi, M. Martelli, and A. Vignoli, Contributions to the spectral theory for nonlinear operators in Banach spaces, Ann. Mat. Pura Aplpl. (4) 118 (1978) 229-294.

[8] P. Hess, On nonlinear mappings of monotone type homotopic to odd operators, J. Funct. Anal. 11 (1972) 138-167.

[9] P.S. Milojević, Fredholm theory and semilinear equations without resonance involving noncompact perturbations, I, II, Applications, Publications de l'Institut Math. 42 (1987) 71-82 and 83-95.

[10] J. Necas, Sur l'alternative de Fredholm pour les operateurs non-lineares avec applications aux problemes aux limites, Ann. Scoula Norm. Sup. Pisa, 23 (1969) 331-345.

[11] R.D. Nussbaum, Degree theory for local condensing maps, J. Math. Anal.Appl. 37 (1972) 741-766.

[12] J. Pejsachowisz, P.J. Rabier, Degree theory for $C^1$ Fredholm mappings of index 0, J. D'Anal. Math., 76 (1998) 289-319.

[13] P.J. Rabier, Nonlinear Fredholm operators with noncompact fibers and applications to elliptic problems on $R^N$, J. Funct. Anal., 187 (2001) 343-367.

[14] P.J. Rabier, Quasilinear elliptic equations on $R^N$ with infinitely many solutions, Nonl. diff. equ. appl., 11 (2004) 311-333.

[15] P.J. Rabier, M.F. Salter, A degree theory for compact perturbations of proper $C^1$ Fredholm mappings of index zero, Abstr. Appl. Anal., 7 (2005) 707-731.

[16] P.J. Rabier, C.A. Stuart, Fredholm and properness properties of quasilinear elliptic equations on $R^N$, Math. Nachr., 231 (2001) 129-168.

[17] P.J. Rabier, C.A. Stuart, Global bifurcation for quasilinear elliptic equations on $R^N$, Math. Z.,237 (2001) 85-124.

[18] I.V. Skrypnik, Methods of Investigation of Nonlinear Elliptic Boundary Value Problems, Nauka, Moscow, 1990.

[19] E. Ziedler, Nonlinear Functional Analysis and Its Applications II/B, Nonlinear Monotone Operators, (1990) Springer-Verlag, New York.