Non-Constructive Galois-Tukey Connections

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Abstract

There are inequalities between cardinal characteristics of the continuum that are true in any model of ZFC, but without a Borel morphism proving the inequality. We answer some questions from Blass [1].

1 Introduction

Vojtáš [7] introduced a framework in which cardinal characteristics of the continuum can be regarded as norms of corresponding relations $\mathbf{A} = (A_-, A_+, A)$ with A_-,$A_+ \subseteq 2^\omega$, $A \subseteq A_- \times A_+$, and the norm

$$||\mathbf{A}|| = \min\{|\mathcal{Z}| \,:\, \mathcal{Z} \subseteq A_+ \,\wedge\, \forall x \in A_- \,\exists z \in \mathcal{Z}\, A(x,z)\}.$$

A Galois-Tukey connection from a relation $\mathbf{B}$ to a relation $\mathbf{A}$, which we call as in [1] a *morphism from* $\mathbf{A}$ *to* $\mathbf{B}$ (— notice the different direction —), is a pair of functions (α, β) such that

$$\begin{aligned} \alpha &: B_- \to A_-, \\ \beta &: A_+ \to B_+, \\ \forall b \in B_- \forall a \in A_+ & \quad (A(\alpha(b), a) \to B(b, \beta(a))). \end{aligned}$$

If there is a morphism from $\mathbf{A}$ to $\mathbf{B}$, then $||\mathbf{B}|| \leq ||\mathbf{A}||$, and indeed the proofs of the inequalities usually exhibit morphisms between the corresponding relations. Following Blass [1], we call inequalities correct if they are true in

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every model of ZFC, and the other ones incorrect. A result of Yipariki in [8] shows, that there may be morphisms corresponding to incorrect inequalities. These morphisms are not absolute between different models of ZFC, of course, and on the other hand many of the morphisms used in the proofs of well-known correct inequalities e.g. of those in the Cichoń diagram, are Borel functions on Borel domains, which we will call *Borel morphisms.*

Some inequalities are proved by Borel morphisms between products or more complicated compositions of the given relations, and Blass [1] conjectures that there are no Borel morphisms between certain unsplitting relations, defined in the next paragraph. We prove a strengthening his conjecture and some related theorems.

We deal with the unsplitting relations: For $n \geq 1$, we have

$$\begin{aligned} \mathbf{R}_n &= (n^\omega, [\omega]^\omega, \{(f,Y) : f \text{ is almost constant on } Y\}), \\ \mathbf{R}_n^\sharp &= (n^\omega, [\omega]^\omega, \{(f,Y) : f \text{ is constant on } Y\}), \\ \mathfrak{r}_n &= ||\mathbf{R}_n||, \\ \mathfrak{r}_n^\sharp &= ||\mathbf{R}_n^\sharp||. \end{aligned}$$

(f is almost constant on Y if there is a finite $Y_0 \subseteq Y$ such that f is constant on $Y \setminus Y_0$.) $\mathfrak{r}_2$ is the usual unsplitting number. It is easy to see that $\mathfrak{r}_m = \mathfrak{r}_n = \mathfrak{r}_n^\sharp = \mathfrak{r}_m^\sharp$ for $m, n \geq 2$.

Blass gives a notion of sequential composition (see 1.3 for the definition) of two copies of $\mathbf{R}_2$, called $\mathbf{R}_2; \mathbf{R}_2$, and a Borel morphism from $\mathbf{R}_2; \mathbf{R}_2$ to $\mathbf{R}_3$. This proves $\mathfrak{r}_2 \geq \mathfrak{r}_3$. The same procedure works also in the $\sharp$-version. Is the notion of sequential composition necessary for the goal to have Borel morphisms? We give an affirmative answer.

Theorem 1.1 *For $n > m$, there is no morphism (α, β) with a Baire measurable α (and arbitrary β) from $\mathbf{R}_m^\sharp$ to $\mathbf{R}_n^\sharp$.*

Theorem 1.2 *For $m \geq 1, n \geq 2$, there is no morphism (α, β) with a Baire measurable α from $\mathbf{R}_m$ to $\mathbf{R}_n^\sharp$.*

The next two theorems say that the notion of category-product and the older notion of component-product are weaker than the sequential composition. Let us first define the three notions:

Definition 1.3 *For two relations* $\mathbf{A}$ *and* $\mathbf{B}$ *the category-product, the component-product and the sequential composition are given by (where* $\dot\cup$ *denotes the disjoint union):*

$$\begin{aligned}
\mathbf{A}\times\mathbf{B} &:= (A_-\dot\cup B_-, A_+\times B_+,\\
&\qquad \{(c,(a,b))\ :\ (c\in A_-\wedge A(c,a))\vee(c\in B_-\wedge B(c,b))\}),\\
\mathbf{A}\times_{comp}\mathbf{B} &:= (A_-\times B_-, A_+\times B_+, \{((c,d),(a,b))\ :\ A(c,a)\wedge B(d,b)\}),\\
\mathbf{A};\mathbf{B} &:= (A_-\times {}^{A_+}B_-, A_+\times B_+, \{((c,\rho),(a,b))\ :\ A(c,a)\wedge B(\rho(a),b)\}).
\end{aligned}$$

Theorem 1.4 *For* $n>m$, $\ell\geq 1$, *there is no morphism* (α,β) *with a Baire measurable* α *from the* ℓ*-fold category- or component-product of copies of* $\mathbf{R}^\sharp_m$ *to* $\mathbf{R}^\sharp_n$.

Theorem 1.5 *For* $m\geq 1, n\geq 2$, $\ell\geq 1$, *there is no morphism* (α,β) *with a Baire measurable* α *from the* ℓ*-fold category- or component-product of copies of* $\mathbf{R}_m$ *to* $\mathbf{R}^\sharp_n$.

Our proofs leave open the following conjecture:

Conjecture 1.6 *For* $n>m$, *there is no morphism* (α,β) *with a Baire measurable* α *(and arbitrary* β*) from* $\mathbf{R}_m$ *to* $\mathbf{R}_n$.

The following theorem is closely related to the conjecture.

Theorem 1.7 $n>m\geq 1$. *There is no pair of functions* α,β *such that* $\alpha\colon n^\omega\to m^\omega$ *is Borel and* $\beta\colon[\omega]^\omega\to[\omega]^\omega$ *and* $\forall f_0,f_1\in n^\omega\ \forall X\in[\omega]^\omega$

$$(\alpha(f_0)\restriction X=\alpha(f_1)\restriction X\to f_0\restriction\beta(X)=^* f_1\restriction\beta(X)).$$

The theorems above are proved in section 2. In the third section, we investigate unsplitting relations with modified domains and get some results on the point where non-existence of morphisms with a Baire measurable first component changes to existence of Borel morphisms.

Notation: We give some notation concerning trees, finite sequences, etc. $\subset$ and $\supset$ denote the proper relations. $D^{<\omega}=\{t\ :\ \exists h\in\omega\ t\colon h\to D\}$. $[D]^\omega=\{X\subseteq D\ :\ |X|=\omega\}$. $T\subseteq n^{<\omega}$ is a tree, if $(\forall t\in T)(\forall s\in n^{<\omega})$ if $s\subseteq t$ then $s\in T$. The h-th level of a tree T, $T\restriction h=T\cap\omega^h$. T is an

ω-tree, if $T \neq \emptyset$ and $(\forall s \in T)(\exists t \in T)\, t \supset s$. $t \in T \subseteq n^{<\omega}$ is a branching point of T if $\exists c_0 \neq c_1 \in n\ t\hat{}c_i \in T$. For an ω-tree T, the set of branches of T is $[T] = \{f : \forall h\ f\restriction h \in T\}$. For $s \in n^{<\omega}$, we have the basic open sets $[s] = \{f \in n^\omega : f \supset s\}$ and the restricted trees $T_s = \{t \in T : t \supseteq s\}$. $\forall^\infty$ means for all but finitely many, $\exists^\infty$ means there are infinitely many, and for $f, g \in n^\omega$, we write $f =^* g$ iff $\forall^\infty h\ f(h) = g(h)$.

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2 The Proofs

Suppose $\alpha\colon n^\omega \to m^\omega$ is Baire measurable, i.e. for every open set C in m^ω there is an open set D in n^ω such that the symmetric difference $(f^{-1}{}''C)\Delta D$ is meager is n^ω. The following three steps can be used in the proofs of all the above theorems, though theorem 1.2 has also a shorter proof avoiding the work in the first step below and applying the Lebesgue density theorem [5] instead.

First step: Restriction to suitable domains.

We use the following lemma of Kuratowski [3]:

Lemma 2.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, and $\mathcal{Y}$ have a countable base. For every Baire measurable function $f\colon \mathcal{X} \to \mathcal{Y}$ there is a comeager set $C \subseteq \mathcal{X}$ such that $f\restriction C$ is continuous (w.r.t. the subspace topology on C induced by $\mathcal{X}$).*

We apply this to α and get a comeager $C \subseteq n^\omega$ such that $\alpha\restriction C$ is continuous. We take nowhere dense H_i, $i \in \omega$, such that $n^\omega \setminus C = \bigcup_{i\in\omega} H_i$.

We now choose by induction on i an increasing sequence $n_0 = 0 < m_0 < n_1 \ldots < n_i < m_i < n_{i+1} < m_{i+1} \ldots$ and $s_i\colon [n_i, m_i) \to n$, $t_i\colon [m_i, n_{i+1}) \to n$.

Suppose n_i is chosen. Then we take $m_i > n_i$ and $s_i\colon [n_i, m_i) \to n$ such that $H_i \cap \{f \in n^\omega : f\restriction[n_i, m_i) = s_i\} = \emptyset$. (The existence of such m_i, s_i is proved in [6], théorème 21.) Now we take $n_{i+1} > m_i$ and $t_i\colon [m_i, n_{i+1}) \to n$ such that $H_i \cap \{f \in n^\omega : f\restriction[m_i, n_{i+1}) = t_i\} = \emptyset$. This ends the induction.

We define

$$\begin{aligned} T_0 &= \{s \in n^{<\omega} \,:\, \forall i(m_i \leq \mathrm{lh}(s) \rightarrow s\restriction[n_i, m_i) = s_i)\}, \\ T_1 &= \{s \in n^{<\omega} \,:\, \forall i(n_{i+1} \leq \mathrm{lh}(s) \rightarrow s\restriction[m_i, n_{i+1}) = t_i)\}. \end{aligned}$$

Since $[T_j] \cap \bigcup_{i\in\omega} H_i = \emptyset$, the $\alpha\restriction[T_j]$ and hence $\alpha\restriction([T_0] \cup [T_1])$ are continuous.

For later use, it is important that the T_j are fully branching on two sets of levels that combine to ω: $\forall k \in \bigcup_{i\in\omega}[n_i, m_i) \forall t \in T_1\restriction k \forall j < n\ t\hat{\ }j \in T_1\restriction(k+1)$, and $\forall k \in \bigcup_{i\in\omega}[m_i, n_{i+1})\ \forall t \in T_0\restriction k\ \forall j < n\ t\hat{\ }j \in T_0\restriction(k+1)$.

Second step

Since $\alpha\restriction([T_0] \cup [T_1])$ is a continuous function with a compact domain, there is an increasing sequence $\langle k_u \,:\, u \in \omega\rangle$ such that there is a function

$$g\colon \bigcup_{j=0,1} \bigcup_{u\in\omega} T_j\restriction k_u \rightarrow m,$$

such that $\forall f \in [T_0] \cup [T_1]\ \forall u \in \omega\ \alpha(f)(u) = g(f\restriction k_u)$.

Third step: Selecting a subset of m that appears densely often in T_j.

$\exists$ subsequence $\langle k_{u_v} \,:\, v \in \omega\rangle$, $M_j \subseteq m$ for $j = 0, 1$, $r_j \in T_j\restriction k_{u_0}$ such that

$$\forall t \in (T_j)_{r_j}\restriction k_{u_v}\ \ g''((T_j)_t\restriction k_{u_{v+1}}) = M_j.$$

This follows from

$$\forall W \in [\omega]^\omega \exists V \in [W]^\omega \exists M \subseteq m\ \exists r_j \in T_j\ \forall t \in (T_j)_{r_j}\ \forall^\infty v \in V\ g''((T_j)_t\restriction v) = M.$$

We set

$$X = \{u_v \,:\, v \in \omega\}.$$

End of the proofs of theorems 1.1 and 1.4

Now let $n > m$. We are looking for an X' such that $\beta(X')$ cannot be defined if it has to respect the morphism property. The following is sufficient for this aim: $\exists X'\ \forall$ infinite $\beta(X')\ \exists f_0, f_1\ \exists \ell \in \beta(X') \setminus \{\min(\beta(X'))\}\ f_0\restriction\ell = f_1\restriction\ell$ and $f_0(\ell) \neq f_1(\ell)$ and $\alpha(f_0)\restriction X'$ and $\alpha(f_1)\restriction X'$ are constant. This is provided by the next lemma.

Lemma 2.2 *$\exists S_j \subseteq T_j$, $g_j : X \to M_j$ such that S_1 has a branching point on each level in $\bigcup_{i\in\omega}[n_i, m_i)$ and S_0 has a branching point in each level in $\bigcup_{i\in\omega}[m_i, n_{i+1})$ and*

$$\forall f \in [S_j]\ \alpha(f)\restriction X = g_j.$$

Proof: Independently for each j, by induction on $v \in \omega$, we construct $S_j\restriction k_{u_v}$. We begin with $S_j\restriction k_{v_0} = r_j$. Suppose $S_j\restriction k_{u_v}$ is constructed, we take an arbitrary r out of this set. We select a good "color" $g_j(u_{v+1}) \in M_j$ such that more than $\frac{1}{\text{card}(M_j)}$ points t in $(T_j)_r\restriction k_{u_{v+1}}$ and at least one point out of each $(T_j)_{r'}\restriction k_{u_{v+1}}, r' \in S_j\restriction k_{u_v}$, fulfill $g(t) = g_j(u_{v+1})$. We put these points into $S_j\restriction k_{u_{v+1}}$.

Every level of $(S_0)_r\restriction k_{u_{v+1}}$ in $[k_{u_v}, k_{u_{v+1}}) \cap \bigcup_{i\in\omega}[m_i, n_{i+1})$ has a branching point, since otherwise in $(S_0)_r\restriction k_{u_{v+1}}$ there would be at most $\frac{1}{n}$ times the maximal number n^z, $z = \text{card}([k_{u_v}, k_{u_{v+1}}) \cap \bigcup_{i\in\omega}[m_i, n_{i+1}))$ points. Similar for S_1. □

We remark that 2.2 proves the weaker version of theorem 1.7 gotten by leaving out the * on the right hand side.

In order to finish the proof of 1.1, we take an infinite $X' \subseteq X$ such that $\text{card}((g_0, g_1)''X') = 1$. Then for all $f \in [S_0] \cup [S_1]$ the restriction $\alpha(f)\restriction X'$ is constant, but there is no infinite $\beta(X')$, even no set with more than three elements $\geq k_{u_0}$, such that all $f \in [S_0] \cup [S_1]$ are constant on $\beta(X')$. Thus theorem 1.1 is proved.

The proof of theorem 1.4 is finished by the following observation:

Suppose $\alpha = (\alpha_0, \alpha_1, \ldots \alpha_{\ell-1}) : n^\omega \to (m^\omega)^\ell$, $X \in [\omega]^\omega$, $\langle X_i : i < \ell\rangle$ a partition of X into infinite parts, $g_j : X \to M_j$, $S_j \subseteq n^{<\omega}$ are as above, $\forall i < \ell\, \forall f \in [S_j]\ \alpha_i(f)\restriction X_i = g_j\restriction X_i$. Such an X' and X'_i are provided by applying the second and the third step to α (thus getting maybe different M_i) and mixing the α_i, $i < \ell$, in a suitable way in lemma 2.2. Then for any $X'_i \in [X_i]^\omega$ with $\text{card}((g_0, g_1)''X'_i) = 1$, there is no infinite Y such that $\forall f \in [S_0] \cup [S_1]$ ($\alpha_i(f)$ is constant on X'_i for all $i < \ell \to f$ is constant on Y).

End of proofs of theorems 1.2 and 1.5

We do not construct any trees at all, only two special branches of $[T_0] \cup$

$[T_1]$. The premise is $n \geq 2$. Then

$$\forall B \in [\omega]^\omega \; \exists j \in 2 \; \forall c \in M_j \; \exists f_0, f_1 \in [T_j] \; \exists u \; \exists \ell \in B \cap [k_u, k_{u+1})$$

$$f_0 \restriction \ell = f_1 \restriction \ell \; \wedge f_0(\ell) \neq f_1(\ell) \wedge \forall w \in X \setminus (u+1) \;\; \alpha(f_0)(w) = \alpha(f_1)(w) = c.$$

Just construct starting with a branching point $\ell \in B$ two branches, similar but easier then lemma 2.2.

An easier way is the following: We choose any closed set $[T]$ such that $\alpha \restriction [T]$ is continuous, and for any $t \in T$, the closed set $[T_t]$ has Lebesgue measure greater than zero. Then we apply the Lebesgue density theorem [5] and get (with the analogous properties of k_{u_v}, M, X, for T):

$$\forall B \in [\omega]^\omega \; \forall c \in M \; \exists f_0, f_1 \in [T] \; \exists u \; \exists \ell \in B \cap [k_u, k_{u+1})$$

$$f_0 \restriction \ell = f_1 \restriction \ell \; \wedge f_0(\ell) \neq f_1(\ell) \wedge \forall w \in X \setminus (u+1) \;\; \alpha(f_0)(w) = \alpha(f_1)(w) = c.$$

The modification of the proof of 1.2 towards a proof of 1.5 is very similar to what we did for 1.4.

End of the proof of theorem 1.7

Now we use again $n > m$ and proceed similar as but more carefully than in the proof of lemma 2.2. We claim:

$$\forall B \in [\omega]^\omega \; \exists B' \in [B]^\omega \; \exists j \in 2 \; \exists f_0, f_1 \in T_j \forall \ell \in B'$$

$$f_0(\ell) \neq f_1(\ell) \wedge \alpha(f_0) \restriction X = \alpha(f_1) \restriction X.$$

This is proved by chosing $f_j \restriction k_{u_v}$ by induction on v. B' is a set containing one point out of each nonempty $B \cap \bigcup_{i \in \omega} [n_i, m_i) \cap [k_{u_v}, k_{u_{v+1}})$, $v \in \omega$, if there are infinitely v such that this is nonempty, otherwise, one point out of each nonempty $B \cap \bigcup_{i \in \omega} [m_i, n_{i+1}) \cap [k_{u_v}, k_{u_{v+1}})$, $v \in \omega$. We use that the proof of lemma 2.2 allows in the successor step the choice of r and take $r = f_0 \restriction k_{u_v}$ and determine $f_0 \restriction k_{u_{v+1}}$ such that in addition to the requirement $g(f_0 \restriction k_{u_{v+1}}) = g(f_1 \restriction k_{u_{v+1}})$ the nonequality $f_0 \restriction k_{u_{v+1}}(\ell) \neq f_1 \restriction k_{u_{v+1}}(\ell)$ is true if $\ell \in B'$ has to be considered in the step from v to $v+1$.

3 Boundaries

In this section we consider some instances of the following generalization of the problem. Given two spaces A_-, B_- of functions with domain ω: Is there a Borel $\alpha\colon B_- \to A_-$ such that there is a $\beta\colon [\omega]^\omega \to [\omega]^\omega$ such that $\forall f \in B_- \, \forall X \in [\omega]^\omega$: If $\alpha(f)$ is constant on X then f is constant on $\beta(X)$.

Our example is a sharp result in the case $n = m+1$: We set $B_-(n) := \{s \in n^\omega \,:\, \exists^{\leq 1} k \; s(k) = n-1\}$ and $B(n) := \{s \in n^{<\omega} \,:\, \exists^{\leq 1} k \; s(k) = n-1\}$.

Theorem 3.1 *Suppose $n > m \geq 1$. We endow $B_-(n)$ with the topology of n^ω. Then there is no Baire measurable $\alpha\colon B_-(n) \to m^\omega$ and $\beta\colon [\omega]^\omega \to [\omega]^\omega$ such that*

$\forall f \in B_-(n) \, \forall X \in [\omega]^\omega :$ *If $\alpha(f)$ is constant on X, then f is constant on $\beta(X)$.*

The proof refers heavily to the method shown in section 2. We give only the parts that are sharper than the steps in the previous section. W.l.o.g., we assume $n = m+1$, otherwise 3.1 is contained in 1.1.

First we choose T_0 and T_1 more carefully than in section 2, such that $m \notin \bigcup_{i\in\omega} \text{range}(s_i) \cup \text{range}(t_i)$: We take H_{2i}, $i \in \omega$, such that $H_{2i} \subseteq m^\omega$ is nowhere dense in m^ω and $\alpha \restriction m^\omega \setminus \bigcup_{i\in\omega} H_{2i}$ is continuous. We take $H_{2i+1} \subseteq B_-(n) \setminus m^\omega$ nowhere dense in $B_-(n) \setminus m^\omega$ such that $\alpha \restriction B_-(n) \setminus (m^\omega \cup \bigcup_{i\in\omega} H_{2i+1})$ is continuous. Then the s_i and t_i can be chosen with the properties as in the previous section and the additional requirement $m \notin \bigcup_{i\in\omega} \text{range}(s_i) \cup \text{range}(t_i)$.

We code the continuous part of α as usual into a function g and we modify the considerations in the third step: For each $j \in 2$, either there are $r_j \in B(n) \setminus m^{<\omega}$, $M_j \neq m$, $\langle k_{u_v} \,:\, v \in \omega\rangle$ with the described properties, or there are $r_j \in m^{<\omega}$, $M_j \subseteq m$, $\langle k_{u_v} \,:\, v \in \omega\rangle$ such that

$$\begin{aligned}\forall v \in \omega \qquad & (\forall t \in (T_j)_{r_j} \restriction k_{u_v} \;\; (g''((T_j)_t \restriction k_{u_{v+1}}) \supseteq M_j) \wedge \\ & \forall t \in (T_j)_{r_j} \restriction k_{u_v} \cap m^{<\omega} \;\; (g''((T_j)_t \restriction k_{u_{v+1}}) = M_j)).\end{aligned}$$

Now we select a homogeneous subtree S_j of T_j by homogenizing on the levels in some set $\{k_{u_v} \,:\, v \in \omega\}$ (notation from section 2). We assume that we are in the second case, the first case is treated in 2.2. We fix some j.

Since T_j has less branching than the trees of section 2, we have to sharpen lemma 2.2. In order order to avoid clumsy indices, we forget the levels without any splitting in T_0 or in T_1 and work with one tree $T = B(n)$: Since $r_j \in m^{<\omega}$ and $m \notin \bigcup_{i\in\omega} \text{range}(s_i) \cup \text{range}(t_i)$, by squashing $(T_j)_{r_j}$ in an obvious manner we get them isomorphic to $B(n)$. Hence it is easy to see that the following lemma finishes the proof of theorem 3.1.

Lemma 3.2 *Suppose $g\colon B(n) \to m$, $\langle k_{u_v} : v \in \omega\rangle$, $M \subseteq m$ are such that*

$$\begin{aligned} \forall v \in \omega \qquad & (\forall t \in T{\restriction}k_{u_v}\ (g''(T_t{\restriction}k_{u_{v+1}}) \supseteq M) \wedge \\ & \forall t \in (T_j)_{r_j}{\restriction}k_{u_v} \cap m^{<\omega}\ (g''((T_j)_t{\restriction}k_{u_{v+1}}) = M)). \end{aligned}$$

As above, we set $X = \{u_v : v \in \omega\}$. Then $\exists S \subseteq B_-(n)$, $g_0\colon X \to M$ such that S has a branching point on each level and

$$\forall f \in [S]\ \forall v \in \omega\ g(f{\restriction}k_{u_v}) = g_0(u_v).$$

Proof: In addition to S having a branching point on each level we require:

$$\forall h \in \omega\ (S{\restriction}h) \cap m^h \neq \emptyset.$$

To start the induction, we take $\emptyset \in S$.

Induction step: We suppose that level ℓ of S is chosen, say $\{s_0, s_1 \dots s_r\}$, and that $r = s_0 \in S{\restriction}\ell \cap m^\ell$. Let $\ell' = k_{u_{v+1}} = \ell + d$ be the next height.

Now the selection of a "right color" $g_0(u_{v+1})$ and the corresponding points in $S{\restriction}\ell'$ is split into two cases.

First case: Among the $t \in B(n)_r{\restriction}\ell'$ of a color $c \in M$ appearing most often in this set, there is some $t \in m^{\ell'}$. We take all the points of such a majority color c into the ℓ'th level of S. In the other $(B(n))_{s_i}$ we take some point or all points of color c into $S{\restriction}\ell'$.

If there were any level between ℓ and $\ell' - 1$ that did not contain a branching point, then there would be strictly less than an $\frac{1}{m}$ of the points in $B(n)_r{\restriction}\ell'$ left in $S_r{\restriction}\ell'$. This is shown by the following calculation:

$$\begin{aligned} \text{card}(B(n)_r{\restriction}\ell') &= m^d + m^0 \cdot m^{d-1} + m \cdot m^{d-2} \dots + m^{d-1} \cdot m^0 \\ &= m^{d-1}(m+d). \end{aligned}$$

If S_r had one level between ℓ and $\ell' - 1$ without any splitting point and were everywhere else maximally splitting (for any subtree, we get less or equal in the following equation), then

$$\mathrm{card}(S_r \restriction \ell') = m^{d-2} \cdot (m + d - 1),$$

which is strictly less than

$$\frac{\mathrm{card}(B(n)_r \restriction \ell'}{m} = m^{d-2} \cdot (m + d)$$

and hence contradicting the choice of the color c.

Second case: Not the first case. Then all colors in M appearing most often in $B(n)_r \restriction \ell'$ do appear in $B(n)_r \restriction \ell' \setminus m^{\ell'}$. Hence in $B(n)_r \restriction \ell' \cap m^{\ell'} = (m^{\ell'})_r$ there are at most $m - 1$ colors and the argumentation for full trees applies. Again we take for any other s_i some point of the chosen color into $S_{s_i} \restriction \ell'$. $\square$

Remark: For example $B_-(m+1)(K) := \{f \in B_-(m+1) : \forall k \geq K \, f(k) \in m\}$, $K \in \omega$ shows that $\{B_- : \exists$ a morphism (α, β) with Baire measurable α from $(m^\omega, [\omega]^\omega,$"is constant on") to $(B_-, [\omega]^\omega,$ "is constant on")$\}$ is not closed under countable unions. A moment's reflection shows that is is closed under finite unions. The same is true for Borel α and continuous α, because for each K, there is a trivial morphism with continuous α to the relation restricted to $B_-(m+1)(K)$.

The example of $B_-(m+1)$ also shows that the non-sharp and the sharp unsplitting relations behave different with respect to the existence of Borel morphisms.

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