The Latest Frege

1. Opening

Many authors believe that the manuscripts Frege wrote in 1924–1925 are not theoretically of interest. They are rather a product of his emotional despair and theoretical dead-end which he reached in the last years of his life. Such is also the judgement of Michael Dummett delivered in his seminal book *Frege: Philosophy of Language*. According to Dummett, "the few fragmentary writings of Frege's final period—1919–1925—are not of high quality: they are interesting chiefly as showing that Frege did, at least at the very end of his life, acknowledge the failure of the logicist programme" (Dummett 1981, p. 664).

In this paper we will try to show that the widely accepted negative assessment of Frege's latest writings is due to a lack of understanding of their true idea. In fact, the change in Frege's mind in the last two or three years of his life was result of long deliberations on a severe tension in his founding intuitions. The change itself made his logico-philosophical project more coherent and, thus, is of utmost theoretical importance.

2. Frege's Intuition-Dualism and its Elimination

The attentive reader of Frege can easily discern a striking dualism in his attitude towards the role of intuition in logic and mathematics. On the one hand, his objective was to construct a deductive system of inference free from intuition. He repeatedly said that "one may not appeal to intuition as a means of proof" (1881a, 32 n.). On the other hand, Frege conceded (on the same page) that "it is permissible to use intuition as a helpful expedient in pinning down [*festhalten*] an idea". In fact, the whole project for a concept-script—which must also demonstrate how the logical proof is done—was built on the latter assumption. Frege's task was to suggest a graphically perfect language, the purpose of which was to *show* how things went in logic.[1]

This position implied a deflationary understanding of logic. Apparently, Frege was convinced that "if our language was logically more perfect, we would perhaps have no further need of logic, or we might read it off [*ablesen*] from the language" (1915, p. 252). In the

[1] This project is not to be confused with the project for a *perfectly grounded language*. According to the latter project, "in science the purpose of a proper name is to designate an object determinately; if this purpose is unfulfilled, the proper name has no justification in science" (1906, 178).

Tractatus, Wittgenstein transformed this conditional into a strict deflationary understanding of logic: “We can actually do without logical propositions; for in a suitable notation we can in fact recognise the formal properties of propositions by mere inspection of the propositions themselves” (Wittgenstein 1922, 6.122).

Apparently, Frege’s concept-script presupposes spatial imagination that cab help by grasping logical objects. In this connection, some authors suggest that Frege’s concept-script conveys a “perceptual model of understanding”:

> Frege’s staunch semantic realism requires that the nature of human “understanding” be interpreted in a manner quite analogous with the interpretation of “seeing” on the (non-intuitive) realist model of visual perception. (Schweizer 1991, p. 264)

In fact, the very use of the concept of grasping (*fassen*) suggests that there is something solid—a figure, a *Gestalt*—that is to take hold of and further to pick it up.

In the present paper we shall demonstrate that Frege’s controversial statements on intuition are a result of the fact that he was concerned with two different types of problems. On the one hand, he was convinced that by thinking, there is no place for perceptual intuition. On the other hand, logical inference, judgement, and deduction are made when we recognise—through a kind of intellectual intuition—some spatially organised structures.

3. The Need for New Symbolism

Frege’s theoretical reason for the project of concept-script was that “our attention is directed by nature to the outside”—to senses. So, we have no other choice but to think in symbols. At least by humans, a “concept is first gained by symbolising it; for since it is, in itself, *imperceptible*, it requires a perceptible representative in order to appear to us” (1881b, p. 84). It is important to notice in this connection that this problem does not concern language only, as Dummett suggests. The whole realm of thought must be expressed in proper symbolism, and language is only one part of this symbolism. Incidentally, Dummett was not the first misled on this point. Wittgenstein, who almost literally repeated Frege on the need to “perceptually represent” thinking, also limited the project for perfect symbolism to language.

Frege’s world of logic—the world of deduction—is objective but not spatial. Amongst its denizens are the numbers. They are concepts. “Spatial predicates are not applicable to them.” (1884, § 61) The same is true about concepts, about the true-values truth and falsehood, and, of course, about logical forms. The problem is that this non-spatial world is—at least by humans—of necessity to be grasped intuitively. Usually, we make this by the way of using language. The language, however, is not invented in order to accomplish scientific, exact purposes. That is why we must invent and introduce an appropriate, more felicitous for our intuition symbolism.

Frege saw this task as his main priority. His motive was the conviction that

> without the great invention of symbols which call to mind that which is absent, invisible, . . . the course of our ideas cannot gain its freedom from this: it would still be limited to that which our hand can fashion, our voice intone. (1881b, p. 83)

On the contrary, "if we produce the symbol of an idea which a perception has called to mind, we create in this way a firm, new focus about which idea gather." (Ibid., pp. 83–4)

It remained unclear, however, what the theoretical ground of Frege was to assume that an intuitive representation of the non-spatial world of logic is possible at all. How exactly do symbols with certain spatial characteristics, call up in our mind something invisible? Is this a process of decoding? If yes, according to which rules is this deciding made? Even more puzzling is Frege's assumption that precisely the *spatial order* of logical signs should demonstrate the *logical order* of logical objects.

In what follows we will see that, in real fact, Frege had no grounds for this presupposition. Realising this, in the last two, or three years of his life, he adopted the view that logical signs and objects are of the same order—of spatial order.

4. The Spatial Character of the New Symbolism

Frege's idea was that the concept-script should serve for "perspicuous [*anschauliche*] representation of the forms of thought" (ibid., p. 89). It should be nothing but an optical instrument with the help of which we could grasp the logical forms without much ado. These very forms are nothing but shapes (gestalts) that can be "generally sharply defined and clearly distinguished" (ibid., p. 87). The perspicuity of symbolism is to be achieved through the spatial relations of the symbols. This motivated Frege to advance the baroque symbolism of his concept. It was an expression of his conviction that

> the spatial relations of written symbols on a two-dimensional writing surface can be employed in far more diverse ways to express inner relationship than the mere following and preceding in

one-dimensional time. . . . In fact, simple sentential ordering in no way corresponds to the diversity of logical relations through which thoughts are interconnected. (Ibid.)

Actually, the whole concept-script of Frege is built referring to spatial relations:

(i) The sign for assertion consists of a content stroke and judgement stroke. Since the content stroke denotes the one-levelled combination (interweaving) of ideas (Vorstellungen),[2] it is symbolised by a *horizontal* stroke. The act of assertion that introduces into judgement another dimension, that of the will, is symbolised by a *vertical* stroke (1879, § 2).

(ii) Two propositions are connected by conditional stroke (*Bedingungsstrich*). Their alternative value is symbolised via posing two content strokes one *above* the other. The conditional stroke relates these opposite contents.

(iii) The generality is expressed by way of *concavity* which comes to symbolise that it treats singular terms intensionaly. This way of symbolising accepts that the logical constants must *show* how they operate; they cannot be defined.

(iv) Also the most elementary operation in Frege's logic are made recognizable through their spartial properties. Thus Frege maintained that "where logic is concerned, it seems that every combination of parts results from completing something that is in need of supplementation [*Ergänzungsbedürftigkeit*]" (1919, p. 254). A typical example here is the concept of function. The "need of supplementation" by functions is denoted by clearly intuitive means: it is expressed by the *space* within the brackets in $f(\quad)$ (1924, pp. 271–2).

We can conclude that all the four most basic operations of Frege's logic are *ineffable*.

5. Frege's *Begriffsschrift* as Ideography

Apparently, Frege's program for conceptual notation is nothing but an *ideography*—an organ on for a graphical representation of ideas. Some of his colleagues, who closely followed him, correctly understood this point. So Peano projected a "construction of graphic

[2] Incidentally, the interweaving of ideas in judgement, as well as of objects in states of affairs (*Sachverhalt*) is expressed by Russell graphically with the " ≬ " sign in the blue-print to the famous "On Denoting"—"On Fundamentals" (1994), and also in a paper published shortly afterwards: "The Theory of Implication" (1906).

symbolism, or ideography, capable of representing all the ideas of logic" (Peano 1973, p. 190). He insisted that this symbolism is not merely a tachygraphy. Russell also adopted a program for correct graphical representation of logic. *Principia Mathematica* offers

> a symbolism especially designed to represent the ideas and processes of deduction which occur in [it]". It "aids the intuition in regions too abstract for the imagination readily to present to the mind the true relation between the ideas employed" (Russell and Whitehead 1910, p. 2)

In fact, the very etymology of Frege's term "concept-script" (*Begriffsschrift*) shows that this was a project for an ideography. It was not a new means for extra-spatial reasoning but a two-dimensional representation of ideas. Frege, namely, borrowed it from Trendelenburg (1856), and Trendelenburg knew it from Carl Hindenburg's "combinatorial school".[3] The latter, in turn, coined the term *Begriffsschrift* on the model of the French term *ideographie* which was introduced at the end of the eighteenth-century by D'Alambert and Condillac. Finally, Frege's *Begriffsschrift* is conventionally translated in French as *Idéographie*.[4] It cannot be a surprise that Frege's *Begriffsschrift* is translated in French as *Idéographie*.

Frege was convinced that his *Begriffsschrift* is nothing but a further development of the symbolism already adopted in science and mathematics. The signs in mathematics, for example, express specific contents. "What [they] still lack is the logical cement [*Mörtel*] that will bind these building stones firmly together." (1881a, p. 13) The cement (logic) and the bricks (scientific/mathematical concepts and theories), however, lie in one, spatially ordered world. Something similar is true about the language of chemistry.

Much of the success of this kind of program depends on how the symbols of the perfect language are designed. Importantly enough, it was precisely on this point that Frege failed. His specific script—the baroque notation of *Begriffsschrift* and *Grundgesetze* (*The Basic Laws of Arithmetic*)—failed to persuade the readers. The new form of symbolism was not adopted by the logicians—also by those who were most sympathetic to his logical conceptions. Russell, for example, preferred Peano's notation, highly appreciating it precisely for its advantages in representing.

Wittgenstein was perhaps the only logician who tried to develop Frege's symbolism further. This is clearly seen in *Tractatus* 6.1203, where he suggested an "intuitive method" for

[3] See, for example, Hindenburg (1803).

[4] Cf. Frege (2000).

recognising an expression as tautology. Unfortunately, it was not adopted by his numerous acolytes. Be this as it may, the later Wittgenstein did not reject it. He used it with confidence, for example, in his lectures of 1935 (see his 1979a, p. 136). This shows the theoretical importance of that idea.

6. Frege's Geometrical Turn in 1923

Main claim of this paper is that exactly this implicit spatial stance of Frege's logic urged him to adopt at the end of his life the geometrical foundation of mathematics, and more precisely, the geometrical nature of mathematical objects. We must bear in mind, however, that Frege realised the geometrical origin of mathematics (and philosophy) only after long deliberations. Only towards the end of his days he saw that the "infinite in the genuine and strictest sense of the word" follows from the geometrical source of knowledge (Frege 1924, p. 273).

Perhaps what made Frege realise the intuitive character of logic was the stress laid on the geometrical character of Frege's own logical symbolising by Wittgenstein during his last visit to Jena in December 1912 and December 1913. Here one must be reminded of Ruben Goodstein's report that Wittgenstein once told him. By his first discussion on logic with Frege, apparently, the visit in December 1912, the latter "wiped the floor with him". Wittgenstein returned to England very disheartened, but a year later he [Wittgenstein] sought another interview with Frege and this time "he wiped the floor with Frege" (Goodstein 1972, p. 272). Peter Geach, who was told only the first part of the anecdote, was convinced (Anscombe and Geach 1961, p. 130) that its second part is "spurious" (Geach 1988, xiv). If we have in mind Geach's pro-Fregean biases, this assessment is not a surprise. Against it it can be pointed out that, as matter of fact, Wittgenstein communicated the story to Goodstein more than 12 years earlier (in 1931–5) than to Geach (1945–7). It is reasonable to expect that Goldstein's story, delivered much earlier than this of Geach, is the correct one.

Peter Hacker believes that the decisive turn in the Frege–Wittgenstein discussion came after the Dec. 1912 visit of Wittgenstein (see Hacker 1996, p. 307). A reason for this is his letter to Russell from 26 December 1912, which reads:

> I had a long discussion with Frege about our *Theory of Symbolism* of which, I think, he roughly understood the general outline. He said he would think the matter over. (1974, p. 17; italics added).

We have good grounds to surmise, however, that Wittgenstein succeeded in intriguing Frege (if not in ultimately persuading him) in the inconsistency of his logical theory by his third visit to him in December 1913. We can guess the content of their discussion from the fact that immediately after they met in Jena again, Wittgenstein formulated the doctrine of "logical showing" in this form: "In 'aRb', 'R' is not a symbol, but [the geometrical fact] that 'R' is between one name and another symbolises" (1979b, p. 109). In fact, this was the main innovation in the "Notes Dictated to G. E. Moore", which cannot be found in "Notes on Logic" dictated in September 1913.

Unfortunately, Frege, who (for example, in his letter from September 16, 1919) hoped "to find something by you [Wittgenstein] what makes complete what I already found. . . to learn to see with your eyes," (Frege 1989, p. 21) was much slower in catching the point of this lesson. Only after 1923 (after he received Wittgenstein's *Tractatus* just printed) did he found the courage to radically change his philosophy of mathematics in accordance with Wittgenstein's remarks. Realising that his concept-script has a spatial character, Frege now adapted the view that the (mathematical) operations it presents are geometrical. Only now was Frege's logic free from its fatal dualism. Its main point became the expressed assumption that both philosophical and mathematical knowledge have geometrical sources. The mathematical deduction is based on intuition.

7. Logical Showing

An advantage of the interpretation presented in this paper is that it suggests a new, perspicuous treatment of the, otherwise, enigmatic theory of "logical showing" adopted by both Frege and Wittgenstein. I the *Tractatus*, Wittgenstein often insisted that the logical properties of symbols "show themselves"; that they are ineffable. More than twenty years ago Peter Geach demonstrated that there are also ineffable points in Frege's logic (1976). About the same time, the same way done by Peter Hacker (1975).

The interpreters of Frege and Wittgenstein find the theory of "logical showing" paradoxical; a "dialectical matter" (Gabriel 1991). They also face insurmountable difficulties by trying to specify why *exactly* is it valid. Thus Geach claims that it comes to light only "when we reflect upon logic" (ibid, p. 56), and Gabriel—by making "categorical differences".

In contrast, according to our interpretation, both Frege and Wittgenstein maintained that what cannot be said in logic, but is *shown* in it, are its main concepts: judgement, function, logical constants. It follows that if we are good enough in fixing the shape (*Gestalt*) of thinking—the *perfect symbolism*—then the whole discipline of logic would become superfluous. The perfect logical symbolism suggested will not be merely "a [sum of] 'winks' with the help of 'pictorial expressions'," as some authors suggest (Gabriel 1991, p. 84), but a thoroughly "perspicuous *representation*" of human thought.[5]

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