The origins of Schwinger's Euclidean Green's functions

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This paper places Julian Schwinger's development of the Euclidean Green's function formalism for quantum field theory in historical context. It traces the techniques employed in the formalism back to Schwinger's work on waveguides during World War II, and his subsequent formulation of the Minkowski space Green's function formalism for quantum field theory in 1951. Particular attention is dedicated to understanding Schwinger's physical motivation for pursuing the Euclidean extension of this formalism in 1958.

Introduction. Schwinger's introduction of the Green's function formalism for characterizing quantum field theories constitutes one of the most influential of his numerous contributions to the physics of elementary particles. He originally developed his Green's function method for waveguide problems during World War II, and then exported it into quantum field theory in a series of papers in 1951. His Minkowski space Green's functions allowed for a more general characterization of the theory than had previously been possible using perturbative techniques. In 1958 Schwinger published *On the Euclidean Structure of Relativistic Field Theory*, in which he introduced a technique for characterizing quantum field theories in Euclidean space rather than in Minkowski space.[1] Both Mehra and Milton[2] and Schweber[3] have provided significant insight into the connection between Schwinger's war work and his introduction of the Minkowski space Green's function formalism. In a retrospective lecture delivered late in his life Schwinger explicitly acknowleged this connection and showed how Green's function methods influenced his work throughout his career (Schwinger, 1993). While Schwinger is also widely credited for producing the first Euclidean formalism for field theory, the historical literature has neglected his motivations for introducing this extension of the formalism. My aim in this paper is to articulate a more complete account of this development.

Schwinger's previous work on Green's functions uniquely prepared him to make this contribution at the technical level.[4] The motivation for establishing the Euclidean extension also contains novel physical reasoning. Aspects

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[1]Schwinger (1958b)

[2]Mehra and Milton (2000)

[3]Schweber (1994)

[4]Late in his life Schwinger acknowledged this when he claimed that "...although it could have appeared any time after 1951, it was 1958 when I published *The Euclidean Structure of Relativistic Field Theory*" (Schwinger, 1993, p. 7). This is a reference to the fact that the 1951 Green's functions papers provided the technical framework for the 1958 Euclidean Green's function paper.

of Schwinger's motivation can be inferred from the publication in which he introduced the formalism. However, Schwinger had considered transformations to Euclidean space in several contexts before using them as the basis for a novel formulation of field theory, and these provide further insight into his reasoning about the role of imaginary time transformations. In this paper I provide evidence that one of these contexts, a talk he delivered in 1957 on dispersion relations to determine the structure of Green's functions, contains the earliest articulation of the central physical insight that motivated the development of a formulation of quantum field theory in Euclidean space.

Schwinger was certainly not the first to transform field theoretic quantities into Euclidean space. Both Dyson and Wick had transformed to Euclidean space during calculations before him. Moreover, during the period in which Schwinger was developing the Green's function formalism, mathematical physicists working in the axiomatic approach to field theory had also become interested in rigorously determining the region of analyticity of Green's functions. In fact, an influential paper of Wightman's demonstrates that the analyticity region of the Green's functions includes the Euclidean region as a special case.[5] The novel aspect of Schwinger's contribution was to provide a complete formalism for field theory which emphasized the importance of the Euclidean region in particular. The characterization of quantum field theories in terms of Euclidean Green's functions, which have come to be known as Schwinger functions, is an important technique for the development of constructive models of the theory. Of the few rigorous models of field theory that have been constructed, many have existence proofs which rely critically on the Schwinger functions. The Euclidean formalism is also the basis for the enormously productive analogy between quantum field theory and statistical mechanics. Despite its eventual importance, the initial reception of the Euclidean formalism was not enthusiastic. Schwinger's characterization of the theory in terms of Minkowski space Green's functions was already widely viewed as abstract and overly formal in comparison to Feynman's graphical techniques. Moreover, the move from an underlying manifold of Minkowski space to a Euclidean space was unintuitive, given that what was being represented in the theory was fields in Minkowski space. For these reasons, Schwinger's motivation for developing the Euclidean formalism calls for explanation.

Understanding the origins of Schwinger's Euclidean Green's functions requires not only an understanding of how he developed the technical apparatus used to capture the formalism, but also his physical motivation for extending it to Euclidean space. The following section discusses the work of Mehra and Milton, and Schweber, on the developments which led to Schwinger's Minkowski space Green's function formalism for quantum field theory. In

[5] Wightman (1956)

particular, it includes discussion of Schwinger's novel application of Green's function methods during World War II, and how he exported the technique into quantum field theory in a series of 1951 papers. In the third section I identify aspects of the motivation that led him to generalize the formalism developed in 1951 to the Euclidean formulation in his work on the CPT theorem and in a talk on dispersion relations delivered in 1957. I present evidence that this was the first context in which Schwinger articulated a critical piece of the motivation for developing the Euclidean formalism. In the fifth section I discuss Schwinger's introduction of the Euclidean formalism in 1958, and connect it to the broader context discussed in the third section. I conclude by reviewing the argument and connecting Schwinger's contribution to two later developments.

Green's functions for waveguides and field theory. The development of quantum field theory stalled during World War II as many theoretical physicists left their research positions to work on projects for the war. A small group of theorists spent the war working on radar technology at the MIT Radiation Laboratory. Hans Bethe produced the initial work on the project and, in 1942, invited several other physicists including Schwinger to collaborate with him. As a graduate student at Columbia, Schwinger's earliest work focused primarily on phenomenological problems in nuclear physics. He had already obtained his first professorship at Purdue when Bethe recruited him to come to the Radiation Laboratory. Before going to MIT, Schwinger was invited to Los Alamos by Oppenheimer. It would have been a natural position for him as much of his early work had been dedicated to nuclear physics.[6] However, Schwinger declined Oppenheimer's offer and spent the years of the war working on waveguide problems for radar equipment at MIT. Schwinger's decision to go to the Radiation Lab instead of Los Alamos had an important effect on the trajectory of his research. His work on waveguide problems influenced his work on field theory after the war in a number of important ways. The applied physics and engineering problems that Schwinger solved at the Radiation Lab were the origin of a calculational technique that he directly imported into quantum field theory after the war. This section briefly explains how Schwinger's work at the Radiation Lab led to the development of his modern use of Green's functions in field theory. Detailed treatments of these developments can be found in work by Mehra and Milton[7] and Schweber[8].[9]

[6] In an interview with Schweber he explained that "I would like to think that I had a gut reaction against [going]. I was probably the only active theoretical nuclear physicist who wasn't there. There must have been some deep instinct to stay" (Schweber, 1994, p. 295).

[7] Mehra and Milton (2000)

[8] Schweber (1994, 2005)

[9] Though it will not be discussed further in this paper, it is worth noting that his work at the Radiation Lab seems to have influenced his perspective on the nature of physical

Before the war the Radiation Lab was staffed almost exclusively with electrical engineers. The directors of the lab looked to theoretical physicists with the onset of the war because a more complete theoretical understanding of the systems being used could save time in the development process. Unlike the systems that the engineers were accustomed to, microwave radio devices had a size on the order of the wavelength of the radiation they produced. Because of this they had to be engineered to transfer energy through metallic waveguides rather than wires. Understanding the properties of such systems required dealing directly with the electromagnetic fields rather than currents and voltages. Solving Maxwell's equations for the fields was complicated by the fact that realistic applications contained many different obstacles to the propagation of the radiation through the waveguides.[10]

Schwinger's task at the Radiation Lab was to develop a framework for understanding the propagation of the radiation through complex geometries involving many obstacles. The method Schwinger developed was based centrally on the use of Green's functions to describe the propagation of the modes in the radiation. Green's theorem provides a connection between volume integrals and surface integrals over volumes. Even before Schwinger's work on waveguides it was commonplace to use Green's functions to solve electrodynamic problems. Schwinger's method was different in that he treated Green's functions as functional operators that define a linear relation between a field inside a region and the boundary conditions for the field on the surface around that region. According to Mehra and Milton, the oldest record of this approach is contained in the 1943 MIT Radiation Laboratory Report 43-44, of which Schwinger was the sole author.[11] He modeled the systems with a modified form of Maxwell's equations.[12] To solve these equations Schwinger defined an electric field Green's function and a magnetic field Green's function, in terms of which the fields could be expressed. This allowed for the calculation of the fields outside of a region for any boundary conditions, by surface integrals over the surface bounding the region. The method could be applied to many different systems by selecting different boundary conditions. Mehra and Milton explain that with this strategy, "...Schwinger rewrote the equations of classical field theory in a form that later served him as a template

theorizing in a way that informed his unique perspective on renormalization theory. For further discussion of this connection see (Kaiser, 2009, pp. 41-42), (Galison, 1997, pp. 820-827), and Schwinger (1983). This perspective lead him to reject operator field theory for his own source theory later in his career. This aspect of Schwinger's thinking has been discussed in Cao (1998), Cao and Schweber (1993), Mehra and Milton (2000), and Mehra et al. (2003).

[10]For further discussion of the general theoretical project at the Radiation Lab see Mehra and Milton (2000, pp. 105-106). For a comprehensive account see Brown (1999).

[11]Mehra and Milton (2000, p. 119)

[12]Levine and Schwinger (1950)

for the future relativistic quantum field theory."[13]

Following the war, Schwinger took a job at Harvard, where his work turned back to nuclear physics and quantum field theory. In 1947 he developed the first covariant formulation of quantum electrodynamics, work for which he would eventually be awarded the Nobel Prize. Schwinger spent much of the next decade growing increasingly focused on the general formalism for quantum field theory. This turn is first evident in *On Gauge Invariance and Vacuum Polarization*, which contains his first use of Green's functions in the field theory context.[14] Later in 1951, Schwinger published a series of two papers, *On the Green's Functions of Quantized Fields*, in which he established the Green's function framework for quantum field theory in detail.[15] His work on applied physics and engineering problems during the war played a critical role in motivating certain aspects of this work on the physics of elementary particles.

The introductory paragraph of the first paper provides a clear statement of Schwinger's intention for the introduction of this new formalism. The characterization of the theory in terms of Green's functions allowed for the particle nature of field excitations to be made manifest. Moreover, he emphasizes that while in the case of interacting fields the calculation of Feynman propagators typically relied on perturbation theory, this was not necessary.[16] He notes that "Although [perturbation theory] may be resorted to for detailed calculations, it is desirable to avoid founding the formal theory of the Green's functions on the restricted basis provided by the assumption of expandability in powers of coupling constants."[17] Schwinger's motivation was to provide a characterization of interacting quantum field systems in which the non-perturbative structure was evident.

In order to accomplish this goal Schwinger used the dynamical principle that he had introduced earlier the same year in *The Theory of Quantized Fields I*.[18] The principle gives a differential characterization of the function that generates the transformation from the the eigenvalues of a complete set of commuting operators, ζ_2'', on one spacelike hypersurface, σ_2, to the eigenvalues, ζ_1', on another hypersurface, σ_1. Using it, Schwinger is able to derive simultaneous differential equations for the two point Green's function. The same procedure can be used to construct the higher Green's functions, and the first paper ends by illustrating this fact.

The second paper appears in the same issue immediately following the

[13]Mehra and Milton (2000, p. 121)

[14]Schwinger (1951a)

[15]Schwinger (1951b,c)

[16]For details of the connection between Schwinger's Green's functions and Feynman's perturbative techniques see Schweber (2005).

[17]Schwinger (1951b, p. 452)

[18]Schwinger (1951d)

first.[19] It begins by explaining an incompleteness in the preceding discussion. In particular, Schwinger notes that throughout the first paper he did not give an explicit construction of the states on σ_1 and σ_2 that are included in the definition of the Green's functions. This information is included in the boundary conditions for the differential equations that define the them. It was thus necessary to find boundary conditions for the Green's functions associated with the vacuum states on σ_1 and σ_2. The second paper is dedicated to accomplishing this task. In producing the modern real space Green's function formalism, Schwinger followed the same pattern that he used in resolving waveguide problems. First he determined a set of simultaneous functional differential equations for Green's functions, and then he turned to the task of determining boundary conditions which would allow him to express their solution in closed form.

Schwinger continues his argument by determining a boundary condition for the Green's functions which ensures that they correspond to the physical propagation of excitations in the field. The Dirac one-particle Green's function is given in terms of vacuum expectation values by:

$$\begin{aligned} G(x,x') &= i\langle\psi(x)\bar{\psi}(x')\rangle, \quad x_0 > x_0', \\ &= -i\langle\bar{\psi}(x')\psi(x)\rangle, \quad x_0 < x_0', \end{aligned} \tag{1}$$

and the variation of $\psi(x)$ in the region around σ_1 is represented by,

$$\psi(x) = e^{iP_0(x_0-X_0)}\psi(X)e^{-iP_0(x_0-X_0)}, \tag{2}$$

for P_0 the energy operator and X a fixed point. Thus, when $x \sim \sigma_1$,

$$G(x,x') = i\langle\psi(X)e^{-i[(P_0-P_0^{vac})(x_0-X_0)]}\bar{\psi}(x')\rangle, \tag{3}$$

where P_0^{vac} is the energy eigenvalue of the vacuum. Since the vacuum is the state of lowest energy, $P_0 - P_0^{vac}$ has no negative eigenvalues, and thus in the region around σ_1, $G(x,x')$ is a function which contains only positive frequencies. He explains that these frequencies correspond to the energies of states of unit positive charge. Similarly, when $x \sim \sigma_2$,

$$G(x,x') = -i\langle\bar{\psi}(x')e^{i[(P_0-P_0^{vac})(x_0-X_0)]}\psi(X)\rangle \tag{4}$$

This, of course, contains only negative frequencies, which correspond to the energies of unit negative charge states. Schwinger then claims that, "We thus encounter Green's functions that obey the temporal analog of the boundary condition characteristic of a source radiating into space" and he notes that

[19] Schwinger (1951c)

both Stückelberg[20] and Feynman[21] had already considered such functions.[22] He argues that the boundary condition characterizing the Green's function for the vacuum states on σ_1 and σ_2 are only dependent on those surfaces in that they need to be in the region of outgoing waves. For this reason he could treat the entire spacetime manifold as the domain of the functions. Upon introducing this simplification, the equations defining the Green's function can be rewritten incorporating the boundary condition corresponding to outgoing waves.

Using this technique Schwinger obtained a system of simultaneous functional differential equations for the vertex function and the polarization functions, as well as the electron and photon one-particle Green's functions.[23] Schwinger concludes the paper with the note that "The details of this theory will be published elsewhere, in a series of articles entitled 'The Theory of Quantized Fields."'[24] He began a third paper for the Green's Function series but it was never completed.[25] The two papers in the Green's function series that did appear established a solid foundation for the use of Green's functions to characterize quantum field theories in Minkowski space.

The path to the Euclidean formalism. In the last section I reviewed how Schwinger used the framework he had established in his work on waveguides to develop the Green's function formalism for quantum field theory. In this section I will consider Schwinger's path to the Euclidean formalism. Dyson and Wick had already used the technique of transforming to imaginary time, developments which Schwinger certainly would have been aware of. In fact, before developing the Euclidean formalism Schwinger had on at least two previous occasions transformed to Euclidean space for calculations. In particular, imaginary time calculations had arisen in his work on the CPT

[20]Stückelberg (1946)
[21]Feynman (1949)
[22]Schwinger (1951c, p. 456)
[23]Additional discussion of the details of this paper can be found in Schweber (2005), and Mehra and Milton (2000).
[24]Schwinger (1951c, p. 459)
[25]It survives as an undated manuscript in the Schwinger archive in the Special Collections at UCLA (Schwinger, b). Much of the paper is dedicated to constructing the n-particle interaction operators by repeated application of a functional differential operator. This manuscript appears to have been used as a first draft for another more polished, yet still incomplete and undated manuscript entitled *Coupled fields*. This paper begins as follows: "This note gives a preliminary account of some aspects of the general theory for a B(ose-Einstein) field coupled with a D(irac-Fermi) field, which is to be published in the series of articles *The theory of quantized fields*" (Schwinger, a, p. 1). Much of the contents of these unpublished manuscripts do appear in this series of papers which Schwinger published between 1951 and 1954 (Schwinger, 1951b,c, 1953a,b, 1954a,b). In these papers Schwinger uses his new Green's function formalism to generate a number of important new results, though they are not central to the development of Euclidean Green's functions and thus will not be discussed here.

theorem and on the use of dispersion relations to determine the structure of Green's functions. This section describes these early applications of transformations to Euclidean space, and presents evidence that a crucial aspect of the motivation for the Euclidean formalism can be traced to Schwinger's work on dispersion relations.

The application of transformations to imaginary time go back at least to Dyson's 1949 work on scattering problems in quantum electrodynamics.[26] In this paper, Dyson analytically continues from real to imaginary energies, which amounts to a transformation from Minkowski to Euclidean space. The motivation for this transformation was simply to avoid the singularities in the propagators that occur on the mass shell. That is, Dyson introduced the transformation as a calculational strategy to improve the behavior of otherwise troublesome expressions. Similar motivations for transforming to Euclidean space can be found in Wick's 1954 paper where he introduces the Wick rotation.[27] The idea of the Wick rotation is to transform the Minkowski space metric,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{5}$$

into the Euclidean metric,

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2, \tag{6}$$

by allowing the Minkowski time coordinate, t, to take on complex values. In this case, the Minkowski metric becomes the Euclidean metric when the time is restricted to the imaginary axis. Problems in Minkowski space can be transformed into problems in Euclidean space by making the substitution $t \to i\tau$. Wick explains that the wave equation obtained by making such a transformation results in an eigenvalue problem which "...presents several advantages in that many of the ordinary mathematical methods become available."[28] Again, Wick's motivation for moving to Euclidean space seems primarily focused on producing mathematical expressions that are more convenient and well behaved for calculations.

The first context in which Schwinger had investigated the transformation of field theoretic quantities to Euclidean space was his paper on the CPT theorem.[29] Mehra and Milton trace this paper back to a priority dispute with Pauli over the first version of the CPT theorem for the case of interacting fields.[30] This may be the first context in which Schwinger introduced imag-

[26]Dyson (1949)
[27]Wick (1954)
[28]Wick (1954, p. 1124)
[29]Schwinger (1958c)
[30]Though the first proofs are typically credited to Lüders and Pauli for Lüders (1954) and Pauli (1955), Schwinger felt he had already provided an equivalent result in Schwinger (1951d). See Mehra and Milton (2000, p. 382) for discussion.

inary time in a calculation. He did so while introducing the transformation properties of a Hermitian field under the Lorentz group. Because the Lorentz group does not possess a finite dimensional unitary matrix representation, the matrices involved in the transformation equation cannot be Hermitian. It is only when Schwinger introduces imaginary time that the matrices become Hermitian. This marked a step toward the Euclidean formalism in that it was one of the first occasions on which Schwinger was lead to consider the connections between representations of the Lorentz and Euclidean groups.[31]

The other context in which Schwinger had considered an imaginary time transformation in field theory was work that he did on dispersion relations for understanding the structure of Green's functions. He presented this work at a Rochester conference in April of 1957, and what appears to be a mimeographed transcript of the talk survives in the Schwinger Papers at the Special Collections of the UCLA library.[32] As Mehra and Milton have noted, this work was the subject of a dispute with Pauli and Källén.[33] They suggest that perhaps because of this negative reaction Schwinger never published a paper on this work. While Schwinger never did publish, this mimeographed transcript contains the earliest articulation of one of central physical insights leading to the Euclidean Green's function paper.

Schwinger began his talk at the Rochester conference by explaining that he was going to demonstrate an approach for finding the structure of Green's functions for quantum field systems. He reiterated the physical significance of these functions, emphasizing that they contain all of the physical information about a system, its energy values, and its scattering properties. He proceeds using his characterization of Green's functions in terms of simultaneous differential equations, and he sets out to determine a boundary condition on those differential equations that properly reflects that the Green's functions correspond to the vacuum expectation value of a time ordered product of field operators. To determine a concrete expression for such a boundary condition he considers a definite time ordering, and reasons as follows. Let x^0 be the time coordinate in the ordering that is greater than all of the others. Then, according to the definition of a time ordered product, a Green's function involving a field operator at that time coordinate must contain the field at x^0 to the left of the field operators at any of the other points in the ordering. He argues that for this to be satisfied, the dependence of the Green's function on the latest of all times must depend only on positive frequencies. He summarizes this reasoning as follows:

[31] Schwinger (1958c, pp. 224-225)

[32] Schwinger (1957a). A lightly edited version of the transcript later appeared in the conference proceedings (Schwinger, 1957b).

[33] Källén, whose talk immediately followed Schwinger's at the conference, did not agree that Schwinger had given the most general form for the dispersion relations. For an account of the dispute see (Mehra and Milton, 2000, pp. 380-381).

"We have, therefore, the boundary condition that the Green's function, in its dependence upon the latest of all times, contains only positive frequencies, and in its dependence upon the earliest of all times, contains only negative frequencies. In effect we have a description in terms of waves which can be considered as moving in the space-time region in such a way that if we have a number of such points in space-time, the waves are moving always out of the region in question. When we are on the boundary of the region in the sense of considering the time coordinate that is later than all the others, the frequencies are positive and the waves move out; if it is the earliest of all times, the frequencies are negative, and the waves move out again. In short, we are dealing with a generalization of the Green's function originally introduced by Feynman which corresponds precisely to the boundary condition of outgoing waves. The waves are in a time sense, running out of the region in question."[34]

Schwinger then turns to replacing the boundary condition corresponding to outgoing waves with a regularity requirement on the Green's functions. In other words, he wanted to find a condition on the regularity of the Green's functions that obtained only in those solutions to the equations characterizing the Green's functions in the case of the boundary condition of outgoing waves. What he found was that the boundary condition of outgoing waves was equivalent to the imposition of the requirement that the Green's function, "...should remain a regular function when you make the time coordinate complex in a specific way, and that you never find an exponential that becomes unlimitedly large."[35] The regularity requirement that has this effect is that when all of the time coordinates are multiplied by the complex number:

$$x^0 \to x^0(1 - i\epsilon), \tag{7}$$

where $\epsilon > 0$, the Green's function remains regular as a function of the time coordinate. Schwinger explains that this requirement "...is fully equivalent to the particular choice of boundary conditions of outgoing waves."[36] He illustrates this with the example of two points,

$$x_1, x_2; \quad x_1^0 > x_2^0; \quad e^{-iP^0(x_1^0 - x_2^0)} \to e^{-iP^0(x_1^0 - x_2^0)(1-i\epsilon)}, \tag{8}$$

and explains that:

You recognize that this substitution, which multiplies equally well

[34]Schwinger (1957a, pp. 2-3)
[35]Schwinger (1957a, pp. 4-5)
[36]Schwinger (1957a, pp. 4-5)

the time interval by $1 - i\epsilon$, forces me, if I am to deal with a quantity that remains bounded no matter how great this positive time difference is, to choose these numbers to be positive only, so that the real part is negative. In short, with this sequence of time differences, the substitution above forces me to pick positive frequencies. On the other hand, it is clear that if the time interval were negative, then I should have to take negative frequencies. So the distinction between positive and negative frequencies, in accordance with the sign of the time difference, is equally well expressed by the requirement of regularity of these Green's functions under the substitution $x^0 \to x^0(1 - i\epsilon)$."[37]

Schwinger then considers how this impacts the invariance properties of Green's functions under Lorentz transformations. He argues that since $G(x, x')$ is a function defined for arbitrary x and x', the only invariant function that can be produced must be produced from the space-time distance between the points, $(x - x')^2$, so that,

$$G(x, x') = G((x - x')^2). \tag{9}$$

He wanted to determine the function of the space-time interval that remains regular when $x^0 \to x^0(1 - i\epsilon)$, and $x^{0'} \to x^{0'}(1 - i\epsilon)$. When these transformations are made the square of the interval is:

$$(x - x')^2 = (X - X')^2 - (x^0 - x^{0'})^2 \to (x - x')^2 + i\epsilon \tag{10}$$

Thus he concludes that: "...the statement is that G is to be a function of the invariant distance which remains regular when the argument is extended into the upper half plane. That's the boundary condition that accompanies the physical choice of outgoing waves."[38] The next section shows how the determination of this regularity requirement marked a critical step toward the Euclidean formulation of field theory that Schwinger produced in the year following the Rochester talk.

Before proceeding to the next section it is interesting to note that Schwinger retrospectively attributed part of the inspiration for this regularity requirement and the Euclidean formalism to his work on waveguides. In an interview with Mehra he explained the physical problem he was trying to solve was to determine which members of the infinite set of boundary conditions to the differential equations characterizing the Green's functions determined physically relevant solutions.[39] He went on to say that:

[37]Schwinger (1957a, pp. 4-5)
[38]Schwinger (1957a, pp. 5-6)
[39]He explained that by the physical solutions he meant solutions that capture "...the fact that the vacuum is the ground state of the system and the lowest state has energy zero, momentum zero, and is a relativistic invariant thing" (Mehra and Milton, 2000, p. 386).

> I recognized somewhere along the line that the condition that the waves move outward could be expressed by an extension into complex space. That is, if you rotated the time axis into a complex space, then the boundary conditions are such that the Green's functions... [are] decreasing exponentials. ... I simply recognized that by moving from real time into complex time in a certain way that would select just the physically acceptable states of the Green's function. In fact it must go back to the electrical engineering days of waveguide stuff because ... in a waveguide if you have a high enough frequency the wave propagates. If the frequency gets too low, it exponentially attenuates. And if you have a general solution, you must always choose the right sign of the square root so it goes down and not up.[40]

Schwinger directly attributes the inspiration for the regularity condition to his work on waveguides. This section has presented evidence that the first place Schwinger explicitly articulated the realization that physical propogation could be captured with the regularity requirement was in his 1957 talk at the Rochester conference.

Euclidean Green's functions. This section provides analysis of the 1958 paper in which Schwinger introduced the Euclidean formalism for quantum field theory.[41] It also connects the argument in the 1958 paper to the insight from the Rochester conference talk discussed in the last section. In addition to the publication where Schwinger introduced the Euclidean formalism, he gave a talk on the same subject at a conference at CERN. A transcript of that talk is published in the conference proceedings.[42] It is identical to the published version with the exception of the addition of an extended opening paragraph. This introduction lays out the novel perspective that leads to the developments in the paper:

> We are all accustomed to the idealization that accompanies the quantum theory of fields in its representation of physical phenomena, i.e. the characteristic quantum mechanical feature of the use of abstract vectors and operators to symbolize physical quantities. But in one respect, at least, the quantum field theory has been conservative. It continues to make use of a classical space-time background, upon which the quantum description is superimposed. I would like to suggest a slight deepening of the abstract basis for the representation of physical phenomena, which is the

[40]Mehra and Milton (2000, p. 386)
[41]Schwinger (1958b)
[42]Schwinger (1958a)

> replacement of the Lorentz or Minkowski space by a Euclidean space.[43]

This departure from the use of a realistic underlying manifold is even more drastic because in quantum mechanical theories the class of admissible states is determined by the underlying spacetime symmetry group. The Lorentz group and the Euclidean group are however, completely different, making Schwinger's proposal quite radical. His insight was that the differences between the groups could be used to select the class of physical states. In particular he notes that:

> . . . while you can certainly take a representation of the Euclidean group and from it derive a representation of the Lorentz group, you will not get all possible representations in this way. What I would like to assert is that while one does not get all of the representations of the Lorentz group, all the representations of physical interest are actually obtained. The essential point to be made is that this possibility of a correspondence between the quantum theory of fields with its underlying Lorentz space, and a mathematical image in a Euclidean space – if one adopts a postulate that one should be able to do this in detail – gives results which go beyond what can be obtained from the present theory of fields.[44]

Schwinger thought that the Euclidean formalism allowed for the selection of the physical solutions to the equations that determine the Green's functions. The imposition of the regularity requirement introduced in the last section was developed to accomplish precisely that purpose.[45]

Another motivation that is suggested in the introduction is that Schwinger viewed the Euclidean formalism as a response to the mathematical problems of field theory. In the paper he explains that:

> . . . by freeing ourselves from the limitations of the Lorentz group, which has produced all the well known difficulties of quantum field theory, one has here a possibility – if this is indeed necessary – of

[43] Schwinger (1958a, p. 134)
[44] Schwinger (1958a, p. 134)
[45] A related but distinct form of this motivation can be seen later in the paper when Schwinger suggests that "...to permit the complete transformation from the Lorentz to the Euclidean metric, every half-integer spin (F.D.) field must carry a charge. Just such a general fermionic charge property, under the name of nucleonic charge or leptonic charge, is either well established experimentally, or has been conjectured on other grounds. The Euclidean formulation may be the proper basis for comprehending this general attribute of F.D. fields" (Schwinger, 1958b, p. 136).

> producing new theories. That is, one has the possibility of constructing new theories in the Euclidean space and then translating them back into the Lorentz system to see what they imply.[46]

This comment suggests a connection to a conference that Schwinger had attended the year before in Lille on 'Les Problèmes Mathématique de la Théorie Quantique des Champs,' which was one of the first to bring together mathematicians and physicists to discuss the problems of quantum field theory.[47] Wightman's talk at the conference reviewed the mathematical problems of field theory and discussed the fact that unlike Euclidean space, whose invariant domains are bounded, Minkowski space has unbounded invariant domains. Schwinger's own contribution to the conference was not entirely successful.[48] Despite this, Schwinger does seem to have been motivated by some of the mathematical concerns about the state of the theory expressed at the conference. In the introduction to his talk at CERN he also noted that "...when one finds formulations that are equivalent, one of these will be distinguished as the one that makes contact with the future theory. All we can do at the moment is to look at all the possible ways of formulating the present theory."[49] Schwinger viewed the Euclidean formalism as empirically equivalent to, but better mathematically behaved, than the Minkowski space theory.

The first task he takes up in the paper is to show that a field theory in Minkowski space can be transformed into a theory in Euclidean space. The Green's functions are the objects of correspondence that allow him to accomplish this. Since the Green's functions contain all of the physical information about the theory, by establishing a connection between the Minkowski space Green's functions and the Euclidean space Green's functions, he is able to capture all physical information about the Euclidean formulation of the theory. He considers a generic Hermitian field, χ, which decomposes into a Bose-Einstein fields, Φ, and a Fermi-Dirac field, Ψ. The Green's functions are defined by the relations:

$$G_+(x_1, \ldots, x_p) = \langle (\chi(x_1) \cdots \chi(x_p))_+ \rangle \epsilon_+(x_1 \cdots x_p), \tag{11}$$

[46]Schwinger (1958b, p. 134)

[47]Michel and Deheuvels (1959)

[48]In an interview with Mehra he recalled that "I gave a lecture on whatever I was thinking about the formulations of field theory at the time. I don't think it was the action principle, but I think I wrote down some symbolic solutions of the field equations involving exponentials of a product of a couple of functional operators and the mathematicians in the audience burst into laughter. That was outrageous, disgraceful. I was a little stunned, so that was not very successful. But the audience was wrong" (Mehra and Milton, 2000, p. 381). Interestingly, a transcript of Schwinger's talk does not appear in the published conference proceedings (Michel and Deheuvels, 1959).

[49]Schwinger (1958a, p. 134)

$$G_-(x_1,\ldots,x_p) = \langle(\chi(x_p)\cdots\chi(x_1))_-\rangle\epsilon_-(x_p\cdots x_1). \tag{12}$$

These definitions are limited in generality because they are valid only for fields whose components are all kinematically independent at a given time.[50] Schwinger closed this gap in generality in a subsequent paper in which he produced a Euclidean formulation of quantum electrodynamics.[51] He proceeds by stipulating that the Green's functions are invariant under proper homogeneous orthochronus Lorentz transformations and observes that their dependence on the space-time coordinates is determined by the energy-momentum vector:

$$\chi(x) = e^{-iPx}\chi e^{iPx}. \tag{13}$$

The vacuum is invariant in the sense that

$$\langle 0| = e^{-iPx} = \langle 0| \quad \text{and} \quad e^{iPx}|0\rangle = |0\rangle. \tag{14}$$

Let $x^{(1)}\cdots x^{(p)}$ be the time ordering of $x_1\cdots x_p$. Then,

$$G_+(x) = \langle\chi e^{iP(x^{(1)}-x^{(2)})}\chi\cdots e^{iP(x^{(p-1)}-x^{(p)})}\chi\rangle\epsilon_+(x_1\cdots x_p), \tag{15}$$

and,

$$G_-(x) = \langle\chi e^{iP(x^{(p)}-x^{(p-1)})}\chi\cdots e^{iP(x^{(2)}-x^{(1)})}\chi\rangle\epsilon_+(x_p\cdots x_1). \tag{16}$$

Since the time dependence of G_+ is generated by the operators,

$$e^{-iP^0(t^{(a)}-t^{a+1})}, \tag{17}$$

it can be seen to contain only positive frequencies. Similarly, G_- is generated by,

$$e^{iP^0(t^{(a)}-t^{a+1})}, \tag{18}$$

so it contains only negative frequencies. Up to this point the existence of the real space Green's functions was simply assumed. He notes that they can be seen to be absolutely convergent expressions when the positive frequency operators in G_+ are replaced with,

$$e^{-iP^0(t^{(a)}-t^{a+1})(1-i\epsilon)}, \tag{19}$$

[50] More specifically, he notes that "In more general situations additional terms are necessary, the function of which is to maintain the non-dependence of the Green's functions on the particular time-like direction employed in the time-ordering, which is otherwise assured by the commutativity or anti-commutativity of fields at points in space-like relation" (Schwinger, 1958a, p. 135).

[51] Schwinger (1959)

and the negative frequency operators in G_- are replaced with,

$$e^{iP^0(t^{(a)}-t^{a+1})(1+i\epsilon)}, \tag{20}$$

where he explains that "...the limit $\epsilon \to +0$ is to be eventually performed."[52] This also ensures the absolute convergence of the expressions for the Green's functions under the more general time substitution:

$$\begin{aligned} G_+ \quad &: \quad t_a \to \tau_a e^{-i\theta} \quad \sin\theta > 0 \\ G_- \quad &: \quad t_a \to \tau_a e^{i\theta} \end{aligned} \tag{21}$$

for $0 < \theta < \pi$. The new time variables τ_a have the same ordering as the t_a. This more general transformation establishes the desired connection to field theory in Euclidean space:

> We adopt a special notation to accompany the particular choice $\theta = 1/2\pi$ which asserts the existence of the functions $G_+(t \to -ix_4)$ and $G_-(t \to +ix_4)$. In this way there emerges a correspondence between the Green's functions in space-time and functions defined on a four-dimensional Euclidean manifold. To the extent that the two Euclidean functions thus obtained are related, there also appears an analytical continuation that connects the two distinct types of space-time Green's functions, $G_\pm$. Conversely, given one of the Euclidean functions, the substitutions $x_4 \to e^{i(\pi/2-\epsilon)}t$ and $x_4 \to e^{-i(\pi/2-\epsilon)}t$ will yield functions having the space-time character of G_+ and G_-, respectively, in the limit as $\epsilon \to +0$.[53]

Schwinger then turns to the task of supplying "... an independent basis for the Euclidean Green's functions, from which has disappeared all reference to the space and time distinctions of the Lorentz metric."[54] His strategy is to take the system of differential equations characterizing a set of Green's functions and then to convert them to the Euclidean metric. He considers the theory defined by the Lagrangian:

$$\mathcal{L} = \frac{1}{4}\left[\chi A^\mu \partial_\mu \chi - \partial_\mu \chi A^\mu \chi\right] + \frac{1}{2}\chi B \chi - \mathfrak{h}_1, \tag{22}$$

which yields the field equations,

$$A^\mu \partial_\mu \chi + B\chi = \frac{\partial_l \mathfrak{h}_1}{\partial \chi}. \tag{23}$$

[52]Schwinger (1958a, p. 135)
[53]Schwinger (1958a, pp. 135-136)
[54]Schwinger (1958a, p. 136)

The commutation relations on a spacelike surface are given by,

$$\left[A^0\chi(x), \chi(x')\right]_\pm = i\delta^0(x - x'). \tag{24}$$

By combining the field equations and the commutation relations he obtains the differential equations for the Green's functions:

$$\begin{aligned}(A^\mu\partial_\mu + B)_1 G_+(x_1 \cdots x_p) + \ldots &= i\delta(x_1 - x_2)G_+(x_3 \cdots x_p) \\ &\quad \pm i\delta(x_1 - x_3)G_+(x_2 \cdots x_p) + \ldots\end{aligned} \tag{25}$$

The terms missing from the left hand side of the equation are those that represent the interaction effects in the field equations. The differential equations for G_- can be constructed in the same way.

Schwinger concludes the paper by indicating a way to replace the characterization of the Euclidean Green's functions in terms of differential equations with a more explicit construction involving a generating function. He notes that "A large variety of equivalent forms can now be devised for the Green's functions, based primarily upon the well-established transformation and representation theory for canonical variables of the first and second kind."[55] However, Schwinger explicitly defers any further application of this technique to particular systems to a later paper.[56] This passage also contains a footnote which promises a more extended discussion of the relevant representation theory in "Quantum Theory of Fields, in: Handbuch der Physik; volume V/2, Berlin, Springer (to be published)."[57] This article never appeared.[58] However,

[55]Schwinger (1958a, p. 134)

[56]In the later paper Schwinger uses the formalism that he had developed to cast quantum electrodynamics in Euclidean form. See Schwinger (1959).

[57](Schwinger, 1958a, p. 134)

[58]Schwinger was invited to contribute an article to the *Handbuch der Physik*, by the editor of the project, Flugge, in February of 1955 (Flugge, 1955b). He was asked to contribute an article on "Quantum Theory of Wave Fields" to volume 5 of the Handbuch. This volume was also scheduled to include a contribution from Pauli, "Prinzipien der Quantenmechanik," as well as from Källén on "Quantenelektrodynamik" (Flugge, 1955a). Schwinger was apparently slow to produce his manuscript, and in March of 1957, Flugge wrote to him to explain that Pauli and Källén's contributions were already prepared for print. In November of the same year Flugge wrote again and noted that Pauli and Källén "... are angry with me that I held up publication of their articles by waiting for yours. I again had to face a rather unpleasant pressure from these two authors who firmly demanded to have volume 5 published right now without your contribution and who also made an indication (to put it mildly) that I never would get a manuscript from you at all" (Flugge, 1957). He goes on to suggest that he publish Pauli and Källén's contributions as volume 5 part I and that Schwinger's contribution appear later as volume 5 part II. Given the citation that Schwinger gave, it seems that he must have agreed to this plan, however, while volume 5 part I did eventually appear with Pauli and Källén's contributions, volume 5 part II never appeared. I have been unable to identify evidence

with this, Schwinger had already introduced a complete Euclidean extension of the formalism for quantum field theory.[59]

Having produced the Euclidean formalism, Schwinger noted that "...the utility of introducing a Euclidean metric has frequently been noticed in connection with various specific problems, but an appreciation of the complete generality of the procedure has been lacking."[60] Recall that Dyson and Wick had already considered similar transformations for other reasons, and that Schwinger had also already considered what amount to Euclidean transformations on at least two occasions. This remark emphasizes that Schwinger viewed his central contribution in *On the Euclidean Structure of Relativistic Field Theory* as providing a complete Euclidean formalism. Following the development of the Euclidean formalism Schwinger did make one immediate application. In particular, he showed that quantum electrodynamics could be cast in Euclidean form.[61] This was his final contribution on the topic.[62]

Conclusion. Two further developments were critical for establishing the modern status of the Euclidean formalism. One is due to Symanzik, who produced a purely Euclidean formulation of quantum field theory.[63] This work secured the firm connection between Euclidean field theory and classical statistical mechanics. The other development concerns the completion of the connection between field theory in Minkowski space and in Euclidean space. More specifically, it remained to be shown when an arbitrary Euclidean space theory determined a physical Minkowski space theory. Schwinger anticipated this problem,[64] which was approached and solved by Osterwalder and Schrader in the context of Wightman's axiomatic formalism nearly ten years after Schwinger introduced the Euclidean extension of his own formalism. They found necessary and sufficient conditions for a field theory defined

that Schwinger ever started preparing as article specifically for the Handbuch. However, Schwinger's 1955 lectures at the Les Houches summer school contain an extended discussion of canonical transformations (Schwinger, 1955). They do not contain the Euclidean connection but this may be what Schwinger had in mind for the Handbuch article.

[59]During the question and answer period, Yamaguchi noted that Nakano was working on a very similar connection between Euclidean field theory and the standard theory. Nakano's paper appeared the following year (Nakano, 1959).

[60]Schwinger (1958a, p. 134)

[61]Schwinger (1959)

[62]In fact, not long after this work Schwinger grew discontented with the operator field formalism in general and produced his own source theory as a candidate replacement. See Cao (1998), Cao and Schweber (1993), and Mehra and Milton (2000) for discussion.

[63]Symanzik (1966)

[64]The discussion period of Schwinger's talk at CERN ended with him noting that, "The question of to what extent you can go backwards, remains unanswered, i.e. if one begins with an arbitrary Euclidean theory and one asks: when do you get a sensible Lorentz theory? This I do not know. The development has been in one direction only; the possibility of future progress comes from the examination of the reverse direction, and that is completely open" (Schwinger, 1958c, p. 140).

in terms of Euclidean Green's functions to have an analytic continuation to a quantum field theory in Minkowski space, defined in terms of Wightman distributions.[65] This development completed the connection that Schwinger had begun to develop, and established a permanent place for Schwinger's Euclidean Green's functions in the constructive field theory literature.

The transformation to Euclidean space now seems quite natural because it has assumed such a central place in modern approaches to quantum field theory. It is perhaps for this reason that the historical literature has not focused on the use of such transformations for defining quantum field theories. However, at the time Schwinger introduced the Euclidean formalism, the move to an underlying Euclidean manifold was a radical one. In this paper I have provided an account of the origins of Schwinger's Euclidean Green's functions. While his development of Minkowski space Green's functions was an important step in this direction, the Euclidean extension did not follow inevitably from this formalism. Instead, the regularity requirement capturing the boundary condition of outgoing waves was an essential aspect of the motivation for considering Euclidean space. I have provided evidence that the first explicit articulation of the regularity condition occurred in Schwinger's 1957 Rochester conference talk. The development of Euclidean space quantum field theory is better understood when viewed in this context.

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[65] Osterwalder and Schrader (1973, 1975)

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