A solution to Curry and Hindley's problem on combinatory strong reduction

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WORKSHOP ON RECENT TRENDS IN PROOF THEORY (University of Bern, July 9-11, 2008)

The problem

- The problem
- 2 Analytic proof systems for combinatory logic and λ -calculus

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- Solution to the problem

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- $oldsymbol{4}$ Proving transitivity elimination for $G_{ext}[\mathbb{X}]$ systems

- The problem
 - Combinatory strong reduction
 - Curry's indirect confluence proof
 - Statement of the problem
- 2 Analytic proof systems for combinatory logic and λ -calculus
- Solution to the problem
- 4 Proving transitivity elimination for $G_{ext}[X]$ systems

Combinatory strong reduction

Primitive combinators: I, K, S

$$\frac{t \succ t}{rt \succ rs} \mu \qquad \frac{t \succ s}{tr \succ sr} \nu \qquad \frac{t \succ r}{t \succ s} \tau$$

$$\frac{t \succ s}{tr \succ rs} \psi \qquad \frac{t \succ r}{t \succ s} \tau$$

$$\frac{t \succ s}{\lambda^* x. t \succ \lambda^* x. s} \xi$$

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Abstraction is defined according to the strong algorithm.

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Remark

The combinator I is taken as primitive just to avoid having a *trivial* example of a term in strong normal form which is not strongly irreducible.

Indeed, notice that SK > KI. So, by defining I := SKK, we would have:

$$I \equiv \mathsf{SKK} \succ \mathsf{KIK} \succ \mathsf{K}(\mathsf{KIK})\mathsf{K} \succ \dots$$





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We shall be concerned with point 1, or better with the proof of $CR(\succ)$.



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- (P2) for $t, s \in \Lambda$: $t \rightarrow_{\beta \eta} s \Rightarrow t_H \succ s_H$,
- (P3) for $t, s \in \mathbf{T}_{\{\mathsf{I.K.S}\}}$: $t =_{c\beta\eta} s \Rightarrow t_{\lambda} =_{\beta\eta} s_{\lambda}$.



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These satisfy:

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: $t =_{c\beta\eta} s \Rightarrow t_{\lambda} =_{\beta\eta} s_{\lambda}$.

Then:

$$\begin{array}{cccc} t =_{c\beta\eta} s & \Rightarrow & t_{\lambda} =_{\beta\eta} s_{\lambda} & \text{by (P3)} \\ & \Rightarrow & \exists r \in \Lambda : \ t_{\lambda} \twoheadrightarrow_{\beta\eta} r \ _{\beta\eta} \twoheadleftarrow s_{\lambda} & \text{by CR}(\twoheadrightarrow_{\beta\eta}) \\ & \Rightarrow & t \succ r_{H} \prec s & \text{by (P2) and (P1)} \end{array}$$

H.B. Curry and R. Feys, Combinatory Logic, Vol. I, 1958

List of "Unsolved problems" in § 6 F.5



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"c. Is it possible to prove the Church-Rosser property directly for strong reduction, without having recourse to transformations between that theory and the theory of λ-conversion? ..."

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Remark

A solution was advanced by K. Loewen in 1968. His proof, however, seems to contain an error — as pointed out in Hindley's MR review (1970).

Hindley's statement of the problem

Problem #1 — TLCA List of Open Problems, http://tlca.di.unito.it/opltlca/

Submitted by Roger Hindley Date: Known since 1958!

Statement. Is there a direct proof of the confluence of $\beta\eta$ -strong reduction?

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The $\beta\eta$ -strong reduction is the combinatory analogue of $\beta\eta$ -reduction in λ -calculus. It is confluent. Its only known confluence-proof is very easy, [Curry and Feys, 1958, 6F, p. 221 Theorem 3], but it depends on the having already proved the confluence of $\lambda\beta\eta$ -reduction. Thus the theory of combinators is not self-contained at present. Is there a confluence proof independent of λ -calculus?

- The problem
- 2 Analytic proof systems for combinatory logic and λ -calculus
 - Synthetic vs analytic equational proof systems
 - G-systems
 - Main results
- 3 Solution to the problem
- igl(4) Proving transitivity elimination for $G_{ext}[\mathbb{X}]$ systems

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(which cannot be dispensed with, except that in trivial cases) **has an inherently synthetic character** in combining derivations, like *modus* **ponens** in Hilbert-style proof systems

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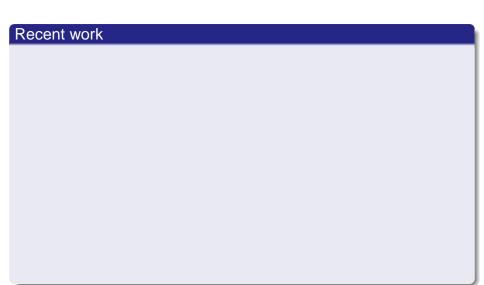
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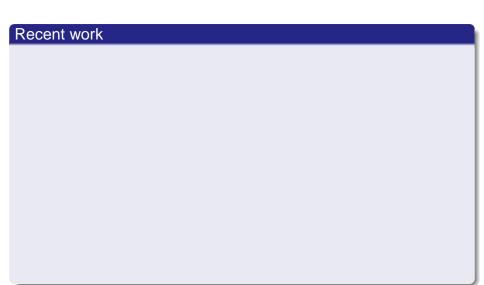
- Naive proof-theoretic arguments are usually impossible (e.g.: syntactic consistency proofs by induction on the length of derivations)
- No kind of "subterm property"
- In general, derivations lack any significant mathematical structure
- As a consequence, 'synthetic' equational calculi do not lend themselves directly to proof-theoretical analysis

Question

Are there significant cases in which it is both *possible* and *useful* to turn a 'synthetic' equational proof system into an **equivalent** 'analytic' proof system,

where the transitivity rule is provably redundant?





Combinatory logic: CL (& generalizations)

P. M., Analytic combinatory calculi and the elimination of transitivity, Arch. Math. Logic 43 (2004), 159-191.

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Extensional Combinatory logic: CL_{ext} (& generalizations)

P. M., A solution to Curry and Hindley's problem on combinatory strong reduction, submitted.

G-systems

Overwiew

synthetic proof-systems

synthetic proof-systems

synthetic proof-systems



equivalent (candidate) analytic proof-systems ("G-systems")

synthetic proof-systems

IL

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(effective) transitivity elimination for G-systems

synthetic proof-systems

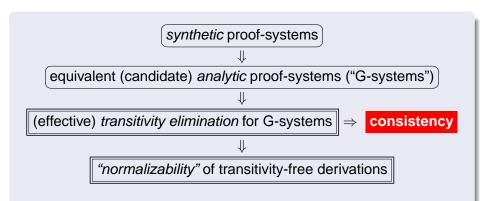
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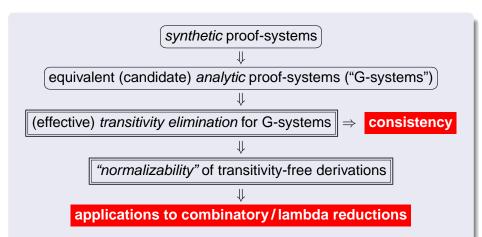
(equivalent (candidate) analytic proof-systems ("G-systems")

↓

(effective) transitivity elimination for G-systems

⇒ consistency





- combinatory axiom schemas $I \beta$ -conversion schema
 - ▶ turned into pairs of suitable introduction rules ▶ □ ▶ □





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- restricted to atomic terms

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- monotony rule(s)
 - ▶ taken in the parallel version

$$\frac{t = s \quad p = q}{tp = sq}_{App}$$

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$$\frac{t = s \quad p = q}{tp = sq}_{App}$$

- extensionality rule (if any)
 - taken in the version

$$\frac{tx = sx}{t = s} E_{xt} \quad \{x \notin V(ts)\}$$

G-systems for full combinatory logic: $\mathbf{G}[\mathbb{C}]$ / $\mathbf{G}_{\mathrm{ext}}[\mathbb{C}]$

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\mathbf{G}[\mathbb{C}] (corresponding to \mathbf{CL})
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left and right combinatory introduction rules for I, K, S



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left and right combinatory introduction rules for I, K, S



$G_{ext}[\mathbb{C}]$ (corresponding to CL_{ext})

+ the extensionality rule [Ext]



and for arbitrary combinatory systems $\mathbb{X} \colon G[\mathbb{X}] \: / \: G_{ext}[\mathbb{X}]$

A **combinatory system** \mathbb{X} is a map, defined on a non-empty set $\mathbf{X} = dom(\mathbb{X})$ of primitive combinators $(\mathsf{F},\mathsf{G}\dots)$, which associates to each $\mathsf{F} \in \mathbf{X}$ a pair $\langle k_\mathsf{F}, d_\mathsf{F} \rangle$ s.t.:

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- k_{F} , the *index* of F under \mathbb{X} , is a non negative integer;
- d_F , the definition of F under \mathbb{X} , is a term with $V(d_F) \subseteq \{v_1, \dots, v_{k_F}\}$.

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Intuitively, for each primitive combinator $F \in X$:

$$X : F \longmapsto Ft_1 \dots t_{k_F} = d_F[v_1/t_1, \dots, v_{k_F}/t_{k_F}]$$
 (AX F)

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$\left[\mathbf{G}[\mathbb{X}] \, / \, \mathbf{G}_{\mathsf{ext}}[\mathbb{X}] \right]$

are defined exactly as $G[\mathbb{C}]/G_{ext}[\mathbb{C}]$, except that the introduction rules for I, K, S are replaced by the rules $[F_I]_{\mathbb{X}}$, $[F_r]_{\mathbb{X}}$, for each $F \in \mathbb{X}$

 $G[\beta]$ (corresponding to $\lambda\beta$)



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"structural rules":

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left and right β-introduction rules



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left and right β-introduction rules

$$G_{\text{ext}}[\beta]$$
 (corresponding to $\lambda\beta\eta$)

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left and right β-introduction rules

(corresponding to $\lambda\beta\eta$) $G_{\rm ext}[\beta]$

+ the extensionality rule [Ext]

Lemma [Equivalence]

G-systems are equivalent to the corresponding synthetic systems

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Main Theorem [*⊤*-elimination]

G-systems admit (effective) transitivity elimination



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Main Theorem [τ -elimination]

G-systems admit (effective) transitivity elimination

Proof (in order of increasing complexity):

• $\mathbf{G}[\mathbb{X}]$ (\mathbb{X} arbitrary)

[PM 04]



Lemma [Equivalence]

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Main Theorem [τ -elimination]

G-systems admit (effective) transitivity elimination

Proof (in order of increasing complexity):

• G[X] (X arbitrary) [PM 04]

• $G_{ext}[X]$ (X linear) [PM 04]

Lemma [Equivalence]

G-systems are equivalent to the corresponding synthetic systems

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G-systems admit (effective) transitivity elimination

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- The problem
- 2 Analytic proof systems for combinatory logic and λ -calculus
- Solution to the problem
 - Extraction Lemma
 - A direct confluence proof
- 4 Proving transitivity elimination for $G_{ext}[X]$ systems

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- $\mathcal{D} \equiv t = t$ [t atomic]
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- $\mathcal{D} \equiv R(\mathcal{D}_1)$ [R a combinatory rule]
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As to the last case, indeed:

$$tx \succ r \prec sx \ [x \notin V(ts)] \Rightarrow_{\mathsf{rule}\ \xi} t \equiv \lambda^* x.tx \succ \lambda^* x.r \prec \lambda^* x.sx \equiv s$$

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This confluence proof for \succ is independent of λ -calculus!



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- $iggle{4}$ Proving transitivity elimination for $\mathbf{G}_{\mathsf{ext}}[\mathbb{X}]$ systems
 - Preliminaries
 - The strategy
 - Steps 1-4



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This strategy doesn't work when the **extensionality rule** is present, coupled with **non linear** combinators.

We show that the following **generalized transitivity rule**

$$\frac{t = \mathbf{s} \quad \Phi[\mathbf{s}] = r}{\Phi[\mathbf{t}] = r} \tau^*$$

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The proof consists of four main steps (in this order):

generalized F-inversion



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- generalized F-inversion
- left τ-elimination
- generalized F-introduction
- elimination of a topmost occurrence of $[\tau^*]$



Step 1: generalized F-inversion Lemma

For any $F \in X$, with $k = k_F$, and any context Φ :

Every τ -free derivation

$$\mathcal{D} \vdash^{-} \Phi \llbracket \mathsf{F} t_1 \dots t_k p_1 \dots p_n \rrbracket = s$$

can effectively be transformed into a τ -free derivation

$$\mathcal{D}^* \vdash^- \Phi \llbracket d_{\mathsf{F}}[t_1, \dots, t_k] p_1 \dots p_n \rrbracket = s$$

which, moreover, is a *right* derivation provided \mathcal{D} is a *right* derivation

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This follows from the following:







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Proof.

By main induction on $s(\mathcal{D})$ and secondary induction on ||t||.



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Proof.

Main induction on $s(\mathcal{D}_2)$, secondary induction on $s(\mathcal{D}_1)$, ternary induction on ||s||, using F-inversion.



Step 3: generalized F-introduction Lemma

For any $F \in X$, with $k = k_F$, and any context Φ :

The following generalized combinatory introduction rules are τ -free admissible:

$$\frac{\Phi\llbracket d_{\mathsf{F}}[t_1,\ldots,t_k]p_1\ldots p_n\rrbracket = s}{\Phi\llbracket \mathsf{F}t_1\ldots t_k p_1\ldots p_n\rrbracket = s} \, {}_{[\mathsf{F}_l^+]} \qquad \frac{s = \Phi\llbracket d_{\mathsf{F}}[t_1,\ldots,t_k]p_1\ldots p_n\rrbracket}{s = \Phi\llbracket \mathsf{F}t_1\ldots t_k p_1\ldots p_n\rrbracket} \, {}_{[\mathsf{F}_r^+]}$$

Moreover, $[F_l^+]$ and $[F_r^+]$ preserve *left-handedness*, resp. *right-handedness*.



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Moreover, $[F_l^+]$ and $[F_r^+]$ preserve *left-handedness*, resp. *right-handedness*.

Proof.

By left τ -elimination.



$$\frac{\Phi[\![\mathsf{F}t_1\dots t_k\overline{p}]\!] = \Phi[\![d_\mathsf{F}[t_1,\dots,t_k]\overline{p}]\!]}{\Phi[\![\mathsf{F}t_1\dots t_k\overline{p}]\!] = s} \Phi[\![d_\mathsf{F}[t_1,\dots,t_k]\overline{p}]\!] = s}{\Phi[\![\mathsf{F}t_1\dots t_k\overline{p}]\!] = s} \text{ Left elim.}$$

*: structural rules + applications of $[F_l]$

For any context Φ:

To each pair of τ -free derivations

$$\mathcal{D}_1 \vdash^- t = \mathbf{s}$$
 and $\mathcal{D}_2 \vdash^- \Phi[\mathbf{s}] = r$

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The proof runs by ω^3 -induction

• main: $s(\mathcal{D}_1)$

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- main: $s(\mathcal{D}_1)$
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- ternary: $h(\mathcal{D}_2)$

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The proof runs by ω^3 -induction

- main: $s(\mathcal{D}_1)$
- secondary: ||s||
- ternary: h(D₂)

taking main cases according to the last inference R of \mathcal{D}_1 .

Case
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M.I.H. + generalized F-inversion



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$$\frac{t = s'}{t = s} \operatorname{F}_{r} \qquad \vdots \\
\frac{\Phi[\![s]\!] = r}{\Phi[\![t]\!] = r} \tau^{*}$$

Case
$$R = [F_r]$$

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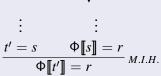
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$$R = [F_l]$$

M.I.H. + generalized F-introduction

$$\frac{\frac{t'=s}{t=s} \mathsf{F}_l \qquad \vdots}{\Phi[\![s]\!]=r} \\
\frac{\Phi[\![t]\!]=r}{\tau^*}$$

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$$\frac{\underline{t'=s}}{t=s} F_{l} \qquad \vdots \\
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 $\Phi[\![t]\!] = r$

Case
$$R = [App]$$

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Case
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Case
$$R = [App]$$

$$\frac{t_{1} = s_{1} \quad t_{2} = s_{2}}{t_{1}t_{2} = s_{1}s_{2}} \xrightarrow{App} \Phi[s_{1}s_{2}] = r} \tau^{*}$$



S.I.H. + context shifts

$$\frac{\underbrace{t_1 = s_1 \quad t_2 = s_2}_{t_1 t_2 = s_1 s_2} App}_{\Phi[\![t_1 t_2]\!] = r} \oplus \underbrace{\Phi[\![s_1 s_2]\!] = r}_{\tau^*}$$

 $\begin{array}{ccc}
\vdots & & \vdots & \vdots \\
t_1 = s_1 & \Phi[s_1 s_2] = r \\
 & \Phi[t_1 s_2] = r
\end{array}$ S.I.H.

 $\frac{\Phi\llbracket t_1 s_2 \rrbracket = r}{\Phi\llbracket t_1 t_2 \rrbracket = r} S.I.H.$

$$\frac{t_{1} = s_{1} \quad t_{2} = s_{2}}{t_{1}t_{2} = s_{1}s_{2}} \xrightarrow{App} \Phi[s_{1}s_{2}] = r} \tau^{*}$$

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\underline{t_1 = s_1} & \Psi \llbracket s_1 \rrbracket = r \\
\hline
\Phi \llbracket t_1 s_2 \rrbracket = r & S.I.H. \\
\hline
\Phi \llbracket t_1 t_2 \rrbracket = r & S.I.H.
\end{array}$$

S.I.H. + context shifts

$$\frac{t_1 = s_1 \quad t_2 = s_2}{t_1 t_2 = s_1 s_2} \underset{App}{App} \quad \vdots \\
\Phi[\![s_1 s_2]\!] = r \\
\tau^*$$

 $\vdots \qquad \vdots \qquad \vdots \\
\underline{t_1 = s_1 \quad \Phi[s_1 s_2] = r} \\
\underline{\Phi[s_2] = r} \quad S.I.H.$ $\Phi[t_1 t_2] = r \quad S.I.H.$

Case
$$R = [Ext]$$

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We have now to look both at

• the last inference R' of \mathcal{D}_2



Case
$$R = [Ext]$$

We have now to look both at

- the last inference R' of \mathcal{D}_2
- the form of the context Φ



$$\begin{array}{ccc}
\vdots \\
\underline{tx = sx}_{Ext} & \vdots \\
\underline{t = s}_{Ext} & \underline{s = r}_{\tau^*}
\end{array}$$

$$\begin{array}{ccc}
\vdots \\
\underline{tx = sx}_{Ext} & \vdots \\
\underline{t = s}_{Ext} & \underline{s = r}_{\tau^*}
\end{array}$$

If Φ is distinct from * we look at R'

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$$R' = [App] / [F_r] / [Ext]$$

Easy, by the ternary I.H.

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$$R' = [App] / [F_r] / [Ext]$$

Easy, by the ternary I.H.

$$R' = [\mathsf{F}_l]$$

More delicate: a "cross-cut" is required.

We use the ternary I.H. followed by an application of the M.I.H.

$$Stsr = tr(sr)$$
 [AXS]

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$$\frac{tr(sr)p_1 \dots p_n = q}{\mathsf{S}tsrp_1 \dots p_n = q} [\mathsf{S}_l]$$

$$\frac{q = tr(sr)p_1 \dots p_n}{q = \mathsf{S}tsrp_1 \dots p_n} [\mathsf{S}_r]$$

where $n \ge 0$, i.e.: the "side terms" p_1, \ldots, p_n may be missing

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where $n \ge 0$, i.e.: the "side terms" p_1, \ldots, p_n may be missing

Combinatory introduction rules for other primitive combinators F:

 $[F_l]$ and $[F_r]$ are defined similarly



 β -introduction rules:

$$(\lambda x.t)r = t[x/r]$$
 [β -conv]

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$$\frac{t[x/r]p_1 \dots p_n = q}{(\lambda x.t)rp_1 \dots p_n = q} {}_{[\beta_l]} \qquad \frac{q = t[x/r]p_1 \dots p_n}{q = (\lambda x.t)rp_1 \dots p_n} {}_{[\beta_r]}$$

where $n \ge 0$, i.e.: the "side terms" p_1, \ldots, p_n may be missing





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where $n \ge 0$, i.e.: the "side terms" p_1, \ldots, p_n may be missing



$$\frac{tp_1 \dots p_n = s}{\mathsf{I}tp_1 \dots p_n = s} [\mathsf{I}_l] \qquad \qquad \frac{s = tp_1 \dots p_n}{s = \mathsf{I}tp_1 \dots p_n} [\mathsf{I}_r] \qquad (n \ge 0)$$

$$\frac{tp_1 \dots p_n = s}{\mathsf{K}trp_1 \dots p_n = s} \left[\mathsf{K}_l \right] \qquad \frac{s = tp_1 \dots p_n}{s = \mathsf{K}trp_1 \dots p_n} \left[\mathsf{K}_r \right] \qquad (n \ge 0)$$

$$\frac{tq(rq)p_1 \dots p_n = s}{\mathsf{S}trqp_1 \dots p_n = s} [\mathsf{S}_l] \qquad \frac{s = tq(rq)p_1 \dots p_n}{s = \mathsf{S}trqp_1 \dots p_n} [\mathsf{S}_r] \qquad (n \ge 0)$$





$$\mathsf{F} t_1 \dots t_{k_\mathsf{F}} = d_\mathsf{F} [t_1, \dots, t_{k_\mathsf{F}}] \quad (\mathsf{AX}\,\mathsf{F})_\mathbb{X}$$

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$$\frac{d_{\mathsf{F}}[t_1,\ldots,t_{k_{\mathsf{F}}}]p_1\ldots p_n=s}{\mathsf{F}t_1\ldots t_{k_{\mathsf{F}}}p_1\ldots p_n=s}\,_{\mathsf{F}t_1} \qquad \frac{s=d_{\mathsf{F}}[t_1,\ldots,t_{k_{\mathsf{F}}}]p_1\ldots p_n}{s=\mathsf{F}t_1\ldots t_{k_{\mathsf{F}}}p_1\ldots p_n}\,_{\mathsf{F}r]_{\mathbb{X}}}$$



