Coherent choice functions without Archimedeanity

Enrique Miranda and Arthur Van Camp

Abstract We study whether it is possible to generalise Seidenfeld et al.'s representation result for coherent choice functions in terms of sets of probability/utility pairs when we let go of Archimedeanity. We show that the convexity property is necessary but not sufficient for a choice function to be an infimum of a class of lexicographic ones. For the special case of two-dimensional option spaces, we determine the necessary and sufficient conditions by weakening the Archimedean axiom.

1 Introduction

In a problem of decision making under uncertainty, a subject's preferences between a set of alternatives can naturally be modelled by means of a so-called *choice function*, that determines those options that are considered admissible to the subject. The rationality of the subject's preferences was studied by Arrow [2] and Uzawa [19], and later axiomatised by Rubin [12]. A feature of this axiomatisation is that a rational choice function always returns a single admissible option, or multiple admissible options that are indifferent to each other.

Nevertheless, when faced with a set of options a choice function may give more than one optimal alternative, and this does not necessarily imply that all these chosen options are indifferent to our subject: they may instead be considered *incomparable*. Coherent choice functions were extended to allow for incomparability between the options by Seidenfeld et al. [15]. Under their axiomatisation, they proved a representation theorem in terms of probability/utility pairs: a choice function C is coherent if

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and only if there is an arbitrary non-empty set S of probability/utility pairs such that $f \in C(A)$ whenever f maximises p-expected u-utility over A for some (p,u) in S.

In [23], we extended the above-mentioned axiomatisation by Seidenfeld et al. [15] to choice functions defined on (abstract) options that form a vector space, rather than horse lotteries, and also let go of two of their axioms: (i) the Archimedean one, because it prevents choice functions from modelling the typically non-Archimedean preferences captured by coherent sets of desirable gambles; and (ii) the convexity axiom, because it turns out to be hard to reconcile with Walley–Sen maximality as a decision rule. By doing this, we obtained a theory of coherent choice functions that includes coherent sets of desirable gambles, and therefore most other imprecise probability models, as particular cases; and that is at the same time more general, because they are not necessarily completely determined by pairwise comparisons between the options.

In spite of these advantages, our coherent choice functions also have the drawback of not leading to a strong belief structure [6]. Such a representation is nevertheless interesting, because it allows choice functions to be constructed using basic building blocks. In [23], we did discuss a few interesting examples of special 'representable' choice functions, such as the ones from a coherent set of desirable gambles via maximality, or those determined by a set of probability measures via E-admissibility.

In the present paper, we add more detail to our previous findings by investigating in more detail the implications of the convexity axiom, while still letting go of Archimedeanity. We show that, if a representation theorem under convexity were indeed possible, it would necessarily involve lexicographic probabilities, as studied by Blume et al. [3], Fishburn [8] and Seidenfeld et al. [14], but that unfortunately such representation is not generally guaranteed. In order to establish this, we derive some interesting properties of coherent choice functions in terms of their so-called rejection sets. Our argument leads us to introduce an additional axiom, which we call *weak Archimedeanity*, which guarantees representation, at least in the case of two-dimensional option spaces.

Our paper is organised as follows. In Section 2, we recall the basics of coherent choice functions on vector spaces of options as introduced in our earlier work [23], and establish a number of properties that will be useful later on. In Section 3, we recall our definition of lexicographic choice functions from [22] and the properties of their associated binary preferences. Then we bring up the representation question of whether a convex coherent choice function is always the infimum of a family of lexicographic choice functions. Our motivation for focusing on them is that (i) they have been connected to a representation of preferences in the context of choices over horse lotteries in [14]; and (ii) that, as we shall show, the subset of *maximal* choice functions, that play a similar role in the case of binary preferences, are not sufficient in the case of choice functions. In order to present our results, we study in quite some detail the particular case of coherent choice functions on a binary possibility space, and show in Section 4 that these can be characterised by means of a so-called *rejection set*. Based on our results, we are able to answer the representation question by showing (i) that convexity is necessary but not sufficient for a coherent choice function to be the infimum of lexicographic choice functions; and (ii) that a necessary and sufficient condition can be obtained, at least in the case of two-dimensional option spaces, by adding what we call *weak Archimedeanity*. The paper concludes with some additional discussion in Section 6. In order to facilitate the reading, we have gathered all the proofs in an Appendix.

2 Coherent choice functions on vector spaces

Consider a real vector space $\mathcal{V}$ provided with the vector addition $+$ and scalar multiplication. We denote by 0 the additive identity. For any subsets A_1 and A_2 of $\mathcal{V}$ and any λ in $\mathbb{R}$, we let $\lambda A_1 := \{\lambda u : u \in A_1\}$ and $A_1+A_2 := \{u+v : u \in A_1 \text{ and } v \in A_2\}$.

Given any subset A of an option space $\mathcal{V}$, we define its *positive hull* $\operatorname{posi}(A)$ as the set of all positive finite linear combinations of elements of A:

$$\operatorname{posi}(A) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, u_k \in A \right\} \subseteq \mathcal{V},$$

and its *convex hull* $\operatorname{CH}(A)$ as the set of convex combinations of elements of A:

$$\operatorname{CH}(A) := \left\{ \sum_{k=1}^{n} \alpha_k u_k : n \in \mathbb{N}, \alpha_k \in \mathbb{R}_{\geq 0}, \sum_{k=1}^{n} \alpha_k = 1, u_k \in A \right\} \subseteq \operatorname{posi}(A) \subseteq \mathcal{V}.$$

A subset A of $\mathcal{V}$ is called a *convex cone* if it is closed under positive finite linear combinations, i.e. if $\operatorname{posi}(A) = A$. A convex cone $\mathcal{K}$ is called *proper* if $\mathcal{K} \cap -\mathcal{K} = \{0\}$. With any proper convex cone $\mathcal{K} \subseteq \mathcal{V}$, we associate an ordering $\preceq_{\mathcal{K}}$ on $\mathcal{V}$, defined for all u and v in $\mathcal{V}$ by $u \preceq_{\mathcal{K}} v \Leftrightarrow v - u \in \mathcal{K}$.

The vector space of options $\mathcal{V}$, ordered by the vector ordering $\preceq_{\mathcal{K}}$, is called an *ordered vector space* $\langle \mathcal{V}, \preceq_{\mathcal{K}} \rangle$. We will refrain from explicitly mentioning the actual proper convex cone $\mathcal{K}$ we are using, and simply write $\mathcal{V}$ to mean the ordered vector space, and use $\preceq$ as a generic notation for the associated vector ordering. Finally, with any vector ordering $\preceq$, we associate the strict partial ordering $\prec$ as follows:

$$u \prec v \Leftrightarrow (u \preceq v \text{ and } u \neq v) \Leftrightarrow v - u \in \mathcal{K} \setminus \{0\} \text{ for all } u, v \text{ in } \mathcal{V}.$$

We call u *positive* if $u \succ 0$, and collect all positive options in the convex cone $\mathcal{V}_{\succ 0} := \mathcal{K} \setminus \{0\}$.

One instance of particular interest for this paper is that where we fix a possibility space $\mathcal{X}$ and let $\mathcal{V}$ be the set of *gambles* or bounded real-valued functions $f : \mathcal{X} \to \mathbb{R}$. In that case, we will use $\mathcal{L}(\mathcal{X})$ to denote this option set, or simply $\mathcal{L}$ when the possibility space is clear from the outset. The vector ordering we shall consider then will be the usual point-wise ordering, defined for any gambles f and g by:

$$(\forall x \in \mathcal{X}) f \leq g \Leftrightarrow f(x) \leq g(x) \tag{1}$$

and, as before, $f \prec g \Leftrightarrow f \preceq g$ and $f \neq g$. We will then denote $\mathcal{L}_{\succ 0} \coloneqq \{f \in \mathcal{L}(\mathcal{X}) : 0 \prec f\}$.

From now on, we assume any ordering $\preceq$, generic but fixed. So we assume that $\mathcal{V}$ is an ordered vector space, with vector ordering $\preceq$. We denote by $\mathcal{Q}(\mathcal{V})$ the set of all non-empty *finite* subsets of $\mathcal{V}$, a strict subset of the power set of $\mathcal{V}$. When it is clear what option space $\mathcal{V}$ we are considering, we will also use the simpler notation $\mathcal{Q}$. Elements A of $\mathcal{Q}$ are the option sets amongst which a subject can choose his preferred options.

Definition 1 A *choice function* C on an option space $\mathcal{V}$ is a map

$$C\colon \mathcal{Q} \to \mathcal{Q} \cup \{\emptyset\} \colon A \mapsto C(A) \text{ such that } C(A) \subseteq A.$$

We collect all the choice functions on $\mathcal{V}$ in $\mathcal{C}(\mathcal{V})$, often denoted as $\mathcal{C}$ when it is clear from the context what the option space is.

The idea underlying this simple definition is that a choice function C selects the set $C(A)$ of 'best' options in the *option set* A. Our definition resembles the one commonly used in the literature [1, 15, 17], except perhaps for an also not entirely unusual restriction to *finite* option sets [9, 13, 16].

Equivalent to a choice function C, we may consider its corresponding *rejection function* R, defined by $R(A) \coloneqq A \setminus C(A)$ for all A in $\mathcal{Q}$. It returns the options $R(A)$ that are rejected—not selected—by C. We collect all the rejection functions on $\mathcal{V}$ in the set $\mathcal{R}(\mathcal{V})$, often denoted as $\mathcal{R}$ when it is clear from the context what the option space is.

We focus on a special class of rejection functions, which we will call *coherent*.

Definition 2 We call a rejection function R on $\mathcal{V}$ *coherent* if for all A, A_1 and A_2 in $\mathcal{Q}$, all u and v in $\mathcal{V}$, and all λ in $\mathbb{R}_{>0}$:

R1. $R(A) \neq A$;
R2. if $u \prec v$ then $u \in R(\{u, v\})$;
R3. a. if $A_1 \subseteq R(A_2)$ and $A_2 \subseteq A$ then $A_1 \subseteq R(A)$;
b. if $A_1 \subseteq R(A_2)$ and $A \subseteq A_1$ then $A_1 \setminus A \subseteq R(A_2 \setminus A)$;
R4. a. if $A_1 \subseteq R(A_2)$ then $\lambda A_1 \subseteq R(\lambda A_2)$;
b. if $A_1 \subseteq R(A_2)$ then $A_1 + \{u\} \subseteq R(A_2 + \{u\})$.

We collect all coherent rejection functions on $\mathcal{V}$ in the set $\bar{\mathcal{R}}(\mathcal{V})$, often simply denoted as $\bar{\mathcal{R}}$ when it is clear from the context which vector space we are using.

These axioms are a subset of the ones introduced in [15], which in our previous work [22] we duly translated from horse lotteries to our abstract options. Our Axiom R2 is slightly more restrictive than its counterpart for horse lotteries considered by Seidenfeld et al. [15], but our other axioms are slightly less restrictive.

One axiom we omit from our coherence definition is the Archimedean one. Typically the preference associated with coherent sets of desirable gambles does not have the Archimedean property (see [23]), so letting go of this axiom is necessary if we want to explore the connection with desirability.

The second axiom that we do not consider as necessary for coherence is what we will call the *convexity axiom*:

R5. if $A \subseteq A_1 \subseteq \mathrm{CH}(A)$ then $R(A_1) \cap A = R(A)$, for all A and A_1 in $\mathcal{Q}$.

The idea behind this axiom is that any gamble that is rejected within an option set A_1 should also be rejected from any smaller option set A resulting from removing non-extreme points from A_1. Albeit an interesting axiom, as noted by Seidenfeld et al. [15] it is incompatible with Walley–Sen maximality [18, 25], in the manner that we will make explicit later on.

An interesting rescaling property that we shall need further on is the following:

Proposition 1 *Let R be a rejection function on $\mathcal{Q}$ satisfying axioms* R3a*,* R4a *and* R5*. Then for all n in $\mathbb{N}$, all $u_1, u_2, \dots, u_n$ in $\mathcal{V}$ and all $\mu_1, \mu_2, \dots \mu_n$ in $\mathbb{R}_{>0}$:*

$$0 \in R(\{0, u_1, u_2, \dots, u_n\}) \Leftrightarrow 0 \in R(\{0, \mu_1 u_1, \mu_2 u_2, \dots, \mu_n u_n\}). \tag{2}$$

If we replace 0 by any non-zero option u, this result need no longer hold.

We have learned from dire experience that in verifying whether a rejection function is coherent, Axiom R3b is often hardest to check. But under various additional conditions, it has a number of equivalent formulations that may simplify this task:

Proposition 2 *Let R be any rejection function on $\mathcal{Q}$, and consider the following statements:*

(i) *R satisfies Axiom* R3b*;*
(ii) $(\forall A \in \mathcal{Q})(\forall u \in R(A))u \in R(\{u\} \cup A \setminus R(A))$*;*
(iii) $(\forall A \in \mathcal{Q})(\forall v \in R(A) \setminus \{0\})(0 \in R(A) \Rightarrow 0 \in R(A \setminus \{v\}))$*;*

Then (i) *implies* (ii) *and* (iii)*. If R satisfies Axiom* R3a*, then* (i) *and* (ii) *are equivalent, and if R satisfies in addition Axiom* R4b*, then* (i)*,* (ii) *and* (iii) *are equivalent.*

Using Proposition 2, we can find an easy characterisation of Axiom R1.

Corollary 1 *Consider any rejection function R that satisfies Axioms* R3b *and* R4b*. Then R satisfies Axiom* R1 *if and only if* $0 \notin R(\{0\})$.

For two choice functions C and C', we call C *not more informative* than C'—and we write $C \sqsubseteq C'$—if $C(A) \supseteq C'(A)$ for all A in $\mathcal{Q}$. The idea behind this is that a more informative choice function selects the admissible options more selectively from within the option set. The relation $\sqsubseteq$ is reflexive, antisymmetric and transitive, so the set $\mathcal{C}$ of all choice functions ordered by $\sqsubseteq$ is a partial order. Moreover, it is actually a complete lattice: given any set $\mathcal{C}' \subseteq \mathcal{C}$ of choice functions, its infimum $\inf \mathcal{C}'$ and supremum $\sup \mathcal{C}'$ exist in $\mathcal{C}$, and are given by $(\inf \mathcal{C}')(A) = \bigcup_{C \in \mathcal{C}'} C(A)$ and $(\sup \mathcal{C}')(A) = \bigcap_{C \in \mathcal{C}'} C(A)$ for all A in $\mathcal{Q}$. This translates naturally to rejection functions.

3 The link with desirability

In [23], we have studied in some detail how the coherent choice functions in the sense of Definition 2 can be related to coherent sets of desirable options (gambles). In the present section, we investigate what remains of this connection when we require in addition that our choice functions should satisfy Axiom R5.

We recall that a set of desirable options D is simply a subset of the vector space $\mathcal{V}$. The underlying idea is that the subject strictly prefers each option u in this set to the status quo 0. As we did for choice functions, we pay special attention to *coherent* sets of desirable options.

Definition 3 A set of desirable options D is called *coherent* if for all u and v in $\mathcal{V}$, and all λ in $\mathbb{R}_{>0}$:
D_1. $0 \notin D$;
D_2. $\mathcal{V}_{\succ 0} \subseteq D$;
D_3. if $u \in D$ then $\lambda u \in D$;
D_4. if $u, v \in D$ then $u + v \in D$.
We collect all coherent sets of desirable options in the set $\bar{\mathcal{D}}$.

More details can be found in [25], [26], [11] and the references therein.

Axioms D_3 and D_4 guarantee that a coherent D is a convex cone. This convex cone induces a strict partial order $\prec_D$ on $\mathcal{V}$, by letting $u \prec_D v \Leftrightarrow 0 \prec_D v - u \Leftrightarrow v - u \in D$, so $D = \{u \in \mathcal{V} : 0 \prec_D u\}$ [7, 11]. D and $\prec_D$ are mathematically equivalent: given one of D or $\prec_D$, we can determine the other unequivocally using the formulas above. When it is clear from the context which set of desirable options D we are working with, we often refrain from mentioning the explicit reference to D in $\prec_D$ and then we simply write $\prec$. One of the axioms says that $< \subseteq \prec_D$.

We can associate a set of desirable options D_R with every given rejection function R by focusing on its binary rejections:

$$u \prec_{D_R} v \Leftrightarrow u - v \in D_R \Leftrightarrow u \in R(\{u, v\}) \text{ for all } u, v \text{ in } \mathcal{V}.$$

For more details, we refer to [23, Section 3]. D_R is a coherent set of desirable options if R is a coherent rejection function. Conversely, if we start out with a coherent set of desirable options D then the set $\{R \in \bar{\mathcal{R}} : D_R = D\}$ of all coherent rejection functions whose binary choices are represented by D, is non-empty, and its smallest, or least informative, element $R_D := \inf\{R \in \bar{\mathcal{R}} : D_R = D\}$ is given by:

$$R_D(A) := \{u \in A : (\exists v \in A) v - u \in D\} = \{u \in A : (\exists v \in A) u \prec v\} \text{ for all } A \text{ in } \mathcal{Q}.$$

It selects all options from A that are dominated under the ordering $\prec_D$, or in other words, its corresponding choice function is based on Walley–Sen maximality.

Proposition 3 ([22, Proposition 11]) *Given any coherent set of desirable options D, then $0 \in R_D(\{0\} \cup A) \Leftrightarrow D \cap A \neq \emptyset$ for all A in $\mathcal{Q}$.*

Although R_D is coherent when D is, it does not necessarily satisfy the additional Axiom R5, as shown in [22, Example 1]; the sets of desirable options D for which R_D does satisfy the convexity axiom are identified in the next proposition.

Proposition 4 ([22, Proposition 12]) *Consider any coherent set of desirable options D. Then the rejection function R_D satisfies Axiom* R5 *if and only if* $\mathrm{posi}(D^c) = D^c$.

This proposition seems to indicate that there is something special about coherent sets of desirable options whose complement is a convex cone too. We give them a special name that will be motivated and explained next.

Definition 4 A coherent set of desirable options D is called *lexicographic* if $\operatorname{posi}(D^c) = D^c$ or, equivalently, if $\operatorname{posi}(D^c) \cap D = \emptyset$. We collect all the lexicographic coherent sets of desirable options in $\bar{\mathcal{D}}_{\mathrm{L}}$.

The set $\bar{\mathcal{D}}_{\mathrm{L}}$ of lexicographic sets of desirable options is non-empty. It includes, for instance, the so-called *maximal* sets of desirable options, see [22], which is the subclass of those coherent sets of desirable options satisfying

$$(\forall u \in \mathcal{V} \setminus \{0\})(u \in D \text{ or } -u \in D). \tag{3}$$

We collect all the coherent sets of desirable options that satisfy Equation (3) above in the set $\hat{\mathcal{D}}$.
The reason why we call the elements of $\bar{\mathcal{D}}_{\mathrm{L}}$ *lexicographic* lurks behind a close connection with the well-studied *lexicographic probability systems*.

Definition 5 A *lexicographic probability system* is an ℓ-tuple $p := (p_1, \dots, p_\ell)$ of probability mass functions on a possibility space $\mathcal{X}$. We associate with this tuple p an expectation operator $E_p := (E_{p_1}, \dots, E_{p_\ell})$, and a (strict) preference relation $\prec_p$ on $\mathcal{L}(\mathcal{X})$, defined by: $f \prec_p g \Leftrightarrow E_p(f) <_{\mathrm{L}} E_p(g)$ for all f and g in $\mathcal{L}$, where, for every h in $\mathcal{L}$, $E_p(h) := (E_{p_1}(h), \dots, E_{p_\ell}(h))$, is an element of an ℓ-dimensional vector space and $<_{\mathrm{L}}$ denotes the lexicographic order, given for any ℓ-dimensional vectors $(x_1, \dots, x_\ell)$ and $(y_1, \dots, y_\ell)$ by:

$$(x_1, \dots, x_\ell) <_{\mathrm{L}} (y_1, \dots, y_\ell) \Leftrightarrow (\exists j \in \{1, \dots, \ell\})(x_j < y_j \text{ and } (\forall i \leq j-1) x_i = y_i)$$

We call ℓ the number of *layers* of the lexicographic probability system.

An important property that a lexicographic probability system p may or may not have, is that of having *no non-trivial Savage-null events*: p has no non-trivial Savage-null events if for every x in $\mathcal{X}$, there is at least one k in $\{1, \dots, \ell\}$ for which $p_k(x) > 0$.

We have showed in [22, Section 5] that any lexicographic set of desirable options D defines a lexicographic probability system with no non-trivial Savage-null events, and *vice versa*. Lexicographic sets of desirable options are therefore an elegant and simple representation of lexicographic probability systems.

To get some feeling for what these lexicographic models represent, we first look at the special case of binary possibility spaces $\{\mathrm{H}, \mathrm{T}\}$, leading to a two-dimensional option space $\mathcal{V} = \mathcal{L}(\{\mathrm{H}, \mathrm{T}\})$ provided with the point-wise order in Equation (1). It turns out that lexicographic sets of desirable options (gambles) are easy to characterise there.

Proposition 5 ([22, Proposition 16]) *All lexicographic coherent sets of desirable gambles on the binary possibility space* $\{\mathrm{H}, \mathrm{T}\}$ *are given by:*

$$\bar{\mathcal{D}}_{\mathrm{L}} := \{D_\rho, D_\rho^{\mathrm{H}}, D_\rho^{\mathrm{T}} : \rho \in (0,1)\} \cup \{D_0, D_1\} = \{D_\rho : \rho \in (0,1)\} \cup \hat{\mathcal{D}},$$

where, for all ρ *in* $(0,1)$*,*

$$
\begin{aligned}
D_\rho &:= \{\lambda(\rho - \mathbb{I}_{\{\mathrm{H}\}}) : \lambda \in \mathbb{R}\} + \mathcal{L}_{>0} = \operatorname{span}(\{\rho - \mathbb{I}_{\{\mathrm{H}\}}\}) + \mathcal{L}_{>0}\\
D_\rho^{\mathrm{H}} &:= D_\rho \cup \{\lambda(\rho - \mathbb{I}_{\{\mathrm{H}\}}) : \lambda \in \mathbb{R}_{<0}\} = D_\rho \cup \operatorname{posi}(\{\mathbb{I}_{\{\mathrm{H}\}} - \rho\})\\
D_\rho^{\mathrm{T}} &:= D_\rho \cup \{\lambda(\rho - \mathbb{I}_{\{\mathrm{H}\}}) : \lambda \in \mathbb{R}_{>0}\} = D_\rho \cup \operatorname{posi}(\{\rho - \mathbb{I}_{\{\mathrm{H}\}}\})\\
D_0 &:= \{f \in \mathcal{L} : f(\mathrm{T}) > 0\} \cup \mathcal{L}_{>0}\\
D_1 &:= \{f \in \mathcal{L} : f(\mathrm{H}) > 0\} \cup \mathcal{L}_{>0}.
\end{aligned}
$$

Definition 6 A coherent rejection function R is called *lexicographic* if $R = R_D$ for some D in $\bar{\mathcal{D}}_{\mathrm{L}}$. We collect all the lexicographic coherent rejection functions in $\bar{\mathcal{R}}_{\mathrm{L}}$.

Proposition 6 ([22, Proposition 23]) *Consider an arbitrary coherent set of desirable options* D*. Then*

$$\inf\{R \in \bar{\mathcal{R}} : R \textit{ satisfies Axiom } \mathrm{R5} \textit{ and } D_R = D\} = \inf\{R_{D'} : D' \in \bar{\mathcal{D}}_{\mathrm{L}} \textit{ and } D \subseteq D'\}.$$

Therefore, they also represent the least informative coherent choice function that satisfies Axiom R5, taking into account that Axiom R5 is preserved when taking infima.

4 No representation of choice functions on a binary space

Lexicographic choice functions seem to fulfil the role of *probability mass* in our theory without any Archimedean axiom. This is in contradistinction with the theory of choice functions on horse lotteries with an Archimedean axiom [15], where the most informative choice functions are those that are induced by probability mass functions. Seidenfeld et al. [15] show that every coherent choice function on horse lotteries (also satisfying their Archimedean axiom) is an infimum of such maximally informative choice functions. This ensures that coherent choice functions with the Archimedean axiom constitute a so-called *strong belief structure* [6][1]. The relevance of such strong belief structures is that they allow for a simple account of conservative inference, as we can essentially work with the maximal models (that is, those that are not dominated by any other model).

Since coherent sets of desirable options are represented by their dominating maximal ones, it is natural to wonder if they fulfil the same representational role for choice functions. Our next example shows that this is not the case. The main

[1] A family of belief models is called a *belief structure* when it is a lattice with respect to some partial order $\leq$, it is closed under infima and it has no top. It is called a *strong* belief structure when in addition any belief model can be obtained as the infima of the maximal models that dominate it.

underlying idea is that maximal sets of desirable options are in some sense too informative, and do not allow to encompass the interactions between the different options that are sometimes embedded into a choice function.

Example 1 Consider the binary space $\{H, T\}$ and let us define the coherent set of desirable gambles $D := \{f \in \mathcal{L} : f(H) + f(T) > 0\}$. Clearly, $D^c = \{f \in \mathcal{L} : f(H) + f(T) \leq 0\}$ is a convex cone, so D is a lexicographic set of desirable gambles, and hence, by Proposition 4, R_D is coherent and satisfies Axiom R5.

Is R_D representable by a subset of $\{R_{\hat{D}} : \hat{D} \in \hat{\mathcal{D}}\}$? To answer this in the negative, consider the option set A in $\mathcal{Q}_0$ that consists of the gambles $\{f, -f, 0\}$, where $f(H) = 1, f(T) = -1$. Then neither f nor $-f$ belongs to D, whence by Proposition 3, $0 \notin R_D(A)$. However, $A \cap \hat{D} \neq \emptyset$, for every $\hat{D}$ in $\hat{\mathcal{D}}$. To see this, it suffices to take into account that any maximal set of desirable options shall include either f or $-f$ because of Equation (3). This means that R_D is not representable by subsets of $\hat{\mathcal{D}}$, even though R_D satisfies Axiom R5. ◊

Thus, a representation in terms of appropriately chosen $\{R_{\hat{D}} : \hat{D} \in \hat{\mathcal{D}}\}$ is impossible. But since we have seen in Proposition 6 that lexicographic rejection functions seem to fulfil at least some representing role in our theory without Archimedeanity, it seems at least possible that there might be a representation result in terms of $\bar{\mathcal{R}}_{\mathrm{L}}$—in terms of lexicographic rejection functions. This brings us to the central question of this section: is, in parallel with the result by Seidenfeld et al. [15], every coherent rejection function R that satisfies the Axiom R5 an infimum of *lexicographic rejection functions*, or in other words, is $R = \inf\{R' \in \bar{\mathcal{R}}_{\mathrm{L}} : R \sqsubseteq R'\}$, or equivalently,

$$R(A) = \bigcap\{R'(A) : R' \in \bar{\mathcal{R}}_{\mathrm{L}} \text{ and } R \sqsubseteq R'\} \text{ for all } A \text{ in } \mathcal{Q}?$$

We will show in this section that, unfortunately and perhaps somewhat surprisingly, *this is generally not the case*, by studying in more detail the special case of coherent rejection functions on two-dimensional option spaces.

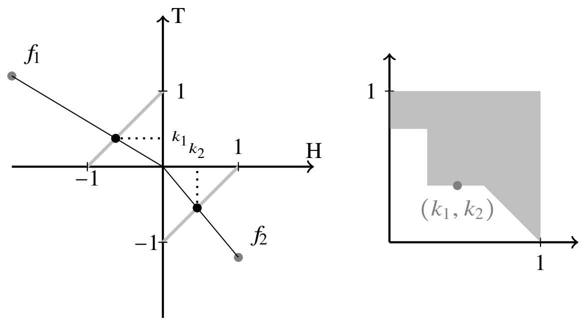
Our counterexample that we will build will be a rejection function on a two-dimensional option space. Therefore, in the remainder of this section, we concentrate on the two-dimensional option space $\mathcal{V} = \mathcal{L}(\mathcal{X})$ of gambles on an uncertain variable that can assume only two possible values $\mathcal{X} := \{\mathrm{H}, \mathrm{T}\}$, which is isomorphic to $\mathbb{R}^2$, and with the order given by Equation (1).

4.1 An equivalent characterisation: rejection sets

As we will see shortly, the coherent rejection functions on the option space $\mathcal{V} = \mathbb{R}^2$ are uniquely determined by what we shall call a *rejection set*, consisting essentially of those option sets that allow us to reject 0 from them. Instead of describing the gambles that reject 0 directly, this new characterisation will rather use Axiom R4a to rescale gambles in the second and fourth quadrants

$$\mathcal{V}_{\mathrm{II}} := \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{H}) < 0 < f(\mathrm{T})\} \text{ and } \mathcal{V}_{\mathrm{IV}} := \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{T}) < 0 < f(\mathrm{H})\}, \tag{4}$$

obtaining variants that can be described more easily. Indeed, every gamble f_1 in $\mathcal{V}_{\mathrm{II}}$ can be uniquely described as $f_1 = \lambda_1(k_1 - 1, k_1)$ with λ_1 in $\mathbb{R}_{>0}$ and k_1 in $(0,1)$, and similarly, every gamble f_2 in $\mathcal{V}_{\mathrm{IV}}$ as $f_2 = \lambda_2(k_2, k_2 - 1)$ with λ_2 in $\mathbb{R}_{>0}$ and k_2 in $(0,1)$, as indicated by the figure below.



Definition 7 Given any coherent rejection function R, we define its *rejection set* $K_R \subseteq [0,1)^2$ as

$$K_R := \{(k_1, k_2) \in [0,1)^2 : 0 \in R(\{(k_1 - 1, k_1), 0, (k_2, k_2 - 1)\})\}.$$

We will call any subset $K \subseteq [0,1)^2$ a rejection set. It will be useful to consider a number of potential properties of rejection sets K:

K1. *monotonicity*: if $(k_1, k_2) \in K$, $k_1' \geq k_1$ and $k_2' \geq k_2$, then also $(k_1', k_2') \in K$, for all (k_1, k_2) and (k_1', k_2') in $[0,1)^2$;

K2. *non-triviality*: $(0,0) \notin K$;

K3. a. for all a, b and c in $[0,1)$ such that $c < a$, $a + b < 1$, $(b, a) \in K$ and $(1 - a, c) \in K$:

$$(x, c) \in K \text{ for all } x \text{ in } (b, 1) \text{ and } (b, y) \in K \text{ for all } y \text{ in } (c, 1);$$

b. for all a and c in $[0,1)$ such that $c < a$, $(0, a) \in K$ and $(1 - a, c) \in K$:

$$(0, c) \in K;$$

c. for all a and b in $[0,1)$ such that $0 < a$, $a+b < 1$, $(b, a) \in K$ and $(1-a, 0) \in K$:

$$(b, 0) \in K;$$

K4. if $k_1 + k_2 > 1$ then $(k_1, k_2) \in K$, for all (k_1, k_2) in $[0,1)^2$.

Properties K2 and K3 imply the following useful property:

Lemma 1 *Consider any rejection set* $K \subseteq [0,1)^2$*. If* K *satisfies Properties* K2 *and* K3*, then for every* $a \in [0,1]$*, either* $(0, a) \notin K$ *or* $(1 - a, 0) \notin K$*.*

The coherence of R—and the extra Axiom R5 and the weaker Condition (2)—implies a number of corresponding properties of its rejection set K_R:

Proposition 7 *Consider any coherent choice function R on $\mathcal{L}(\{\mathrm{H},\mathrm{T}\})$. Then its rejection set K_R satisfies Properties* K1 *and* K2*. Furthermore, if R satisfies Condition* (2)*, then K_R also satisfies Property* K3*. Finally, if R satisfies Axiom* R5 *then K_R also satisfies Properties* K3 *and* K4*.*

Conversely, we now show how to associate a rejection function with any rejection set $K \subseteq [0,1)^2$. Taking into account Property K2, we only consider sets K that do not contain 0.

Definition 8 Given any subset $K \subseteq [0,1)^2 \setminus \{0\}$, we define its corresponding rejection function R_K as follows. We let

$$R_K(\{0\}) = \emptyset. \tag{5}$$

Next, for any A in $\mathcal{Q}_0$, we let $0 \in R_K(A \cup \{0\})$ if at least one of the following conditions holds:

$$A \cap \mathcal{L}_{>0} \neq \emptyset \tag{6}$$

$$(\exists \lambda_1 \in \mathbb{R}_{>0}, (k_1, 0) \in K)\lambda_1(k_1 - 1, k_1) \in A \tag{7}$$

$$(\exists \lambda_2 \in \mathbb{R}_{>0}, (0, k_2) \in K)\lambda_2(k_2, k_2 - 1) \in A \tag{8}$$

$$(\exists \lambda_1, \lambda_2 \in \mathbb{R}_{>0}, (k_1, k_2) \in K \cap (0,1)^2)\{\lambda_1(k_1 - 1, k_1), \lambda_2(k_2, k_2 - 1)\} \subseteq A, \tag{9}$$

and finally, we allow for $R(A)$ to contain non-zero gambles by imposing the following condition:

$$(\forall A \in \mathcal{Q})(\forall f \in A) f \in R_K(A) \Leftrightarrow 0 \in R_K(A - \{f\}). \tag{10}$$

The intuition behind this is that the elements of K of the type $(k_1, 0)$ or $(0, k_2)$ determine gambles—$(k_1 - 1, k_1)$ and $(k_2, k_2 - 1)$, respectively—that allow us to reject 0; the other possibility of rejecting 0 is by means of the combined action of a gamble in the second quadrant—$(k_1 - 1, k_1)$ for k_1 in $(0,1)$—and one in the fourth quadrant—$(k_2, k_2 - 1)$ for k_2 in $(0,1)$.

Alternatively, we can summarise Conditions (7)–(9) as

$$(\exists \lambda_1, \lambda_2 \in \mathbb{R}_{>0}, (k_1, k_2) \in K)\{\lambda_1(k_1-1, k_1), \lambda_2(k_2, k_2-1)\} \subseteq A \cup \{(-1,0), (0,-1)\}. \tag{11}$$

Lemma 2 *For any $K \subseteq [0,1)^2 \setminus \{0\}$ and any A in $\mathcal{Q}$, at least one of the Conditions* (7)–(9) *holds if and only if Condition* (11) *holds.*

Now that we know how to associate with a rejection set K a rejection function R_K, let us determine which conditions on K ensure the coherence of R_K. We begin by showing that a number of coherence axioms follow directly from the definition, irrespective of the choice of the rejection set $K \subseteq [0,1)^2 \setminus \{0\}$:

Proposition 8 *Consider any subset $K \subseteq [0,1)^2 \setminus \{0\}$. Then the rejection function R_K given by Definition 8 satisfies Axioms* R2*,* R3a*,* R4a*,* R4b *and Condition* (2)*.*

If in addition K satisfies Properties K1–K3*, then the rejection function R_K given by Definition 8 satisfies Axioms* R3b *and* R1*.*

In other words, given any subset K of $[0,1)^2 \setminus \{0\}$ that satisfies Properties K1–K3, the rejection function R_K given by Definition 8 is coherent and satisfies Property (2).

We conclude from the preceding discussion that any coherent rejection function determines a rejection set via Definition 7, which, in turn, can be used to determine a rejection function via Definition 8. Our next proposition shows that these two procedures commute, or, in other words, that a coherent rejection function is uniquely determined by its associated rejection set, and the other way around. In order to get there, we first establish the following lemma:

Lemma 3 *Consider any coherent rejection function R on $\mathcal{L}(\{\mathrm{H},\mathrm{T}\})$ that satisfies Condition* (2)*. Consider the option sets $\{f_1,\dots,f_m\} \subseteq \mathcal{V}_{\mathrm{II}}$ and $\{g_1,\dots,g_n\} \subseteq \mathcal{V}_{\mathrm{IV}}$, for some m and n in $\mathbb{N}$. Then the following equivalences hold for any i in $\arg\max\{\frac{f_k(T)}{f_k(T)-f_k(H)} : k \in \{1,\dots,m\}\}$ and any j in $\arg\max\{\frac{v_k(H)}{v_k(H)-v_k(T)} : k \in \{1,\dots,n\}\}$:*

(i) $0 \in R(\{0,f_1,\dots,f_m,g_1,\dots,g_n\}) \Leftrightarrow 0 \in R(\{0,f_i,g_j\})$;
(ii) $0 \in R(\{0,g_1,\dots,g_n\}) \Leftrightarrow 0 \in R(\{0,g_j\})$;
(iii) $0 \in R(\{0,f_1,\dots,f_m\}) \Leftrightarrow 0 \in R(\{0,f_i\})$.

Incidentally, Proposition 1 ensures that this lemma applies in particular to coherent rejection functions that satisfy Axiom R5.

Proposition 9 *For any coherent rejection function R on $\mathcal{L}(\{\mathrm{H},\mathrm{T}\})$ that satisfies Condition* (2)*, $R = R_{K_R}$. Conversely, for any rejection set K satisfying Properties* K1–K3*, $K = K_{R_K}$.*

To conclude our preliminary discussion of the relation between rejection sets and rejection functions, we characterise the conditions under which the rejection function R_K determined by a rejection set K satisfies the 'convexity' Axioms R5. We begin with a lemma that will simplify the argument.

Lemma 4 *Consider (k_1,k_2) in $[0,1)^2$. Let $A := \{(k_1-1,k_1),0,(k_2,k_2-1)\}$, then*

$$\mathrm{posi}(A) = \begin{cases} B + \mathcal{L}_{\geq 0} & \text{if } k_1 + k_2 > 1 \\ B & \text{if } k_1 + k_2 = 1 \\ B + \mathcal{L}_{\leq 0} & \text{if } k_1 + k_2 < 1, \end{cases}$$

where $B := \{\lambda(k_1-1,k_1) : \lambda \in \mathbb{R}_{\geq 0}\} \cup \{\lambda(k_2,k_2-1) : \lambda \in \mathbb{R}_{\geq 0}\}$.

In particular, it follows from this result for $A = \{(k_1-1,k_1),0,(k_2,k_2-1)\}$ that

$$\mathrm{posi}(A) \cap \mathcal{L}_{>0} = \emptyset \Leftrightarrow k_1 + k_2 \leq 1, \text{ for all } (k_1,k_2) \text{ in } [0,1)^2. \tag{12}$$

Proposition 10 *Consider any rejection set $K \subseteq [0,1)^2 \setminus \{0\}$ that satisfies Properties* K1–K3*, and the corresponding rejection function R_K on $\mathcal{L}(\{\mathrm{H},\mathrm{T}\})$. Then the following two statements are equivalent:*
(i) *R_K satisfies Axiom* R5*,*
(ii) *K satisfies Property* K4.

The results in this section so far can be succinctly summarised as follows:

Theorem 1 *Consider a two-dimensional option space $\mathcal{V} := \mathcal{L}(\{\mathrm{H},\mathrm{T}\})$ with the order given by Equation* (1)*. There is a one-to-one correspondence between coherent rejection functions on $\mathcal{V}$ satisfying Condition* (2) *and subsets of $[0,1)^2$ satisfying Properties* K1–K3.

Moreover, there is a one-to-one correspondence between coherent rejection functions on $\mathcal{V}$ satisfying Axiom R5 *and subsets of $[0,1)^2$ satisfying Properties* K1–K4.

4.2 Counterexample

Let us call *lexicographic* rejection set a rejection set corresponding to a lexicographic choice function. In order to find a rejection set that is no infimum of such lexicographic rejection sets, we first need to find out what these lexicographic rejection sets look like. Recall from Proposition 5 that all the lexicographic coherent sets of desirable gambles on a binary possibility space $\{\mathrm{H},\mathrm{T}\}$ are given by

$$\bar{\mathcal{D}}_{\mathrm{L}} := \{D_\rho, D_\rho^{\mathrm{H}}, D_\rho^{\mathrm{T}} : \rho \in (0,1)\} \cup \{D_0, D_1\} = \{D_\rho : \rho \in (0,1)\} \cup \hat{\mathcal{D}},$$

and the lexicographic rejection functions on $\mathcal{L}(\{\mathrm{H},\mathrm{T}\})$ are $\bar{\mathcal{R}}_{\mathrm{L}} = \{R_D : D \in \bar{\mathcal{D}}_{\mathrm{L}}\}$. We determine the corresponding rejection sets. For any D in $\bar{\mathcal{D}}_{\mathrm{L}}$, we let K_D be the rejection set that corresponds to the rejection function R_D. For any ρ in $(0,1)$ and $(k_1,k_2) \in [0,1)^2$, observe that

$$\begin{aligned}(k_1,k_2) \in K_{D_\rho} &\Leftrightarrow 0 \in R_{D_\rho}(\{(k_1-1,k_1), 0, (k_2,k_2-1)\})\\ &\Leftrightarrow \{(k_1-1,k_1),(k_2,k_2-1)\} \cap D_\rho \neq \emptyset\\ &\Leftrightarrow (k_1-1,k_1) \in D_\rho \text{ or } (k_2,k_2-1) \in D_\rho\\ &\Leftrightarrow k_1 > \rho \text{ or } k_2 > 1-\rho,\end{aligned} \tag{13}$$

and similarly,

$$(k_1,k_2) \in K_{D_\rho^{\mathrm{H}}} \Leftrightarrow (k_1-1,k_1) \in D_\rho^{\mathrm{H}} \text{ or } (k_2,k_2-1) \in D_\rho^{\mathrm{H}} \Leftrightarrow k_1 > \rho \text{ or } k_2 \geq 1-\rho, \tag{14}$$

and

$$(k_1,k_2) \in K_{D_\rho^{\mathrm{T}}} \Leftrightarrow (k_1-1,k_1) \in D_\rho^{\mathrm{T}} \text{ or } (k_2,k_2-1) \in D_\rho^{\mathrm{T}} \Leftrightarrow k_1 \geq \rho \text{ or } k_2 > 1-\rho. \tag{15}$$

Finally, also for D_0 and D_1,

$$(k_1, k_2) \in K_{D_0} \Leftrightarrow (k_1 - 1, k_1) \in D_0 \text{ or } (k_2, k_2 - 1) \in D_0 \Leftrightarrow k_1 > 0 \tag{16}$$

and

$$(k_1, k_2) \in K_{D_1} \Leftrightarrow (k_1 - 1, k_1) \in D_1 \text{ or } (k_2, k_2 - 1) \in D_1 \Leftrightarrow k_2 > 0. \tag{17}$$

We are now, finally, ready to provide an example of a rejection set that satisfies Properties K1–K4—or a coherent rejection function that satisfies Axiom R5—but is no intersection of lexicographic rejection sets.

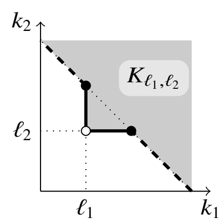
Example 2 Consider any ℓ_1 and ℓ_2 in $(0, 1)$ such that $\ell_1 + \ell_2 < 1$, and the rejection set $K_{\ell_1,\ell_2} \subseteq [0, 1)^2$ depicted in the figure below, and defined by

$$K_{\ell_1,\ell_2} \coloneqq \left\{(k_1, k_2) \in [0, 1)^2 : k_1 + k_2 > 1 \text{ or } (k_1, k_2) > (\ell_1, \ell_2)\right\}. \tag{18}$$

We show that it corresponds to a rejection function that is coherent and satisfies Property R5. By Theorem 1 it suffices to show that K_{ℓ_1,ℓ_2} satisfies Properties K1–K4. That it satisfies Properties K1, K2 and K4 is clear from its definition. We show that it also satisfies Property K3. Note that $(0, a) \notin K_{\ell_1,\ell_2}$ and $(1 - a, 0) \notin K_{\ell_1,\ell_2}$ for all a in $[0, 1]$, so the Properties K3b and K3c are trivially satisfied for K_{ℓ_1,ℓ_2}. It therefore only remains to prove that Property K3a is satisfied for K_{ℓ_1,ℓ_2}. Consider any a, b and c in $[0, 1)$ such that $c < a$, $a + b < 1$, $(b, a) \in K_{\ell_1,\ell_2}$ and $(1 - a, c) \in K_{\ell_1,\ell_2}$. We need to show that then

$$(x, c) \in K_{\ell_1,\ell_2} \text{ for all } x \text{ in } (b, 1) \text{ and } (b, y) \in K_{\ell_1,\ell_2} \text{ for all } y \text{ in } (c, 1),$$

so consider any x in $(b, 1)$ and y in $(c, 1)$. Since $(b, a) \in K_{\ell_1,\ell_2}$ and $a + b < 1$, Equation (18) tells us that $(b, a) > (\ell_1, \ell_2)$, so $x > b \geq \ell_1$. Similarly, since $(1 - a, c) \in K_{\ell_1,\ell_2}$ and $c < a$ (or equivalently, $1 - a + c < 1$), Equation (18) tells us that $(1 - a, c) > (\ell_1, \ell_2)$, so $y > c \geq \ell_2$. Then $(x, c) > (\ell_1, \ell_2)$ and $(b, y) > (\ell_1, \ell_2)$, whence indeed $(x, c) \in K_{\ell_1,\ell_2}$ and $(b, y) \in K_{\ell_1,\ell_2}$. So we see that K_{ℓ_1,ℓ_2} satisfies Properties K1–K4. It therefore corresponds to a coherent and 'convex' rejection function.



We show that K_{ℓ_1,ℓ_2} is no intersection of lexicographic rejection sets. Assume *ex absurdo* that it is an intersection $\bigcap K_{\mathcal{D}'}$ of some non-empty collection of lexicographic rejection sets $K_{\mathcal{D}'} \coloneqq \{K_D : D \in \mathcal{D}'\}$, with $\mathcal{D}' \subseteq \bar{\mathcal{D}}_{\mathrm{L}}$. Then, since $(\ell_1, \ell_2) \notin K_{\ell_1,\ell_2}$, there must be some D in $\mathcal{D}'$ such that $(\ell_1, \ell_2) \notin K_D$. There are a number of possibilities: (i) $D = D_\rho$ for some ρ in $(0, 1)$, (ii) $D = D^{\mathrm{H}}_\rho$ for some ρ

in $(0,1)$, or (iii) $D = D^{\mathrm{T}}_{\rho}$ for some ρ in $(0,1)$—$D \in \{D_0, D_1\}$ is impossible since (ℓ_1, ℓ_2) belong to both K_{D_0} [by Equation (16)] and K_{D_1} [by Equation (17)].

In case (i), since $(\ell_1, \ell_2) \notin K_{D_\rho}$, we infer from Equation (13) that $\ell_1 \leq \rho$ and $\ell_2 \leq 1 - \rho$, or in other words, that $\rho \in [\ell_1, 1 - \ell_2]$. From $\ell_1 + \ell_2 < 1$, we infer that $\ell_1 < \rho$ or $\ell_2 < 1 - \rho$. We consider the case that $\ell_1 < \rho$; if $\ell_2 < 1 - \rho$, a symmetrical argument leads to a similar result. From Equation (13) we infer, using $\ell_2 \leq 1 - \rho$, that on the one hand $(\rho, \ell_2) \notin K_{D_\rho}$. On the other hand, we infer from $(\rho, \ell_2) > (\ell_1, \ell_2)$ that $(\rho, \ell_2) \in K_{\ell_1,\ell_2}$, by Equation (18). This leads us to conclude that $K_{\ell_1,\ell_2} \neq K_{D_\rho}$.

In case (ii), then, since $(\ell_1, \ell_2) \notin K_{D^{\mathrm{H}}_{\rho}}$, we infer from Equation (14) that $\ell_1 \leq \rho$ and $\ell_2 < 1 - \rho$, or in other words, that $\rho \in [\ell_1, 1 - \ell_2)$. This implies that $\ell_2 < \frac{1-\rho+\ell_2}{2} < 1 - \rho$: indeed, $\frac{1-\rho+\ell_2}{2}$ is a convex mixture of ℓ_2 and $1 - \rho$. From Equation (14), we infer, using $\frac{1-\rho+\ell_2}{2} < 1 - \rho$, that on the one hand $(\ell_1, \frac{1-\rho+\ell_2}{2}) \notin K_{D^{\mathrm{H}}_{\rho}}$. On the other hand, we infer from $(\ell_1, \frac{1-\rho+\ell_2}{2}) > (\ell_1, \ell_2)$ that $(\ell_1, \frac{1-\rho+\ell_2}{2}) \in K_{\ell_1,\ell_2}$, by Equation (18). This leads us to conclude that $K_{\ell_1,\ell_2} \neq K_{D^{\mathrm{H}}_{\rho}}$.

In case (iii), a completely symmetrical argument leads to the conclusion that $K_{\ell_1,\ell_2} \neq K_{D^{\mathrm{H}}_{\rho}}$.

This tells us that none of the remaining possibilities can hold, a contradiction. ◊

Thus, the rejection function that corresponds to K_{ℓ_1,ℓ_2} is coherent and satisfies Property R5 by Theorem 1, but it is no infimum of lexicographic rejection functions. This answers the initial question in this section—is $R = \inf\{R' \in \bar{\mathcal{R}}_{\mathrm{L}} : R \sqsubseteq R'\}$ for every coherent rejection function R that satisfies Property R5?—in the negative: in the restrictive case of two possible outcomes, we have found a counterexample.

5 Weak Archimedeanity

In order to find an additional requirement that guarantees representation, at least in the binary case, let us further analyse the properties of rejection sets.

Definition 9 Consider any $K \subseteq [0,1)^2$ satisfying Properties K1–K4. We define the following two maps $\pi_1 \colon [0,1) \to [0,1]$ and $\pi_2 \colon [0,1) \to [0,1]$:

$$\pi_1(z) \coloneqq \inf\{a \in \mathbb{R} : (z,a) \in K\} \text{ and } \pi_2(z) \coloneqq \inf\{a \in \mathbb{R} : (a,z) \in K\}$$

for all z in $[0,1)$. Here, we let $\inf \varnothing \coloneqq 1$, so that $\pi_1(z) = 1$ if $(z,\ell) \notin K$ for all ℓ in $[0,1)$, and $\pi_2(z) = 1$ if $(\ell,z) \notin K$ for all ℓ in $[0,1)$.

Note that $\pi_1(z) \in [0, 1-z]$ and $\pi_2(z) \in [0, 1-z]$ for every z in $[0,1)$, because $\{(k_1,k_2) \in [0,1)^2 : k_1 + k_2 > 1\} \subseteq K$. Since K is assumed to be increasing,

$$(\forall z \in [0,1))\big((\forall y \in (\pi_1(z),1))(z,y) \in K\big) \text{ and } \big((\forall x \in (\pi_2(1-z),1))(x,1-z) \in K\big).$$

Proposition 11 *π_1 and π_2 are non-increasing. Moreover, for any z in $[0,1)$:*

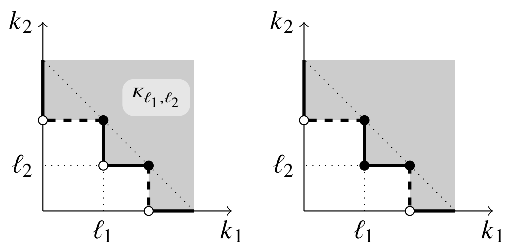
- *If* $\pi_1(z) < 1 - z$ *then* $\pi_1(z) = \pi_1(t)$ *for all* t *in* $(z, 1 - \pi_1(z))$*;*
- *If* $\pi_2(z) < 1 - z$ *then* $\pi_2(z) = \pi_2(t)$ *for all* t *in* $(z, 1 - \pi_2(z))$.

Next we introduce the notion of *weak Archimedeanity* that, as we shall show, shall be instrumental in characterising those coherent choice functions that are the infima of a family of lexicographic choice functions.[2] We begin by giving the definition in terms of rejection sets:

Definition 10 A rejection set K is called *weakly Archimedean* when it satisfies

$$\big(\forall (k_1, k_2) \in (0,1)^2, \forall k_1' \in (k_1, 1), \forall k_2' \in (k_2, 1)\big)$$
$$(k_1 + k_2 < 1, (k_1', k_2) \in K, (k_1, k_2') \in K \Rightarrow (k_1, k_2) \in K).$$

The term Archimedeanity is reminiscent of some type of continuity in the preferences encompassed by the rejection set, and indeed implies that the rejection set must be closed in one particular case: when an element can be approximated from the top and from the right within the rejection set. Our definition is inspired by the idea of excluding rejection functions such as the one in our Example 2, that, as we have shown, is no infimum of lexicographic choice functions. Indeed, weak Archimedeanity rules out the rejection functions that have the rejection set K_{ℓ_1,ℓ_2} from the left figure in its basis.



Observe that $\ell_1 + \ell_2 < 1$, $(k_1', \ell_2) \in K$ and $(\ell_1, k_2') \in K$ for every k_1' in $(\ell_1, 1)$ and k_2' in $(\ell_2, 1)$, but $(\ell_1, \ell_2) \notin K$. Weak Archimedeanity implies that $(\ell_1, \ell_2) \in K$, as depicted on the right figure. In terms of choice functions, the notion of weak Archimedeanity becomes the following:

Definition 11 A rejection function R on a binary possibility space is called *weakly Archimedean* when it satisfies

$$(\forall u \in \mathcal{V}_{\mathrm{II}}, \forall v \in \mathcal{V}_{\mathrm{IV}})$$
$$\big(\mathrm{posi}(\{u, v\}) \cap \mathcal{V}_{\geq 0} = \emptyset, (\forall \epsilon \in \mathbb{R}_{>0})(0 \in R(\{u + \epsilon, 0, v\}) \text{ and } 0 \in R(\{u, 0, v + \epsilon\}))$$
$$\Rightarrow 0 \in R(\{u, 0, v\})\big). \quad (19)$$

[2] We want to caution the reader that there are other non-equivalent definitions of weak Archimedeanity: for instance, the one given by Zaffalon and Miranda in [27, Defintion 19] and [28] for binary comparison of the options, differs from ours, even when we restrict our definition to binary option sets.

The intuition here is that if an increase, no matter how small, of one of the two options allows us to reject the zero option, then the options u, v by themselves should also let us reject 0. Recall that if $\{H, T\}$ is a basis of the vector space $\mathcal{V}$, then $\mathcal{V}_{\mathrm{II}}, \mathcal{V}_{\mathrm{IV}}$ are given by Equation (4), and that $\mathcal{V}_{\geq 0} \coloneqq \{f \in \mathcal{L}(\{\mathrm{H}, \mathrm{T}\}) : f(\mathrm{H}), f(\mathrm{T}) \geq 0\}$.

Proposition 12 *Consider some two-dimensional vector space $\mathcal{V}$ with basis $\{H, T\}$. Consider a coherent rejection function R on $\mathcal{V}$ satisfying Condition* (2) *and its associated rejection set K_R. Then K_R is weakly Archimedean if and only if R is weakly Archimedean.*

Weak Archimedeanity is closed under infima.

Proposition 13 *Consider some two-dimensional vector space $\mathcal{V}$ with basis $\{H, T\}$. Consider an arbitrary collection $\mathcal{R}'$ of coherent rejection functions on $\mathcal{V}$ that satisfy Condition* (2)*. If every rejection function in $\mathcal{R}'$ is weakly Archimedean, then so is* $\inf \mathcal{R}'$*. Similarly, given an arbitrary collection of rejection sets $\{K_i : i \in I\}$ satisfying Properties* K1–K3*, if every K_i is weakly Archimedean, then so is* $\inf\{K_i : i \in I\}$.

From this it follows that weak Archimedeanity is necessary for a coherent choice function to be the infimum of lexicographic ones:

Corollary 2 *Consider some two-dimensional vector space $\mathcal{V}$ with basis $\{H, T\}$. Any infimum of lexicographic rejection functions on $\mathcal{V}$ is weakly Archimedean.*

Next, we are going to establish that weak Archimedeanity is not only necessary, but also sufficient, for a coherent choice function to be the infimum of a family of lexicographic ones. We begin with an auxiliary result. It basically tells us that whenever $k_1 + k_2 < 1$ and $(k_1, k_2) \notin K$ then there is some $(z, 1-z)$ dominating (k_1, k_2) such that $(z-\epsilon, 1-z-\epsilon) \notin K$ for all ϵ in $\mathbb{R}_{>0}$.

Proposition 14 *Consider any rejection set K that satisfies Properties* K1–K3 *and that is weakly Archimedean, and consider any (k_1, k_2) in $[0,1)^2$ such that $k_1 + k_2 < 1$ and $(k_1, k_2) \notin K$. Then $\pi_1(z) = 1 - z$ or $\pi_2(1-z) = z$ for some z in $[k_1, 1-k_2]$.*

Definition 12 Given a rejection set K that satisfies Properties K1–K4, we define $\mathcal{D}' \subseteq \bar{\mathcal{D}}_{\mathrm{L}}$ as $\mathcal{D}' \coloneqq \bigcup\{D^x : x \in [0,1)\}$, where, for all x in $(0,1)$:

$$D^x \coloneqq \begin{cases} \{D_x\} & \text{if } (x, 1-x) \notin K \\ \{D_x^H\} & \text{if } (x, 1-x) \in K, (\forall \epsilon \in \mathbb{R}_{>0})(x, 1-x-\epsilon) \notin K \\ & \quad \text{and } (\forall \epsilon \in \mathbb{R}_{>0})(x-\epsilon, 1-x) \in K \\ \{D_x^T\} & \text{if } (x, 1-x) \in K, (\forall \epsilon \in \mathbb{R}_{>0})(x, 1-x-\epsilon) \in K \\ & \quad \text{and } (\forall \epsilon \in \mathbb{R}_{>0})(x-\epsilon, 1-x) \notin K \\ \{D_x^H, D_x^T\} & \text{if } (x, 1-x) \in K, (\forall \epsilon \in \mathbb{R}_{>0})(x, 1-x-\epsilon) \notin K \\ & \quad \text{and } (\forall \epsilon \in \mathbb{R}_{>0})(x-\epsilon, 1-x) \notin K \end{cases}$$

and

$$D^0 := \begin{cases} \{D_0^H\} & \text{if } (\forall \epsilon \in \mathbb{R}_{>0})(0, 1-\epsilon) \notin K \text{ and } (\exists \epsilon \in \mathbb{R}_{>0})(1-\epsilon, 0) \in K \\ \{D_1^T\} & \text{if } (\forall \epsilon \in \mathbb{R}_{>0})(1-\epsilon, 0) \notin K \text{ and } (\exists \epsilon \in \mathbb{R}_{>0})(0, 1-\epsilon) \in K \\ \{D_0^H, D_1^T\} & \text{if } (\forall \epsilon \in \mathbb{R}_{>0})(0, 1-\epsilon) \notin K \text{ and } (1-\epsilon, 0) \notin K \\ \varnothing & \text{if } (\exists \epsilon \in \mathbb{R}_{>0})(0, 1-\epsilon) \in K \text{ and } (1-\epsilon, 0) \in K. \end{cases}$$

Using this collection of sets of desirable options, we define a coherent rejection function $\inf\{R_D : D \in \mathcal{D}'\}$, whose rejection set we call K':

$$K' := \bigcap_{D \in \mathcal{D}'} K_{R_D} = \{(k_1, k_2) \in [0,1)^2 : 0 \in \bigcap_{D \in \mathcal{D}'} R_D(\{(k_1-1, k_1), 0, (k_2, k_2-1)\})\}.$$

Our next theorem shows that every rejection set that is weakly Archimedean, and therefore every rejection function on a binary possibility space that is weakly Archimedean, is an infimum of lexicographic choice functions.

Theorem 2 *Given a rejection set K that satisfies Properties* K1–K4 *and that is weakly Archimedean, we let $\mathcal{D}'$ be the collection of lexicographic coherent set of desirable options as in Definition 12 and K' the rejection set that corresponds to the rejection function* $\inf\{R_D : D \in \mathcal{D}'\}$*. Then $K' = K$, and hence K is an infimum of lexicographic rejection sets.*

We see then that it is weak Archimedeanity, and not condition R5 alone, that allows us to characterise those coherent choice functions on binary possibility spaces that are the infimum of lexicographic ones.

6 Discussion

There are several open problems deriving from this work. First and foremost, we should extend our characterisation of the infimum of lexicographic choice functions to higher-dimensional option spaces. One difficulty here is that a representation akin to the one we have given in terms of rejection sets seems hard, because we will not be able to reduce the choices to option sets of either two or three gambles, as we have done here: it can be checked that, even in the case of a space of three elements, there is no upper bound on the cardinality of the option sets characterising our choices [10]. Another matter is that for general possibility spaces coherent sets of desirable options may have much more complex structures than the ones in the binary case. In particular, lexicographic sets of gambles will no longer be either maximal or strictly desirable; while this provides us with some additional expressive power, it also complicates the technical developments.

On the other hand, we would also like to combine our results with those in [21, 23, 24], by investigating the notion of indifference and the process of conditioning with lexicographic choice functions. In particular, this should allow us to link our work with that on conditioning lexicographic probabilities by Blume [3].

Finally, we should compare our work with the recent axiomatisation proposed by De Bock & De Cooman [5] that, by including one extra rationality axiom, leads to a subfamily of coherent choice functions that are represented in terms of sets of desirable gambles. This approach excludes (renders incoherent) choice functions such as those in [20, Example 16]. In the binary case, we can deduce from our results in this paper that the additional rationality axiom in [5] can be derived from our notion of weak Archimedeanity; for more general spaces it has been established in [4, Theorem 19] that convexity becomes sufficient and not only necessary for a choice function that is coherent under their axiomatisation to be the infimum of a family of lexicographic ones.

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Appendix: Proofs

Proof (of Proposition 1) It suffices to prove the direct implication. To this end, consider any $u_1, \dots, u_n$ in $\mathcal{V}$ and $\mu_1, \dots, \mu_n$ in $\mathbb{R}_{>0}$, and assume that $0 \in R(\{0, u_1, \dots, u_n\})$. Let $\mu^* := \min\{\mu_1, \dots, \mu_n\} \in \mathbb{R}_{>0}$, $A := \{0, \mu_1 u_1, \dots, \mu_n u_n\}$ and $A' := \{0, \mu^* u_1, \dots, \mu^* u_n\}$; we need to show that then $0 \in R(A)$. Using Axiom R4a we infer that $0 \in R(A')$, and using Axiom R3a also that $0 \in R(A_1)$, with $A_1 := A \cup A'$. Note that $\mu^* u_k \in \mathrm{CH}(\{0, \mu_k u_k\})$ for every k in $\{1, \dots, n\}$, whence $A \subseteq A_1 \subseteq \mathrm{CH}(A)$. Therefore, by applying Axiom R5 we find indeed that $0 \in R(A)$.□

Proof (of Proposition 2) *** Quique, this is my try based on the proof of Proposition 25 from the thesis: *** We will first show that (i) implies (ii) and (iii). That (i) implies (ii) follows immediately from Axiom R3b [with $\tilde{A} := R(A) \setminus \{u\}$, $\tilde{A}_1 := R(A)$ and $\tilde{A}_2 := A$]. That (i) implies (iii) follows immediately from Axiom R3b [with $\tilde{A} := \{v\}$, $\tilde{A}_1 := \{0, v\}$ and $\tilde{A}_2 := A$].

We will now assume that R satisfies Axiom R3a and show that then (ii) implies (i). To this end, consider any A, A_1 and A_2 in $\mathcal{Q}$ in and assume that $A \subseteq A_1 \subseteq R(A_2)$. Then in particular $u \in R(A_2)$, and therefore, using (ii), $u \in R(\{u\} \cup A_2 \setminus R(A_2))$, for every u in $A_1 \setminus A$. Applying Axiom R3a, we infer that $u \in R(A_2 \setminus A)$ for every u in $A_1 \setminus A$, whence indeed $A_1 \setminus A \subseteq R(A_2 \setminus A)$.

To finish the proof, we will assume that R additionally satisfies Axiom R4b, and show that then (iii) implies (i). To this end, infer first that (iii) implies, using Axiom R4b, that

$$(\forall A \in \mathcal{Q})(\forall u \in R(A))(\forall v \in R(A) \setminus \{u\}) u \in R(A \setminus \{v\}), \tag{20}$$

which is easily seen once we realise that Axiom R4b implies that $u \in R(A)$ is equivalent to $0 \in R(A - \{u\})$, for any A in $\mathcal{Q}$ and u in A. So assume that R satisfies Equation (20); we will prove that it satisfies Axiom R3b. Let $A := \{u_1, \dots, u_n\}$,

$A_1 := A \cup \{v_1, \dots, v_m\}$ and $A_2 := A_1 \cup \{w_1, \dots, w_r\}$, where $n \in \mathbb{N}$ and $m, r \in \mathbb{Z}_{\geq 0}$, and assume that $A_1 \subseteq R(A_2)$. Consider any j in $\{1, \dots, m\}$, then we have to prove that $v_j \in R(\{v_1, \dots, v_m, w_1, \dots, w_r\}) = R(A_2 \setminus \{u_1, \dots, u_n\})$. Since $\{u_1, u_2\} \subseteq R(A_2)$ and $\{v_j, u_1\} \subseteq R(A_2)$, it follows from Equation (20) that $\{u_2, v_j\} \subseteq R(A_2 \setminus \{u_1\})$, whence, again using Equation (20), $v_j \in R(A_2 \setminus \{u_1, u_2\})$. Also, $\{u_1, u_3\} \subseteq R(A_2)$, whence $u_3 \in R(A_2 \setminus \{u_1\})$ using Equation (20). Since we already know that also $u_2 \in R(A_2 \setminus \{u_1\})$, we infer that $u_3 \in R(A_2 \setminus \{u_1, u_2\})$, again using Equation (20). In turn, this implies that $v_j \in R(A_2 \setminus \{u_1, u_2, u_3\})$. We can go on in this way until we reach the desired statement, that $v_j \in R(A_2 \setminus \{u_1, \dots, u_n\})$, after a finite number of steps.

*** Quique, this is the old version of the proof: *** That (i) implies (ii) and (iii) follows from Axiom R3b.

To prove that (ii) implies (i) under Axiom R3a, consider any A, A_1 and A_2 in $\mathcal{Q}$ and assume that $A \subseteq A_1 \subseteq R(A_2)$. Then, for every u in $A_1 \setminus A$, in particular $u \in R(A_2)$, and therefore, using (ii), $u \in R(\{u\} \cup A_2 \setminus R(A_2))$. Applying Axiom R3a, we infer that $u \in R(A_2 \setminus A)$ for every u in $A_1 \setminus A$, whence indeed $A_1 \setminus A \subseteq R(A_2 \setminus A)$.

To see that (iii) implies (ii) when R satisfies additionally Axiom R4b, note that Axiom R4b implies that $u \in R(A)$ is equivalent to $0 \in R(A - \{u\})$, for any A in $\mathcal{Q}$ and u in A. Applying then (iii) a finite number of times we deduce that it implies (ii), which from the previous statement is equivalent to (i) under Axiom R3a. Quique, I don't see how we can apply this, because the option set for which will apply it will change every time, doesn't it? I think that it will every application be a smaller option set. □

Proof (of Corollary 1) That the first statement implies the second is immediate. To establish the converse, we will prove the contraposition. Assume that R does not satisfy Axiom R1. Therefore, we have that $A = R(A)$ for some A in $\mathcal{Q}$. Consider any u in A, then by Proposition 2(ii) we find that $u \in R(\{u\} \cup A \setminus R(A)) = R(\{u\} \cup A \setminus A) = R(\{u\})$. By Axiom R4b therefore indeed $0 \in R(\{0\})$. □

Proof (of Lemma 1) If $a = 0$ then $(0, a) = (0, 0) \notin K$ by Property K2. Analogously, if $a = 1$ then $(1 - a, 0) = (0, 0) \notin K$ by Property K2. Assume therefore that $a \in (0, 1)$, and assume *ex absurdo* that both $(0, a)$ and $(1 - a, 0)$ are elements of K. Use Property K3b to infer that $(0, 0) \in K$, which contradicts Property K2. □

Proof (of Proposition 7) We first prove that K_R satisfies Property K1. Consider any (k_1, k_2) in K_R, and any (k_1', k_2') in $[0, 1)^2$ such that $k_1' \geq k_1$ and $k_2' \geq k_2$. Then $(k_1, k_2) \in K_R$ simply means that $0 \in R(\{(k_1 - 1, k_1), 0, (k_2, k_2 - 1)\})$, and $k_1' \geq k_1$ and $k_2' \geq k_2$ implies that $(k_1' - 1, k_1') \geq (k_1 - 1, k_1)$ and $(k_2' - 1, k_2') \geq (k_2 - 1, k_2)$. [22, Proposition 2] tells us that then $0 \in R(\{(k_1' - 1, k_1'), 0, (k_2', k_2' - 1)\})$, whence indeed $(k_1', k_2') \in K_R$.

To prove that K_R satisfies Property K2, assume *ex absurdo* that $0 \in K_R$, or equivalently, that $0 \in R(\{(-1, 0), 0, (0, -1)\})$. Since $(-1, 0) < 0$, we infer from Axiom R2 that $(-1, 0) \in R(\{(-1, 0), 0\})$, and therefore also that $(-1, 0) \in R(\{(-1, 0), 0, (0, -1)\})$, by Axiom R3a. A similar argument leads from $(0, -1) < 0$ to $(0, -1) \in R(\{(-1, 0), 0, (0, -1)\})$. This implies that $\{(-1, 0), 0, (0, -1)\} = R(\{(-1, 0), 0, (0, -1)\})$, which contradicts Axiom R1.

Next, assume that R satisfies Condition (2). To prove that K_R then satisfies Property K3, we first prove that it satisfies Property K3a. Consider any a, b and c in $[0,1)$ and assume that $c < a$, $a + b < 1$, and that (b, a) and $(1 - a, c)$ belong to K_R. We are going to prove that $(b, y) \in K_R$ for every y in $(c, 1)$; the proof that also $(x, c) \in K_R$ for every x in $(b, 1)$ is similar. Consider any λ in $\mathbb{R}_{>0}$, then Condition (2) guarantees that $0 \in R(\{(b-1, b), 0, \lambda(a, a-1)\})$ and $0 \in R(\{\lambda(-a, 1-a), 0, (c, c-1)\})$. By Axiom R4b, we then find that $-\lambda(a, a-1) \in R(\{(b - \lambda a - 1, b - \lambda a + \lambda), -\lambda(a, a-1), 0\})$ and $\lambda(a, a-1) \in R(\{0, \lambda(a, a-1), (c + \lambda a, c + \lambda a - \lambda - 1)\})$, and applying Axiom R3a then leads to $\{-\lambda(a, a-1), \lambda(a, a-1)\} \subseteq R(\{(b - \lambda a - 1, b - \lambda a + \lambda), -\lambda(a, a-1), 0, \lambda(a, a-1), (c + \lambda a, c + \lambda a - \lambda - 1)\})$. This, together with Axiom R3a, implies that $\{-\lambda(a, a-1), 0, \lambda(a, a-1)\} \subseteq R(\{(b - \lambda a - 1, b - \lambda a + \lambda), -\lambda(a, a-1), 0, (c, c-1), \lambda(a, a-1), (c + \lambda a, c + \lambda a - \lambda - 1)\})$. Applying Axiom R3b implies that $-\lambda(a, a-1)$ is included in $R(\{(b - \lambda a - 1, b - \lambda a + \lambda), -\lambda(a, a-1), (c, c-1), (c + \lambda a, c + \lambda a - \lambda - 1)\})$ and by Axiom R4b this implies that $0 \in R(\{(b-1, b), 0, (c + \lambda a, c + \lambda a - \lambda - 1), (c + 2\lambda a, c + 2\lambda a - 2\lambda - 1)\})$. Let us call $u := (c + \lambda a, c + \lambda a - \lambda - 1)$ and $v := (c + 2\lambda a, c + 2\lambda a - 2\lambda - 1)$, and $\mu_1 := \frac{1}{c+\lambda a}$ and $\mu_2 := \frac{1}{c+2\lambda a}$; these real numbers are both positive since $0 \le c < a$ and $\lambda > 0$. Then $0 \in R(\{(b-1, b), 0, u, v\})$, and $0 \in R(\{(b-1, b), 0, \mu_1 u, \mu_2 v\})$ by Condition (2). But $\mu_1 u < \mu_2 v$ since $\mu_1 u = (1, \frac{c+\lambda a-\lambda-1}{c+\lambda a})$ and $\mu_2 v = (1, \frac{c+2\lambda a-2\lambda-1}{c+2\lambda a})$, and $\frac{c+\lambda a-\lambda-1}{c+\lambda a} < \frac{c+2\lambda a-2\lambda-1}{c+2\lambda a}$ using the assumptions. Then $\mu_1 u \in R(\{\mu_1 u, \mu_2 v\})$ by Axiom R2, whence $\{0, \mu_1 u\} \subseteq R(\{(b-1, b), 0, \mu_1 u, \mu_2 v\})$ by Axiom R3a. Then $0 \in R(\{(b-1, b), 0, \mu_2 v\})$ by Axiom R3b, and $0 \in R(\{(b-1, b), 0, \mu_3 v\})$ by Condition (2) with $\mu_3 = \frac{1}{2\lambda+1} > 0$, whence $\left(b, \frac{c+2\lambda a}{1+2\lambda}\right) \in K_R$. Now, by varying λ in $\mathbb{R}_{>0}$ the number $\frac{c+2\lambda a}{1+2\lambda}$ can take any value in the interval (c, a). We conclude that $(b, y) \in K_R$ for every $y \in (c, 1)$, after also recalling that we have already proved that K_R satisfies Property K1.

To prove that K_R satisfies Property K3b, assume that $0 \le c < a < 1$, $(0, a) \in K_R$ and $(1 - a, c) \in K_R$. Because K_R already satisfies Property K3a [with in particular $b := 0$], we know that $(x, c) \in K_R$ for every x in $(0, 1)$ and $(0, y) \in K_R$ for every y in $(c, 1)$. We have to show that $(0, c) \in K_R$. To this end, consider the gambles $u := (\frac{1-c}{2} - 1, \frac{1-c}{2})$ and $v := (c, c-1)$. Because in particular $(x, c) \in K_R$ for $x = \frac{1-c}{2} \in (0, 1)$, we have that $0 \in R(\{u, 0, v\})$. Similarly, because in particular $(0, y) \in K_R$ for $y = \frac{1+c}{2} \in (c, 1)$, we have that $0 \in R(\{(-1, 0), 0, -u\})$. Since also $(-1, 0) \in R(\{(-1, 0), 0\})$—and therefore $(-1, 0) \in R(\{(-1, 0), 0, -u\})$ by Axiom R3a—because $(-1, 0) < 0$ and by Axiom R2, this leads us to conclude that $\{(-1, 0), 0\} \subseteq R(\{(-1, 0), 0, -u\})$, and therefore also $0 \in R(\{0, -u\})$ by Axiom R3b. Hence, $u \in R(\{u, 0\})$, by Axiom R4b, and therefore $u \in R(\{u, 0, v\})$, by Axiom R3a. Hence $\{0, u\} \subseteq R(\{u, 0, v\})$, so Axiom R3b leads to $0 \in R(\{0, v\})$. Now Axiom R3a implies that indeed $(0, c) \in K_R$, so Property K3b is satisfied. Property K3c can be shown to hold in a similar way.

To conclude, assume that R satisfies Axiom R5. Since this implies that Condition (2) holds by Proposition 1, we already know that Property K3 is satisfied, so it only remains to prove that K_R satisfies Property K4. Consider any (k_1, k_2) in $[0,1)^2$ such that $k_1 + k_2 > 1$. Then $\left(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\right) > 0$, whence $0 \in R(\{0, (\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2})\})$ by Axiom R2. By Axiom R3a, we get $0 \in R(\{(k_1-1,k_1), 0, (k_2,k_2-1), (\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2})\})$. Since $(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}) \in \mathrm{CH}(\{(k_1-1,k_1),(k_2,k_2-1)\})$, Axiom R5 leads us to conclude that $0 \in R(\{(k_1-1,k_1), 0, (k_2,k_2-1)\})$, so indeed $(k_1,k_2) \in K_R$. □

Proof (of Lemma 2) If Condition (7) holds, then $\{\lambda_1(k_1-1,k_1),(0,-1)\} \subseteq A \cup \{(-1,0),(0,-1)\}$, so Condition (11) holds with $\lambda_2 := 1$ and $k_2 := 0$. Similarly, if Condition (8) holds, then $\{(-1,0),\lambda_2(k_2,k_2-1)\} \subseteq A \cup \{(-1,0),(0,-1)\}$, so Condition (11) holds with $\lambda_1 := 1$ and $k_1 := 0$. If Condition (9) holds, then Condition (11) holds trivially.

Conversely, assume that Condition (11) holds. If both $k_1 \neq 0$ and $k_2 \neq 0$, then Condition (9) holds trivially, so assume that either $k_1 = 0$ or $k_2 = 0$—they cannot both be zero, because $0 \notin K$. So assume that $k_1 = 0$ and $k_2 > 0$, then we infer from the assumption that $\{\lambda_1(-1,0),\lambda_2(k_2,k_2-1)\} \subseteq A \cup \{(-1,0),(0,-1)\}$. Since $k_2 > 0$ implies that $\lambda_2(k_2,k_2-1) \neq (-1,0)$ and $\lambda_2(k_2,k_2-1) \neq (0,-1)$ for any choice of $\lambda_2 > 0$, it must be that $\lambda_2(k_2,k_2-1) \in A$, so Condition (8) holds. The case $k_2 = 0$ and $k_1 > 0$ is similar. □

Proof (of Proposition 8) For Axiom R2, consider any f and g in $\mathcal{L}$ such that $f < g$. Then $0 < g - f$, so we infer from Condition (6) that $0 \in R_K(\{0, g-f\})$, and then from Condition (10) that indeed $f \in R_K(\{f,g\})$.

For Axiom R3a, assume that $A_1 \subseteq R_K(A_2)$ and $A_2 \subseteq A$. Then we need to prove that $A_1 \subseteq R_K(A)$. Consider any $f \in A_1$, then also $f \in A_2$ and $f \in A$, so we can let $A_2' := A_2 - \{f\}$ and $A' := A - \{f\}$, where $A_2' \subseteq A'$. We then infer from Condition (10) that $0 \in R_K(A_2')$, which means that at least one of the Conditions (6)–(9) holds. But any of these conditions implies that also $0 \in R_K(A')$. Condition (10) then guarantees that $f \in R_K(A)$ and therefore that, indeed, $A_1 \subseteq R_K(A)$.

That Axioms R4a and R4b are satisfied follows from Conditions (6)–(10).

For Condition (2), consider any option set $A = \{f_1, \dots, f_n\} \in \mathcal{Q}$, where n is a natural number, and any positive real numbers $\mu_1, \dots, \mu_n$. Assume that $0 \in R_K(\{0\} \cup A)$. First of all, if $f_i \in \mathcal{L}_{>0}$ for some i in $\{1,\dots,n\}$, then also $\mu_i f_i \in \mathcal{L}_{>0}$ since $\mu_i > 0$, whence indeed $0 \in R_K(\{0, \mu_1 f_1, \dots, \mu_n f_n\})$, by Condition (6). So assume that $f_i \notin \mathcal{L}_{>0}$ for all i in $\{1,\dots,n\}$. There are now only three possibilities. The first is that there are λ_1 in $\mathbb{R}_{>0}$ and $(k_1,0)$ in K such that $\lambda_1(k_1-1,k_1) = f_i$ for some i in $\{1,\dots,n\}$. Then $(\lambda_1\mu_i)(k_1-1,k_1) = \mu_i f_i \in \{\mu_1 f_1, \dots, \mu_n f_n\}$, and Condition (7) guarantees that indeed $0 \in R_K(\{0, \mu_1 f_1, \dots, \mu_n f_n\})$. The second possibility is that there are λ_2 in $\mathbb{R}_{>0}$ and $(0,k_2)$ in K such that $\lambda_2(k_2,k_2-1) = f_j$ for some j in $\{1,\dots,n\}$. Then $(\lambda_2\mu_j)(k_2,k_2-1) = \mu_j f_j \in \{\mu_1 f_1, \dots, \mu_n f_n\}$, and Condition (8) guarantees that indeed $0 \in R_K(\{0, \mu_1 f_1, \dots, \mu_n f_n\})$. And the final possibility is that there are λ_1 and λ_2 in $\mathbb{R}_{>0}$ and (k_1,k_2) in $K \cap (0,1)^2$ such that $\lambda_1(k_1-1,k_1) = f_i$ and $\lambda_2(k_2,k_2-1) = f_j$ for some i and j in $\{1,\dots,n\}$. Then $(\lambda_1\mu_i)(k_1-1,k_1) = \mu_i f_i$ and $(\lambda_2\mu_j)(k_2,k_2-1) = \mu_j f_j$, and Condition (9) guarantees that indeed $0 \in R_K(\{0, \mu_1 f_1, \dots, \mu_n f_n\})$.

Assume now that K satisfies in addition Properties K1–K3. We begin by proving that R_K satisfies Axiom R3b. Assume *ex absurdo* that it does not, then Proposition 2 guarantees that there are A in $\mathcal{Q}$ and g in $A \setminus \{0\}$ such that $\{0, g\} \subseteq R_K(A)$ and $0 \notin R_K(A \setminus \{g\})$.

Because $0 \in R_K(A)$, we infer from Definition 8 and Lemma 2 that there are two possibilities: (i) $A \cap \mathcal{L}_{>0} \neq \emptyset$, or (ii) $\{\lambda_1(k_1 - 1, k_1), \lambda_2(k_2, k_2 - 1)\} \subseteq A \cup \{(-1, 0), (0, -1)\}$ for some λ_1 and λ_2 in $\mathbb{R}_{>0}$ and some (k_1, k_2) in K.

We first deal with case (i). Here we can assume without loss of generality that $A \cap \mathcal{L}_{>0} = \{g\}$ because, otherwise $A \setminus \{g\} \cap \mathcal{L}_{>0} \neq \emptyset$ and we could apply Condition (6) to conclude that $0 \in R_K(A \setminus \{g\})$, a contradiction. We will use the notation $g = (x, y) > 0$. Because also $g \in R_K(A)$, Condition (10) guarantees that $0 \in R_K(A - \{g\})$, and a similar argument as before shows that there are now two possibilities: (i.a) $(A - \{g\}) \cap \mathcal{L}_{>0} \neq \emptyset$; and (i.b) $\{\lambda_3(k_3 - 1, k_3), \lambda_4(k_4, k_4 - 1)\} \subseteq (A - \{g\}) \cup \{(-1, 0), (0, -1)\}$ for some λ_3 and λ_4 in $\mathbb{R}_{>0}$ and some (k_3, k_4) in K. But in fact (i.a) is impossible, because it would contradict our earlier conclusion that $A \cap \mathcal{L}_{>0} = \{g\}$. So we can restrict our attention to case (i.b) with $(A - \{g\}) \cap \mathcal{L}_{>0} = \emptyset$. There are now 3 possibilities: (i.b.1) $k_3 \neq 0 \neq k_4$ corresponding to Condition (9), (i.b.2) $k_3 = 0 \neq k_4$ corresponding to Condition (8), and (i.b.3) $k_3 \neq 0 = k_4$ corresponding to Condition (7)—$k_3 = 0 = k_4$ is impossible because $0 \notin K$. It is possible to show that each of these three cases leads eventually to $0 \in R_K(A \setminus \{g\})$, a contradiction.

We now turn to case (ii), where we assume that $A \cap \mathcal{L}_{>0} = \emptyset$ and that there are λ_1 and λ_2 in $\mathbb{R}_{>0}$ and (k_1, k_2) in K such that $\{\lambda_1(k_1 - 1, k_1), \lambda_2(k_2, k_2 - 1)\} \subseteq A \cup \{(-1, 0), (0, -1)\}$. Here we distinguish between three possibilities: (ii.a) $g \notin \{\lambda_1(k_1 - 1, k_1), \lambda_2(k_2, k_2 - 1)\}$, (ii.b) $g = \lambda_1(k_1 - 1, k_1)$, and (ii.c) $g = \lambda_2(k_2, k_2 - 1)$.

But we see at once that case (ii.a) is impossible, because it implies by Condition (11) that $0 \in R_K(A \setminus \{g\})$, a contradiction. So we now concentrate on the cases (ii.b) and (ii.c), where it is by the way obvious that indeed $A \cap \mathcal{L}_{>0} = \emptyset$.

We begin with the discussion of case (ii.b). We first of all claim that now $k_1 > 0$. Indeed, if $k_1 = 0$ then $(k_1, k_2) = (0, k_2) \in K$, and Property K2 implies that $k_2 > 0$. Since we know that in this case $\lambda_2(k_2, k_2 - 1) \in A \setminus \{g\}$ [since $g = \lambda_1(k_1 - 1, k_1) \neq \lambda_2(k_2, k_2 - 1)$], Condition (8) guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

So we may assume that $k_1 > 0$, and the assumption that $g \in R_K(A)$, or in other words, that $0 \in R_K(A - \{g\})$, leaves us with two possibilities: that (ii.b.1) $(A - \{g\}) \cap \mathcal{L}_{>0} \neq \emptyset$, or that (ii.b.2) $\{\lambda_3(k_3 - 1, k_3), \lambda_4(k_4, k_4 - 1)\} \subseteq (A - \{g\}) \cup \{(-1, 0), (0, -1)\}$ for some λ_3 and λ_4 in $\mathbb{R}_{>0}$ and (k_3, k_4) in K.

For case (ii.b.1), there is some $h \coloneqq (x', y') > 0$ such that $f \coloneqq g + h \in A$. Since the second component $\lambda_1 k_1 + y'$ of f is positive and $f \notin \mathcal{L}_{>0}$, we find that f must lie in the second quadrant, and therefore its first component $\lambda_1 k_1 - \lambda_1 + x'$ is negative: $\lambda_1 k_1 < \lambda_1 - x'$ and therefore $\lambda_3^* \coloneqq \lambda_1 - x' + y' > 0$. If we now let $k_3^* \coloneqq \frac{\lambda_1 k_1 + y'}{\lambda_1 - x' + y'}$, then $f = \lambda_3^*(k_3^* - 1, k_3^*)$. Moreover, $k_3^* < 1$ because this is equivalent to $\lambda_1 k_1 - \lambda_1 + x' < 0$, which we have already found to be true. Similarly, $k_3^* \geq k_1$ because this is equivalent to $x' k_1 + y'(1 - k_1) \geq 0$. Then $(k_3^*, k_2) \in K$ because $(k_1, k_2) \in K$ and K is increasing [Property K1]. Since we now know that $\{\lambda_3^*(k_3^* - 1, k_3^*), \lambda_2(k_2, k_2 - 1)\} \subseteq A \setminus \{g\}$, Condition (9) guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

For case (ii.b.2), $\{g + \lambda_3(k_3 - 1, k_3), g + \lambda_4(k_4, k_4 - 1)\} \subseteq A \cup \{g + (-1, 0), g + (0, -1)\}$, or in other words, $\{(\lambda_1 k_1 + \lambda_3 k_3 - \lambda_1 - \lambda_3, \lambda_1 k_1 + \lambda_3 k_3), (\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4)\} \subseteq A \cup \{g + (-1, 0), g + (0, -1)\}$ We claim that here $k_3 < k_1$.

To prove this, assume *ex absurdo* that $k_3 \geq k_1$, then also $k_3^* := \frac{\lambda_1 k_1 + \lambda_3 k_3}{\lambda_1 + \lambda_3} \geq k_1 > 0$. Moreover, $k_3^* < 1$ because it is a convex combination of $k_1 < 1$ and $k_3 < 1$, and therefore $(k_3^*, k_2) \in [0,1)^2 \setminus \{0\}$ and $(k_3^*, k_2) \geq (k_1, k_2)$. Then $(k_3^*, k_2) \in K$ because $(k_1, k_2) \in K$ and K is increasing [Property K1]. Moreover, if we also let $\lambda_3^* := \lambda_1 + \lambda_3 > 0$, then $\lambda_3^*(k_3^* - 1, k_3^*) = g + \lambda_3(k_3 - 1, k_3) \in A \cup \{g + (-1,0), g + (0,-1)\}$, and since we know that $\lambda_3(k_3 - 1, k_3) \notin \{(-1,0), 0, (0,-1)\}$ [because $\lambda_3 > 0$ and $k_3 \geq k_1 > 0$], this leads us to conclude that $\{\lambda_3^*(k_3^* - 1, k_3^*), \lambda_2(k_2, k_2 - 1)\} \subseteq A \setminus \{g\}$, so Condition (9) together with $(k_3^*, k_2) \in K$ guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

Since $k_3 < k_1$ rules out the possibility that $k_1 = 0$, we find that $k_1 > 0$ as an intermediate result. In the remainder of this case (ii.b), note that nothing depends on whether $k_2 = 0$ or $k_2 > 0$. We can now distinguish between three *distinct* possibilities: (ii.b.2.1) $k_3 > 0$ and $k_4 > 0$, (ii.b.2.2) $k_3 = 0$ and $k_4 > 0$, and (ii.b.2.3) $k_3 > 0$ and $k_4 = 0$, which correspond to Conditions (9), (8) and (7), respectively—$k_3 = 0 = k_4$ is impossible because $0 \notin K$.

In case (ii.b.2.1) we see that $\{\lambda_3(k_3 - 1, k_3), \lambda_4(k_4, k_4 - 1)\} \cap \{(-1,0), 0, (0,-1)\} = \emptyset$, and therefore $\{(\lambda_1 k_1 + \lambda_3 k_3 - \lambda_1 - \lambda_3, \lambda_1 k_1 + \lambda_3 k_3), (\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4)\} \subseteq A \setminus \{g\}$. We distinguish between two possibilities, which will determine in what quadrants these points lie: $\lambda_4 \leq \lambda_1$ and $\lambda_4 > \lambda_1$.

If $\lambda_4 \leq \lambda_1$, then we establish, reasoning ex absurdo, that $k_4 \leq 1 - k_1$. Once we have this, because K is increasing [Property K1], we infer from $(k_3, k_4) \in K$ that $(k_3, 1 - k_1) \in K$. We distinguish between two further possibilities: $k_1 + k_2 < 1$ and $k_1 + k_2 \geq 1$.

If $k_1 + k_2 < 1$ then we can use Property K3a with $a = 1 - k_1$, $b = k_3$ and $c = k_2$. Observe that $a + b = 1 - k_1 + k_3 < 1$ because $k_3 < k_1$, that $c = k_2 < 1 - k_1 = a$ by assumption, that $(b, a) = (k_3, 1 - k_1) \in K$ has been proved above, and that $(1 - a, c) = (k_1, k_2) \in K$ also by assumption, whence $(\forall k_3' \in (k_3, 1))(k_3', k_2) \in K$. In particular, let $k_3' := \frac{\lambda_1 k_1 + \lambda_3 k_3}{\lambda_1 + \lambda_3}$. Then $k_3' > \min\{k_1, k_3\} = k_3 > 0$, where the first inequality follows from $\lambda_1 > 0$ and $\lambda_3 > 0$, and the equality from $k_3 < k_1$. Moreover, $k_3' < 1$ because it is a convex combination of $k_1 < 1$ and $k_3 < 1$. Hence $k_3' \in (k_3, 1)$ and therefore $(k_3', k_2) \in K$. If we now let $\lambda_3' := \lambda_1 + \lambda_3 > 0$, then we see that $\lambda_3'(k_3' - 1, k_3') = (\lambda_1 k_1 + \lambda_3 k_3 - \lambda_1 - \lambda_3, \lambda_1 k_1 + \lambda_3 k_3) \in A \setminus \{g\}$, whence also $\{\lambda_3'(k_3' - 1, k_3'), \lambda_2(k_2, k_2 - 1)\} \subseteq A \setminus \{g\}$, and Condition (9) now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

If $k_1 + k_2 \geq 1$ then we have that $k_2 \geq 1 - k_1 \geq k_4$. Also $k_3^* := \frac{\lambda_1 k_1 + \lambda_3 k_3}{\lambda_1 + \lambda_3} > \min\{k_1, k_3\} = k_3 > 0$, where the first inequality follows from $\lambda_1 > 0$ and $\lambda_3 > 0$, and the equality from $k_1 > k_3$. Moreover, $k_3^* < 1$ because it is a convex combination of $k_1 < 1$ and $k_3 < 1$. This tells us that $(k_3^*, k_2) \in [0,1)^2 \setminus \{0\}$ and $(k_3^*, k_2) > (k_3, 1 - k_1)$. We then find that $(k_3^*, k_2) \in K$ because $(k_3, 1 - k_1) \in K$ and K is increasing [Property K1]. If we now let $\lambda_3^* := \lambda_1 + \lambda_3 > 0$ then we find that $\lambda_3^*(k_3^* - 1, k_3^*) = (\lambda_1 k_1 + \lambda_3 k_3 - \lambda_1 - \lambda_3, \lambda_1 k_1 + \lambda_3 k_3) \in A \setminus \{0\}$, and therefore also $\{\lambda_3^*(k_3^* - 1, k_3^*), \lambda_2(k_2, k_2 - 1)\} \subseteq A \setminus \{g\}$, and Condition (9) now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

If $\lambda_4 > \lambda_1$, then we establish, again reasoning ex absurdo, that $k_4 \leq 1 - k_1$. Once we have this, using that K is increasing, we infer from $(k_3, k_4) \in K$ that $(k_3, 1 - k_1) \in K$.

We now have the same two possibilities $k_1 + k_2 < 1$ and $k_1 + k_2 \geq 1$ as before, and for each of them, we can construct a contradiction in exactly the same way as for the case when $\lambda_4 \leq \lambda_1$.

This shows that we always arrive at a contradiction in case (ii.b.2.1).

In case (ii.b.2.2) we see that $\lambda_4(k_4, k_4 - 1) \notin \{(-1,0), 0, (0,-1)\}$, and therefore $(\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4) \in A \setminus \{g\}$. We distinguish between two possibilities, which will determine in what quadrant this point lies: $\lambda_4 \leq \lambda_1$ or $\lambda_4 > \lambda_1$.

If $\lambda_4 \leq \lambda_1$, then we claim that $k_4 \leq 1 - k_1$. To prove this, assume *ex absurdo* that $k_4 > 1 - k_1$, so $k_1 + k_4 - 1 > 0$. If $\lambda_1 = \lambda_4$, then $(\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4) = \lambda_1(k_1 + k_4 - 1, k_1 + k_4 - 1) > 0$, a contradiction, so we may assume that $\lambda_4 < \lambda_1$. We now wonder in what quadrant the vector $(\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4) \neq 0$ lies. We infer from $k_1 > 0, \lambda_1 > \lambda_4 > 0$ and $k_1 + k_4 > 1$ that $\lambda_1 k_1 + \lambda_4 k_4 - \lambda_4 > \lambda_4(k_1 + k_4) - \lambda_4 > 0$. Since $A \cap \mathcal{L}_{>0} = \emptyset$, we find that $(\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4)$ must lie in the second quadrant, and therefore its first component $\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1$ must be negative: $\lambda_1 k_1 + \lambda_4 k_4 < \lambda_1$. This tells us that $k_4^* := \frac{\lambda_1 k_1 + \lambda_4 k_4 - \lambda_4}{\lambda_1 - \lambda_4} < 1$. Moreover, $k_4^* > k_1$ because this is equivalent to $k_4 > 1 - k_1$. Hence $(k_4^*, k_2) \in [0,1)^2 \setminus 0$ and $(k_4^*, k_2) > (k_1, k_2)$. This tells us that $(k_4^*, k_2) \in K$ because $(k_1, k_2) \in K$ and K is increasing [Property K1]. If we now let $\lambda_4^* := \lambda_1 - \lambda_4 > 0$, then we see that $\lambda_4^*(k_4^* - 1, k_4^*) = (\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4) \in A \setminus \{g\}$. Hence also $\{\lambda_4^*(k_4^* - 1, k_4^*), \lambda_2(k_2, k_2 - 1)\} \subseteq A \setminus \{g\}$, and Condition (9) now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

So we see that $0 < k_4 \leq 1 - k_1 < 1$, so $(0, 1 - k_1) \in [0,1)^2 \setminus \{0\}$ and $(0, 1 - k_1) \geq (0, k_4)$ and hence, because K is increasing [Property K1], we infer from $(0, k_4) = (k_3, k_4) \in K$ that also $(0, 1 - k_1) \in K$. We distinguish between two further possibilities: $k_1 + k_2 < 1$ and $k_1 + k_2 \geq 1$.

If $k_1 + k_2 < 1$ then we can use Property K3b with $a = 1 - k_1$ and $c = k_2$. Observe that $c = k_2 < 1 - k_1 = a$ by assumption, that $(0, a) = (0, 1 - k_1) \in K$ was derived above, and that $(1 - a, c) = (k_1, k_2) \in K$ also by assumption, and therefore we find that $(0, k_2) \in K$. Since $\lambda_2(k_2, k_2 - 1) \in A \setminus \{g\}$, Condition (8) now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

If $k_1 + k_2 \geq 1$ then we have that $k_2 \geq 1 - k_1 \geq k_4$. Then $(0, k_2) \in K$ because $(0, k_4) \in K$ and K is increasing [Property K1]. Since $\lambda_2(k_2, k_2 - 1) \in A \setminus \{g\}$, Condition (8) now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

If $\lambda_4 > \lambda_1$, then we claim that, here too, $k_4 \leq 1 - k_1$. To prove this, assume *ex absurdo* that $k_4 > 1 - k_1$. We wonder in what quadrant the vector $(\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4)$ lies. Infer from $0 < 1 - k_1 < k_4$ and $0 < \lambda_1 < \lambda_4$ that $\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1 > \lambda_1(k_1 + k_4) - \lambda_1 > 0$. Since $A \cap \mathcal{L}_{>0} = \emptyset$, we find that the vector $(\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4)$ must lie in the fourth quadrant, and therefore its second component $\lambda_1 k_1 + \lambda_4 k_4 - \lambda_4$ must be negative: $\lambda_1 k_1 + \lambda_4 k_4 < \lambda_4$. This tells us that $k_4^* := \frac{\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1}{\lambda_4 - \lambda_1} < 1$. Moreover, $k_4^* > k_4$ because this is equivalent to $k_4 > 1 - k_1$. Hence $(0, k_4^*) \in [0,1)^2 \setminus \{0\}$ and $(0, k_4^*) > (0, k_4)$. This tells us that $(0, k_4^*) \in K$ because $(0, k_4) \in K$ and K is increasing [Property K1]. If we now let $\lambda_4^* := \lambda_4 - \lambda_1 > 0$, then we see that $\lambda_4^*(k_4^*, k_4^* - 1) = (\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4) \in A \setminus \{g\}$, and Condition (8) now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

So we see that $0 < k_4 \leq 1 - k_1 < 0$, and hence, because K is increasing, we infer from $(k_3, k_4) \in K$ that $(k_3, 1 - k_1) \in K$. We now have the same two possibilities $k_1 + k_2 < 1$ and $k_1 + k_2 \geq 1$ as before, and for each of them, we can construct a contradiction in exactly the same way as for the case when $\lambda_4 \leq \lambda_1$.

We conclude that case (ii.b.2.2) always leads to a contradiction.

In case (ii.b.2.3) we see that $\lambda_3(k_3 - 1, k_3) \notin \{(-1, 0), 0, (0, -1)\}$, and therefore $(\lambda_1 k_1 + \lambda_3 k_3 - \lambda_1 - \lambda_3, \lambda_1 k_1 + \lambda_3 k_3) \in A \setminus \{g\}$, or if we let $\lambda_3^* := \lambda_1 + \lambda_3 > 0$ and $k_3^* := \frac{\lambda_1 k_1 + \lambda_3 k_3}{\lambda_1 + \lambda_3} > 0$, $\lambda_3^*(k_3^* - 1, k_3^*) \in A \setminus \{g\}$. Observe that also $k_3^* < 1$ because it is a convex combination of $k_1 < 1$ and $k_3 < 1$. This tells us that $(k_3^*, 0) \in [0, 1)^2 \setminus \{0\}$. Moreover, we have that $k_3^* > \min\{k_1, k_3\} = k_3 > 0$ [the strict inequality holds because $\lambda_1 > 0$ and $\lambda_3 > 0$, and the equality holds because $k_1 > k_3$. Hence $(k_3^*, 0) > (k_3, 0)$ and therefore $(k_3^*, 0) \in K$, because also $(k_3, 0) \in K$ and K is increasing [Property K1]. Since $\lambda_3^*(k_3^* - 1, k_3^*) \in A \setminus \{g\}$, Condition (7) now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

We have now found a contradiction in cases (ii.b.2.1)–(ii.b.2.3), which tells us that case (ii.b.2) always leads to a contradiction. Since case (ii.b.1) also led to a contradiction, we may conclude that case (ii.b) always leads to a contradiction.

The discussion of the last remaining case (ii.c) is completely similar to that of case (ii.b): we can distinguish between similar cases, and in each of them we can construct a contradiction in the same manner, by exchanging the roles of k_1 and k_2, and of k_3 and k_4.

Since we have now arrived at a contradiction in all possible cases, we conclude that R_K indeed satisfies Axiom R3b.

We finish the proof by establishing that R_K also satisfies Axiom R1. Since we have already shown that R_K satisfies Axiom R4b [see Proposition 8] and Axiom R3b [see the argumentation above], by Corollary 1 it suffices to show that $0 \notin R_K(\{0\})$. By Condition (5), this is indeed the case. □

Proof (of Lemma 3) We only prove the first equivalence; the proofs for the second and the third equivalences are analogous. It suffices to establish the direct implication, since the converse follows from Axiom R3a.

Call $\lambda_k := f_k(T) - f_k(H) > 0$ and $\ell_k := \frac{f_k(T)}{f_k(T) - f_k(H)} \in [0, 1)$ for every k in $\{1, \ldots, m\}$, and $\lambda'_k := g_k(H) - g_k(T) > 0$ and $\ell'_k := \frac{g_k(H)}{g_k(H) - g_k(T)} \in [0, 1)$ for every k in $\{1, \ldots, n\}$. Then $0 \in R(\{0, f_1, \ldots, f_m, g_1, \ldots, g_n\}) \Leftrightarrow 0 \in R(\{0, (\ell_1 - 1, \ell_1), \ldots, (\ell_m - 1, \ell_m), (\ell'_1, \ell'_1 - 1), \ldots, (\ell'_n, \ell'_n - 1)\})$, using Condition (2). Let $\mathcal{I} := \{k \in \{1, \ldots, m\} : \ell_k = \ell_i\}$ and $\mathcal{J} := \{k \in \{1, \ldots, n\} : \ell'_k = \ell'_j\}$. Then $(\ell_k - 1, \ell_k) = R(\{(\ell_i - 1, \ell_i), (\ell_k - 1, \ell_k)\})$ by Axiom R2, and then also $(\ell_k - 1, \ell_k) \in R(\{0, (\ell_1 - 1, \ell_1), \ldots, (\ell_m - 1, \ell_m), (\ell'_1, \ell'_1 - 1), \ldots, (\ell'_n, \ell'_n - 1)\})$ by Axiom R3a, for all k in $\{1, \ldots, m\} \setminus \mathcal{I}$. In a similar way, we find that $\{0\} \cup \{(\ell_k - 1, \ell_k) : k \in \{1, \ldots, m\} \setminus \mathcal{I}\} \cup \{(\ell'_{k'}, \ell'_{k'} - 1) : k' \in \{1, \ldots, n\} \setminus \mathcal{J}\} \subseteq R(\{0, (\ell_1 - 1, \ell_1), \ldots, (\ell_m - 1, \ell_m), (\ell'_1, \ell'_1 - 1), \ldots, (\ell'_n, \ell'_n - 1)\})$. Then Axiom R3b implies that $0 \in R(\{0\} \cup \{(\ell_k - 1, \ell_k) : k \in \mathcal{I}\} \cup \{(\ell'_{k'}, \ell'_{k'} - 1) : k' \in \mathcal{J}\}) = R(\{0, (\ell_i - 1, \ell_i), (\ell'_j, \ell'_j - 1)\})$, whence indeed $0 \in R(\{0, f_i, g_j\})$, by Condition (2). □

Proof (of Proposition 9) For the first statement, assume that R is coherent and satisfies Condition (2). Then we infer from Proposition 7 that K_R satisfies Properties K1–K3, and therefore Proposition 8 guarantees that R_{K_R} is coherent and satisfies Condition (2) as well. To prove that $R = R_{K_R}$, we consider any A in $\mathcal{Q}$ and f in A, and show that $f \in R(A) \Leftrightarrow f \in R_{K_R}(A)$. Since both R and R_{K_R} satisfy Axiom R4b [Proposition 8], we can assume without loss of generality that $f = 0$.

For the direct implication, assume that $0 \in R(A)$. If $A \cap \mathcal{L}_{>0} \neq \emptyset$ then $0 \in R_{K_R}(A)$ by Condition (6). If $A \cap \mathcal{L}_{>0} = \emptyset$ then $0 \in R(A)$ implies that $g(\mathrm{H}) > 0$ or $g(\mathrm{T}) > 0$ for some g in A. If we use the notation $\mathcal{V}_{\mathrm{II}} \cap A = \{g_1, \dots, g_m\}$ and $\mathcal{V}_{\mathrm{IV}} \cap A = \{g'_1, \dots, g'_n\}$ with m and n in $\mathbb{Z}_{\geq 0}$, this tells us that $\max\{n, m\} > 0$. Also, we may assume without loss of generality that $A \cap \mathcal{L}_{<0} = \emptyset$. By Lemma 3 we infer that there are three possibilities:

(i) $0 \in R(\{0, \tilde{g}, \tilde{g}'\})$, and hence $0 \in R(\{0, h, h'\})$;
(ii) $0 \in R(\{0, \tilde{g}\})$, and hence $0 \in R(\{0, h\})$;
(iii) $0 \in R(\{0, \tilde{g}'\})$, and hence $0 \in R(\{0, h'\})$;

where we let, to ease the notation, $h \coloneqq \frac{1}{\tilde{g}(\mathrm{T}) - \tilde{g}(\mathrm{H})}\tilde{g}$ and $h' \coloneqq \frac{1}{\tilde{g}'(\mathrm{H}) - \tilde{g}'(\mathrm{T})}\tilde{g}'$. For each of these possible cases, we find respectively:

(i) $(h(\mathrm{T}), h'(\mathrm{H})) \in K_R$, which tells us that $0 \in R_{K_R}(\{0, \tilde{g}, \tilde{g}'\})$;
(ii) $(h(\mathrm{T}), 0) \in K_R$, from which we infer that $0 \in R_{K_R}(\{0, \tilde{g}\})$ by Condition (8);
(iii) $(0, h'(\mathrm{H})) \in K_R$, from which we infer that $0 \in R_{K_R}(\{0, \tilde{g}'\})$ by Condition (7).

In all three cases we can now conclude that, indeed, $0 \in R_{K_R}(A)$, by Axiom R3a.

For the converse implication, assume that $0 \in R_{K_R}(A)$. If $A \cap \mathcal{L}_{>0} \neq \emptyset$, then $0 \in R(A)$ by Axioms R2 and R3a, so assume that $A \cap \mathcal{L}_{>0} = \emptyset$. If Condition (7) holds, then there is some k_1 in $(0, 1)$ and some λ_1 in $\mathbb{R}_{>0}$ such that $(k_1, 0) \in K_R$ and $\lambda_1(k_1 - 1, k_1) \in A$. The first statement means that $0 \in R(\{(k_1 - 1, k_1), 0, (0, -1)\})$, whence, after applying a familiar combination of Axioms R2, R3a and R3b, also $0 \in R(\{(k_1 - 1, k_1), 0\})$. Applying Condition (2), the second statement, and Axiom R3a now leads us to deduce that indeed $0 \in R(A)$.

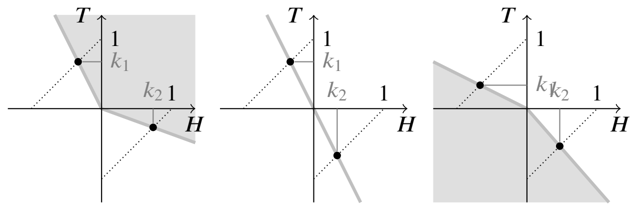
The remaining possibility is that either Condition (8) or Condition (9) holds. The proof in this case is similar. This concludes the proof of the first statement.

For the second statement, assume that K satisfies Properties K1–K3, then we infer from Proposition 8 that R_K is coherent and satisfies Condition (2). Proposition 7 then guarantees that K_{R_K} satisfies Properties K1–K3 as well. To show that $K = K_{R_K}$, consider any (ℓ_1, ℓ_2) in $[0, 1)^2 \setminus \{0\}$. First assume that $(\ell_1, \ell_2) \in K_{R_K}$, meaning that $0 \in R_K(\{(\ell_1 - 1, \ell_1), 0, (\ell_2, \ell_2 - 1)\})$, by the definition of a rejection set of a rejection function. We have to prove that this implies that $(\ell_1, \ell_2) \in K$. The definition of R_K [Definition 8] now tells us that Condition (6), Condition (7), Condition (8), or Condition (9) must obtain, with $A \coloneqq \{(\ell_1 - 1, \ell_1), (\ell_2, \ell_2 - 1)\}$. Since $(\ell_1, \ell_2) \in [0, 1)^2 \setminus \{0\}$, we infer that Condition (6) cannot be fulfilled, and we therefore have three remaining: (a) Condition (7), (b) Condition (8), or (c) Condition (9) is satisfied.

In case (a) there are λ_1 in $\mathbb{R}_{>0}$ and $(k_1, 0)$ in K such that $\lambda_1(k_1 - 1, k_1) \in A$. But, because $A = \{(\ell_1 - 1, \ell_1), (\ell_2, \ell_2 - 1)\}$ with $(\ell_1, \ell_2) \in [0, 1)^2 \setminus \{0\}$, this implies that $\lambda_1 = 1$ and $k_1 = \ell_1$. This guarantees that $(\ell_1, 0) \in K$ and, since K is increasing [Property K1], indeed also that $(\ell_1, \ell_2) \in K$. The proof in cases (b) and (c) is similar.

Conversely, assume that $(\ell_1, \ell_2) \in K$, then Condition (11) guarantees that in particular $0 \in R_K(\{(\ell_1 - 1, \ell_1), 0, (\ell_2, \ell_2 - 1)\})$, which implies that $(\ell_1, \ell_2) \in K_{R_K}$. □

Proof (of Lemma 4) Visual proof: see the three possible situations depicted below.



$k_1 + k_2 > 1$ $k_1 + k_2 = 1$ $k_1 + k_2 < 1$

□

Proof (of Proposition 10) We first prove that (i)⇒(ii). Assume that R_K satisfies Axiom R5, and consider any (k_1, k_2) in $[0,1)^2 \setminus \{0\}$ such that $k_1 + k_2 > 1$. It then follows that $(k_1, k_2) \in (0,1)^2$, and also that $\left(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\right) > 0$, whence $0 \in R_K(\{0, \left(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\right)\})$ by Condition (6). By Proposition 8, R_K satisfies Axiom R3a, whence $0 \in R_K(\{(k_1-1, k_1), 0, (k_2, k_2-1), (\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2})\})$. Also, $\left(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\right) \in \mathrm{CH}(\{(k_1 - 1, k_1), (k_2, k_2 - 1)\})$. But then Axiom R5 implies that $0 \in R_K(\{(k_1 - 1, k_1), 0, (k_2, k_2 - 1)\})$, whence indeed $(k_1, k_2) \in K$.

Next, we prove that (ii)⇒(i). Consider arbitrary A and A_1 in $\mathcal{Q}$ such that $A \subseteq A_1 \subseteq \mathrm{CH}(A)$, and let us show that $R_K(A_1) \cap A \subseteq R_K(A)$. Let $A := \{f_1, \dots, f_n\}$ and $A_1 := A \cup \{f_{n+1}, \dots, f_{n+k}\}$ for some n and k in $\mathbb{N}$. Assume that $f_i \in R_K(A_1)$ for some i in $\{1, \dots, n\}$. We then have to prove that $f_i \in R_K(A)$. We can assume without loss of generality that $f_i = 0$, because also $A - \{f_i\} \subseteq A_1 - \{f_i\} \subseteq \mathrm{CH}(A) - \{f_i\} = \mathrm{CH}(A - \{f_i\})$. To ease the notation along, let $\ell_k := \frac{f_k(\mathrm{T})}{f_k(\mathrm{T}) - f_k(\mathrm{H})}$ and $\lambda_k := f_k(\mathrm{T}) - f_k(\mathrm{H})$ for every k such that $f_k \in \mathcal{V}_{\mathrm{II}}$ [there might be no such k] and verify that $\lambda_k > 0$ and $f_k = \lambda_k(\ell_k - 1, \ell_k)$ for every gamble f_k in $A \cap \mathcal{V}_{\mathrm{II}}$. Similarly, for every k in $\{1, \dots, n\}$ such that $f_k \in \mathcal{V}_{\mathrm{IV}}$ [there might be no such k], let $\ell_k := \frac{f_k(\mathrm{H})}{f_k(\mathrm{H}) - f_k(\mathrm{T})}$ and $\lambda_k := f_k(\mathrm{H}) - f_k(\mathrm{T})$; then $\lambda_k > 0$ and $f_k = \lambda_k(\ell_k, \ell_k - 1)$ for every gamble f_k in $A \cap \mathcal{V}_{\mathrm{IV}}$.

First of all, we see that $A \cap \mathcal{L}_{>0} \neq \emptyset$ implies that indeed $0 \in R_K(A)$, by Condition (6). We may therefore in the remainder of this proof assume that $A \cap \mathcal{L}_{>0} = \emptyset$. Next, we observe that $\mathrm{CH}(A) \cap \mathcal{L}_{>0} \neq \emptyset$ also implies that $0 \in R_K(A)$. This can be proven ex absurdo by observing that it implies that $A \cap \mathcal{V}_{\mathrm{II}} \neq \emptyset$ and $A \cap \mathcal{V}_{\mathrm{IV}} \neq \emptyset$ and applying suitably condition (ii).

Now, since we have assumed that $f_i = 0 \in R_K(A_1)$, Definition 8 tells us that there are four possibilities: one of the four Conditions (6)–(9) must hold for A_1.

Condition (6) for A_1 amounts to $A_1 \cap \mathcal{L}_{>0} \neq \emptyset$, contradicting our assumption that $\mathrm{CH}(A) \cap \mathcal{L}_{>0} = \emptyset$, because $A_1 \subseteq \mathrm{CH}(A)$.

If Condition (9) holds for A_1, then $\{\lambda_1^*(k_1^* - 1, k_1^*), \lambda_2^*(k_2^*, k_2^* - 1)\} \subseteq A_1$ for some λ_1^* and λ_2^* in $\mathbb{R}_{>0}$ and (k_1^*, k_2^*) in $K \cap (0,1)^2$. Let $h_1 := \lambda_1^*(k_1^* - 1, k_1^*)$ and $h_2 := \lambda_2^*(k_2^*, k_2^* - 1)$. Then $A \cap \mathcal{V}_{\mathrm{II}} \neq \emptyset$ and $A \cap \mathcal{V}_{\mathrm{IV}} \neq \emptyset$, so we may assume again without loss of generality that f_1 is a gamble in $\arg\max\left\{\frac{h(\mathrm{T})}{h(\mathrm{T}) - h(\mathrm{H})} : h \in A \cap \mathcal{V}_{\mathrm{II}}\right\}$

and that f_2 is a gamble in $\arg\max\left\{\frac{h(\mathrm{H})}{h(\mathrm{H})-h(\mathrm{T})} : h \in A \cap \mathcal{V}_{\mathrm{IV}}\right\}$. Since we have assumed that $\mathrm{CH}(A) \cap \mathcal{L}_{>0} = \emptyset$, we see that $\mathrm{CH}(\{h_1, 0, h_2\}) \cap \mathcal{L}_{>0} = \emptyset$—and therefore also $\mathrm{posi}(\{h_1, 0, h_2\}) \cap \mathcal{L}_{>0} = \emptyset$—whence, by Equation (12), $k_1^* + k_2^* \leq 1$. If $(k_1^*, k_2^*) = (\ell_k, \ell_m)$ for some k and m in $\{1, \ldots, n\}$ such that $f_k \in \mathcal{V}_{\mathrm{II}}$ and $f_m \in \mathcal{V}_{\mathrm{IV}}$, then $0 \in R_K(A)$ by Condition (9). If this is not the case, then we distinguish between three possibilities: (i) $k_1^* \neq \ell_k$ for all k in $\{1, \ldots, n\}$ such that $f_k \in \mathcal{V}_{\mathrm{II}}$ and $k_2^* = \ell_m$ for some m in $\{1, \ldots, n\}$ such that $f_m \in \mathcal{V}_{\mathrm{IV}}$, (ii) $k_1^* = \ell_k$ for some k in $\{1, \ldots, n\}$ such that $f_k \in \mathcal{V}_{\mathrm{II}}$ and $k_2^* \neq \ell_m$ for all m in $\{1, \ldots, n\}$ such that $f_m \in \mathcal{V}_{\mathrm{IV}}$, and (iii) $k_1^* \neq \ell_k$ for all k in $\{1, \ldots, n\}$ such that $f_k \in \mathcal{V}_{\mathrm{II}}$ and $k_2^* \neq \ell_m$ for all m in $\{1, \ldots, n\}$ such that $f_m \in \mathcal{V}_{\mathrm{IV}}$.

In case (i), we already find that $\lambda(k_2^*, k_2^* - 1) \in A$ for some λ in $\mathbb{R}_{>0}$. If $k_1^* \leq \ell_1$, then $(k_1^*, k_2^*) \in K$ implies that $(\ell_1, k_2^*) \in K$ because K is increasing. Since we know that $f_1 = \lambda_1(\ell_1 - 1, \ell_1) \in A$, this guarantees that $0 \in R_K(A)$, by Condition (9). If $k_1^* > \ell_1$, then we claim that necessarily also $\ell_1 + \ell_2 > 1$, and therefore $(\ell_1, \ell_2) \in K$ by Property K4, so indeed $0 \in R_K(A)$ by Condition (9). To see that $\ell_1 + \ell_2 > 1$, assume *ex absurdo* that (a) $\ell_1 + \ell_2 < 1$ or (b) $\ell_1 + \ell_2 = 1$; it is not difficult to show that both these cases lead to a contradiction.

In case (ii), a completely similar argument leads us to conclude that $0 \in R_K(A)$ here as well.

In case (iii) there are, again, three possibilities: (α) $k_1^* < \ell_1$ and $k_2^* < \ell_2$, so $(\ell_1, \ell_2) \in K$ because K is increasing, and therefore $0 \in R_K(A)$ by Condition (9); (β) $k_1^* > \ell_1$ and $k_2^* < \ell_2$, and its symmetric counterpart $k_1^* < \ell_1$ and $k_2^* > \ell_2$; and (γ) $k_1^* > \ell_1$ and $k_2^* > \ell_2$, and therefore $\ell_1 + \ell_2 < k_1^* + k_2^* \leq 1$, so $\ell_1 + \ell_2 < 1$ and Lemma 4 guarantees that $h_1 \notin \mathrm{posi}(\{f_1, 0, f_2\}) = \mathrm{posi}(A)$, and therefore *a fortiori* $h_1 \notin \mathrm{CH}(A)$, a contradiction. It therefore suffices to consider case (β), and show that $k_1^* > \ell_1$ and $k_2^* < \ell_2$ implies that $0 \in R_K(A)$, since the case that $k_1^* < \ell_1$ and $k_2^* > \ell_2$ can be covered by a completely symmetrical argument. So assume that $k_1^* > \ell_1$ and $k_2^* < \ell_2$. Since $h_1 \in \mathrm{CH}(A) \subseteq \mathrm{posi}(A)$, Lemma 4 and $k_1^* > \ell_1$ guarantee that necessarily $\ell_1 + \ell_2 > 1$, so $(\ell_1, \ell_2) \in K$ by Property K4, and therefore once again $0 \in R_K(A)$, by Condition (9).

The proof when Conditions (8) or (7) hold is similar to that for Condition (9). □

Proof (of Proposition 11) We will prove that π_1 is non-increasing; the proof that π_2 is non-increasing is completely analogous. Assume *ex absurdo* that $\pi_1(z') > \pi_1(z)$ for some z and z' in $[0, 1)$ such that $z' > z$. Then, by the definition of π_1, we have $(\forall y \in (\pi_1(z), 1))(z, y) \in K$. Because K is increasing, we find $(\forall y \in (\pi_1(z), 1))(z', y) \in K$, and hence in particular $(\forall y \in (\pi_1(z), \pi_1(z')))(z', y) \in K$, a contradiction.

Consider now $z \in (0, 1)$. Let us prove the first statement; the proof of the second one is completely analogous. Recall that $(z, y) \in K$ for all y in $(\pi_1(z), 1)$, by the definition of π_1. Call $\delta := 1 - z - \pi_1(z) > 0$. Since K is increasing, we infer that for all ϵ in $(0, \delta)$, $(z, 1 - z - \epsilon) \in K$. On the other hand, by definition of π_1 it follows that $(z + \epsilon, y') \in K$ for all ϵ in $(0, \delta)$ and y' in $(\pi_1(z + \epsilon), 1)$. We call $b = z$, $a = 1 - z - \epsilon$ and $c = y'$. Note that $a + b = 1 - z - \epsilon + z < 1$ and $c = y' < 1 - z - \epsilon = a$ for any y' in $(\pi_1(z + \epsilon), 1 - z - \epsilon) \subseteq (\pi_1(z + \epsilon), 1)$. To see that $\pi_1(z + \epsilon) < 1 - z - \epsilon$, assume *ex absurdo* that $\pi_1(z + \epsilon) \geq 1 - z - \epsilon$, then $\pi_1(z) \geq 1 - z - \epsilon$ by the first statement, indeed a contradiction with the fact that $\epsilon < \delta$. We use Property K3 to infer that $(z, y') \in K$ for all y in $(\pi_1(z+\epsilon), 1)$ and ϵ in $(0, \delta)$. Infer that $\pi_1(z) \leq \pi_1(z+\epsilon)$, and since π_1 is non-increasing by the first part, we conclude that $\pi_1(z) = \pi_1(z+\epsilon)$, for all ϵ in $(0, \delta)$. Therefore, $\pi_1(z) = \pi_1(t)$ for all t in $(z, 1-\pi_1(z))$. □

Proof (of Proposition 12) We first prove necessity. Assume that R is such that K_R is weakly Archimedean, and consider any u in $\mathcal{V}_{\text{II}}$ and v in $\mathcal{V}_{\text{IV}}$ such that $\text{posi}(\{u, v\}) \cap \mathcal{V}_{\geq 0} = \emptyset$, and $0 \in R(\{u+\epsilon, 0, v\})$ and $0 \in R(\{u, 0, v+\epsilon\})$ for all ϵ in $\mathbb{R}_{>0}$. Then, due to Proposition 1, we find that $\forall \epsilon \in \mathbb{R}_{>0}, 0 \in R(\{(k_1-1, k_1)+\epsilon, 0, (k_2, k_2-1)\})$ and $0 \in R(\{(k_1-1, k_1), 0, (k_2, k_2-1)+\epsilon\})$ for $k_1 := \frac{u(T)-u(H)}{u(T)} \in (0,1)$ and $k_2 := \frac{v(H)-v(T)}{v(H)} \in (0,1)$. In particular, we find that $\forall k_1' \in (k_1, 1), k_2' \in (k_2, 1)$, $0 \in R(\{(k_1'-1, k_1'), 0, (k_2, k_2-1)\}$ and $0 \in R(\{(k_1-1, k_1), 0, (k_2', k_2'-1)\}))$, whence $(k_1', k_2) \in K_R$ and $(k_1, k_2') \in K_R$ for all k_1' in $(k_1, 1)$ and k_2' in $(k_2, 1)$, by Definition 7. Also, it can be checked that $k_1 + k_2 < 1$. The weak Archimedeanity of K_R implies that $(k_1, k_2) \in K_R$ by Definition 10, whence $0 \in R(\{(k_1-1, k_1), 0, (k_2, k_2-1)\})$. In turn, that implies by Proposition 1 that $0 \in R(\{u, 0, v\})$.

We now turn to sufficiency. Assume that R satisfies Equation (19) and consider any (k_1, k_2) in $(0,1)^2$ such that $k_1 + k_2 < 1$ and $(k_1', k_2) \in K_R$ and $(k_1, k_2') \in K_R$ for all k_1' in $(k_1, 1)$ and k_2' in $(k_2, 1)$. Then $\forall k_1' \in (k_1, 1), k_2' \in (k_2, 1), 0 \in R(\{(k_1'-1, k_1'), 0, (k_2, k_2-1)\})$ and $0 \in R(\{(k_1-1, k_1), 0, (k_2', k_2'-1)\})$, whence $\forall \epsilon \in \mathbb{R}_{>0}$, $0 \in R(\{(k_1-1, k_1)+\epsilon, 0, (k_2, k_2-1)\})$ and $0 \in R(\{(k_1-1, k_1), 0, (k_2, k_2-1)+\epsilon\})$ by [22, Proposition 2]. Clearly, $(k_1-1, k_1) \in \mathcal{V}_{\text{II}}$ and $(k_2, k_2-1) \in \mathcal{V}_{\text{IV}}$. Due to Equation (12), $\text{posi}(\{(k_1-1, k_1), (k_2, k_2-1)\}) \cap \mathcal{V}_{\geq 0} = \emptyset$. Then, using Equation (19), we find that $0 \in R(\{(k_1-1, k_1), 0, (k_2, k_2-1)\})$, or in other words, that $(k_1, k_2) \in K_R$. □

Proof (of Proposition 13) From the correspondence between weak Archimedeanity for rejection functions and rejection sets (Proposition 12) as well as Proposition 9, it suffices to establish the result for rejection sets. Recalling that in that case the infima of the rejection sets corresponds to their intersection, we deduce from the definition that if K_i is weakly Archimedean for every i in I, also $\inf\{K_i : i \in I\}$ is weakly Archimedean. □

Proof (of Corollary 2) Taking into account Proposition 13, it suffices to show that any lexicographic rejection function is weakly Archimedean. Assume *ex absurdo* that this is not the case for some rejection function R on $\mathcal{V}$. By Proposition 12, this means that its associated rejection set K_R is not weakly Archimedean. Thus, there are u in $\mathcal{V}_{\text{II}}$ and v in $\mathcal{V}_{\text{IV}}$ such that $\text{posi}(\{u, v\}) \cap \mathcal{V}_{\geq 0} = \emptyset$ and $\forall \epsilon \in \mathbb{R}_{>0}$, $(0 \in R(\{u+\epsilon, 0, v\}) \cap R(\{u, 0, v+\epsilon\}))$ while $0 \notin R(\{u, 0, v\})$. Let D_R be the lexicographic set of desirable options associated with R. It follows that $u \notin D_R$ and $v \notin D_R$, and as a consequence that $u+\epsilon \in D_R$ and $v+\epsilon \in D_R$ for every ϵ in $\mathbb{R}_{>0}$. If we denote by P_{D_R} the linear prevision induced by D_R, given by $P_{D_R}(f) := \sup\{\mu : f - \mu \in D_R\}$, it follows that $P_{D_R}(u) = P_{D_R}(v) = 0$. Since by assumption $u \in \mathcal{V}_{\text{II}}$ and $v \in \mathcal{V}_{\text{IV}}$, it follows that there must be some α in $(0,1)$ such that $\alpha u + (1-\alpha)v = 0$, a contradiction with the assumption $\text{posi}(\{u, v\}) \cap \mathcal{V}_{\geq 0} = \emptyset$. □

Proof (of Proposition 14) Assume *ex absurdo* that $\pi_1(z) \neq 1-z$ and $\pi_2(1-z) \neq z$, and hence $\pi_1(z) < 1-z$ and $\pi_2(1-z) < z$, for all z in $[k_1, 1-k_2]$. Then we use Proposition 11 to infer that in particular $\pi_1(k_1) = \pi_1(t)$ for all t in $(k_1, 1-\pi_1(k_1))$. There are two possibilities: (i) $\pi_1(k_1) > k_2$ or (ii) $\pi_1(k_1) \leq k_2$.

If (i) $\pi_1(k_1) > k_2$ we look at $\pi_1(1-\pi_1(k_1))$. By the definition of π_1, we find $(1-\pi_1(k_1), y) \in K$ for all y in $(\pi_1(1-\pi_1(k_1)), 1)$. Moreover, since $\pi_1(k_1) \in [0, 1-k_1]$ by the definition of π_1, we find that $\pi_1(k_1) \in (k_2, 1-k_1]$ and hence $1-\pi_1(k_1) \in [k_1, 1-k_2)$. By the assumption that $\pi_1(z) < 1-z$ for all z in $[k_1, 1-k_2]$, we find that $\pi_1(1-\pi_1(k_1)) < \pi_1(k_1)$. We also look at $\pi_2(\pi_1(k_1))$. By the definition of π_2, we find $(x, \pi_1(k_1)) \in K$ for all x in $(\pi_2(\pi_1(k_1), 1))$. By the assumption that $\pi_2(1-z) < z$ for all z in $[k_1, 1-k_2]$, we find that $\pi_2(\pi_1(k_1)) < 1-\pi_1(k_1)$. Call $a = \pi_1(k_1)$, $b = x$ and $c = y$ for x in $(\pi_2(\pi_1(k_1)), 1)$ and y in $(\pi_1(1-\pi_1(k_1)), 1)$. Use Property K3 to infer that $(x, y') \in K$ and $(x', y) \in K$ for all x greater than but close enough to $\pi_2(\pi_1(k_1))$, y greater than but close enough to $\pi_1(1-\pi_1(k_1))$, x' in $(x, 1)$ and y' in $(y, 1)$. Hence by weak Archimedeanity $(x, y) \in K$ for all x in $(\pi_2(\pi_1(k_1)), 1)$ and y in $(\pi_1(1-\pi_1(k_1)), 1)$. Now, take $x = \frac{\pi_2(\pi_1(k_1))+1-\pi_1(k_1)}{2}$ and $y = \frac{\pi_1(1-\pi_1(k_1))+\pi_1(k_1)}{2}$ to infer that $(\frac{\pi_2(\pi_1(k_1))+1-\pi_1(k_1)}{2}, \frac{\pi_1(1-\pi_1(k_1))+\pi_1(k_1)}{2}) \in K$, and take any t in $(\frac{\pi_2(\pi_1(k_1))+1-\pi_1(k_1)}{2}, 1-\pi_1(k_1))$ and infer that $\pi_1(t) \leq \pi_1(\frac{\pi_2(\pi_1(k_1))+1-\pi_1(k_1)}{2}) < \pi_1(k_1)$. That is a contradiction with the assumption that $\pi_1(t) = \pi_1(k_1)$ for all t in $(k_1, 1-\pi_1(k_1))$.

So we may assume that (ii) $\pi_1(k_1) \leq k_2$ is the case. Infer that then $(k_1, y) \in K$ for all y in $(k_2, 1)$ by the definition of π_1. Using a similar argument, we can infer that $(x, k_2) \in K$ for all x in $(k_1, 1)$. We use now the assumption that K is weakly Archimedean (Definition 10) to infer that $(k_1, k_2) \in K$, a contradiction. □

Proof (of Theorem 2) We first show that $K \subseteq K'$. Consider any (k_1, k_2) in $[0,1)^2$ such that $(k_1, k_2) \notin K'$. Then there must be some D' in $\mathcal{D}'$ such that $0 \notin R_{D'}(\{(k_1 - 1, k_1), 0, (k_2, k_2 - 1)\})$. There are a number of possibilities:

- If $D' = D_x$ for some x in $(0,1)$, then $(x, 1-x) \notin K$, $k_1 \leq x$ and $k_2 \leq 1-x$ by Equation (13), whence also $(k_1, k_2) \notin K$, taking into account that K is increasing.
- If $D' = D_x^H$ for some x in $(0,1)$, then $(x, 1-x) \in K$ by Equation (14), $(\forall \epsilon \in \mathbb{R}_{>0})(x, 1-x-\epsilon) \notin K$, $k_1 \leq x$ and $k_2 < 1-x$. This means that there is some x in $[k_1, 1-k_2)$ such that $(x, 1-x) \in K$ and $(\forall \epsilon \in \mathbb{R}_{>0})(x, 1-x-\epsilon) \notin K$, whence $((\exists x \in [k_1, 1-k_2))((x, 1-x-\frac{1-k_2-x}{2}) = (x, \frac{1-x+k_2}{2}) \notin K)) \Rightarrow (k_1, k_2) \notin K$.
- If $D' = D_x^T$, we follow a similar reasoning to conclude that $(k_1, k_2) \notin K$.
- If $D' = D_0^H$, then $k_1 = 0$, and $(\forall \epsilon \in \mathbb{R}_{>0})(0, 1-\epsilon) \notin K$, and therefore $(k_1, k_2) = (0, k_2) \notin K$.
- Finaly, if $D' = D_1^T$, we follow a reasoning similar to that in the previous point and derive that $(k_1, k_2) = (k_1, 0) \notin K$.

We now turn to showing $K' \subseteq K$. Consider any (k_1, k_2) in $[0,1)^2$ such that $(k_1, k_2) \notin K$. By Proposition 7, $k_1 + k_2 \leq 1$. There are two possibilities: either (i) $k_1 + k_2 = 1$ or (ii) $k_1 + k_2 < 1$. If (i) $k_1 + k_2 = 1$ then k_1 in $(0,1)$ and hence $D_{k_1} \in \mathcal{D}'$ because $(k_1, 1-k_1) = (k_1, k_2) \notin K$. Then infer $0 \notin R_{D_{k_1}}(\{(k_1 - 1, k_1), 0, (k_2, k_2 - 1)\})$ by

Equation (13), whence $0 \notin \bigcap_{D\in\mathcal{D}'} R_D(\{(k_1-1,k_1),0,(k_2,k_2-1)\})$ and hence $(k_1,k_2)\notin K'$. So we may assume that (ii) $k_1+k_2<1$. We now use Proposition 14 to infer that $\pi_1(z)=1-z$ or $\pi_2(1-z)=z$ for some z in $[k_1,1-k_2]$. There are four possible cases: (a) $\pi_1(k_1)=1-k_1$; (b) $\pi_2(k_2)=1-k_2$; (c) $\pi_1(z)=1-z$ for some z in $(k_1,1-k_2)$; and (d) $\pi_2(1-z)=z$ for some z in $(k_1,1-k_2)$. In any of them it is not difficult to prove that $(k_1,k_2)\notin K'$. □

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