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# CARDINAL INVARIANTS AND THE COLLAPSE OF THE CONTINUUM BY SACKS FORCING

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**Abstract.** We study cardinal invariants of systems of meager hereditary families of subsets of $\omega$ connected with the collapse of the continuum by Sacks forcing $\mathbb{S}$ and we obtain a cardinal invariant $\mathfrak{h}_\omega$ such that $\mathbb{S}$ collapses the continuum to $\mathfrak{h}_\omega$ and $\mathfrak{h} \leq \mathfrak{h}_\omega \leq \mathfrak{b}$. Applying the Baumgartner-Dordal theorem on preservation of eventually narrow sequences we obtain the consistency of $\mathfrak{h} = \mathfrak{h}_\omega < \mathfrak{b}$. We define two relations $\preceq_0^*$ and $\preceq_1^*$ on the set $({}^\omega\omega)_{\mathrm{Fin}}$ of finite-to-one functions which are Tukey equivalent to the eventual dominance relation of functions such that if $\mathcal{F} \subseteq ({}^\omega\omega)_{\mathrm{Fin}}$ is $\preceq_1^*$-unbounded, well-ordered by $\preceq_1^*$, and not $\preceq_0^*$-dominating, then there is a nonmeager $p$-ideal. The existence of such a system $\mathcal{F}$ follows from Martin's axiom. This is an analogue of the results of [3], [9, 10] for increasing functions.

**§0. Introduction.** The question when Sacks forcing $\mathbb{S}$ collapses cardinals arose after the proof of Baumgartner and Laver [4] that adding $\omega_2$ Sacks reals by countable support iteration to a model of CH one gets a model in which Sacks forcing collapses the continuum to $\omega_1$. Rosłanowski and Shelah [12] proved that Sacks forcing collapses the continuum to the dominating number $\mathfrak{d}$ which was a confirmation of the hypothesis of Carlson and Laver [5]. Shortly after Peter Simon [13] proved the collapse of the continuum by $\mathbb{S}$ to the unbounded number $\mathfrak{b}$ (see also [11, Theorem 3.1 (1)] for a simplification of the proof). On the other hand, Judah, Miller and Shelah [8] proved the consistency of Martin's axiom together with the collapse of the continuum by Sacks forcing to $\omega_1$. This indicates that the previously mentioned results on the collapse of cardinals by Sacks forcing are far from the complete answer. In connection with Martin's axiom we can ask about (definable) cardinal invariants to which Sacks forcing collapses the continuum and which are equal to the continuum under Martin's axiom. From another point of view we can ask about a simple principle violating Martin's axiom which implies the collapse of the continuum by Sacks forcing to $\omega_1$.

In the present paper we continue this study. In Section 1 to every perfect tree $p \subseteq {}^{<\omega}2$—condition of the Sacks forcing $\mathbb{S}$—we associate several hereditary meager families of subsets of $\omega$. These families are naturally connected with various ways of measurement of rapidity of branching in perfect trees. This way we obtain

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several natural systems $\mathscr{H}$ of hereditary meager sets $I \subseteq \mathscr{P}(\omega)$ and we introduce several cardinal invariants for them which are monotone with respect to a preordering $\preceq$ of $\mathscr{H}$'s. Later in Section 2 it is shown that some of these cardinal invariants characterize the unbounded number $\mathfrak{b}$ or the dominating number $\mathfrak{d}$. The most important cardinal from them is $\kappa(\mathscr{H})$. It is shown that $\kappa(\mathscr{H}) \geq \mathfrak{h}$ for many families $\mathscr{H}$ where $\mathfrak{h}$ is the distributivity number of $\mathscr{P}(\omega)/\mathrm{Fin}$, but there is a system $\mathscr{H} = \{I_p^1 : p \in \mathbb{S}\}$ of meager ideals for which $\kappa(\mathscr{H})$ is exactly the cardinal to which $\mathbb{S}$ collapses the continuum (Theorem 1.11). This is the substance of the above mentioned Simon's proof.

In Section 3 the following property of hereditary sets is isolated and studied: A set $I \subseteq \mathscr{P}(\omega)$ is said to be $\omega$-small if there is a countable set $B \subseteq [\omega]^\omega$ such that for every $x \in I$ there is $b \in B$ such that $|x \cap b| < \omega$. For the family $\mathscr{M}_\omega$ of $\omega$-small hereditary subsets of $\mathscr{P}(\omega)$, the cardinal $\kappa(\mathscr{M}_\omega)$ has the form $\mathfrak{h}_\omega$ which can be considered as a generalization of the distributivity number $\mathfrak{h} = \mathfrak{h}_1$. By monotonicity of the cardinal invariant $\kappa(\mathscr{H})$, we have $\mathfrak{h} \leq \mathfrak{h}_\omega \leq \mathfrak{b}$ and by applying the preservation theorem of Baumgartner and Dordal for eventually narrow sequences in a finite support iterated forcing we obtain the consistency of $\mathfrak{h}_\omega < \mathfrak{b}$ (Theorem 3.6). It follows that $\mathfrak{h}_\omega$ is an upper bound for the collapse of the continuum by $\mathbb{S}$ which is consistently strictly smaller than $\mathfrak{b}$.

Baumgartner and Dordal [3] have proved that under Martin's axiom there exists a well-ordered family of increasing functions which is not a dominating family. This result was used in [9, 10] for a construction of a nonmeager $p$-ideal. Finite-to-one functions can be in some sense considered as a generalization of increasing functions. The advantage of this generalization is that the domain and range of functions need not be ordered. Section 4 deals with an analogue of these results for finite-to-one functions although the results are expressed in an equivalent language using partitions of $\omega$ into finite sets.

Our notation is standard and it is more or less compatible with that of [2, 6, 7].

**§1. Meager hereditary families.** A *tree* is a set $p \subseteq {}^{<\omega}2$ such that (1) $\emptyset \in p$, (2) if $t \in p$ and $s \subseteq t$, then $s \in p$, and (3) $s^\frown 0 \in p$ or $s^\frown 1 \in p$ for every $s \in p$. A tree $p$ is *perfect* if for every $s \in p$ there is $t \in {}^{<\omega}2$ such that $s \subseteq t$, $t^\frown 0 \in p$, and $t^\frown 1 \in p$ (i.e., $t$ is a *splitting* node of $p$). If $p$ is a tree and $s \in p$, then $(p)_s = \{t \in p : t \subseteq s$ or $s \subseteq t\}$ is a tree. We denote the set of splitting nodes and the set of levels containing splitting nodes for a given tree $p$ by

$$\mathrm{sp}(p) = \{s \in p : s^\frown 0 \in p \text{ and } s^\frown 1 \in p\},$$
$$\mathrm{br}(p) = \{n \in \omega : \mathrm{sp}(p) \cap {}^n2 \neq \emptyset\}.$$

For a tree $p$ and a set $a \subseteq \omega$ the tree $p[a] \subseteq p$ is defined by induction (see [13]):

1. $\emptyset \in p[a]$.
2. If $s \in p[a]$ and $|s| = n$, then $s^\frown 0 \in p[a]$ if and only if $s^\frown 0 \in p$, and $s^\frown 1 \in p[a]$ if and only if $s^\frown 1 \in p$ and $n \in a$ or $s^\frown 0 \notin p$.

Clearly, $p[a]$ is a maximal tree $q \subseteq p$ with $\mathrm{br}(q) \subseteq a$.

Trees provide countable coding of closed sets in ${}^\omega 2$, namely, for every closed set $A$ in ${}^\omega 2$ there is a unique tree $p \subseteq {}^{<\omega}2$ such that $A = [p]$ where $[p] = \{x \in {}^\omega 2 : (\forall n \in \omega)\ x{\restriction}n \in p\}$. The projection of a set $A \subseteq {}^\omega 2$ by $a \subseteq \omega$ is the set

$A\restriction a = \{x\restriction a : x \in A\}$ which is a subset of ${}^{a}2$. Sometimes it is easier to manipulate with trees than with closed sets. We shall denote by $\mathbb{S}$ the set of all perfect trees $p \subseteq {}^{<\omega}2$ ordered by inclusion.

The next lemma summarizes some properties of the defined operations.

LEMMA 1.1. *Let $p, q$ be trees and $a, b$ be subsets of $\omega$.*

1. $\mathrm{br}(p) \subseteq a$ *if and only if* $p[a] = p$.
2. $a \subseteq b$ *implies* $p[a] \subseteq p[b]$. *In particular,* $p[a] \subseteq p = p[\omega]$.
3. $p[a][b] = p[a \cap b] = p[a] \cap p[b]$.
4. $p[a] \cap q[b] \subseteq (p \cap q)[a \cap b]$; *if* $p[a] \cap q[b]$ *is a tree, then the equality holds.*
5. $p[a] \cup p[b] \subseteq p[a \cup b]$
6. *If* $\mathrm{br}(p) \subseteq a$, *then the projection of* $[p]$ *to* $[p]\restriction a$ *is one-to-one.*
7. *If $p$ is a perfect tree and* $\mathrm{br}(p) \subseteq a \cup b$, *then there is a perfect tree $q \subseteq p$ such that* $\mathrm{br}(q) \subseteq a$ *or* $\mathrm{br}(q) \subseteq b$.
8. *If $p$ is a perfect tree and* $\mathrm{sp}(p) \subseteq c \cup d$ *for some $c, d \subseteq {}^{<\omega}2$, then there is a perfect tree $q \subseteq p$ such that* $\mathrm{sp}(q) \subseteq c$ *or* $\mathrm{sp}(q) \subseteq d$.
9. *If $A \subseteq {}^{\omega}2$ is perfect, $a \in [\omega]^{\omega}$, and $A\restriction a$ is uncountable, then there is a perfect set $B \subseteq A$ such that the projection of $B$ onto $B\restriction(a \setminus n)$ is one-to-one for all $n \in \omega$.*

PROOF. Assertions (1)–(5) follow from definitions.

(6) Let $x, y \in [p]$ be distinct and let $n$ be minimal such that $x(n) \neq y(n)$. Then $x\restriction n = y\restriction n$ is a splitting node of $p$ and hence $n \in a$. Therefore $x\restriction a \neq y\restriction a$.

(7) is a special case of (8) for $c = \bigcup_{n\in a} {}^{n}2$ and $d = \bigcup_{n \in b} {}^{n}2$.

(8) If $\mathrm{sp}(p) \subseteq c \cup d$, then either there is $s \in p$ such that $\mathrm{sp}((p)_s) \subseteq c$, or $\mathrm{sp}((p)_s) \cap d \neq \emptyset$ for all $s \in p$. In the former case let $q = (p)_s$ and then $\mathrm{sp}(q) \subseteq c$. In the latter case let us choose inductively $t_s \in \mathrm{sp}(p) \cap d$ for $s \in {}^{<\omega}2$ so that $t_s{}^\frown i \subseteq t_{s^\frown i}$ for $i \in \{0, 1\}$ and let $q$ be the perfect tree with $\mathrm{sp}(q) = \{t_s : s \in {}^{<\omega}2\}$. Then $\mathrm{sp}(q) \subseteq d$.

(9) Let $\{k_n : n \in \omega\}$ be an increasing enumeration of $a$ and let $a_n = \{k_j : j \geq n\}$. By induction we define a system of perfect sets $B_s \subseteq A$ for $s \in {}^{<\omega}2$ so that

(i) $B_{s^\frown 0}$, $B_{s^\frown 1}$ are disjoint subsets of $B_s$, the diameter of $B_s$ is $\leq 2^{-n}$ for $s \in {}^{n}2$, $B_\emptyset = A$, and

(ii) $\{B_s\restriction a_n : s \in {}^{n}2\}$ is a disjoint system of uncountable subsets of ${}^{a_n}2$.

Let us assume that $B_s$ for $s \in {}^{n}2$ have been constructed. By (ii), for every $s \in {}^{n}2$ we can fix $i_s \in \{0, 1\}$ such that $B'_s = \{x \in B_s\restriction a_n : x(k_n) = i_s\}$ is uncountable and let $C_s = B'_s\restriction a_{n+1}$. Let us choose perfect sets $C_s^0, C_s^1 \subseteq C_s$ so that the system $\{C_s^i : s \in {}^{n}2 \text{ and } i \in \{0, 1\}\}$ is disjoint and let $B_{s^\frown i}$ be a perfect subset of the set $\{x \in B_s : x\restriction a_{n+1} \in C_s^i\}$ of diameter $\leq 2^{-(n+1)}$.

By (i) the system $\{B_s : s \in {}^{<\omega}2\}$ is a fusion sequence and hence the set $B = \bigcap_{n\in\omega} \bigcup_{s \in {}^{n}2} B_s$ is perfect and $B \subseteq A$. We claim that the projection of $B$ onto $B\restriction a_n$ is one-to-one for each $n$. To see this let $x, y \in B$ be distinct and $n \in \omega$. There are $m \geq n$ and distinct $s_x, s_y \in {}^{m}2$ such that $x \in B_{s_x}$ and $y \in B_{s_y}$. Then $x\restriction a_n \neq y\restriction a_n$ because, by (ii), $B_{s_x}\restriction a_m \cap B_{s_y}\restriction a_m = \emptyset$. ⊣

REMARK 1.2. These remarks are concerned with assertions of Lemma 1.1.

(1) The assumption that $p[a] \cap q[b]$ is a tree in (4) is necessary for the inclusion $\supseteq$ and cannot be omitted. Namely, there are $p$, $q$, $a$, $b$ (here, $p$, $q$ must be distinct,

by (2)) such that $p[a] \cap q[b] \subsetneq (p \cap q)[a \cap b]$: Let $p = {}^{<\omega}2$, $q = ({}^{<\omega}2)_{\langle 1 \rangle}$, $a = b = \omega \setminus \{0\}$. Then $p[a] = ({}^{<\omega}2)_{\langle 0 \rangle}$, $q[b] = q = ({}^{<\omega}2)_{\langle 1 \rangle}$, $p[a] \cap p[b] = \{\emptyset\}$ is not a tree and $(p \cap q)[a \cap b] = q[b] = ({}^{<\omega}2)_{\langle 1 \rangle}$.

(2) In the inclusion (5), the equality $p[a] \cup p[b] = p[a \cup b]$ does not hold in general: There are $p$, $a$, $b$ such that $p[a \cup b]$ contains a perfect tree while neither $p[a]$ nor $p[b]$ contains a perfect tree: Let $a$, $b$ be any infinite disjoint subsets of $\omega$ such that $\omega \setminus (a \cup b)$ is infinite, $0 \in a$, and $1 \in b$. Let $p$ be the set of all $s \in {}^{<\omega}2$ such that $s(i) = 0$ whenever $i \in a \cup b$, $i \neq 0$, and $s(0) = 0$ or $s(1) = 0$. Then $p[a]$, $p[b]$ do not contain perfect subtrees while $p[a \cup b]$ contains the perfect tree $({}^{<\omega}2)_{\langle 1,1 \rangle}$.

(3) In the inclusion $[p[a]]\restriction a \subseteq [p]\restriction a$ the equality does not hold in general. In fact, there are a perfect tree $p$ and $a \in [\omega]^{\omega}$ such that $[p]\restriction a = {}^{a}2$ and $|[q[a]]| = 1$ for every tree $q \subseteq p$: To see this take infinite $a \subseteq \omega$ such that $0 \notin a$ and $\omega \setminus a$ is infinite and let $p$ be the set of all $s \in {}^{<\omega}2$ such that $s(i) = s(i-1)$ for every positive $i \in (\operatorname{dom} s) \cap a$.

A family $I \subseteq \mathscr{P}(X)$ is an *ideal* on a set $X$ if (1) $\emptyset \in I$ and $X \notin I$, (2) $I$ is *hereditary* (which means that $a \in I$ whenever $a \subseteq b$ for some $b \in I$), and (3) $a \cup b \in I$ for every $a, b \in I$.

Let $I$ be a hereditary set on $\omega$, let $f : \omega \to \omega$, and let $a \subseteq \omega$. Let us denote

$$f(I) = \{x \subseteq \omega : f^{-1}(x) \in I\}, \quad f^{-1}(I) = \{x \subseteq \omega : f``x \in I\},$$
$$I\restriction a = \{x \in I : x \subseteq^* a\}$$

where $a \subseteq^* b$ means that $a \setminus b$ is finite.

Notice that $f(I)$ and $I\restriction a$ are hereditary families and they are ideals whenever $I$ is an ideal. Also, $f^{-1}(f(I)) \subseteq I$ and $f(f^{-1}(I)) = I$. If $I$ is an ideal on $\omega$, then $[\omega]^{<\omega} \subseteq f(I)$ if and only if $f^{-1}(\{n\}) \in I$ for every $n \in \omega$.

A function $f : \omega \to \omega$ is *finite-to-one* if $f^{-1}(\{n\})$ is finite for every $n \in \omega$. We shall denote by $({}^{\omega}\omega)_{\mathrm{Fin}}$ the family of finite-to-one functions.

The topology on $\mathscr{P}(\omega)$ has a clopen base consisting of sets $[s] = \{x \subseteq \omega : x \cap \operatorname{dom} s = \{i : s(i) = 1\}\}$ for $s \in {}^{<\omega}2$, i.e., $\mathscr{P}(\omega)$ is homeomorphic to the Cantor space ${}^{\omega}2$ via characteristic functions.

Lemma 1.3. *Let $I \subseteq \mathscr{P}(\omega)$ be a hereditary family. The following conditions are equivalent*:

1. *$I$ is a meager subset of $\mathscr{P}(\omega)$.*
2. *There is a disjoint sequence of finite sets $\{a_n\}_{n=0}^{\infty}$ such that $\bigcup_{n \in x} a_n \notin I$ for all $x \in [\omega]^{\omega}$.*
3. *There is an increasing sequence of natural numbers $\{k_n\}_{n=0}^{\infty}$ with $k_0 = 0$ such that $\bigcup_{n \in x}[k_n, k_{n+1}) \notin I$ for all $x \in [\omega]^{\omega}$.*
4. *There is a finite-to-one $f : \omega \to \omega$ such that $f(I) \subseteq [\omega]^{<\omega}$.*

Proof. (2) $\to$ (1) Let us assume that $(\forall x \in [\omega]^{\omega}) \bigcup_{n \in x} a_n \notin I$. Equivalently, $I \cap \bigcap_{n \in \omega} \bigcup_{m > n} \{y \subseteq \omega : a_m \subseteq y\} = \emptyset$ because $I$ is hereditary. However, this condition says that $I$ is disjoint from a $G_\delta$ dense set and hence $I$ is meager.

The implication (3) $\to$ (2) is trivial and for implications (3) $\to$ (4) and (4) $\to$ (2) it is enough to apply $f^{-1}(\{n\}) = [k_n, k_{n+1})$ and $a_n = f^{-1}(\{n\})$, respectively.

(1) $\to$ (3) If $I$ is meager, then $I \subseteq \bigcup_{k \in \omega} F_k$ for some nowhere dense sets $F_k$ such that $F_k \subseteq F_{k+1}$ for all $k \in \omega$. For $k \in \omega$ let $h(k)$ be the least $m > k$ for which

there is a nonconstant function $\varphi_{k,m} : [k,m) \to 2$ such that $[s \cup \varphi_{k,m}] \cap F_k = \emptyset$ for all $s \in {}^k2$. By induction let us define $k_0 = 0$, $k_{n+1} = h(k_n)$, and let $a_n = \{i \in [k_n, k_{n+1}) : \varphi_{k_n,k_{n+1}}(i) = 1\}$. Then $\{a_n\}_{n=0}^{\infty}$ is an infinite disjoint family of finite nonempty sets and $F_{k_n}$ is disjoint from the set $A_n = \{x \subseteq \omega : x \cap [k_n, k_{n+1}) = a_n\}$ because for every $x \in A_n$ there is $s \in {}^{k_n}2$ such that $x \in [s \cup \varphi_{k_n,k_{n+1}}]$. Let $x \in [\omega]^\omega$ and $k \in \omega$ be arbitrary. There is $n \in x$ with $k_n \geq k$. Hence, $a = \bigcup_{i \in x} a_i$ is in $A_n$ and $a \notin F_k$ because $F_k \subseteq F_{k_n}$. It follows that $\bigcup_{n \in x} a_n \notin I$ for all $x \in [\omega]^\omega$. As $a_n \subseteq [k_n, k_{n+1})$ and $I$ is hereditary, condition (3) holds, too. ⊣

Now we define some hereditary families via the structure of perfect sets. Let $p$ be a perfect tree. In the next definitions the variable $q$ varies on perfect trees:

$$J_p = \{x \subseteq \omega : |[p]\restriction x| \leq \omega\}, \tag{1.1}$$

$$I_p^1 = \{x \subseteq {}^{<\omega}2 : (\forall q \subseteq p)\ \mathrm{sp}(q) \nsubseteq x\}, \tag{1.2}$$

$$I_p^2 = \{x \subseteq \omega : (\forall q \subseteq p)\ \mathrm{br}(q) \nsubseteq x\}, \tag{1.3}$$

$$I_p^3 = \{x \subseteq \omega : |[p[x]]| \leq \omega\}, \tag{1.4}$$

$$I_{\mathbb{S}} = \{x \subseteq {}^{<\omega}2 : (\forall p \in \mathbb{S})\ \mathrm{sp}(p) \nsubseteq x\}. \tag{1.5}$$

$$\mathrm{NWD}_p = \{x \subseteq {}^{<\omega}2 : x \cap \mathrm{sp}(p) \text{ is nowhere dense in } (p, \supseteq)\} \tag{1.6}$$

Let $\pi : {}^{<\omega}2 \to \omega$ be the finite-to-one function defined by $\pi(s) = |s|$. We define

$$I_p^0 = \pi^{-1}(J_p). \tag{1.7}$$

Clearly, $J_p$ and $\mathrm{NWD}_p$ are ideals. By (7) and (8) of Lemma 1.1, $I_p^1$ and $I_p^2$ are ideals. It follows that $I_p^0$ is an ideal, too. $I_{\mathbb{S}} = I_p^1$ for $p = {}^{<\omega}2$ and hence $I_{\mathbb{S}}$ is an ideal. By Remark 1.2 (2), $I_p^3$ need not be an ideal but it is a hereditary family. Notice that $\pi(I_{\mathbb{S}}) = [\omega]^{<\omega}$ and for $p = {}^{<\omega}2$, $J_p = I_p^2 = I_p^3 = [\omega]^{<\omega}$ and $I_p^0 = [{}^{<\omega}2]^{<\omega}$.

By Lemma 1.1 (9) it follows that

$$J_p = \{x \subseteq \omega : (\forall q \subseteq p) \text{ the restriction } [q]\restriction x \text{ is not one-to-one}\}. \tag{1.1$'$}$$

Lemma 1.4. *$\pi(I_p^0) = J_p \subseteq \pi(I_p^1) = I_p^2 \subseteq I_p^3$ and $I_p^0 \subseteq I_p^1 \subseteq \mathrm{NWD}_p$.*

Proof. By (1.1′) and Lemma 1.1 (6) we have $J_p \subseteq I_p^2$. The equality $\pi(I_p^1) = I_p^2$ is by the equivalence $\mathrm{br}(q) \subseteq x$ if and only if $\mathrm{sp}(q) \subseteq \pi^{-1}(x)$ for $x \subseteq \omega$. By Lemma 1.1 (1) we have

$$I_p^2 = \{x \subseteq \omega : (\forall q \subseteq p)\ |[q[x]]| \leq \omega\} \tag{1.3$'$}$$

and therefore $I_p^2 \subseteq I_p^3$. The inclusion $J_p \subseteq \pi(I_p^1)$ implies $I_p^0 = \pi^{-1}(J_p) \subseteq I_p^1$. We prove $I_p^1 \subseteq \mathrm{NWD}_p$.

If $x \subseteq {}^{<\omega}2$ and $x \notin \mathrm{NWD}_p$, then there is $s_0 \in x \cap \mathrm{sp}(p)$ such that for every $s \in p$ with $s_0 \subseteq s$ there is $s' \in x \cap \mathrm{sp}(p)$ with $s \subseteq s'$. This enables an inductive definition of a perfect tree $q \subseteq p$ with stem $s_0$ such that $\mathrm{sp}(q) \subseteq x$ and hence $x \notin I_p^1$. ⊣

In general, the inclusions in Lemma 1.4 are strict. For example, by (1.3′) and Remark 1.2 (3), $J_p$ is usually distinct from $I_p^2$.

Lemma 1.5. *$J_p$, $I_p^1$, $I_p^2$, $I_p^3$ are meager in $\mathcal{P}(\omega)$, and $I_p^0$, $\mathrm{NWD}_p$ are meager in $\mathcal{P}({}^{<\omega}2)$.*

PROOF. By Lemma 1.4 it is enough to prove that $I_p^3$ and $\mathrm{NWD}_p$ are meager. Let $f_p \in {}^\omega\omega$ be such that for every $s \in p$ with $|s| = f_p(n)$ there exists a splitting node $t \supseteq s$ in $p$ with $|t| < f_p(n+1)$. The sets $a_n = [f_p(n), f_p(n+1))$ for $n \in \omega$ are pairwise disjoint and for every $x \in [\omega]^\omega$ we have $\bigcup_{n\in x} a_n \notin I_p^2$ because $p[\bigcup_{n\in x} a_n]$ is a perfect tree. This proves that $I_p^2$ is meager. To prove that $\mathrm{NWD}_p$ is meager take for $a_n$ the $n$th splitting level of $p$. ⊣

We will prove in Lemma 3.4 that all these hereditary sets belong to a smaller class than the class of meager hereditary sets.

By next lemma every ideal $I_p^0$ is isomorphic to some $J_q$ where the isomorphism is given by a fixed bijection from ${}^{<\omega}2$ onto $\omega$.

LEMMA 1.6. *For every $p \in \mathbb{S}$ and every $f \in ({}^\omega\omega)_{\mathrm{Fin}}$ there is $q \in \mathbb{S}$ such that $J_q = f^{-1}(J_p)$.*

PROOF. Set $q = \{s \in {}^{<\omega}2 : (\exists t \in p)\ s \subseteq f \circ t\}$ where $(f \circ t)(k) = t(f(k))$. Then $[q] = \{f \circ x : x \in [p]\}$ and $[q]\restriction a = \{(f\restriction a) \circ y : y \in [p]\restriction f``a\}$ for every $a \subseteq \omega$. As the composition operation is one-to-one it follows that $a \in J_q$ if and only if $f``a \in J_p$. ⊣

LEMMA 1.7. *$\mathbb{S}$ is isomorphic to a dense subset of $\mathscr{P}({}^{<\omega}2)/I_\mathbb{S}$.*

PROOF. Let $[x]_\mathbb{S}$ denote the equivalence class determined by the set $x \subseteq {}^{<\omega}2$ modulo the ideal $I_\mathbb{S}$. The mapping $\varphi : \mathbb{S} \to \mathscr{P}({}^{<\omega}2)/I_\mathbb{S}$ defined by $\varphi(p) = [\mathrm{sp}(p)]_\mathbb{S}$ is a dense embedding. ⊣

REMARK 1.8. Let $B$ be an almost disjoint family on $\omega$ of size $2^\omega$. If $I$ is a hereditary meager subset of $\mathscr{P}(\omega)$ and $f$ is a finite-to-one function such that $f(I) \subseteq [\omega]^{<\omega}$, then $A = \{f^{-1}(x) : x \in B\}$ is almost disjoint family of size $2^\omega$ and $A \cap I = \emptyset$. Therefore meager ideals are not c.c.c..

For $a \subseteq \omega$ and $A \subseteq \mathscr{P}(\omega)$ by $a \perp A$ we mean $|a \cap b| < \omega$ for all $b \in A$. If $I \subseteq \mathscr{P}(\omega)$ is a hereditary set, then $a \perp I$ if and only if $[a]^\omega \cap I = \emptyset$. A matrix is a set of maximal almost disjoint families on $\omega$.

Now we introduce some cardinal invariants for subsets $\mathscr{H}$ of the system

$$\mathscr{M} = \{I \subseteq \mathscr{P}(\omega) : I \text{ is a hereditary meager set}\}.$$

For arbitrary $\mathscr{H} \subseteq \mathscr{M}$ we define

$$\kappa(\mathscr{H}) = \min\{|\mathscr{A}| : \mathscr{A} \text{ is a matrix and } (\forall I \in \mathscr{H})(\exists A \in \mathscr{A})\ |A \setminus I| = 2^\omega\},$$

$$\kappa'(\mathscr{H}) = \min\{|\mathscr{F}| + |\mathscr{A}| : \mathscr{F} \subseteq ({}^\omega\omega)_{\mathrm{Fin}}, \mathscr{A} \text{ is a matrix, and } (\forall I \in \mathscr{H})(\exists f \in \mathscr{F})(\exists A \in \mathscr{A})\ |A \setminus f(I)| = 2^\omega\},$$

$$\lambda(\mathscr{H}) = \min\{|\mathscr{F}| + |\mathscr{A}| : \mathscr{F} \subseteq ({}^\omega\omega)_{\mathrm{Fin}}, \mathscr{A} \text{ is a matrix, and } (\forall I \in \mathscr{H})(\exists f \in \mathscr{F})(\exists a \in \textstyle\bigcup\mathscr{A})\ a \perp f(I)\},$$

$$\lambda'(\mathscr{H}) = \min\{|\mathscr{F}| : \mathscr{F} \subseteq ({}^\omega\omega)_{\mathrm{Fin}} \text{ and } (\forall I \in \mathscr{H})(\exists f \in \mathscr{F})(\exists a \in [\omega]^\omega)\ a \perp f(I)\},$$

$$\mu(\mathscr{H}) = \min\{|\mathscr{F}| : \mathscr{F} \subseteq ({}^\omega\omega)_{\mathrm{Fin}} \text{ and } (\forall I \in \mathscr{H})(\exists f \in \mathscr{F})\ \omega \perp f(I)\}.$$

Sometimes it will be useful to consider the system $\mathscr{M}(Q)$ of meager hereditary subsets of $\mathscr{P}(Q)$ for an infinite countable set $Q$. It is easy to rewrite the definitions of the above cardinals in this more general context by means of finite-to-one functions $f : Q \to \omega$ (for $\kappa'$, $\lambda$, $\lambda'$, $\mu$) and by means of almost disjoint families on $Q$ (for $\kappa$).

Let $Q$ and $Q'$ be any countable infinite sets and let $\mathscr{H} \subseteq \mathscr{M}(Q)$ and $\mathscr{H}' \subseteq \mathscr{M}(Q')$. We define $\mathscr{H} \preceq \mathscr{H}'$ if and only if there is a finite-to-one function $h : Q \to Q'$ such that $(\forall I \in \mathscr{H})(\exists I' \in \mathscr{H}')\ h(I) \subseteq I'$ where $h(I) = \{x \subseteq Q' : h^{-1}(x) \in I\}$.

LEMMA 1.9. *The defined cardinal invariants are monotone with respect to the relation* $\preceq$. *Hence,* $\mathscr{H} \preceq \mathscr{H}'$ *implies* $\square(\mathscr{H}) \leq \square(\mathscr{H}')$ *for* $\square \in \{\kappa, \kappa', \lambda, \lambda', \mu\}$.

PROOF. Without loss of generality let $Q = Q' = \omega$. If $f, h \in ({}^{\omega}\omega)_{\mathrm{Fin}}$, $a \subseteq \omega$, and $A, I, J \subseteq \mathscr{P}(\omega)$, then the following holds:

1. $a \in h(I)$ if and only if $h^{-1}(a) \in I$.
2. $A$ is almost disjoint if and only if $\{h^{-1}(a) : a \in A\}$ is almost disjoint.
3. $h \circ f \in ({}^{\omega}\omega)_{\mathrm{Fin}}$, where $(h \circ f)(n) = f(h(n))$.
4. $I \subseteq J$ implies $f(I) \subseteq f(J)$.

For monotonicity of $\kappa$ apply (1) and (2) and for monotonicity of $\kappa'(\mathscr{H})$, $\lambda(\mathscr{H})$, $\lambda'(\mathscr{H})$, $\mu(\mathscr{H})$ apply (3) and (4). $\dashv$

Using relation $\preceq$ from Lemma 1.9 the inclusions of Lemma 1.4 can be rephrased as follows:

LEMMA 1.10.

1. $\{I_p^0 : p \in \mathbb{S}\} \preceq \{I_p^1 : p \in \mathbb{S}\} \preceq \{I_p^2 : p \in \mathbb{S}\} \preceq \{I_p^3 : p \in \mathbb{S}\}$.
2. $\{I_p^0 : p \in \mathbb{S}\} \preceq \{J_p : p \in \mathbb{S}\} \preceq \{I_p^2 : p \in \mathbb{S}\}$.
3. $\{I_p^1 : p \in \mathbb{S}\} \preceq \{\mathrm{NWD}_p : p \in \mathbb{S}\}$. $\dashv$

A matrix in $\mathbb{S}$ is a system $\mathscr{A}$ of antichains in $\mathbb{S}$. $\mathscr{A}$ is a *shattering matrix* if every element of $\mathbb{S}$ is compatible with continuum many elements of some antichain $A \in \mathscr{A}$; $\mathscr{A}$ is a *weakly shattering matrix* if every element of $\mathbb{S}$ is compatible with continuum many elements of $\bigcup \mathscr{A}$; $\mathscr{A}$ is a *base matrix* if $\bigcup \mathscr{A}$ is dense in $\mathbb{S}$. The least cardinal to which Sacks forcing collapses the continuum we denote by $\mathrm{sh}(\mathbb{S})$. The cardinal $\mathrm{sh}(\mathbb{S})$ is equal to the minimal size of a shattering matrix in $\mathbb{S}$, to the minimal size of a weakly shattering matrix, and to the minimal size of a base matrix. Moreover, $\mathrm{sh}(\mathbb{S}) \leq \mathrm{cf}\, 2^{\omega}$ (see e.g., [11]).

Let us note that we use the word matrix in two different meanings, one as a matrix in $\mathbb{S}$ and the other as a matrix in $\mathscr{P}(\omega)/\mathrm{Fin}$. We assume that the reader will guess the intended meaning from the context of the occurrence.

The motivation for introducing the above cardinals comes from Simon's proof of the fact that Sacks forcing collapses the continuum to $\mathfrak{b}$ (see [13, 11]). Let us note that this proof corresponds to the system of meager hereditary sets $\mathscr{H} = \{I_p^3 : p \in \mathbb{S}\}$ and its basic idea is behind the proof of the inequalities $\mathrm{sh}(\mathbb{S}) \leq \kappa(\{I_p^3 : p \in \mathbb{S}\}) \leq \mathfrak{b}$. The first inequality is a consequence of the next theorem and the second is proved later by Theorem 2.5 or Theorem 2.6.

THEOREM 1.11. $\mathfrak{h} \leq \kappa(\{J_p : p \in \mathbb{S}\})$ *and* $\kappa(\{I_p^1 : p \in \mathbb{S}\}) = \mathrm{sh}(\mathbb{S})$.

PROOF. Let $\mathscr{A}$ be a matrix of size $< \mathfrak{h}$. There is $a \in [\omega]^{\omega}$ such that $(\forall A \in \mathscr{A})$ $(\exists b \in A)\ a \subseteq^* b$. Let $p = \{s \in {}^{<\omega}2 : (\forall i \in (\mathrm{dom}\, s) \setminus a)\ s(i) = 0\}$. Then $J_p = \{x \in \mathscr{P}(\omega) : |x \cap a| < \omega\}$ and $|A \setminus J_p| = 1$ for every $A \in \mathscr{A}$. This proves that $\mathfrak{h} \leq \kappa(\{J_p : p \in \mathbb{S}\})$.

Let $\mathscr{A}$ be a matrix satisfying the condition in the definition of $\kappa(\{I_p^1 : p \in \mathbb{S}\})$ (note that the underlying set for antichains and hereditary families is the countable set ${}^{<\omega}2$ instead of $\omega$). Then $\mathbb{S} = \bigcup_{A \in \mathscr{A}} \mathbb{S}_A$ where $\mathbb{S}_A = \{p \in \mathbb{S} : |A \setminus I_p^1| = 2^{\omega}\}$. For every $A \in \mathscr{A}$, since $|\mathbb{S}_A| \leq \mathfrak{c}$, we can fix a one-to-one function $\pi_A : \mathbb{S}_A \to A$ such that $\pi_A(p) \notin I^1_p$. Now for every $p \in \mathbb{S}_A$ let us fix a perfect tree $\rho_A(p) \subseteq p$ such that $\mathrm{sp}(\rho_A(p)) \subseteq \pi_A(p)$. Then $B_A = \{\rho_A(p) : p \in \mathbb{S}_A\}$ is an antichain in $\mathbb{S}$ refining $\mathbb{S}_A$ and hence $\{B_A : A \in \mathscr{A}\}$ is a base matrix. It follows that $\mathrm{sh}(\mathbb{S}) \leq \kappa(\{I^1_p : p \in \mathbb{S}\})$.

Now we prove $\kappa(\{I^1_p : p \in \mathbb{S}\}) \leq \mathrm{sh}(\mathbb{S})$. Let $\mathscr{B} = \{B_\alpha : \alpha < \mathrm{sh}(\mathbb{S})\}$ be a shattering matrix for $\mathbb{S}$. By [8, Lemma 1.1] or [11, Theorem 2.4] every antichain $B_\alpha$ can be refined to a maximal antichain $B'_\alpha$ so that $\{[p] : p \in B'_\alpha\}$ is a disjoint family of sets. Then $A_\alpha = \{\mathrm{sp}(p) : p \in B'_\alpha\}$ is an almost disjoint family of subsets of ${}^{<\omega}2$ for all $\alpha < \mathrm{sh}(\mathbb{S})$. Let $p \in \mathbb{S}$ be arbitrary. As $\mathscr{B}'$ is a shattering matrix for $\mathbb{S}$ there is $\alpha$ such that $p$ is compatible with continuum many elements of $B'_\alpha$. Then $|A_\alpha \setminus I^1_p| = 2^\omega$ because $p, q \in \mathbb{S}$ are compatible if and only if $\mathrm{sp}(q) \notin I^1_p$. ⊣

For some estimations we will need the following property of $\mathfrak{b}$.

Lemma 1.12. *Let $\mathscr{F} \subseteq ({}^\omega\omega)_{\mathrm{Fin}}$ and $|\mathscr{F}| < \mathfrak{b}$. There is an increasing sequence of natural numbers $\{n_k\}_{k=0}^\infty$ such that*

$$(\forall f, g \in \mathscr{F})(\forall^\infty n \in \omega)(\exists k \in \omega)\ f^{-1}(\{n\}) \subseteq g^{-1}([n_k, n_{k+2})).$$

Proof. For $f, g \in \mathscr{F}$ let $h_{f,g}(n) = \min\{m \in \omega : (\forall k \in \omega)\ f^{-1}(\{k\}) \cap g^{-1}(n) \neq \emptyset \to f^{-1}(\{k\}) \subseteq g^{-1}(m)\}$ (recall that $n = \{0, 1, \ldots, n-1\}$). Let $h \in {}^\omega\omega$ eventually dominates the system of functions $\{h_{f,g} : f, g \in \mathscr{F}\}$ and let $h(n) > n$ for all $n$. Let us define $n_0 = 0$ and $n_{k+1} = h(n_k)$. Now, if $f, g \in \mathscr{F}$, then for all but finitely many $n \in \omega$, if $k$ is minimal such that $f^{-1}(\{n\}) \cap g^{-1}(n_{k+1}) \neq \emptyset$, then $f^{-1}(\{n\}) \subseteq g^{-1}(h_{f,g}(n_{k+1})) \subseteq g^{-1}(h(n_{k+1})) = g^{-1}(n_{k+2})$, and hence, $f^{-1}(\{n\}) \subseteq g^{-1}(n_{k+2}) \setminus g^{-1}(n_k) = g^{-1}([n_k, n_{k+2}))$. ⊣

Lemma 1.13. $\lambda'(\{I^0_p : p \in \mathbb{S}\}) \geq \mathfrak{b}$.

Proof. Let $\mathscr{F}$ be a family of finite-to-one functions $f : {}^{<\omega}2 \to \omega$ and $|\mathscr{F}| < \mathfrak{b}$. We find $p \in \mathbb{S}$ such that $(\forall f \in \mathscr{F})(\forall a \in [\omega]^\omega)(\exists b \in [a]^\omega)\ b \in f(I^0_p)$.

By Lemma 1.12 there is an increasing sequence $\{n_k\}_{k=0}^\infty$ such that

$$(\forall f \in \mathscr{F})(\forall^\infty n \in \omega)(\exists k \in \omega)\ f^{-1}(\{n\}) \subseteq \pi^{-1}([n_k, n_{k+2})) \tag{1.8}$$

where $\pi(s) = |s|$ for $s \in {}^{<\omega}2$. By induction let us define a perfect tree $p \subseteq {}^{<\omega}2$ so that the following two conditions are satisfied:

1. For every $k \in \omega$ there is at most one $s \in p$ such that $n_k \leq |s| < n_{k+2}$ and $s^\frown 1 \in p$.
2. $s^\frown 1 \in p$ if and only if $s$ is a splitting node of $p$.

We show that $p$ has the required properties. Let $f \in \mathscr{F}$ and $a \in [\omega]^\omega$ be given. By (1.8) and by refining $a$ if necessary we can assume that for every $i \in a$ there is $k(i) \in \omega$ so that $f^{-1}(\{i\}) \subseteq \pi^{-1}([n_{k(i)}, n_{k(i)+2}))$ and the intervals $[n_{k(i)}, n_{k(i)+2})$ for $i \in a$ are pairwise disjoint. For $i \in a$ let us fix $s_i \in p$ so that $n_{k(i)} \leq |s_i| < n_{k(i)+2}$ and so that $s_i \in \mathrm{sp}(p)$ whenever possible. For $i \in a$ and $y \in [p]$ the following conditions hold:

3. $y(\pi(s_i)) = 1$ if and only if $s_i \in \mathrm{sp}(p)$ and $s_i \subseteq y$.
4. $y{\restriction}(\pi``f^{-1}(\{i\})) \neq 0$ if and only if $y(\pi(s_i)) = 1$.

Using König's lemma we can find an infinite set $b \subseteq a$ and $x \in {}^\omega 2$ such that $s_i \cap x \subsetneq s_j \cap x$ for $i < j$ and $i, j \in b$. Then, by (3) and (4) we get

5. $(\exists^\infty i \in b)\ y{\restriction}(\pi``f^{-1}(\{i\})) \neq 0$ if and only if $y = x$.

It follows that $[p]\restriction\pi``f^{-1}(b)$ is countable. Therefore $f^{-1}(b) \in \pi^{-1}(J_p)$ and hence $b \in f(I_p^0)$. $\dashv$

REMARK 1.14. Notice that definitions of $\lambda'(\mathscr{H})$ and $\mu(\mathscr{H})$ were obtained from definition of $\lambda(\mathscr{H})$ by removing $\mathscr{A}$. By removing $\mathscr{F}$ from $\lambda(\mathscr{H})$ in order to obtain a definition of an invariant, a need to look for a matrix $\mathscr{A}$ of minimal size having the property

$$(\forall I \in \mathscr{H})(\exists a \in \bigcup\mathscr{A})\ a \perp I$$

arrives. However, there is no such $\mathscr{A}$ if $\mathscr{H}$ contains an $I$ such that $(\forall a \in [\omega]^\omega)(\exists b \in [a]^\omega)\ b \in I$ (for example the ideal $I = I_{\mathbb{S}}$ has this property). In particular, there is no such $\mathscr{A}$ whenever $\{I_p^0 : p \in \mathbb{S}\} \preceq \mathscr{H}$ (see the proof of Lemma 1.13).

LEMMA 1.15. *If* $\{I_p^0 : p \in \mathbb{S}\} \preceq \mathscr{H}$, *then* $\kappa(\mathscr{H}) = \kappa'(\mathscr{H}) \leq \lambda(\mathscr{H}) = \lambda'(\mathscr{H}) \leq \mu(\mathscr{H})$.

PROOF. If $\mathscr{A}$ satisfies the condition for $\kappa(\mathscr{H})$, then $\mathscr{A}$ satisfies also the condition for $\kappa'(\mathscr{H})$ for $\mathscr{F} = \{\mathrm{id}_\omega\}$ containing only the identity on $\omega$. Therefore $\kappa'(\mathscr{H}) \leq \kappa(\mathscr{H})$. For the inverse inequality it is enough to observe that if $A$ is an almost disjoint family and $f \in ({}^\omega\omega)_{\mathrm{Fin}}$, then $A' = \{f^{-1}(a) : a \in A\}$ is almost disjoint and $|A \setminus f(I)| = |A' \setminus I|$ for every $I$.

Let $(\mathscr{F}, \mathscr{A})$ be a witness for $\lambda(\mathscr{H})$. For $A \in \mathscr{A}$ let $A'$ be a refinement of $A$ such that $|\{b \in A' : b \subseteq^* a\}| = 2^\omega$ for all $a \in A$. Let $\mathscr{A}' = \{A' : A \in \mathscr{A}\}$. Then $(\mathscr{F}, \mathscr{A}')$ is a witness for $\kappa'(\mathscr{H})$. Therefore $\kappa'(\mathscr{H}) \leq \lambda(\mathscr{H})$.

Applying a base matrix of $\mathscr{P}(\omega)/\mathrm{Fin}$ of size $\mathfrak{h}$ in definition of $\lambda(\mathscr{H})$ we can see that $\lambda'(\mathscr{H}) \leq \lambda(\mathscr{H}) \leq \lambda'(\mathscr{H}) + \mathfrak{h}$. As $\mathfrak{h} \leq \mathfrak{b}$ and $\lambda'(\mathscr{H}) \geq \mathfrak{b}$, the equalities hold.

The inequality $\lambda'(\mathscr{H}) \leq \mu(\mathscr{H})$ is trivial. $\dashv$

In the next section we prove that $\lambda(\mathscr{H}) = \mathfrak{b}$ and $\mu(\mathscr{H}) = \mathfrak{d}$.

## §2. Some characterizations of $\mathfrak{b}$ and $\mathfrak{d}$.

In general, a binary relation is a triple of sets $\mathbb{A} = (A_-, A_+, A)$ where $A \subseteq A_- \times A_+$; in the case when $A_- = A_+ = X$ we represent the binary relation $\mathbb{A}$ as a pair $(X, A)$.

Let $\mathfrak{b}(\mathbb{A})$ and $\mathfrak{d}(\mathbb{A})$ denote the minimal cardinality of an unbounded family and of a dominating family for a relation $\mathbb{A}$, respectively, i.e.,

$$\mathfrak{b}(\mathbb{A}) = \min\{|Z| : Z \subseteq A_- \text{ and } (\forall y \in A_+)(\exists z \in Z)\ \neg A(z, y)\},$$
$$\mathfrak{d}(\mathbb{A}) = \min\{|Z| : Z \subseteq A_+ \text{ and } (\forall x \in A_-)(\exists z \in Z)\ A(x, z)\}.$$

A morphism between relations $\mathbb{A}$ and $\mathbb{B}$ is a pair of functions $\varphi_- : A_- \to B_-$ and $\varphi_+ : B_+ \to A_+$ such that

$$(\forall a \in A_-)(\forall b \in B_+)\ B(\varphi_-(a), b) \to A(a, \varphi_+(b)).$$

If there is a morphism between $\mathbb{A}$ and $\mathbb{B}$, then $\mathfrak{b}(\mathbb{B}) \leq \mathfrak{b}(\mathbb{A})$ and $\mathfrak{d}(\mathbb{A}) \leq \mathfrak{d}(\mathbb{B})$.

For the existence of a morphism between $\mathbb{A}$ and $\mathbb{B}$ it is sufficient to have a function $\varphi_- : A_- \to B_-$ such that for every bounded set $X \subseteq B_-$ the inverse image $\varphi_-^{-1}(X)$ is a bounded subset of $A_-$ (let us recall that $X \subseteq B_-$ is bounded if there is $y \in B_+$ such that $B(x, y)$ for every $x \in X$). We shall call such a function $\varphi_-$ a Tukey embedding (then $\varphi_+(b) \in A_+$ can be defined as a bound of the set $\varphi_-^{-1}(\{x \in B_- : B(x, b)\})$

for $b \in B_+$). The relations $\mathbb{A}$ and $\mathbb{B}$ are said to be Tukey equivalent if there are Tukey embeddings from $\mathbb{A}$ to $\mathbb{B}$ and from $\mathbb{B}$ to $\mathbb{A}$.

For more details on morphisms and Tukey embeddings see [1].

Let $\mathscr{P}_{\mathrm{Fin}}$ denote the family of all partitions of $\omega$ into nonempty finite sets. Here, by a partition we mean a disjoint family of sets covering $\omega$ not containing the empty set. By technical reasons we represent these partitions as countable infinite sequences of sets. The indexing sets for these partitions can be arbitrary infinite countable sets (in most of the considered cases they are infinite subsets of $\omega$).

For $a, b \in \mathscr{P}_{\mathrm{Fin}}$ and $\psi \in {}^{\omega}\omega$, $\psi \geq 1$, we define

$$a \preceq^*_{\psi} b \text{ if and only if } (\forall^{\infty} n)(\exists^{\psi(n)} k)\ a_k \subseteq b_n,$$

$$a \preceq^*_{0} b \text{ if and only if } (\forall^{\infty} (n,k))\ b_n \nsubseteq a_k$$

where $\exists^n k$ means that there are $n$ many $k$.

All relations $\preceq^*_{\psi}$ with $\psi \geq 1$ are transitive and the relation $\preceq^*_1$ is moreover reflexive.

Every $f \in ({}^{\omega}\omega)_{\mathrm{Fin}}$ defines a partition $\{f^{-1}(\{n\}) : n \in \operatorname{rng}(f)\} \in \mathscr{P}_{\mathrm{Fin}}$ together with its indexing: This representation of elements of $\mathscr{P}_{\mathrm{Fin}}$ we use to define the corresponding relations on $({}^{\omega}\omega)_{\mathrm{Fin}}$ which we denote by the same symbol $\preceq^*_{\psi}$, i.e., for $f, g \in ({}^{\omega}\omega)_{\mathrm{Fin}}$ and for $\psi \geq 1$ or $\psi = 0$ we define

$$f \preceq^*_{\psi} g \text{ if and only if } \{f^{-1}(\{n\}) : n \in \operatorname{rng}(f)\} \preceq^*_{\psi} \{g^{-1}(\{n\}) : n \in \operatorname{rng}(g)\}.$$

We will need the following transformations between partitions and increasing functions: For a strictly increasing function $f \in {}^{\omega}\omega$ such that $f(0) > 0$ let $c(f) = \{c_n(f)\}_{n=0}^{\infty}$ be a partition of $\omega$ into intervals defined by $c_0(f) = [0, f(0))$ and $c_{n+1}(f) = [f(n), f(n+1))$. Conversely, for $a \in \mathscr{P}_{\mathrm{Fin}}$ we define a strictly increasing function $\varphi(a) \in {}^{\omega}\omega$ by $\varphi(a)(n) = \min\{k > \varphi(a)(n-1) : (\exists i)\ a_i \subseteq [\varphi(a)(n-1), k)\}$ where we set $\varphi(a)(-1) = 0$. Then $a \preceq_1 c(\varphi(a))$ and $\varphi(c(f)) = f$.

Lemma 2.1. *The following conditions hold*:

1. $a \preceq^*_{\psi} b$ *and* $b \preceq^*_1 c$ *implies* $a \preceq^*_{\psi} c$.
2. $a \preceq^*_2 b$ *implies* $a \preceq^*_0 b$.
3. *If* $a \preceq^*_1 b$, *then* $(\forall^{\infty} n)\ \varphi(a)(n) < \varphi(b)(2n)$.
4. *If* $f, h \in {}^{\omega}\omega$ *are strictly increasing*, $f(0) > 0$, $h(0) > 0$, *and* $c(f) \preceq^*_{\psi} c(h)$, *then* $(\forall^{\infty} n)\ f(n) < h(2n)$.

Proof. (1) and (2) are trivial.

(3) Assume that $a \preceq^*_1 b$, i.e., there is $m_0$ such that $(\forall m \geq m_0)(\exists k)\ a_k \subseteq b_m$. Let $n_0 = 1 + \max \bigcup_{m<m_0} b_m$. Since $n_0 \leq \varphi(b)(n_0 - 1)$ we have $(\forall n \geq 0)(\exists m \geq m_0)$ $b_m \subseteq c_{n+n_0}(\varphi(b))$. It follows that $\varphi(a)(n) \leq \varphi(b)(n+n_0)$ for all $n$ and hence $(\forall^{\infty} n)$ $\varphi(a)(n) < \varphi(a)(2n)$.

(4) The case $\psi \geq 1$ follows by (3) because $\varphi(c(f)) = f$ and $\varphi(c(h)) = h$. It remains to prove the case $\psi = 0$. Assume that $f$, $h$ are strictly increasing and $c(f) \preceq^*_0 c(h)$, i.e., $(\forall^{\infty}(n,k) \in \omega \times \omega)\ [h(k), h(k+1)) \nsubseteq [f(n), f(n+1))$. It follows that $(\forall^{\infty} n)\ |\operatorname{rng} h \cap [f(n), f(n+1))| \leq 1$ and so there is $n_0$ such that $f(n) \leq h(n + n_0)$ for all $n$. Consequently, $(\forall^{\infty} n)\ f(n) < h(2n)$. ⊣

Theorem 2.2. $(({}^{\omega}\omega)_{\mathrm{Fin}}, \preceq^*_{\psi})$ *is Tukey equivalent to* $({}^{\omega}\omega, \leq^*)$ *for every* $\psi \geq 1$ *and for* $\psi = 0$. *Consequently*, $\mathfrak{b} = \mathfrak{b}(({}^{\omega}\omega)_{\mathrm{Fin}}, \preceq^*_{\psi})$ *and* $\mathfrak{d} = \mathfrak{d}(({}^{\omega}\omega)_{\mathrm{Fin}}, \preceq^*_{\psi})$.

PROOF. We find Tukey embeddings $\alpha : ({}^\omega\omega, \leq^*) \to (\mathscr{P}_{\mathrm{Fin}}, \preceq^*_\psi)$ and $\beta : (\mathscr{P}_{\mathrm{Fin}}, \preceq^*_\psi) \to ({}^\omega\omega, \leq^*)$.

For $f \in {}^\omega\omega$ let $f'(n) = \max\{f(k) : k \leq n\} + n + 1$ and we set $\alpha(f) = c(f')$. Assume that $\mathscr{F} \subseteq \mathscr{P}_{\mathrm{Fin}}$ is $\preceq^*_\psi$-bounded, i.e., there is $a \in \mathscr{P}_{\mathrm{Fin}}$ such that $b \preceq^*_\psi a$ for all $b \in \mathscr{F}$. As $a \preceq^*_1 c(\varphi(a))$, by Lemma 2.1 (1), $b \preceq^*_\psi c(\varphi(a))$ for all $b \in \mathscr{F}$. Therefore $c(f') \preceq^*_\psi c(\varphi(a))$ for all $f \in \alpha^{-1}(\mathscr{F})$. Then, by Lemma 2.1 (4), $(\forall^\infty n)$ $f(n) \leq f'(n) < \varphi(a)(2n)$. Therefore $\alpha^{-1}(\mathscr{F})$ is bounded in $({}^\omega\omega, \leq^*)$.

Set $\beta = \varphi$. Let $\mathscr{F} \subseteq {}^\omega\omega$ be $\leq^*$-bounded by a strictly increasing function $h \in {}^\omega\omega$ such that $h(n) > n$ for all $n$. For $\psi \in {}^\omega\omega$, $\psi \geq 1$ let us define by induction $h^*(0) = 0$ and $h^*(n+1) = h(h^*(n) + \psi(n))$. We prove that $\beta^{-1}(\mathscr{F})$ is bounded by $c(h^*)$ in $(\mathscr{P}_{\mathrm{Fin}}, \preceq^*_\psi)$.

Let $a \in \beta^{-1}(\mathscr{F}) = \varphi^{-1}(\mathscr{F})$. There is $m$ such that $\varphi(a)(k) \leq h(k)$ for all $k \geq m$. Let $n_0$ be such that $h^*(n_0) > \varphi(a)(m)$. Given $n \geq n_0$ let $k$ be minimal such that $h^*(n) \leq \varphi(a)(k)$. Then $m < k \leq h^*(n)$ because $k - 1 \leq \varphi(a)(k-1) < h^*(n)$. Hence $[\varphi(a)(k), \varphi(a)(k + \psi(n))) \subseteq [h^*(n), h^*(n+1))$ because $\varphi(a)(k + \psi(n)) \leq h(h^*(n)+\psi(n)) = h^*(n+1)$. Therefore $a \preceq^*_1 c(\varphi(a)) \preceq^*_\psi c(h^*)$ and so $a \preceq^*_\psi c(h^*)$ for all $a \in \beta^{-1}(\mathscr{F})$. The case $\psi = 0$ follows by Lemma 2.1 (2) from the case $\psi = 2$. ⊣

Let us recall that the additivity $\mathrm{add}(\mathscr{F})$ and the cofinality $\mathrm{cof}(\mathscr{F})$ of a family of sets $\mathscr{F}$ are the minimal size of an unbounded family and the minimal size of a dominating family, respectively, in the partially ordered set $(\mathscr{F}, \subseteq)$.

COROLLARY 2.3. $\mathrm{add}(\mathscr{M}) = \mathfrak{b}$ *and* $\mathrm{cof}(\mathscr{M}) = \mathfrak{d}$.

PROOF. For $a \in \mathscr{P}_{\mathrm{Fin}}$ let $I(a) = \{x \subseteq \omega : (\forall^\infty n)\ a_n \setminus x \neq \emptyset\}$. By Lemma 1.3, $I \in \mathscr{M}$ if and only if there is $a \in \mathscr{P}_{\mathrm{Fin}}$ such that $I \subseteq I(a)$. We show that for $a, b \in \mathscr{P}_{\mathrm{Fin}}$, $I(a) \subseteq I(b)$ if and only if $a \preceq^*_1 b$. Then the equalities $\mathrm{add}(\mathscr{M}) = \mathfrak{b}$ and $\mathrm{cof}(\mathscr{M}) = \mathfrak{d}$ follow by Theorem 2.2.

The implication $a \preceq^*_1 b \to I(a) \subseteq I(b)$ follows directly by definitions. To prove the inverse implication let us assume that $a \not\preceq^*_1 b$. Then the set $x = \{n \in \omega : (\forall k)\ a_k \setminus b_n \neq \emptyset\}$ is infinite. By induction on $i \in \omega$ let us choose $n_i \in x$ so that the sets $\{k : a_k \cap b_{n_i} \neq \emptyset\}$ for $i \in \omega$ are pairwise disjoint and denote $y = \bigcup_{i\in\omega} b_{n_i}$. Then $y \notin I(b)$ and $y \in I(a)$ because $a_k \setminus y \neq \emptyset$ for all $k$. Therefore $I(a) \not\subseteq I(b)$. ⊣

THEOREM 2.4. *Let* $\mathscr{M}_0$ *be the set of all meager ideals on* $\omega$. *Then* $\mathrm{add}(\mathscr{M}_0) = 2$ *and* $\mathrm{cof}(\mathscr{M}_0) = 2^{2^\omega}$.

PROOF. It is enough to prove that there is a system of $2^{2^\omega}$ meager ideals on a countable set $Q$ such that no pair of them has a common upper bound in $\mathscr{M}_0$. The standard construction of an independent system of $2^{2^\omega}$ subsets of a countable set is the key for the proof: Let $Q_n = \{(n, s) : s \in {}^{\mathscr{P}(n)}2\}$ and let $Q = \bigcup_{n\in\omega} Q_n$. Then $|Q_n| < \omega$ and $|Q| = \omega$. For $x \subseteq \omega$ and $i \in 2$ let $A_{x,i} = \{(n, s) \in Q : s(x \cap n) = i\}$. Then $\{A_{x,i} : x \in \mathscr{P}(\omega) \text{ and } i \in 2\}$ is an independent system of size $2^{2^\omega}$ on $Q$, i.e., $A_{x,1} = Q \setminus A_{x,0}$ and for every finite set $B \subseteq \mathscr{P}(\omega)$ and $\varphi : B \to 2$ the intersection $\bigcap_{x\in B} A_{x,\varphi(x)}$ is infinite. Then for every function $f : \mathscr{P}(\omega) \to 2$ the system $\{A_{x,f(x)} : x \in \mathscr{P}(\omega)\}$ generates an ideal on $Q$ which we denote by $I_f$. Hence, if $a \in I_f$, then there is a finite set $B \subseteq \mathscr{P}(\omega)$ such that $a \subseteq \bigcup_{x\in B} A_{x,f(x)} = Q \setminus \bigcap_{x\in B} A_{x,1-f(x)}$. Let $n_0$ be the least $n$ such that $x \cap n \neq x' \cap n$ whenever $x, x' \in B$ and $x \neq x'$. For $n \geq n_0$ let $s_n : \mathscr{P}(n) \to 2$ be defined by $s_n(v) = 1 - f(v)$

if $v \in \{x \cap n : x \in B\}$ and $s_n(v) = 0$ otherwise. Then $(n, s_n) \notin \bigcup_{x \in B} A_{x,f(x)}$. It follows that the set $\{n : Q_n \subseteq a\}$ is finite for every $a \in I_f$ and so the ideals $I_f$ are all meager. If $f, f' : \mathscr{P}(\omega) \to 2$ are distinct, i.e., $f(x) \neq f'(x)$ for some $x \in \mathscr{P}(\omega)$, then $A_{x,f(x)} \in I_f$ and $Q \setminus A_{x,f(x)} = A_{x,f'(x)} \in I_{f'}$. Therefore $I_f$ and $I_{f'}$ have no common upper bound in $\mathscr{M}_0$. ⊣

THEOREM 2.5. *If* $\{I_p^0 : p \in \mathbb{S}\} \preceq \mathscr{H} \preceq \mathscr{M}$, *then* $\lambda(\mathscr{H}) = \mathfrak{b}$ *and* $\mu(\mathscr{H}) = \mathfrak{d}$.

PROOF. For $g \in ({}^\omega\omega)_{\mathrm{Fin}}$ the set $I_g = \{x \subseteq \omega : (\forall^\infty k \in \operatorname{rng}(g))\, g^{-1}(\{k\}) \setminus x \neq \emptyset\}$ is a meager hereditary set on $\omega$ (compare with proof of Corollary 2.3). Notice that for $a \in [\omega]^\omega$ and $f, g \in ({}^\omega\omega)_{\mathrm{Fin}}$,

$$\begin{aligned} a \perp f(I_g) &\Leftrightarrow (\forall b \in [a]^\omega)\, f^{-1}(b) \notin I_g \\ &\Leftrightarrow (\forall b \in [a]^\omega)(\exists^\infty n \in b)(\exists k \in \operatorname{rng}(g))\, g^{-1}(\{k\}) \subseteq f^{-1}(\{n\}) \\ &\Leftrightarrow (\forall^\infty n \in a)(\exists k \in \operatorname{rng}(g))\, g^{-1}(\{k\}) \subseteq f^{-1}(\{n\}). \end{aligned}$$

The second equivalence holds because we can find infinite set $b' \subseteq b$ such that the sets $\{k : g^{-1}(\{k\}) \cap f^{-1}(\{n\}) \neq \emptyset\}$ for $n \in b'$ are disjoint. Therefore

$$\begin{aligned} g \preceq_1^* f &\Leftrightarrow \operatorname{rng}(f) \perp f(I_g), \\ f \not\preceq_0^* g &\Leftrightarrow (\exists a \in [\omega]^\omega)\, a \perp f(I_g). \end{aligned}$$

Let $\mathscr{F}$ be any system of functions $f \in ({}^\omega\omega)_{\mathrm{Fin}}$ such that $\operatorname{rng}(f) = \omega$. It is easy to see that in definitions of cardinals $\lambda'(\mathscr{H})$ and $\mu(\mathscr{H})$ it is enough to consider such families. As every meager hereditary set on $\omega$ is a subset of some $I_g$, the last two equivalences say

$$\begin{aligned} &\mathscr{F} \text{ is } \preceq_1^*\text{-dominating} \Leftrightarrow (\forall I \in \mathscr{M})(\exists f \in \mathscr{F})\, \omega \perp f(I), \\ &\mathscr{F} \text{ is } \preceq_0^*\text{-unbounded} \Leftrightarrow (\forall I \in \mathscr{M})(\exists f \in \mathscr{F})(\exists a \in [\omega]^\omega)\, a \perp f(I). \end{aligned}$$

Then, by Theorem 2.2, $\mu(\mathscr{M}) = \mathfrak{d}$ and $\lambda'(\mathscr{M}) = \mathfrak{b}$. By Lemma 1.13 and Lemma 1.9 we then obtain $\lambda(\mathscr{H}) = \mathfrak{b}$. To finish the proof of $\mu(\mathscr{H}) = \mathfrak{d}$, by Lemma 1.9 it is enough to prove $\mathfrak{d} \leq \mu(\{I_p^0 : p \in \mathbb{S}\})$.

For an infinite set $a \subseteq \omega$ the set $p_a = \{s \in {}^{<\omega}2 : (\forall i \in (\operatorname{dom} s) \setminus a)\, s(i) = 0\}$ is a perfect tree. Then $J_{p_a} = \{x \subseteq \omega : |x \cap a| < \omega\}$ and $I_{p_a}^0 = \pi^{-1}(J_{p_a}) = \{x \subseteq {}^{<\omega}2 : |x \cap \pi^{-1}(a)| < \omega\} \subseteq I_{p_a}^1$. Let $\mathscr{F}$ be a system of finite-to-one functions $f : {}^{<\omega}2 \to \omega$ of size $\mu(\{I_p^0 : p \in \mathbb{S}\})$ satisfying the condition in definition of $\mu(\{I_p^0 : p \in \mathbb{S}\})$. Then for every $a \in [\omega]^\omega$ there is $f \in \mathscr{F}$ such that $f(I_{p_a}^0) = [\omega]^{<\omega}$ and hence $(\forall^\infty n \in \omega)\, \pi^{-1}(a) \cap f^{-1}(\{n\}) \neq \emptyset$. For $f \in \mathscr{F}$ we define $m_0^f = 0$, $m_{n+1}^f = \min\{m \in \omega : (\forall k \leq n)\, \pi``f^{-1}(\{m\}) \cap \pi``f^{-1}(\{m_k^f\}) = \emptyset\}$, $h_f(n) = \max \bigcup \{\pi``f^{-1}(\{m\}) : m \leq m_{2n}^f\}$. We show that $\{h_f : f \in \mathscr{F}\}$ is a dominating family in ${}^\omega\omega$ and hence $d \leq |\mathscr{F}|$. Let $h \in {}^\omega\omega$ be strictly increasing. Then $a = \operatorname{rng}(h)$ is an infinite subset of $\omega$ and hence there is $f \in \mathscr{F}$ and $k \in \omega$ such that $(\forall n \geq k)\, \pi^{-1}(\operatorname{rng}(h)) \cap f^{-1}(\{n\}) \neq \emptyset$. Then for $n \geq k$, $h_f(n)$ dominates at least $n + 1$ different values of $\operatorname{rng}(h)$ (from each set $\pi``f^{-1}(\{m_i^f\})$ for $k \leq i \leq 2n$ at least one value) and hence $h_f(n) \geq h(n)$. This proves that $\mathfrak{d} \leq \mu(\{I_p^0 : p \in \mathbb{S}\})$. ⊣

THEOREM 2.6. $\kappa(\mathscr{M}_0) = \kappa(\mathscr{M}) = \mathfrak{b}$.

PROOF. It is enough to prove that $\mathfrak{b} \leq \kappa(\mathscr{M}_0)$ because $\kappa(\mathscr{M}_0) \leq \kappa(\mathscr{M}) \leq \lambda(\mathscr{M}) = \mathfrak{b}$. Let $\mathscr{A}$ be a matrix of size $< \mathfrak{b}$. Let $C \subseteq \bigcup \mathscr{A}$ be a maximal family such that $\bigcap C_0$ is infinite for all $C_0 \in [C]^{<\omega}$. Let $B$ be the closure of $C$ on finite intersections. Hence, if $A \in \mathscr{A}$ then either $|A \cap B| \leq 1$ or for every $x \in A$ there is $y \in B$ such that $x \cap y$ is finite. Let $I$ be the ideal generated by $\{\omega \setminus x : x \in B\}$. $I$ is the union of less than $\mathfrak{b}$ meager hereditary sets $I_x = \{y \subseteq \omega : |y \cap x| < \omega\}$ for $x \in B$. Therefore, by Corollary 2.3, $I$ is meager. But $|A \setminus I| \leq 1$ for all $A \in \mathscr{A}$. This proves that $\mathfrak{b} \leq \kappa(\mathscr{M}_0)$. ⊣

§3. **Small hereditary sets.** For a given system $B \subseteq [\omega]^\omega$ we say that a set $I \subseteq \mathscr{P}(\omega)$ is *$B$-small* if for every $x \in I$ there is $b \in B$ such that $|x \cap b| < \omega$. If $B$ is closed on finite modifications of its elements, then we can require $x \cap b = \emptyset$ in the definition of a small set. Notice that the closure of a $B$-small set on subsets is again a $B$-small set. The largest $B$-small set with respect to the inclusion is the set $I_{B\text{-small}} = \{x \subseteq \omega : (\exists b \in B)\ |b \cap x| < \omega\}$. We say that $I \subseteq \mathscr{P}(\omega)$ is *$\gamma$-small* if there is a set $B \subseteq [\omega]^\omega$ of size $\leq \gamma$ such that $I$ is $B$-small.

LEMMA 3.1. *A $B$-small set is meager whenever $|B| < \mathfrak{b}$.*

PROOF. Let $I$ be a (hereditary) $B$-small set. Let $f_b \in ({}^\omega\omega)_{\text{Fin}}$ for $b \in B$ be such that $f^{-1}(\{n\}) \cap b \neq \emptyset$ for every $n \in \omega$. Since $|B| < \mathfrak{b}$, there is $f \in ({}^\omega\omega)_{\text{Fin}}$ such that $f_b \preceq_1^* f$ for all $b \in B$. Then $f(I) \subseteq [\omega]^{<\omega}$ and hence $I$ is meager. ⊣

EXAMPLE 3.2. (1) Let $A$ be an almost disjoint family of subsets of $\omega$ and let $I(A) = \{x \subseteq \omega : (\exists X \in [A]^{<\omega})\ x \subseteq^* \bigcup X\}$. The ideal $I(A)$ is $\omega$-small. To see this take any set $B \in [A]^\omega$.

(2) There is a meager hereditary set $I$ which is not $\gamma$-small for $\gamma < 2^\omega$. Let $I = \{x \subseteq {}^{<\omega}2 : (\forall^\infty n \in \omega)\ {}^n2 \setminus x \neq \emptyset\}$. Let us assume that $B \subseteq [{}^{<\omega}2]^\omega$ is such that $(\forall x \in I)(\exists b \in B)\ |x \cap b| < \omega$. For $f \in {}^\omega 2$ let $x_f = \{f \restriction n : n \in \omega\}$. As ${}^{<\omega}2 \setminus x_f \in I$ there is $b \in B$ such that $b \subseteq^* x_f$. As $\{x_f : f \in {}^\omega 2\}$ is almost disjoint it follows that $|B| = 2^\omega$.

(3) There is a meager ideal $I$ which is not $\omega$-small. Let $\{a_n : n \in \omega\}$ be a partition of $\omega$ such that $|a_n| = n$ for all $n$ and let $I = \{x \subseteq \omega : (\exists k)(\forall n)\ |x \cap a_n| \leq k\}$. Let $B \subseteq [\omega]^\omega$ be countable. There is a disjoint system $\{c_b : b \in B\} \subseteq [\omega]^\omega$ such that $c_b \subseteq \{k : b \cap a_k \neq \emptyset\}$. Set $x = \{\min(b \cap a_k) : b \in B \text{ and } k \in c_b\}$. Then $x \in I$ because $|x \cap a_k| \leq 1$ for all $k$, but $|x \cap b| = \omega$ for all $b \in B$.

These examples show that if $I$ is $\gamma$-small and $f$ finite-to-one, then $f(I') \subseteq I$ does not imply that $I'$ is $\gamma$-small even if $I = [\omega]^{<\omega}$.

LEMMA 3.3. *If $f \in ({}^\omega\omega)_{\text{Fin}}$ and $I$ is $B$-small, then $f(I)$ is $f(B)$-small and $f^{-1}(I)$ is $f^{-1}(B)$-small, where $f(B) = \{f``b : b \in B\}$ and $f^{-1}(B) = \{f^{-1}(b) : b \in B\}$. Hence, $I$ is $\gamma$-small if and only if $f^{-1}(I)$ is $\gamma$-small because $I = f(f^{-1}(I))$ and $B = f(f^{-1}(B))$.* ⊣

The following lemma strengthens Lemma 1.5.

LEMMA 3.4. *$J_p, I_p^0, I_p^1, I_p^2, I_p^3$, and $\mathrm{NWD}_p$ are $\omega$-small hereditary sets for all $p \in \mathbb{S}$.*

PROOF. By Lemma 1.4 and Lemma 3.3 it is enough to prove that $\mathrm{NWD}_p$ and $I_p^3$ are $\omega$-small. For $s \in p$ let $b_s = \{t \in \mathrm{sp}(p) : s \subseteq t\}$. If $x \in \mathrm{NWD}_p$ then there is $s \in p$ such that $x \cap b_s = \emptyset$ and hence the sequence $\{b_s : s \in p\}$ witnesses that $\mathrm{NWD}_p$ is $\omega$-small.

Now we prove that $I_p^3$ is $\omega$-small. Let us choose a perfect tree $q \subseteq p$ such that (i) $|\mathrm{sp}(q) \cap {}^n2| \leq 1$ for all $n \in \omega$ and (ii) for every $s \in q$, $s^\frown 0 \in q$ if and only if $s^\frown 0 \in p$. By (ii), $q[x] \subseteq p[x]$ for $x \subseteq \omega$ and hence $I_p^3 \subseteq I_q^3$. Therefore it is enough to prove that $I_q^3$ is $\omega$-small. For $s \in q$ let $b_s$ be the set of all $n \geq |s|$ for which there is a splitting node $t \in \mathrm{sp}(q) \cap {}^n2$ on the leftmost branch in $q$ going through $s$. Let as assume that a set $x \subseteq \omega$ has nonempty intersections with all sets $b_s$, $s \in q$. Let $\{s_t : t \in {}^{<\omega}2\} \subseteq \mathrm{br}(q)$ be such that $|s_\emptyset| \in x \cap b_\emptyset$ and $|s_{t^\frown i}| \in x \cap b_{s_t{}^\frown i}$ for $t \in {}^{<\omega}2$ and $i \in \{0,1\}$. Let $r$ be the perfect tree for which $\mathrm{sp}(r) = \{s_t : t \in {}^{<\omega}2\}$. Then $r \subseteq q[x]$ and hence $x \notin I_q^3$. This proves that $I_q^3$ is $\omega$-small. ⊣

For $\gamma < 2^\omega$ let $\mathscr{M}^\gamma$ denote the family of $\gamma$-small hereditary subsets of $\mathscr{P}(\omega)$. Let us denote

$$\begin{aligned}\mathfrak{h}_\gamma &= \min\{|\mathscr{A}| : \mathscr{A} \text{ is a matrix, and } (\forall B \in [[\omega]^\omega]^{\leq\gamma})(\exists A \in \mathscr{A})\\ &\qquad |\{x \in A : (\forall b \in B)\ |b \cap x| = \omega\}| = 2^\omega\},\\ \mathfrak{h}'_\gamma &= \min\{|\mathscr{A}| : \mathscr{A} \text{ is a matrix, and } (\forall B \in [[\omega]^\omega]^{\leq\gamma})\\ &\qquad \textstyle\sum_{A\in\mathscr{A}} |\{x \in A : (\forall b \in B)\ |b \cap x| = \omega\}| = 2^\omega\}.\end{aligned}$$

LEMMA 3.5. *Let $\gamma < 2^\omega$.*

1. $\gamma < \mathfrak{h}'_\gamma \leq \mathfrak{h}_\gamma$.
2. $\mathfrak{h} = \mathfrak{h}_1 \leq \mathfrak{h}_\gamma = \kappa(\mathscr{M}^\gamma) \leq \mathfrak{b}$ *for* $1 \leq \gamma < \mathfrak{b}$. *In particular,* $\sup_{\gamma<\mathfrak{b}} \mathfrak{h}_\gamma = \mathfrak{b}$.
3. $\mathrm{sh}(\mathbb{S}) \leq \kappa(\{\mathrm{NWD}_p : p \in \mathbb{S}\}) \leq \mathfrak{h}_\omega$.
4. $\mathfrak{h}_\gamma = \mathfrak{h}'_\gamma$ *whenever* $1 < 2^\gamma \leq 2^\omega$.

PROOF. (1) For a matrix $\mathscr{A}$ of size $\gamma$ take $B$ with $|B| \leq \gamma$ such that $|B \cap A| = 1$ for all $A \in \mathscr{A}$. Then $|A \setminus I_{B\text{-small}}| \leq 1$ for $A \in \mathscr{A}$.

(2) If $\gamma < \mathfrak{b}$, then by Lemma 3.1, $\mathfrak{h}_\gamma = \kappa(\mathscr{M}^\gamma) \leq \kappa(\mathscr{M}) = \mathfrak{b}$.

(3) By Lemma 3.4, $\kappa(\{\mathrm{NWD}_p : p \in \mathbb{S}\}) \leq \kappa(\mathscr{M}^\omega) = \mathfrak{h}_\omega$.

(4) Let $\mathscr{A}$ be a matrix on $\omega$ of size $\mathfrak{h}'_\gamma$ such that $\sum_{A\in\mathscr{A}} |A \setminus I_{B\text{-small}}| = 2^\omega$ for every $B \subseteq [\omega]^\omega$ of size $\leq \gamma$. As $2^\gamma \leq 2^\omega$, we can assign in a one-to-one way to every $B \in [[\omega]^\omega]^{\leq\gamma}$ a pair $(x_B, A_B)$ such that $x_B \in A_B \in \mathscr{A}$ and $x_B \notin I_{B\text{-small}}$. By Lemma 3.1, Remark 1.8, and the discussion at the beginning of this section, for every such $B$ there is an almost disjoint family $X_B \subseteq [x_B]^\omega$ of size $2^\omega$ such that $X_B \cap I_{B\text{-small}} = \emptyset$. Now, $A' = \bigcup\{X_B : A_B = A\}$ is almost disjoint for every $A \in \mathscr{A}$ and the matrix $\{A' : A \in \mathscr{A}\}$ proves the inequality $\mathfrak{h}_\gamma \leq \mathfrak{h}'_\gamma$. ⊣

Let $P = \{(s,f) \in {}^{<\omega}\omega \times {}^\omega\omega : s \text{ and } f \text{ are strictly increasing}\}$. $P$ is ordered by $(s,f) \leq (t,g)$ if $s \supseteq t$, $(\forall n \in \omega)\ f(n) \geq g(n)$ and $(\forall n \in \mathrm{dom}(s) \setminus \mathrm{dom}(t))$ $s(n) > g(n)$. Let $P_\alpha$ be the result of the finite-support iteration of length $\alpha$ where $P_{\beta+1} \cong P_\beta * \dot{Q}_\beta$ where $\dot{Q}_\beta$ is $P$ defined in $V^{P_\beta}$.

A sequence $\langle a_\xi : \xi < \lambda\rangle$ of subsets of $\omega$ is an *eventually splitting sequence* if $(\forall a \in [\omega]^\omega)(\exists \xi < \lambda)(\forall \eta > \xi)\ |a \cap a_\eta| = |a - a_\eta| = \omega$. A sequence $\langle a_\xi : \xi < \lambda\rangle$ of subsets of $\omega$ is *eventually narrow* if $(\forall a \in [\omega]^\omega)(\exists \xi < \lambda)(\forall \eta > \xi)\ |a - a_\eta| = \omega$.

Note that $\langle a_\xi : \xi < \lambda\rangle$ is an eventually splitting sequence if and only if the sequence $\langle b_\xi : \xi < \lambda\rangle$ is eventually narrow where $b_{2\xi} = a_\xi$ and $b_{2\xi+1} = \omega \setminus a_\xi$.

THEOREM 3.6 (Baumgartner-Dordal [3]). *Any eventually narrow sequence remains eventually narrow in $V^{P_\alpha}$.* ⊣

For a real $r \in {}^{\omega}2$ and $a \subseteq \omega$ let $a * r = \{n : r(n) = 1 \text{ and } \sum_{k<n} r(k) \in a\}$. If $A \subseteq \mathscr{P}(\omega)$ then we denote $A*r = \{a*r : a \in A\}$. Clearly, $(a \cap b)*r = (a*r) \cap (b*r)$ and hence if $A$ is an almost disjoint family, then so is $A * r$.

LEMMA 3.7. *If $r \in {}^{\omega}2$ is a Cohen real over $V$, then $|b \cap (a * r)| = |b \setminus (a * r)| = \omega$ for all $a, b \in [\omega]^{\omega} \cap V$.*

PROOF. The sets $D_{m,a,b} = \{s \in {}^{<\omega}2 : (\exists n > m)\, s(n) = 1, n \in b, \text{and} \sum_{k<n} s(k) \in a\}$ and $E_{m,b} = \{s \in {}^{<\omega}2 : (\exists n > m)\, n \in b \text{ and } s(n) = 0\}$ are open dense in $({}^{<\omega}2, \supseteq)$ for all $m \in \omega$ and $a, b \in [\omega]^{\omega}$. ⊣

By the next theorem the inequality $\mathfrak{h}_{\omega} < \mathfrak{b}$ is consistent with ZFC.

THEOREM 3.8. *Let $\lambda$ be arbitrary regular cardinal number with $\omega_1 \leq \lambda \leq 2^{\omega}$. Then $V^{P_{\lambda}} \models$ "$\mathfrak{h} = \mathfrak{h}_{\omega} = \mathfrak{h}'_{\omega} = \omega_1$, $\mathfrak{b} = \lambda$, and $\mathfrak{h}'_{\gamma} = \gamma^+$ for all $\omega \leq \gamma < \mathfrak{b}$".*

PROOF. Let us fix an almost disjoint family $A \subseteq [\omega]^{\omega} \cap V$ of size $2^{\omega} = (2^{\omega})^{V_{\lambda}}$. Let $r_{\xi} \in {}^{\omega}2 \cap V^{P_{\omega\xi}}$ be a Cohen real added on the limit stage $\omega\xi$ of the finite-support iteration. Let us fix a cardinal number $\gamma$ with $\omega \leq \gamma < \lambda$. By Lemma 3.7, for every $a \in A$ the sequence $\langle a * r_{\xi} : \xi < \gamma^+\rangle$ is eventually splitting in $V^{P_{\gamma^+}}$ and by Theorem 3.6 it is eventually splitting also in $V^{P_{\lambda}}$. In particular, if $a \in A$ and $B \in [[\omega]^{\omega}]^{\leq\gamma} \cap V^{P_{\lambda}}$, then there is $\xi$ such that $(\forall b \in B)\, |b \cap (a * r_{\xi})| = \omega$. Therefore $\sum_{\xi<\gamma^+} |\{x \in A * r_{\xi} : (\forall b \in B)\, |b \cap x| = \omega\}| = 2^{\omega}$ and so $\mathfrak{h}'_{\gamma} = \gamma^+$ in $V^{P_{\lambda}}$. ⊣

As $\max\{\mathfrak{h}, \mathrm{sh}(\mathbb{S})\} \leq \mathfrak{h}_{\omega}$ it is natural to ask the following:

QUESTION 3.9.

1. Is $\mathfrak{h}_{\omega} \leq \mathrm{cf}\, 2^{\omega}$?
2. Is $\max\{\mathfrak{h}, \mathrm{sh}(\mathbb{S})\} = \mathfrak{h}_{\omega}$?
3. Is $\mathfrak{h} = \mathfrak{h}_{\omega}$?

Clearly, (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

Let us recall that

$$\mathrm{NWD}_p = \{x \subseteq {}^{<\omega}2 : (\forall s \in \mathrm{sp}(p))(\exists t \in \mathrm{sp}(p))\; s \subseteq t \text{ and } x \cap b_t \cap \mathrm{sp}(p) = \emptyset\}$$

for $p \in \mathbb{S}$ where $b_s = \{t \in {}^{<\omega}2 : s \subseteq t\}$ for $s \in {}^{<\omega}2$. If $p = {}^{<\omega}2$, then we simply write NWD. We know that $I_{\mathbb{S}} \subseteq I^1_p \subseteq \mathrm{NWD}_p$ for every $p \in \mathbb{S}$. For a sequence $A = \langle a_s : s \in {}^{<\omega}2\rangle$ of infinite subsets of $\omega$ and an ideal $I \subseteq \mathscr{P}(\omega)$ we define $I//A = \{a \subseteq {}^{<\omega}2 : (\exists x \in I)(\forall s \in a)\, |x \cap a_s| = \omega\}$. Notice that ${}^{<\omega}2 \notin I//A$ if and only $I$ is $\{a_s : s \in {}^{<\omega}2\}$-small. We say that $I$ is a *perfectly small* ideal if there is a sequence $A$ such that $I//A = \mathrm{NWD}$. A perfectly small ideal is $\omega$-small.

LEMMA 3.10. *Every ideal $I$ with $I_{\mathbb{S}} \subseteq I \subseteq \mathrm{NWD}_p$ is perfectly small for all $p \in \mathbb{S}$.*

PROOF. Let us fix $p \in \mathbb{S}$ and let $\{t_s : s \in {}^{<\omega}2\}$ be the enumeration of $\mathrm{sp}(p)$ such that $t_{s^\frown i} \subseteq t_s{}^\frown i$ for all $s \in {}^{<\omega}2$ and $i = 0, 1$. Let $a_s = b_{t_s} \cap \mathrm{sp}(p)$ and let $A = \langle a_s : s \in {}^{<\omega}2\rangle$.

We first prove that $\mathrm{NWD} \subseteq I//A$ whenever $I_{\mathbb{S}} \subseteq I$. Let $a \in \mathrm{NWD}$. Then also $\overline{a} = \{s \in {}^{<\omega}2 : (\exists t \in a)\; s \subseteq t\}$ is in NWD. Let $S$ be the set of all $s \in {}^{<\omega}2 \setminus \overline{a}$ which are minimal with respect to the inclusion. Then $S$ is a maximal antichain in $({}^{<\omega}2, \subseteq)$ and for every $s \in a$ there is $s' \in S$ such that $s \subseteq s'$. For every $s \in S$ let us choose an infinite branch $x_s \subseteq p$ with $t_s \in x_s$ and let $x = \bigcup_{s \in S} x_s$. Clearly, $x \in I_{\mathbb{S}} \subseteq I$. For every $s \in a$, $x \cap a_s \subseteq x \cap a_{s'} = x_{s'} \cap a_{s'}$ is infinite. Therefore $a \in I//A$ and so $\mathrm{NWD} \subseteq I//A$.

Now we prove that $\mathrm{NWD} \subseteq I//A$. Let $a \in I//A$. Then also $\overline{a} \in I//A$ where $\overline{a} = \{s : (\exists t \in a)\ s \subseteq t\}$ and let $x \in I \subseteq \mathrm{NWD}_p$ be such that $x \cap a_s \neq \emptyset$ for all $s \in \overline{a}$. We claim that $\overline{a} \in \mathrm{NWD}$. Otherwise there exists $s_0$ such that every $s \supseteq s_0$ is in $\overline{a}$. But as $x \in \mathrm{NWD}_p$, there exists $s \supseteq s_0$ such that $x \cap b_{t_s} \cap \mathrm{sp}(p) = x \cap a_s = \emptyset$. This contradiction proves that $\overline{a} \in \mathrm{NWD}$ and hence $\mathrm{NWD} \subseteq I//A$. ⊣

QUESTION 3.11. Let $\mathscr{H}$ be the set of perfectly small ideals on $\omega$. How large is $\kappa(\mathscr{H})$?

## §4. A nonmeager $p$-ideal.

J. E. Baumgartner and P. Dordal [3] assuming Martin's axiom have proved that there exists a well-ordered unbounded family of increasing functions which is not a dominating family. In [9, 10] the authors have proved that if there is a well-ordered unbounded family of increasing functions which is not a dominating family, then there is a nonmeager $p$-ideal. In this section we prove analogical results for $\mathscr{P}_{\mathrm{Fin}}$.

Let us recall that an ideal $I \subseteq \mathscr{P}(\omega)$ is a $p$-ideal if for every sequence of sets $x_n \in I$ for $n \in \omega$ there is $x \in I$ such that $x_n \subseteq^* x$ for all $n$.

Notice that if $\mathscr{F} \subseteq \mathscr{P}_{\mathrm{Fin}}$ is $\preceq_1^*$-dominating, then $\mathscr{F}$ is $\preceq_0^*$-dominating:. Let $a \in \mathscr{P}_{\mathrm{Fin}}$ be such that $a \not\preceq_0^* b$ (i.e., $(\exists^\infty(n,k))\ b_n \subseteq a_k$) for all $b \in \mathscr{F}$. Set $a'_n = a_{2n} \cup a_{2n+1}$. If $n$ is such that $(\exists k)\ b_n \subseteq a_k$, then $(\forall k)\ a'_k \not\subseteq b_n$. Therefore $a' \not\preceq_1^* b$ for all $b \in \mathscr{F}$.

THEOREM 4.1. *Let $\mathscr{F} \subseteq \mathscr{P}_{\mathrm{Fin}}$ be a $\preceq_1^*$-unbounded family which is well-ordered by $\preceq_1^*$. Let $\{a_n\}_{n=0}^\infty \in \mathscr{P}_{\mathrm{Fin}}$ be not $\preceq_0^*$-dominated by $\mathscr{F}$. Then $I = \{x \subseteq \omega : (\exists b \in \mathscr{F})(\forall^\infty k \in x)(\forall n)\ b_n \not\subseteq a_k\}$ is a nonmeager p-ideal.*

PROOF. Assume that $I$ is meager and we obtain a contradiction. Then there is $h \in ({}^\omega\omega)_{\mathrm{Fin}}$ such that $h(I) \subseteq [\omega]^{<\omega}$. Define $a' \in \mathscr{P}_{\mathrm{Fin}}$ by $a'_m = \bigcup\{a_k : k \in h^{-1}(\{m\})\}$. We prove that $\mathscr{F} \preceq_1^* a'$. Let $b \in \mathscr{F}$. Then $x_b = \{k : (\forall n)\ b_n \not\subseteq a_k\} \in I$. Hence $h^{-1}(\{m\}) \setminus x_b \neq \emptyset$ for all but finitely many $m \in \omega$ and let $k_{b,m} \in h^{-1}(\{m\}) \setminus x_b$. As $k_{b,m \notin x_b}$ there is $n$ such that $b_n \subseteq a_{k_{b,m}} \subseteq a'_m$. Hence $b \preceq_1^* a'$. This is a contradiction and hence $I$ is nonmeager. $I$ is a $p$-ideal because $x_b \subseteq^* x_{b'}$ whenever $b, b' \in \mathscr{F}$ and $b \preceq_1^* b'$. ⊣

THEOREM 4.2. *If Martin's axiom holds, then there exists $\mathscr{F} \subseteq \mathscr{P}_{\mathrm{Fin}}$ which is a $\preceq_1^*$-unbounded family, well-ordered by $\preceq_1^*$, and not $\preceq_0^*$-dominating.*

PROOF. Let DS denote the family of all finite disjoint sequences of finite subsets of $\omega$. Let us fix $a \in \mathscr{P}_{\mathrm{Fin}}$ and let $H = \{b \in \mathscr{P}_{\mathrm{Fin}} : a \not\preceq_0^* b\} = \{b \in \mathscr{P}_{\mathrm{Fin}} : (\exists^\infty(n,k))\ b_n \subseteq a_k\}$. We can construct $\mathscr{F} \subseteq H$ by repeatedly using Lemma 4.3. ⊣

LEMMA 4.3. *Assume that Martin's axiom holds. Let $\mathscr{F} \subseteq H$ be well-ordered by $\preceq_1^*$ and let $|\mathscr{F}| < 2^\omega$. If $b \in \mathscr{P}_{\mathrm{Fin}}$, then there exists $d \in H$ such that $\mathscr{F} \preceq_1^* d$ and $d \not\preceq_0^* b$.*

PROOF. Let $Q = \{(s, A) : s \in \mathrm{DS},\ A \in [\mathscr{F}]^{<\omega}\}$ be ordered by $(s, A) \leq (s', A')$ if $s \supseteq s'$, $A \subseteq A'$, and $(\forall i \in \mathrm{dom}(s) \setminus \mathrm{dom}(s'))(\forall c \in A')(\exists u)\ c_u \subseteq s(i)$. $Q$ is $\sigma$-centered. By Lemma 4.4, for every $c \in \mathscr{F}$ and $k \in \omega$ the set $D_{c,k} = \{(t, A) : c \in A,\ (\exists i, j \geq k)\ t(j) \subseteq a_i,\ (\exists i, j \geq k)\ b_i \subseteq t(j)$, and $k \subseteq \bigcup \mathrm{rng}\, s\}$ is dense. Let $G \subseteq Q$ be a $\{D_{c,k} : c \in \mathscr{F}, k \in \omega\}$-generic filter. Let $d = \bigcup\{s : (\exists A)\ (s, A) \in G\}$. Then $d \in \mathscr{P}_{\mathrm{Fin}}$ by (1), $\mathscr{F} \preceq_1^* d$ by (2), $d \in H$ by (3), and $d \not\preceq_0^* b$ by (4). ⊣

LEMMA 4.4. *Let $A \in [H]^{<\omega}$ be such that $A \preceq_1^* f$ for some $f \in H$. Let $b \in H$, $s \in \mathrm{DS}$, and $k \in \omega$. There exists $t \in \mathrm{DS}$ such that*

1. $s \subsetneq t$ *and* $k \subseteq \bigcup \operatorname{rng} t$.
2. $(\forall i \in \operatorname{dom}(t) \setminus \operatorname{dom}(s))(\forall c \in A)(\exists u)\ c_u \subseteq t(i)$.
3. $(\exists i \geq k)(\exists j)\ t(j) \subseteq a_i$.
4. $(\exists i \geq k)(\exists j)\ b_i \subseteq t(j)$.

PROOF. We can find $i \geq k$ and $j \in \omega$ such that $a_i \cap \bigcup \operatorname{rng} s = \emptyset$, $f_j \subseteq a_i$, and $(\forall c \in A)(\exists u)\ c_u \subseteq f_j$. Let us denote $w = f_j$. For each $c \in \mathscr{F}$ let us fix $i(c) \in \omega$ such that $(\exists u)\ c_u \subseteq f_{i(c)}$ and $f_{i(c)} \cap (w \cup \bigcup \operatorname{rng} s) = \emptyset$. Let $i_0 \geq k$ be such that $b_{i_0} \cap (w \cup \bigcup \operatorname{rng} s) = \emptyset$ and let $n = \operatorname{dom} s$. We define $t \supseteq s$ with $\operatorname{dom} t = n + 2$ as follows: $t(n) = \bigcup_{c \in A} f_{i(c)} \cup b_{i_0} \cup k \setminus (w \cup \bigcup \operatorname{rng} s)$ and $t(n+1) = w$. ⊣

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