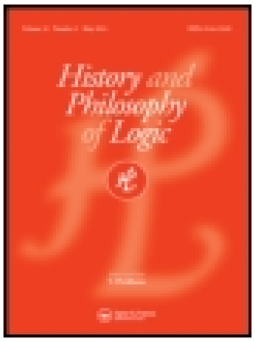

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Another Side of Categorical Propositions: The Keynes–Johnson Octagon of Oppositions

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ABSTRACT
The aim of this paper is to make sense of the Keynes–Johnson octagon of oppositions. We will discuss Keynes' logical theory, and examine how his view is reflected on this octagon. Then we will show how this structure is to be handled by means of a semantics of partition, thus computing logical relations between matching formulas with a semantic method that combines model theory and Boolean algebra.

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1. Introduction

The following proposes a historical and formal approach to what is known nowadays as Keynes' logical octagon. While the literature around Keynes' logical works is not tremendous as it stands, many more studies have been developed about his pioneering work on complementary categorical propositions. On the one hand, *Dekker 2015* recalled that one passage of Aristotle's *De Interpretatione* foreshadowed the possibility of dealing with categorical propositions whose subject term is negated. But Aristotle never included this case into his own syllogistic, where subject terms are always affirmed. On the other hand, a number of contemporary works extended the famous square of opposition into a logical cube (*Dekker 2015*; *Demey and Smessaert 2018*; *Dubois et al. 2020*), without always mentioning the central contribution to Keynes in this respect by revealing the 'other side' of categorical propositions (whenever the subject term is denied). Our point is not to implement these formal developments but, rather, to present a complete analysis of Keynes' octagon in three steps: first, a historical introduction to Keynes' logical achievements (Section 1); second, a look at the logical structure that was pioneered by Keynes (Section 2); and third, a formal semantics suggested to make sense of the logical relations between the eight propositions of Keynes' octagon.

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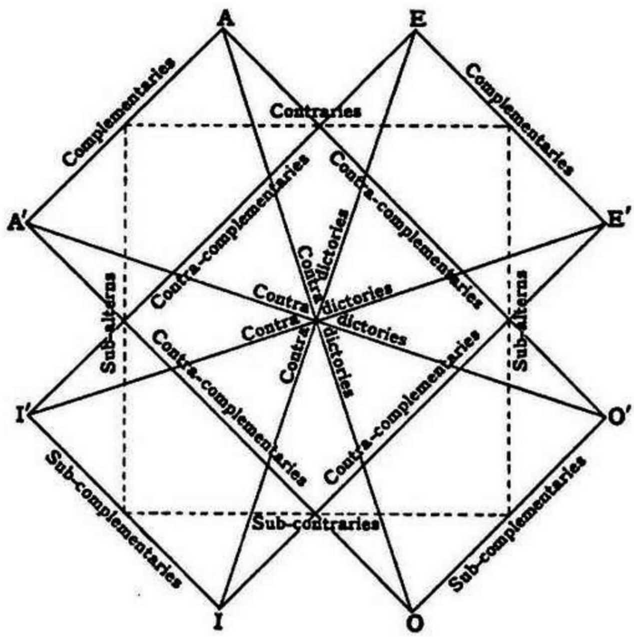

Figure 1. Logical octagon in (*Keynes 1894*, p. 113).

2. Keynes' Logical Octagon

In the third edition of his well-known textbook *Studies and Exercises in Formal logic* (*1894*), the British logician John Neville Keynes included an octagon of oppositions that has long been unnoticed (Figure 1).

Keynes acknowledged the help of William E. Johnson in the making of this octagon (*Keynes 1894*, p. 113). Later on, Johnson published a similar octagon in his own treatise (Figure 2).

More recently, Edward A. Hacker also introduced his own version of the octagon. Hacker briefly discussed Johnson's octagon and stated that its purpose was 'to show the relations of independence, and [that] it is not at all clear as to how one is to read the other logical relations' (*Hacker 1975*, p. 352).

The aim of this paper is to make sense of the Keynes-Johnson octagon of oppositions between matching propositions.[1] For this purpose, we will briefly recall Keynes theory of propositions. Indeed, this octagon should be understood within Keynes' logical theory where categorical propositions involve terms, S and P, and their complements, S' and P' (S' stands for not-S, and P' for not-P). As such, Keynes goes beyond the traditional typology of categorical propositions as found in the traditional square. It must be reminded that Keynes worked on a 'generalization of logical processes in their application to complex inferences', entering into rivalry with logicians such as George Boole and John Venn who

[1] Any two categorical propositions are said 'matching' whenever they include the same subject and predicate terms.

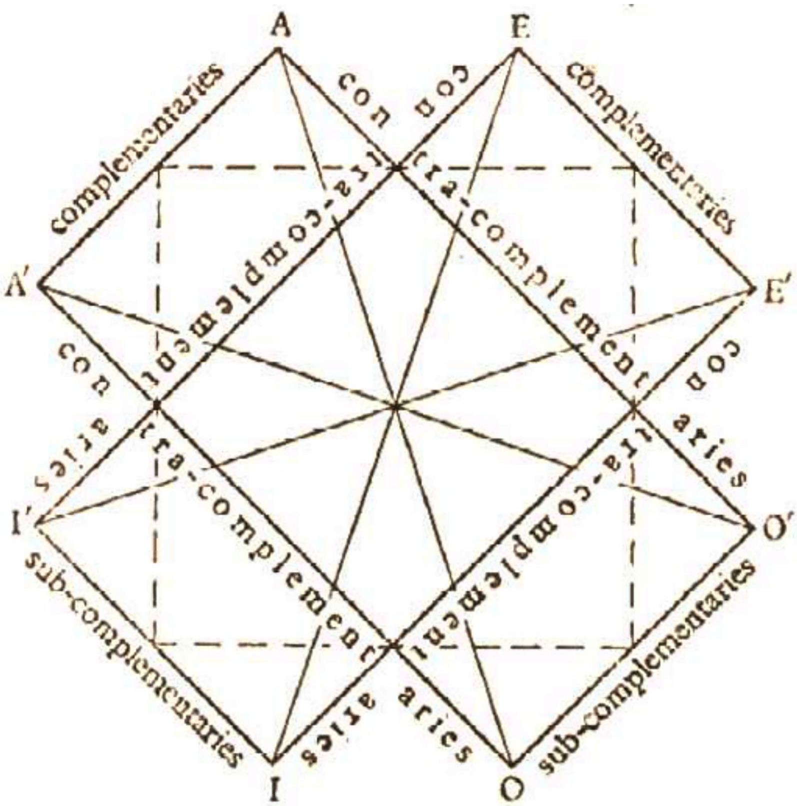

Figure 2. Logical octagon in (*Johnson 1921*, p. 142).

worked complex problems with the use of symbolic methods.[2] Keynes considered his work to include

> the first systematic attempt that has been made to deal with formal reasonings of the most complicated character without the aid of mathematical symbols and without abandoning the ordinary non-equational and predicative form of proposition.[3]

Keynes' attempt was quite successful though symbolic logicians suspected that he would never have been in position to invent his non-symbolic methods if he did not previously learn and practice symbolic methods. Whatever, Keynes' treatise, first published in 1884 (with three subsequent editions in 1887, 1894 and 1906), should be read as an attempt to expand syllogistic beyond the traditional scope to deal with complex propositions and complex problems.[4] As such, the octagon must be seen, not just as a logical curiosity but

[2] Boole developed a mathematical theory of logic in his *Laws of Thought* (1854) Since propositions are expressed in the form of equations, logic problems are reduced to systems of equations. Such problems consisted in determining the conclusion that follows from a set of premises. This is achieved by eliminating undesired or superfluous terms (*Green 1991*). A dispute occurred among Boole's followers regarding the best notation to tackle such problems. William S. Jevons and Venn maintained the equational form while others (Charles S. Peirce, Ernst Schröder, Hugh MacColl, etc.) favored a notation based on inclusion (*Peckhaus 1989*; *Moktefi 2019*). Some traditional logicians, such as John Cook Wilson, opposed to the use of symbolism and argued that such problems may be solved without this mathematical apparatus (*Marion and Moktefi 2014*).

[3] *Keynes 1884*, p. vii.

[4] The treatment of such complex problems played an important role in the competition among logicians regarding the elimination problem. Indeed, traditional logic problems were generally reduced to syllogisms (involving 3 terms) or series

rather as a serious illustration of Keynes' project. In a way, the octagon stands in Keynes' theory the same way the 'square' stands in traditional logic.

Keynes' octagon reflects his logical theory and the desire to expand traditional syllogistic. Handling negative terms is an important part of this project. In accordance with many of his contemporaries (e.g. Lewis Carroll) but in opposition with some others (e.g. John Venn), Keynes considered that contradictory terms should be visually regarded on the same footing since the distinction between *A* and not-*A* is conventional. Another instance of Keynes' expansion of syllogistic to include negative terms might be observed in his treatment of Euler diagrams. Keynes attempted to adapt Euler's scheme in order to handle negative terms. Hence, he identified and represented diagrammatically 7 possible relations between classes *S*, not-*S*, *P*, and not-*P* (Figure 3).

Keynes' relations (as one might call them) have been obtained simply by substituting two subcases for each Gergonne's relation between any two classes *S* and *P*: one in which the outer region is empty, and one where it is not. For instance, Gergonne's case where the classes *S* and *P* are completely disjoint engender two subcases, depending on whether there is 'something' outside or nothing.

Among the 10 relations obtained by distinguishing two sub-cases for each of the 5 Gergonne's relations, Keynes excluded 3 cases where either not-*S* or not-*P* (or both) was empty, thereby keeping 7 relations (*Keynes 1894*, pp. 140–6) only.[5] As curious as Keynes' diagrams might seem, they are necessary if one wants to deal rigorously with logical problems involving negative terms with the use of Euler-type diagrams. In making his octagon, Keynes, with the help of Johnson, proceeds similarly. His generalization led him to 'amplify the list of formal relations recognized in the square of opposition and also to extend the meaning of certain terms' (*Keynes 1894*, p. 89). The octagon is the result of this inquiry.

3. The Structure of the Logical Octagon

After introducing the historical roots of the logical octagon from Keynes' writings, let us scrutinize its content and logical properties in the following. A main distinction is to be made between propositions with respect to their subject-terms and predicate-terms. The original propositions **A**,**E**,**I**,**O** relate to propositions whose subject-term is always affirmed, whereas the additional propositions $\mathbf{A}'$,$\mathbf{E}'$,$\mathbf{I}'$,$\mathbf{O}'$ are those whose subject-term is always negated. We will explain the interrelations between both kinds of categorical propositions, ultimately.

of syllogisms. However, Boole and his followers worked on the general solution of problems involving any number of terms and any number of propositions. In this line, several logicians (Venn, Allan Marquand, Alexander Macfarlane, Lewis Carroll, etc.) invented diagrammatic schemes that would allow tackling problems involving more than 3 terms (*Moktefi and Edwards 2011*). It was precisely Keynes' goal to show that such complex problems can be solved without the appeal to mathematical notations.

[5] Joseph D. Gergonne introduced his relations in *1817*, p. 193: 'Let us presently examine the diverse circumstances in which two ideas, when compared to each other, can be relatively to their extension' (translated by the author). See also *Grattan-Guinness 1977*. In his attempt, Gergonne did not consider negative terms. Arthur Schopenhauer undertook a similar task in 1818 and got closer to Keynes' set of relations (*Moktefi 2020*). Keynes's dismissal of the cases where classes or their negatives are empty reflects the difficulties that early logicians faced with empty classes. Venn alluded the problem by introducing buffer-units, known as compartments, that may be empty, in which case, the corresponding classes are said not to exist. Carroll defined imaginary classes that contain non-existing individuals but did not pursue the idea consistently. In his steps, MacColl developed a logic of fictions and entered into controversy with Bertrand Russell on this subject (*Radford 1995*; *Abeles and Moktefi 2011*).

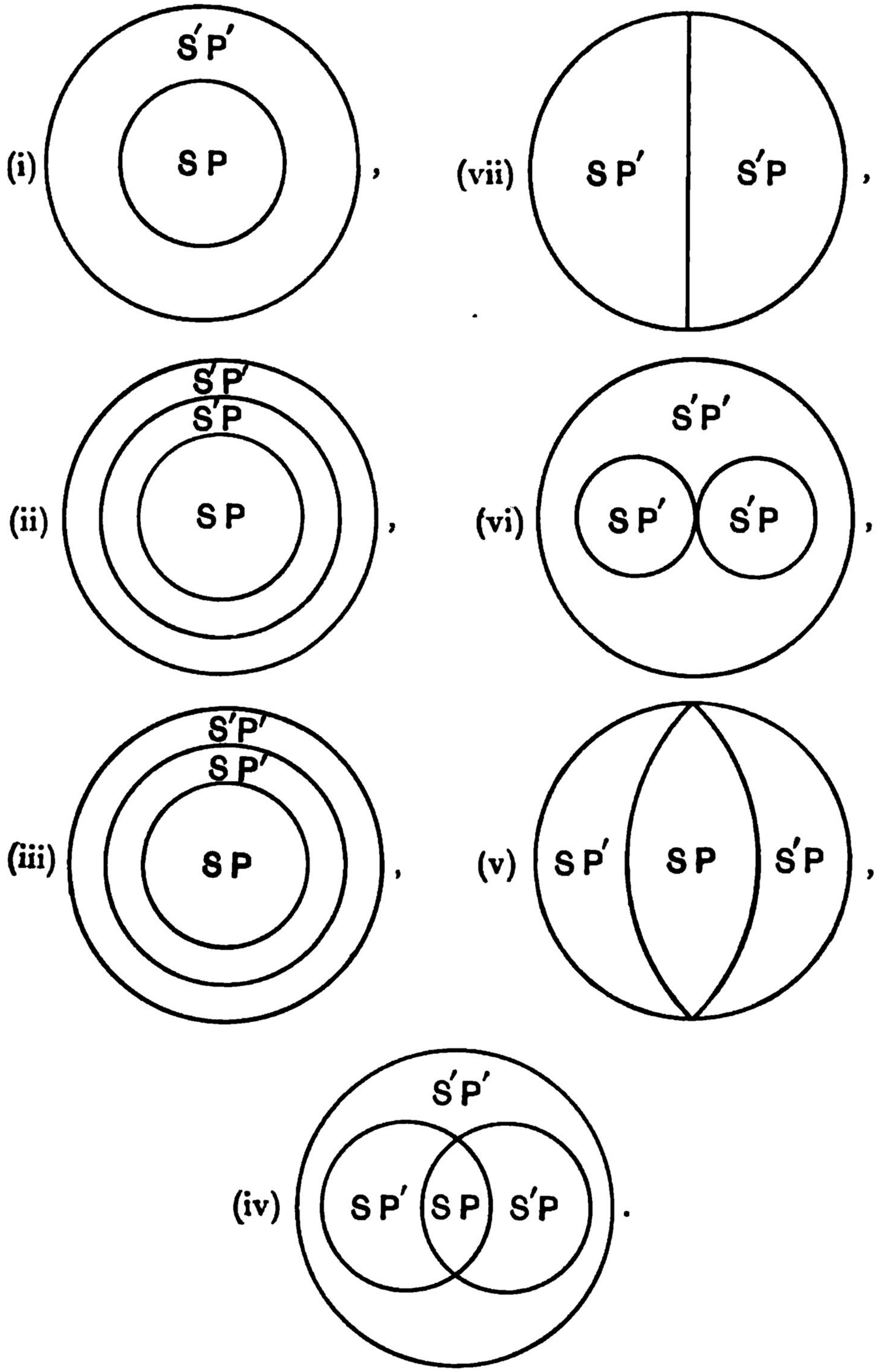

Figure 3. Logical relations in (*Keynes 1894*, p. 142).

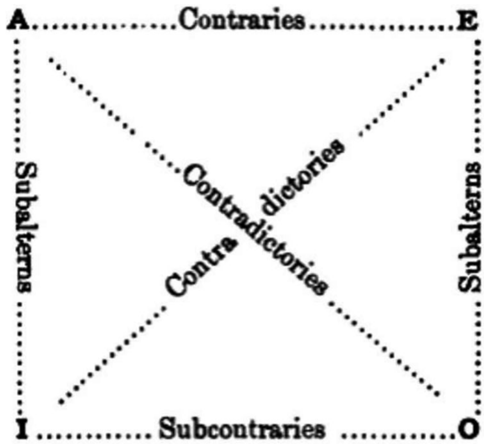

Figure 4. Logical square of opposition in (*Keynes 1894*, p. 81).

3.1. Relations Between Propositions with Subject and Predicate Fixed

To understand Keynes' generalization, let us start with the square itself. In traditional form, propositions are formed by predication. It includes one subject term among two (either S or P), and one other predicate term (P or S, respectively), wherein a predicate (either P or S) is attributed or denied to all or part of a subject (either S or P, respectively): every proposition has one quantity (either universal or particular) and one quality (either affirmative or negative). Hence, once the subject and predicate are fixed, you have $2 \times 2 \times 1 = 4$ types of propositions.

$$\mathbf{A} : S\mathbf{a}P; \mathbf{E} : S\mathbf{e}P; \mathbf{O} : S\mathbf{o}P; \mathbf{I} : S\mathbf{i}P.$$

Consequently, you have 6 combinations with the 4 classical relations: contradiction, contrariety, subcontrariety, and subalternation. These are those one finds in the traditional square, as reproduced by Keynes himself in his treatise (Figure 4):

It is important to keep in mind that these 4 relations are the relations between distinct propositions where subject and predicate are fixed. Indeed, there are other possible relations between two given propositions whatever. For instance, they might just be equivalent, or independent.

3.2. Relations Between Propositions with Two Terms Whatever

Let us now assume that we are interested in propositions with two distinct terms whatever (that might be subject or predicate). Since the subject and predicate are not fixed, we have the following possibilities: any of two subject terms (S, or P) and one that has not occurred as subject terms (the other term, whether affirmed or negated). So we have here $2 \times 2 \times 2 = 8$ types of propositions:

$$\mathbf{A} : S\mathbf{a}P; \mathbf{E} : S\mathbf{e}P; \mathbf{O} : S\mathbf{o}P; \mathbf{I} : S\mathbf{i}P; \mathbf{A}' : P\mathbf{a}S; \mathbf{E}' : P\mathbf{e}S; \mathbf{O}' : P\mathbf{o}S; \mathbf{I}' : P\mathbf{i}S$$

Since $P\mathbf{e}S$ is equivalent to $S\mathbf{e}P$ and $P\mathbf{i}S$ to $S\mathbf{i}P$ (by conversion), we actually have 6 distinct propositions and, consequently, 15 possible combinations. Most can be defined by using

Figure 5. coextensive classes (*Keynes 1894*, p. 131).

Figure 6. *S* is strictly included into *P* (*Keynes 1894*, p. 131).

the 4 oppositions already identified when subject and predicate are fixed. However, 4 logical relations connect independent propositions: **A** and **A**′; **A** and **O**′; **O** and **A**′; **O** and **O**. The treatment of these 4 combinations is the main difference between Keynes–Johnson and Hacker. The later did not make sense of them and did not represent them on his diagram. By contrast, Keynes represented these on his octagon. More importantly, he defined different types of independence relations depending on what the propositions in each combination say about the terms *S* and *P* (*Keynes 1894*, p. 131).

Let us first examine the combination **A**-**A**′. These two propositions taken together state that the classes *S* and *P* are coextensive, i.e. have the same extension (Figure 5). Keynes calls them *complementary* propositions.

The second pair **A**-**O′** states that *S* is included into *P* but does not exhaust it. So, *S* is strictly included in *P* (Figure 6). Keynes calls the propositions of the pair *contra-complementary* propositions.

The propositions of the third pair **O**-**A**′ are also *contra-complementary*, with *P* being strictly included into *S*. Finally, the fourth pair **O**-**O**′ asserts that *S* and *P* are neither coextensive nor included into one another. So they must correspond to one of the two combinations (Figure 7). Keynes calls the propositions of the pair *sub-complementary* propositions.

Hence, independent propositions can be complementary, contra-complementary, or sub-complementary. These relations exhaust the relations between the classes *S* and *P*, since

Figure 7. *S* and *P* are neither coextensive nor included into one another (*Keynes 1894*, p. 131).

all 5 possible relations (known as Gergonne's relations) have been covered. According to Keynes, these technical terms have been suggested to him by Johnson (*Keynes 1894*, p. 100).

So, now we have defined all 15 relations between the 6 distinct propositions connecting two terms whatever. These can be represented by a hexagon of opposition. Keynes did not provide such a graphical representation, but it can be easily done.

3.3. Relations Between Propositions Connecting Any Two Terms and Their Contradictories

Now that we identified the different types of relations between two propositions whatever according to Keynes, namely: equivalence, contradiction, contrariety, sub-contrariety, subalternation, and independence (complementarity, sub-complementarity, or contra-complementarity), we can more easily make sense of the Keynes–Johnson octagon of opposition, where propositions connect any two terms and their contradictories.

Let the terms be *S* and *P*, their contradictories not-*S* and not-*P*. By adding negative terms to the affirmative ones, we thus have the following possibilities: any of four subject terms (*S*, not-*S*, *P*, or not-*P*) and any of two ones that have not occurred as subject terms (the other term, whether affirmed or negated). So we have here $2 \times 4 \times 2 \times 2 = 32$ types of propositions, but only 8 nonequivalent propositions as shown in Keynes' list (Figure 8):

There are 28 relations connecting those 8 non-equivalent propositions. It is these relations that are represented by the octagon (Figure 1).

One might think that several such relations are not represented: for instance, those between **A** and **E**, **A** and **E**′, etc. However, we should keep in mind a graphical convention adopted by Keynes in this diagram. The disconnected lines connecting two sides actually condense each 4 relations connecting the terms of those sides. Let us for instance observe the discontinuous line connecting sides: **AA**′ and **EE**′. It actually represents 4 relations: **A**-**E**, **A**-**E**′, **A**′-**E**, **A**′-**E**′. All these pairs are formed by contrary propositions, so that Keynes uses just one discontinuous line that might represent any of the 4 pairs. This graphical convention is common in diagrammatic reasoning, notably in Keynes' other diagrams. Still, one might regret that Keynes did not represent properly all relations. Again, this can be easily done to get a complete octagon of opposition with the following symbols for the corresponding logical relations: *ct* (contrariety), *cd* (contradiction), *sct* (subcontrariety), *sb* (subalternation), *cm* (complementarity), *ccm* (contracomplementarity), *scm* (subcomplementarity) (Figure 9).

	(i)	(ii)	(iii)	(iv)
		Universals		
A	SaP	= SeP′	= P′eS	= P′aS′
A′	S′aP′	= S′eP	= PeS′	= PaS
E	SaP′	= SeP	= PeS	= PaS′
E′	S′aP	= S′eP′	= P′eS′	= P′aS
		Particulars		
O	SoP	= SiP′	= P′iS	= P′oS′
O′	S′oP′	= S′iP	= PiS′	= PoS
I	SoP′	= SiP	= PiS	= PoS′
I′	S′oP	= S′iP′	= P′iS′	= P′oS

Figure 8. Categorical propositions in (*Keynes 1894*, p. 110).

All one has to do now is to identify the type of relation connecting any two corners in the octagon, among the relations previously identified: equivalence, contradiction, contrariety, sub-contrariety, subalternation and independence (whether it be complementarity, sub-complementarity or contracomplementarity). Actually, only the equivalence relation is missing from the diagram. This suggests that Keynes' understanding of the concept of opposition might be largely understood as non-equivalence. On the other hand, it is possible also that he understood them as any relation whatever but just did not bother to include equivalence; indeed, this would have obliged him (superfluously) to construct a 32-opposition structure by including the other 3 propositional forms equivalent to each of the 8 non-equivalent propositions of the octagon.

4. A Semantics for the Logical Octagon

We propose now a Boolean semantics in order to *show*[6] the logical relations between all categorical propositions. For this purpose, we take the meaning of any formula to be formulated in terms of the truth-conditions of its components, i.e. the set of models in which the latter hold. As the Keynes–Johnson octagon lies in a bivalent frame where any proposition is either true or false, this means that, for any proposition A and any model w_i, either A or its negation (but not both) belongs to w_i. This also amounts to applying a valuation function v to A and resulting either to the value 1 (for truth, i.e. whenever A belongs to the corresponding model) or the value 0 (for falsity, i.e. whenever A does not belongs to the

[6] Literally speaking, since the following semantic method consists in determining the logical relation between any matching propositions by comparing their characteristic bits (or Boolean values) pairwisely.

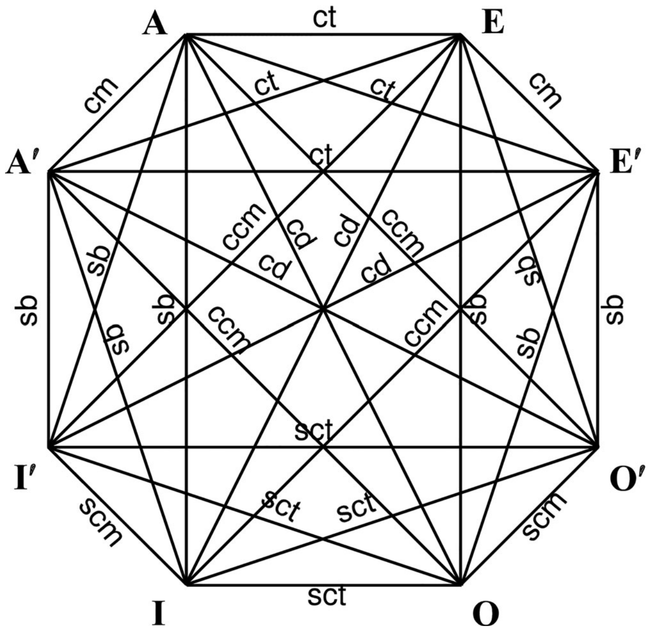

Figure 9. An exhaustive logical octagon in (*Keynes 1894*).

corresponding model):

$$v(w_i, A) = 1 \text{ iff } A \in w_i, \text{ and } v(w_i, A) = 0 \text{ iff } A \notin w_i$$
$$v(w_i, A) = 0 \text{ iff } v(w_i, \neg A) = 1.$$

Let us consider what the components of categorical propositions are and in which models these formulas hold, before turning to the logical relations between them.

4.1. Logical Forms

Every categorical proposition consists in predicating something from something. By using First-Order Logic (henceforth: FOL) as a formal language, the previous formulas of traditional logic may be rephrased in terms of predicate constants S,P and individual variable x: any predication of the form $\mathbf{X} = S\mathbf{x}P$ means that some/every x that is S is/is not P. The eight categorical propositions of the Keynes–Johnson octagon are rephrased in the following table, accordingly.

	FOL
A	$\neg(\exists x)(Ax \wedge \neg Bx)$
E	$\neg(\exists x)(Ax \wedge \neg Bx)$
I	$(\exists x)(Ax \wedge Bx)$
O	$(\exists x)(Ax \wedge \neg Bx)$
A′	$\neg(\exists x)(\neg Ax \wedge Bx)$
E′	$\neg(\exists x)(\neg Ax \wedge \neg Bx)$
I′	$(\exists x)(\neg Ax \wedge \neg Bx)$
O′	$(\exists x)(\neg Ax \wedge Bx)$

Indeed, it turns out that some of the above components corresponds to models of the Euler diagrams. These can be made clearer through the uniform Venn diagrams, in which either one of the four areas of the universe is shaded (or not) in black whenever it corresponds to the affirmation (or denial) of each one of the above four ordered literals: Some S is P, Some S is not-P, Some not-S is P, and Some not-S 12 is not-P. In the below diagram, the left-side circle corresponds to the class of Ss and the right-side circle to the class of Ps. Every model includes four main parts: the class of Ss, the class of Ps, the class of Ss and Ps, and the surrounding class of whatever is neither S nor P (Figure 10).

$$\mathbf{A} = \neg(\exists x)(Ax \wedge \neg Bx) \equiv (x)(Ax \rightarrow Bx)$$
$$\mathbf{A}' = \neg(\exists x)(\neg Ax \wedge Bx) \equiv (x)(\neg Ax \rightarrow \neg Bx)$$

and the same holds for the negative universals $\mathbf{E}$ and $\mathbf{E}'$:

$$\mathbf{E} = \neg(\exists x)(Ax \wedge Bx) \equiv (x)(Ax \rightarrow \neg Bx)$$
$$\mathbf{E}' = \neg(\exists x)(\neg Ax \wedge \neg Bx) \equiv (x)(\neg Ax \rightarrow Bx)$$

Despite the failure to translate complementaries in terms of each other, both species of categorical proposition find their truth-conditions in the same set of models. We will also see that other related sets of categorical propositions: contra-complementaries and sub-complementaries, are derived from complementaries by varying the affirmation of negation of their quantifiers. The three sets of complementaries, contra-complementaries and sub-complementaries constitute a common set of *independent* propositions, which is not a syntactic but, rather, a semantic notion. This justifies the next transition from the syntactic to the semantic approach to categorical propositions.

4.2. Models

A model is a complete and consistent set of propositions that hold in it, and we already tackled the kinds of models assumed by Keynes in his study of the logical octagon. More generally, the set of models that make sense of categorical propositions corresponds to a finite set of two propositions. Let p and q any such propositions that are either true or false in a model. Starting from the *proto-proposition*

$$\pm p \cap \pm q,$$

where $\pm$ means that the corresponding component may be either affirmed ($+$) or denied ($-$), the affirmation or denial of the two components may then be combined into an arrangement of $2^2 = 4$ different models: those where both p and q and affirmed, $\{p, q\}$; those where p is affirmed and q is denied: $\{p\}$; those where p is denied and q is affirmed, $\{q\}$; and, finally, those in which both p and q are denied, $\{\}$.

(1) $p \cap q$
(2) $p \cap \bar{q}$
(3) $\bar{p} \cap q$
(4) $\bar{p} \cap \bar{q}$

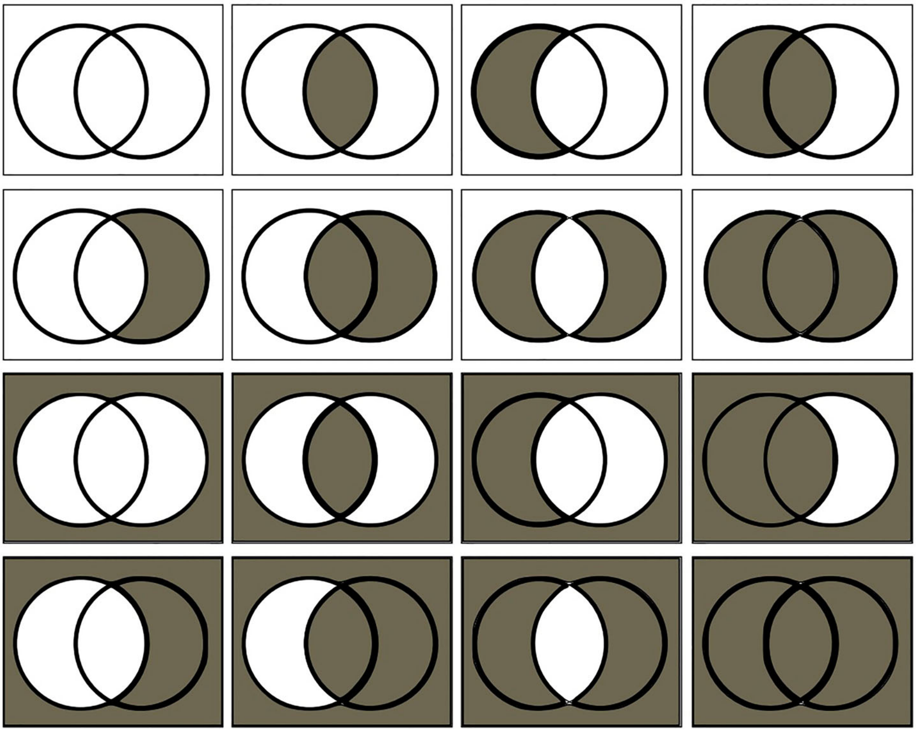

Figure 10. The 16 logical relations between 2 classes, from full domain w_1 (at the bottom right) to empty domain w_{16} (on the top corner left).

(1)–(4) are Conjunctive Normal Forms (henceforth: CNFs) whose disjunction leads to the $2^{2^2} = 16$ Disjunctive Normal Forms (henceforth: DNFs) of classical logic, that is, the set of binary operators that can be represented with the help of truth-tables:

$$\pm(p \cap q) \cup \pm(p \cap \overline{q}) \cup \pm(\overline{p} \cap q) \cup \pm(\overline{p} \cap \overline{q})$$

Each binary operator of conjunction corresponds to either one resulting DNF, as exemplified by the following definition of conjunction $\wedge$ in terms of disjunct proto-propositions:

$$p \wedge q =_{df} (p \cap q) \cup \overline{(p \cap \overline{q})} \cup \overline{(\overline{p} \cap q)} \cup \overline{(\bar{p} \cap \bar{q})}$$

The same constructive process may be applied to categorical propositions, insofar as their subject-terms and predicate-terms correspond to the two propositions p and q. In terms of FOL, this is made more precise as follows: let p symbolize the proposition that some arbitrary individual, say a, is S, and let q symbolize the proposition that a is P. Then the corresponding proto-proposition for categorical propositions is a predication about a,

$$\pm Sa \cap \pm Pa$$

leads to an analogous set of $2^2 = 4$ CNFs for categorical propositions,

(5) $Sa \cap Pa$

(6) $Sa \cap \bar{P}a$
(7) $\bar{S}a \cap Pa$
(8) $\bar{S}a \cap \bar{P}a$

Thus (5)–(8) are categorical proto-propositions, i.e. the basic components in terms of which each of the 8 vertices of the Keynes–Johnson are to be expressed in the form of a DNF where the value a is generalized into an arbitrary individual variable x:

$$\pm\exists x(Sx \cap Px) \cup \pm\exists x(Sx \cap \bar{P}x) \cup \pm\exists x(\bar{S}x \cap Px) \cup \pm\exists x(\bar{S}x \cap \bar{P}x)$$

Indeed, it turns out that some of the above components corresponds to models of the Euler diagrams. These can be made clearer through the uniform Venn diagrams, in which either one of the four areas of the universe is shaded (or not) in gray whenever it corresponds to the affirmation (or denial) of each one of the above four ordered literals: Some S is P, Some S is not-P, Some not-S is P, and Some not-S is not-P. In the below diagram, the left-side circle corresponds to the class of Ss and the right-side circle to the class of Ps. Every model includes four main parts: the class of Ss, the class of Ps, the class of Ss and Ps, and the surrounding class of whatever is neither S nor P (Figure 10).

$$\begin{aligned}
w_1 &= \{\exists x(Sx \cap Px), \exists x(Sx \cap \bar{P}x), \exists x(\bar{S}x \cap Px), \exists x(\bar{S}x \cap \bar{P}x)\}\\
w_2 &= \{\exists x(Sx \cap Px), \exists x(Sx \cap \bar{P}x), \exists x(\bar{S}x \cap Px), \overline{\exists x(\bar{S}x \cap \bar{P}x)}\}\\
w_3 &= \{\exists x(Sx \cap Px), \exists x(Sx \cap \bar{P}x), \overline{\exists x(\bar{S}x \cap Px)}, \exists x(\bar{S}x \cap \bar{P}x)\}\\
w_4 &= \{\exists x(Sx \cap Px), \overline{\exists x(Sx \cap \bar{P}x)}, \exists x(\bar{S}x \cap Px), \exists x(\bar{S}x \cap \bar{P}x)\}\\
w_5 &= \{\overline{\exists x(Sx \cap Px)}, \exists x(Sx \cap \bar{P}x), \exists x(\bar{S}x \cap Px), \exists x(\bar{S}x \cap \bar{P}x)\}\\
w_6 &= \{\exists x(Sx \cap Px), \exists x(Sx \cap \bar{P}x), \overline{\exists x(\bar{S}x \cap Px)}, \overline{\exists x(\bar{S}x \cap \bar{P}x)}\}\\
w_7 &= \{\exists x(Sx \cap Px), \overline{\exists x(Sx \cap \bar{P}x)}, \exists x(\bar{S}x \cap Px), \overline{\exists x(\bar{S}x \cap \bar{P}x)}\}\\
w_8 &= \{\overline{\exists x(Sx \cap Px)}, \exists x(Sx \cap \bar{P}x), \exists x(\bar{S}x \cap Px), \overline{\exists x(\bar{S}x \cap \bar{P}x)}\}\\
w_9 &= \{\exists x(Sx \cap Px), \overline{\exists x(Sx \cap \bar{P}x)}, \overline{\exists x(\bar{S}x \cap Px)}, \exists x(\bar{S}x \cap \bar{P}x)\}\\
w_{10} &= \{\overline{\exists x(Sx \cap Px)}, \exists x(Sx \cap \bar{P}x), \overline{\exists x(\bar{S}x \cap Px)}, \exists x(\bar{S}x \cap \bar{P}x)\}\\
w_{11} &= \{\overline{\exists x(Sx \cap Px)}, \overline{\exists x(Sx \cap \bar{P}x)}, \exists x(\bar{S}x \cap Px), \exists x(\bar{S}x \cap \bar{P}x)\}\\
w_{12} &= \{\exists x(Sx \cap Px), \overline{\exists x(Sx \cap \bar{P}x)}, \overline{\exists x(\bar{S}x \cap Px)}, \overline{\exists x(\bar{S}x \cap \bar{P}x)}\}\\
w_{13} &= \{\overline{\exists x(Sx \cap Px)}, \exists x(Sx \cap \bar{P}x), \overline{\exists x(\bar{S}x \cap Px)}, \overline{\exists x(\bar{S}x \cap \bar{P}x)}\}\\
w_{14} &= \{\overline{\exists x(Sx \cap Px)}, \overline{\exists x(Sx \cap \bar{P}x)}, \exists x(\bar{S}x \cap Px), \overline{\exists x(\bar{S}x \cap \bar{P}x)}\}\\
w_{15} &= \{\overline{\exists x(Sx \cap Px)}, \overline{\exists x(Sx \cap \bar{P}x)}, \overline{\exists x(\bar{S}x \cap Px)}, \exists x(\bar{S}x \cap \bar{P}x)\}\\
w_{16} &= \{\overline{\exists x(Sx \cap Px)}, \overline{\exists x(Sx \cap \bar{P}x)}, \overline{\exists x(\bar{S}x \cap Px)}, \overline{\exists x(\bar{S}x \cap \bar{P}x)}\}
\end{aligned}$$

Now we are in position to represent the truth-conditions of categorical propositions by means of a *bitstring*, that is, an ordered set of Boolean values associated to every model.

Using the Boolean valuation function $v(A) \mapsto \{1,0\}$ that associates either truth (1-bit) or falsehood (value 0-bit) to every A, every model w_i in which A holds is such that $v(w_i, A) = 1$, and $v(w_i, A) = 0$ otherwise.

It is important to recall that not any of the above 16 models belongs to the corresponding valuation of categorical propositions: only 7 of these are retained by Keynes, as was mentioned earlier.[7] Following Figure 3, the ensuing correspondence between Euler and Venn diagrams: (i) = w_9, (ii) = w_4, (iii) = w_3, (iv) = w_1, (v) = w_2, (vi) = w_5, (vii) = w_8, leads to the following general bitstring that affords the ordered value of every categorical proposition A into an reordered set of restricted models W^*:

$$v(W^*, A) = \{w_1(A), w_2(A), w_3(A), w_4(A), w_5(A), w_8(A), w_9(A)\}$$

together with the corresponding bitstring for each single categorical proposition:

A	$v(W^*, A)$
A	0001001
E	0000110
I	1111001
O	1110110
A$'$	0010001
E$'$	0100010
I$'$	1101101
O$'$	1101110

Ultimately, we are going to see that these valuations confirm the expected logical relations that hold in the Keynes–Johnson octagon.

4.3. Logical Relations

The logical octagon that was proposed by Keynes–Johnson and Hacker on page 2 include a total of 28 logical relations, as depicted in (Figure 11) by the exhaustive octagon where each vertex is related to any other one. It corresponds to a combination of two squares of opposition, that is, two diagrams establishing a number of logical dependencies between the truth-values of its vertices. These logical relations are: contrariety, contradiction, subcontrariety, and subalternation, which can be defined in terms of the compossible truth-values of their relata. Thus, for any propositions A,B and any (consistent and complete) model w_i:

A and B are *contrary* to each other iff $v(w_i, A) = 1$ entails $v(w_i, B) = 0$.

A and B are *contradictory* iff $v(w_i, A) = 1$ entails $v(w_i, B) = 0$ and $v(w_i, A) = 0$ entails $v(w_i, B) = 1$.

A and B are *subcontrary* iff $v(w_i, A) = 0$ entails $v(w_i, B) = 1$.

B is *subaltern* to A iff $v(w_i, A) = 1$ entails $v(w_i, B) = 1$ and $v(w_i, B) = 0$ entails $v(w_i, A) = 0$.

The set of logical relations figured on page 2 match with the above characteristic bistrings of the categorical propositions, insofar that their ordered bits literally 'show' the

[7] For a restricted study of categorical propositions in these 7 models, due to the existential import, see e.g. *Chatti and Schang 2013*; for an unrestricted study of these in any of the 16 models, including a usually unnoticed case of 'negative import', see *Schang and Englebretsen 2022*.

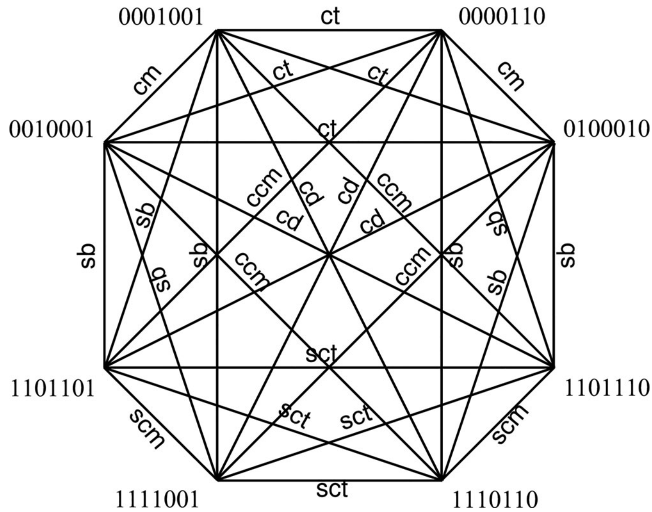

Figure 11. Categorical propositions and their corresponding bitstrings.

semantic dependence that obtains between formulas. Thus, it can be checked that, for any ordered pair of formulas A, B:

- A and B are *contraries* in that affirming the one implies denying the other, i.e. their respective bitstrings never include any 1-bit with respect to the same model: {**A**,**E**}, {**A**,**E**′}, {**E**,**A**′}, {**A**′,**E**′};
- A and B are *contradictories* in that affirming the one implies denying the other and vice versa, i.e. their respective bitstrings never include either 1-bits or 0-bits in the same model: {**A**,**O**}, {**E**,**I**}, {**A**′,**O**′}, {**E**′,**I**′};
- A and B are *subcontraries* in that denying the same implies affirming the other, i.e. their respective bitstrings never include either 0-bits in the same model: {**I**,**O**}, {**I**,**O**′}, {**O**,**I**′}, {**I**′,**O**′};
- B is subaltern to A in that affirming A implies affirming B and denying B implies denying A, i.e. the bitstring of B never includes 0-bits wherever the bitstring of A has a 1-bit and the bitstring of A always includes 0-bits wherever the bitstring of B has a 0-bit: {**A**,**I**}, {**A**,**I**′}, {**E**,**O**}, {**E**,**O**′}, {**A**′,**I**}, {**A**′,**O**′}, {**E**′,**O**}, {**E**′,**I**′}.

The remaining 8 logical relations of the octagon are not of the same sort, because they do not express any dependence between the truth-values of their relata. Hence the following definition of logical *independence*, in semantic terms: for any formulas A and B and any model w_i,

A and B are logically *independent* from each other iff the truth-value of either of A and B does not entail anything about the truth-value of either of B and A.

This is clearly shown in the bitstrings of the complementary propositions, {**A**,**A**′} and {**E**,**E**′}. Among this residual set of independent relations, the remaining ones are those that Keynes calls ‘contra-complementaries’ and ‘subcomplementaries’. These are not to be defined either in terms of constrained truth-values, due to their being mutually independent from each other. Rather, they correspond to those following combinations of dependent and independent relations such that, for any formulas *A*, *B*:

- *A* and *B* are *contra-complementaries* iff the contradictory of *A* is the complementary of *B*;
- *A* and *B* are *subcomplementaries* iff they are the respective subalterns of complementaries.

This leads to those corresponding pairs of 4 contra-complementaries, where the first relatum is always the complementary of the second relatum: {**A**,**O**′},{**E**,**I**′},{**A**,**O**′}, and {**A**,**O**′}. Finally, the last 2 pairs are subcomplementaries where each relatum is subaltern to the respective ordered complementary: {**I**,**I**′},{**O**,**O**′}. This functional definition of subcomplementary is analogous to that of subalternation proposed in *Schang (2018)*: subcomplementaries are the subalterns *of* complementaries, just as subalterns are the contradictories *of* contraries.[8] But unlike the latter, subcomplementaries cannot be reduced completely in semantic terms of truth-conditions since complementarity remains a irreducibly syntactic notion. The fact that complementaries cannot be defined semantically explains why the derived relations of contra-complementarity and sub-complementarity cannot be as well, whose definiens are a mixture of syntactic terms (complementary) and semantic terms (contradictory, and subaltern).

A final representation of the logical relations yields the diagram (Figure 11) where the Roman letters are replaced by the characteristic bitstrings of the corresponding categorical propositions. The reader is invited to compare it with the above Boolean redefinitions of all logical relations, let these be cases of dependence or independence.

5. Conclusion

We proposed a historical and formal analysis of Keynes–Johnson octagon of oppositions between two kinds of categorical proposition: Aristotelian, whose subject term *S* is always affirmed; Keynesian, whose subject term *S* is always negated. In the first two parts of the paper, the history concerned its introduction and formalization through the references of *Keynes 1894*, *Johnson 1921* and *Hacker 1975* in basic terms of classes. In the third part of the paper, the logical octagon was explored syntactically and semantically with the modern methods of first-order logic and model theory. The final result and expectedly unprecedented contribution[9] is a *bitstring semantics* (see *Schang 2012*, *2018*), which completes the account offered by *Demey and Smessaert 2018* in syntactic terms of duality and literally ‘shows’ the truth-conditions of categorical propositions in terms of ordered model sets every proposition belongs to or not. By this way, we hope to get two main results: a clarification of the logical octagon by means of its formal reconstruction; a generalization of the theory of oppositions, from the initial dependence to the final independence

[8] Given that subalterns are defined as the *contradictories of contraries*, this also entails that subcomplementaries are the *contradictories of the contraries* of complementaries. For a Boolean calculus of such iterated logical relations in terms of opposition-forming operators applied to bitstrings (in the wide sense of opposition as any relation of logical dependence), see *Schang 2018*.

[9] Bistring semantics has been also developed in *Demey and Smessaert 2018*, however. The original contribution of the present semantic method concerns the Boolean calculus of logical relations between matching propositions.

relations between any well-formed formulas. The semantic method will be applied later on to other logical contexts like modal formulas or polyadic relations, beyond the present subject-matter of categorical propositions.

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References

Abeles, F. F, and Moktefi, A. 2011. 'Hugh MacColl and Lewis Carroll: crosscurrents in geometry and logic', *Philosophia Scientiae*, **15** (1), 55–76.

Boole, G. 1854. *An Investigation of the Laws of Thought*, London: Walton & Maberly.

Chatti, S., and Schang, F. 2013. 'The cube, the square, and the problem of existential import', *History and Philosophy of Logic*, **34** (2), 101–32.

Dekker, P. J. E. 2015. 'Not *only* barbara', *Journal of Logic, Language and Information*, **24**, 95–129.

Demey, L., and Smessaert, H. 2018. 'Aristotelian and duality relations beyond the square of opposition', in: P. Chapman et al. (eds.), *Diagrammatic Representation and Inference, Diagrams 2018*, Volume 10871, pp. 640–56.

Dubois, D., Prade, H., and Rico, A. 2020. 'Structures of opposition and comparisons: Boolean and gradual cases', *Logica Universalis*, **14** (1), 115–49.

Gergonne, J. D. 1817. 'Essai de dialectique rationnelle', *Annales de Mathématiques Pures et Appliquées*, **7**, 189–228.

Grattan-Guinness, I. 1977. 'The Gergonne relations and the intuitive use of Euler and Venn diagrams', *International Journal of Mathematical Education in Science and Technology*, **8** (1), 23–30.

Green, J. 1991. 'The problem of elimination in the algebra of logic', in T. Drucker (ed.), *Perspectives in the History of Mathematical Logic*, Basel: Birkhäuser, pp. 1–9.

Hacker, E. A. 1975. 'The octagon of opposition', *Notre Dame Journal of Formal Logic*, **16** (3), 352–3.

Johnson, W. E. 1921. *Logic: Part I*, London: Macmillan.

Keynes, J. N. 1884. *Studies and Exercises in Formal Logic*, 1st ed., London: Macmillan.

Keynes, J. N. 1894. *Studies and Exercises in Formal Logic*, 3rd ed., London: Macmillan.

Marion, M., and Moktefi, A. 2014. 'La logique symbolique en débat á Oxford á la fin du XIXe siécle: les disputes logiques de Lewis Carroll et John Cook Wilson', *Revue d'Histoire des Sciences*, **67** (2), 185–205.

Moktefi, A. 2019. 'The social shaping of modern logic', in D. Gabbay et al. (eds.), *Natural Arguments: A Tribute to John Woods*, London: College Publications, pp. 503–20.

Moktefi, A. 2020. 'Schopenhauer's Eulerian diagrams', in J. Lemanski (ed.), *Mathematics, Logic and Language in Schopenhauer*, Basel: Birkhaüser, pp. 111–27.

Moktefi, A., and Edwards, A. W. F. 2011. 'One more class: Martin Gardner and logic diagrams', in M. Burstein (ed.), *A Bouquet for the Gardener*, New York: LCSNA, pp. 160–74.

Peckhaus, V. 1989. 'Hugh MacColl and the German algebra of logic', *Nordic Journal of Philosophical Logic*, **3**, 17–34.

Radford, C. 1995. 'MacColl, Russell, the existential import of propositions, and the null-class', *The Philosophical Quarterly*, **45** (180), 316–31.

Schang, F. 2012. 'Abstract logic of opposition', *Logic and Logical Philosophy*, **21**, 415–38.

Schang, F. 2018. 'End of the square?', *South American Journal of Logic*, **4** (2), 1–21.

Schang, F., and Englebretsen, G. 2022. 'The non-standard forms of categorical propositions', (submitted draft).