Axiomatizing Relativistic Dynamics using Formal Thought Experiments

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Abstract

Thought experiments are widely used in the informal explanation of Relativity Theories; however, they are not present explicitly in formalized versions of Relativity Theory. In this paper, we present an axiom system of Special Relativity which is able to grasp thought experiments formally and explicitly. Moreover, using these thought experiments, we can provide an explicit definition of relativistic mass based only on kinematical concepts and we can geometrically prove the Mass Increase Formula $m_0 = m_k \cdot \sqrt{1 - v^2}$ in a natural way, without postulates of conservation of mass and momentum.

1 Introduction

David Hilbert's still open 6th problem is about to provide a foundation of Physics similar to that of Mathematics. The search for this foundation means to find suitable formal axiomatic systems in which we can prove the formal counterparts of the predictions of Physics.

Why is Hilbert's problem still important? Because, the role of basic assumptions and basic concepts in Physics is at least as important as in Mathematics. Therefore, we would like to have a clear and well-structured understanding of these concepts and assumptions.

As part of this project we would like to support predictions of Physics with precise proofs. This fact also motivates us to use mathematical logic because mathematical logic is currently the best framework in which we can present the most precise proofs.

Using the formal language of mathematical logic, we can clarify the tacit assumptions and opaque notions, as well as we can provide precise proofs for the predictions of Physics.

Another advantage of using mathematical logic is the powerful device of *model theory*: using these tools we are not only able to decide whether a proof is correct or not, but to *discover the exact boundaries* of our theories. For example, we can *prove* that if a statement is *unprovable*.

Here come the methods of *reverse mathematics* into the picture. Using model theoretical tools we are able to examine the exact dependencies of the axioms, what is more: we can find more and more fundamental, sufficient conditions to prove an important statement. For example, [4] showed that the Mass Increase Theorem can be proved from conservation of the centerline of mass without using the conservation of mass and linear momentum.[1] This means also that the Mass Increase Theorem is true even in those models in which the conservation of mass or linear momentum *fails* (but the conservation of centerline of mass is valid).

This reverse mathematical perspective will be also important in this paper: we base our dynamics on an even more general foundation than what was used in [4].

At the very beginning of such a foundation, we have to choose *a* mathematical logic. And we have to choose wisely: not all of them are suitable for axiomatization. We have to choose one which is rich enough to formulate Physics, but not *too rich* to obscure some basic assumptions by making them "unknowable" because it decides them in the meta level, see [1, §Why FOL?], [31, §11]. The standard choice is *classical first-order logic*. For example, all of [2], [3], [7], [8], [15], [22], [28] choose first-order logic to axiomatize relativity theories.

However, thought experiments, which are a natural and common tool in the everyday practice of Physics, do not fit very well in these classical framework,

[1]Another good example is that faster than light motion of particles *per se* is logically independent from both relativistic kinematics [32] and relativistic dynamics [25]. For an axiomatic approach defining coordinate systems moving faster than light, see [20].

they seem to use more than models. In section 2 we show that thought experiments are good candidates for being transformations between classical models. One could say, that this is not surprising at all: as real experiments change the reality, the thought experiments change *the models* of reality. The need for this research was already articulated in [4, §6] and [5, pp.6-7].

Anyhow, there is a logic capable of expressing thought experiments, and is rich and *safe* enough to provide axiomatic bases for relativity theories. This is the first-order logic of 'possible worlds': the *first-order modal logic*. This paper is not the first one connecting modal logic and relativity theories. [14], [30], [29] use modalities locally to axiomatize the causal ordering of events in Minkowski spacetimes, and [18] uses first-order modal logic to eliminate the explicit use of reference frames. We use the modalities to express thought experimentation, i.e., transforming classical models of Special Relativity, more explicitly to distinguish axioms referring to fundamental physical laws and axioms postulating fundamental properties of thought experiments.

1.1 Results

The main results of this paper can be summarized as follows:

- We prove standard predictions of special relativity by *formal thought experiments* in a *natural* way, very close to the informal explanation. The motivation of formal thought experiments will be presented in section 2.

- We develop a first-order modal logic axiomatization of relativistic kinematics and dynamics in which it is possible to distinguish between actual and potential object. This will be done in section 3 and 4.

- We define mass *explicitly* using thought experiments in subsection 4.1.

- We prove the relativistic Mass Increase Formula

$$\mathrm{m}_0(b) = \mathrm{m}_k(b) \cdot \sqrt{1 - \mathrm{v}_k(b)^2}. \tag{1}$$

  in subsection 4.3 (Thm. 11, p.35) using thought experimentation.

2 On the Formalization of Thought Experiments

To explore the nature of the thought experiments present in the discourse of relativity physics, we show a typical argument about that the simultaneity of events is not absolute (i.e., is observer dependent): the train and platform thought experiment.

Our main assumption about the physical reality is a simple consequence of Einstein's two original postulates [12]:

The speed of light is constant for each observer. (AxPhObs)

"Theorem" 1. Simultaneity is not absolute.

"Proof" 1. Consider a train and a train station, such that the train is passing by the station with constant speed. Suppose that Alice is on the train, while Bob is standing on the station. We assume that Alice is sitting in the middle of the train according to Bob. We now show that there could be two events simultaneous according to Bob, which are not simultaneous for Alice.

To do so, let us make a thought experiment: Imagine that two lightnings strike the two ends of the train *simultaneously for Bob*.

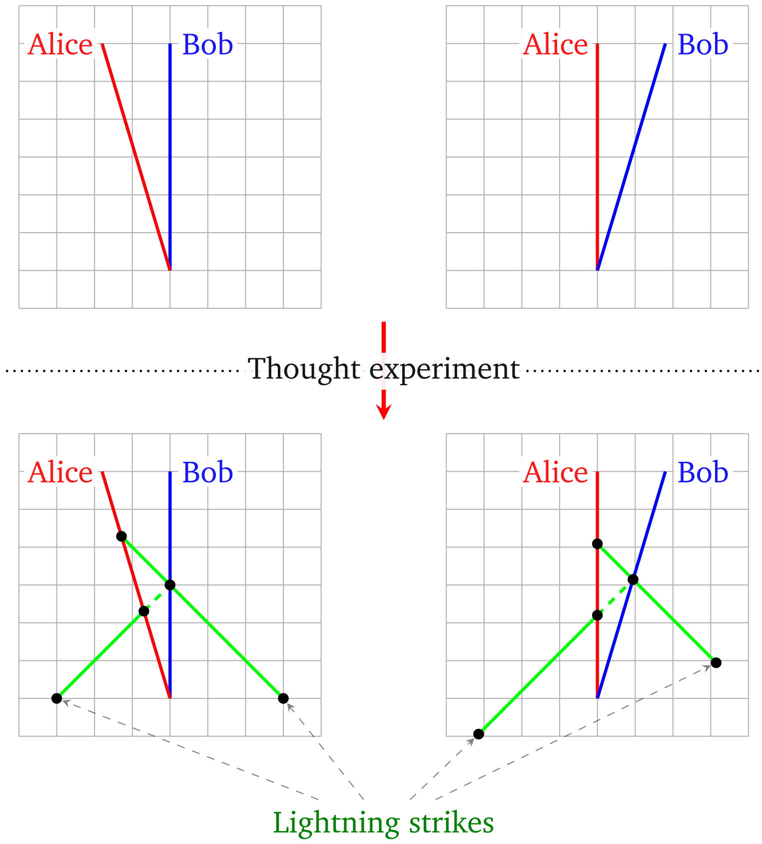
By the fact that the speed of light is constant for Bob (AxPhObs), the light of the flash in front of Alice reaches her *first*, and (if the train is slower than light[2]) the light from her back reaches Alice *second*. The physical reality is the same for both Alice and Bob; therefore, Alice also observes the light signals in different events. We can assume that Alice is sitting on the middle of the train according to her as well.[3] Since the speed of light is constant also for Alice according to (AxPhObs), and the two flashes occur equidistantly with respect to her, the flash in front of her occur at a different time than the one behind according to Alice, see Fig. 1. So we proved that there could be two events simultaneous for Bob but not for Alice, so the simultaneity of events is not absolute. "Q.E.D."

In the previous informal proof, we used a lot of natural but tacit assumptions.

[2]The statement "no inertial observer can go faster than light" follows from the basic assumptions we use in this proof, so we can use it. For a precise proof, see [6].

[3]This basic statement can be proved using the very same assumptions as we use in this proof.

Figure 1: Illustration of the thought experiment showing the observer-dependence of simultaneity



Observations: The physical reality is the same for each observer (AxEv)[4], and the observers coordinatize themselves in the origin (AxSelf).

Mathematics: Since we used notions such as *distance* and *speed*, we relied on some axioms of the real numbers.

However, this proof does not work for Alice and Bob if the two flashes are not "possible" in the very special space-time locations as we used in the proof: they occurred simultaneously for Bob, they were equidistant with respect to Alice, and they were oriented in the direction of the movement of Alice. This introduction of photons is a very good example for what we usually call a *thought experiment*. So in this sense we relied on a thought experimentation axiom:

[4]Later we will introduce a modal version of this assumption, see AxMEv on p.18

Light Signal Sending Thought Experiments: In every space-time location, in every direction, *it is possible to send out* a light signal. (AxPhExp)

Why postulating the existence of "possible" photons are legitimate? For example, because the notion as simultaneity should be independent from the actual existence of some photons, i.e., the simultaneity should be understood in terms of possible events.

However, in what sense the two flashes in the example are "possible"? Logical consistency is a good starting point: the two flashes are possible because their existence does not violate the other axioms we use (e.g., AxPhObs or Einstein's more general postulates). Then the step of introduction of the two photons can be interpreted as a *transformation* of a model of the (classical) axioms into a *very similar* model of the same axioms: this model is more only in the aspect that it has two extra photons in the locations where we need them to be. So a model transformation which expands the model with two photons obeying to AxPhObs is a good candidate for a formal counterpart of the light signal sending thought experiment AxPhExp. This gives the idea that physical thought experiments should be formalized as transformations of classical models.

2.1 Logic for thought experiments

Non-trivial transformations of models are always understood between two different models. However, the truth of a formula must be based on only one model (otherwise it is not a *model*). Either way, if we would like to formalize the notion of thought experiment, the truth of corresponding formulas should be based on two models: on the model *before* and *after* the transformation.

The solution comes from *modal logic*. A modal model is a *set* of classical models connected with a relation. This relation can be the representative of thought experimentation, i.e., model transformation. While the thought experimentation-free (classical) formulas are evaluated in the usual way, we introduce the (modal) formulas $\Diamond\varphi$ with the intended meaning of "there is a transformed model in which φ is true" or "there is a thought experiment such that φ."

Formally: In the classical model w of the modal model $\mathfrak{M}$ the formula $\Diamond\varphi$ is true iff there is a ("transformed") model $v\text{Я}w$ in which φ is true.[5]

3 Kinematics

3.1 Language

Since we will reduce the notion of mass to kinematical notions, the language and models of Dynamics will be very similar to that of Kinematics'. The only difference will be the presence of an individual constant naming the *mass-standard body* to determine the standard unit of mass. Therefore, we discuss the language and models of Dynamics now.

Our main predicate is about coordinatization:

W(k, b, t, x, y, z): "Observer k coordinatizes body b
at the space-time location (t, x, y, z)."

We will use mathematical variables $x, y, z, t, x_1, \ldots$ to denote numbers, e.g., coordinates, and physical variables $b, c, d, k, l, h, m, \ldots$ to denote bodies and observers. We will assume that every observer is a body but not the other way around. For this differentiation we introduce a predicate for inertial observers:

IOb(k): "k is an inertial observer," where k is a physical term.

Since we stay in Special Relativity, in the rest of this paper we omit the expression "inertial."

Light signals play an important role in Relativity Theories; so we introduce a primitive predicate for them as well:

Ph(k): "k is a light signal," where k is a physical term.

Our only non-variable primitive physical term is the mass-standard ε which will play a central role in Dynamics in section 4.

[5]Note that the starting idea, that thought experiments should be understood as tests for logical consistency, is fulfilled.The truth of $\Diamond\varphi$ involves also *classical* logical consistency with the *classical* axioms. If $\Diamond\varphi$ is true, then there is a (transformed) classical model in which φ is true. Since the classical axioms must be true in each world of the modal model, they are also true in the transformed model. That means that φ is consistent.

In the case of mathematics we use the usual $+$ and $\cdot$ basic operations and the ordering $\leq$.

To form complex formulas we use the usual classical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$ to express "not", "and", "or", "if-then", "for all", "there exists", respectively. We use the following abbreviations to simplify our formulas:

$$(\exists b \in \varphi)\psi \stackrel{def.}{\Longleftrightarrow} \exists b(\varphi(b) \wedge \psi), \qquad (\forall b \in \varphi)\psi \stackrel{def.}{\Longleftrightarrow} \forall b(\varphi(b) \rightarrow \psi).$$

For the same reason, we refer to n-tuples using the vector notation:

$$\forall \bar{x}\varphi(\bar{x}) \stackrel{def.}{\Longleftrightarrow} \forall x_1, \ldots, x_n\, \varphi(x_1, \ldots, x_n).$$

Our only non-classical connective is the modal operator $\Diamond$ with the intended meaning that "there is a thought experiment according to which…" or "the actual model can be transformed in a way such that…". We define a dual operator $\Box\varphi$ as $\neg\Diamond\neg\varphi$; hence $\Box\varphi$ is true iff "φ is *invariant* under model transformations/thought experiments." Therefore, an axiom of the form $\Box\varphi$ means that "we use only those thought experiments according to which φ is *invariant*."

3.2 Semantics

A model for MSpecRel:

$$\mathfrak{M} = \langle \mathfrak{Q}, \mathfrak{P}, W^{\mathfrak{M}} \rangle \quad \text{where} \quad \begin{array}{l} \mathfrak{Q} = \langle Q, +^{\mathfrak{M}}, \cdot^{\mathfrak{M}}, \leq^{\mathfrak{M}} \rangle, \\ \mathfrak{P} = \langle S, R, D, IOb^{\mathfrak{M}}, Ph^{\mathfrak{M}}, \varepsilon^{\mathfrak{M}} \rangle. \end{array}$$

Here $\mathfrak{Q}$ is the mathematical and *classical* (Tarskian) part of the model:

$$+^{\mathfrak{M}}, \cdot^{\mathfrak{M}} : Q^2 \rightarrow Q, \qquad \leq^{\mathfrak{M}} \subseteq Q^2,$$

and $\mathfrak{P}$ is the physical and *modal* part of the model. The set S is the set of *possible worlds*, which is a nonempty set used for naming the *classical* first-order models.

R is a reflexive binary relation on S called the *alternative-relation*. The purpose of this relation is to select those possible worlds which can be reached from the actual world by thought experiments. The precise calibration of this relation will be done by axioms containing modal operators $\Box$ and $\Diamond$.

D is a function assigning to each $w \in S$ a (possibly empty) set D_w. These sets are considered as the domain of physical quantification, or simply the set of *existing or "actual" physical objects* in the world w. The *possible objects* are the objects that are "actual-in-some-possible-world":

$$\mathrm{U} \stackrel{def.}{=} \bigcup_{w \in \mathrm{S}} \mathrm{D}_w.$$

$\mathrm{IOb}^{\mathfrak{M}}$ and $\mathrm{Ph}^{\mathfrak{M}}$ are modal predicates for *observers* and *photons*. Since the sets of observers and photons can vary in different worlds, the modal predicates are functions assigning subsets of U to each world w:

$$\mathrm{IOb}^{\mathfrak{M}}, \mathrm{Ph}^{\mathfrak{M}} : \mathrm{S} \to \mathscr{P}(\mathrm{U}).$$

Function $\varepsilon^{\mathfrak{M}}$ assigns a possible object, the one and only (and not necessarily existing) *mass-standard* for each $w \in \mathrm{S}$ in a way that the denotation of ε cannot vary between R connected worlds (i.e., it is a so-called *rigid designator*):

$$\varepsilon^{\mathfrak{M}} : \mathrm{S} \to \mathrm{U} \text{ and } w\mathrm{R}v \Rightarrow \varepsilon^{\mathfrak{M}}_w = \varepsilon^{\mathfrak{M}}_v.$$

Finally $\mathrm{W}^{\mathfrak{M}}$ is the "hybrid" modal and classical predicate for coordinatization. This is also a function, since the world-views can vary from world to world:

$$\mathrm{W}^{\mathfrak{M}}_w \subseteq \mathrm{D}^2_w \times \mathrm{Q}^4.$$

Assignments. Let σ_{Q} be an assignment of the classical part of the model in the classical sense, i.e., a function assigning the elements of Q to the mathematical variables. In the case of the physical and modal parts, let an assignment σ_{U} mapping *possible* individuals to the physical variables. Then a two-sorted assignment for a model of MSpecRel:

$$\sigma(x) \stackrel{def.}{=} \begin{cases} \sigma_{\mathrm{Q}}(x) & \text{if } x \text{ is a mathematical variable,} \\ \sigma_{\mathrm{U}}(x) & \text{if } x \text{ is a physical variable.} \end{cases}$$

We define the x-variant assignments in the usual way:

$$\sigma \stackrel{x}{\equiv} \tau \stackrel{def.}{\iff} \text{for all } y \neq x : \sigma(y) = \tau(y).$$

Terms. The denotation of terms are defined in the usual way:

$$t^{\mathfrak{M},w,\sigma} \stackrel{def.}{=} \begin{cases} \sigma(t) & \text{if } t \text{ is a variable,} \\ f_i^{\mathfrak{M},w,\sigma}(t_1^{\mathfrak{M},w,\sigma}, \dots, t_n^{\mathfrak{M},w,\sigma}) & \text{if } t = f_i(t_1, \dots, t_n). \end{cases}$$

Truth. To define truth, we introduce the following notation:

$$\mathfrak{M}, w \models \varphi[\sigma].$$

We read this in the following way: φ is true in the world w of the modal model $\mathfrak{M}$ according to an assignment σ. The precise definition is given by recursion: The truth of the atomic sentences made by $=$ and W:

$$\begin{aligned} \mathfrak{M}, w \models \mathrm{W}(k, b, \bar{x})[\sigma] \quad &\overset{def.}{\iff} \quad \langle k^{\mathfrak{M}}, b^{\mathfrak{M}}, \bar{x}^{\mathfrak{M}} \rangle \in \mathrm{W}_w^{\mathfrak{M}}, \\ \mathfrak{M}, w \models t_1 = t_2[\sigma] \quad &\overset{def.}{\iff} \quad t_1^{\mathfrak{M}} = t_2^{\mathfrak{M}}. \end{aligned}$$

The truth of the other atomic formulas is defined similarly. The truth of formulas connected by $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined in the usual way; however, the truth of the quantified and modalized formulas are special:

$$\begin{aligned} \mathfrak{M}, w \models \exists x \varphi[\sigma] \quad &\overset{def.}{\iff} \quad \begin{cases} \text{there exists a } \tau \equiv^x \sigma \text{ such that} \\ \mathfrak{M}, w \models \varphi[\tau], \end{cases} \\ \mathfrak{M}, w \models \exists b \varphi[\sigma] \quad &\overset{def.}{\iff} \quad \begin{cases} \text{there exists a } \tau \equiv^b \sigma \text{ such that} \\ \tau(b) \in \mathrm{D}_w \text{ and } \mathfrak{M}, w \models \varphi[\tau], \end{cases} \\ \mathfrak{M}, w \models \Diamond \varphi[\sigma] \quad &\overset{def.}{\iff} \quad \begin{cases} \text{there exists a } w' \in \mathrm{S} \text{ such that} \\ w \mathrm{R} w' \text{ and } \mathfrak{M}, w' \models \varphi[\sigma]. \end{cases} \end{aligned}$$

Note that in the case of the physical sort, we quantify over D_w, i.e., over the *actually existing bodies*. The possible existing bodies are only accesible using modalities, such as $\Diamond\exists b$, $\Box\Diamond\Diamond\forall b$, etc.

A formula is said to be true in a model, $\mathfrak{M} \models \varphi$ iff it is true in all of its worlds according to any assignment.

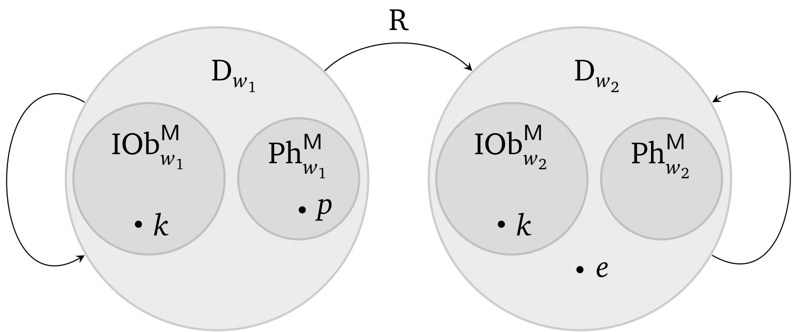
The worlds we use are not ordinary classical models, because the classical axiom schema of universal instantiation ($\forall b \varphi(b) \rightarrow \varphi(t/b)$) is false in them. To show this, we give a simple example: Consider the following model illustrated on Fig. 2:

$$\langle \mathbb{R}, \mathrm{S}, \mathrm{R}, \mathrm{D}, \mathrm{IOb}^{\mathfrak{M}}, \mathrm{Ph}^{\mathfrak{M}}, \varepsilon^{\mathfrak{M}}, \mathrm{W}^{\mathfrak{M}} \rangle$$

- $\mathbb{R}$ is the field of real numbers.
- There are only two worlds w_1 and w_2, i.e., $\mathrm{S} = \{w_1, w_2\}$, such that w_2 is a transformed version of w_1, and both worlds are transformed versions of themselves:

$$\mathrm{R} = \{\langle w_1, w_2 \rangle, \langle w_1, w_1 \rangle, \langle w_2, w_2 \rangle\}.$$

Figure 2: An example for a first-order modal model of our language



- In both worlds there exist only two entities: $D_{w_1} = \{k, p\}, D_{w_2} = \{k, e\}$. So the possible entities are $U = \{k, p, e\}$.

- k is an observer in both worlds, $IOb_{w_1} = IOb_{w_2} = \{k\}$, p is a photon in w_1, $Ph_{w_1} = \{p\}, Ph_{w_2} = \emptyset$, e is the mass-standard of w_1 *and* w_2, i.e., $e = \varepsilon^{\mathfrak{M}}_{w_1} = \varepsilon^{\mathfrak{M}}_{w_2}$ (they cannot differ, since $w_1 R w_2$).

- k sees itself in the origin in both worlds, k coordinatize p moving from 0 in the direction of its x-axis in the world w_1, e is stationary for k in w_2.

$$W^{\mathfrak{M}}_{w_1} = \left\{ \langle a, b, t, x, y, z \rangle \; : \; k = a \text{ and } \begin{array}{l} \text{if } b = k \text{ then } x = y = z = 0, \\ \text{if } b = p \text{ then } t = x, y = z = 0, \end{array} \right\}$$

$$W^{\mathfrak{M}}_{w_2} = \left\{ \langle a, b, t, x, y, z \rangle \; : \; k = a \text{ and } \begin{array}{l} \text{if } b = k \text{ then } x = y = z = 0, \\ \text{if } b = e \text{ then } x = 2, y = z = 0, \end{array} \right\}$$

Let us now consider formula $(\exists b)\, b = \varepsilon$ expressing that ε exists. For expressing existence this way, we use the following abbreviation:

$$E(c) \overset{def.}{\iff} (\exists b)\, b = c.$$

Now $E(\varepsilon)$ is true in w_2 but not in w_1, since $\varepsilon^{\mathfrak{M}}_{w_1} = e \notin D_{w_1}$. However, the formula $\forall b E(b)$ is true in w_1, since $E(b)$ and $E(p)$ are true, but since for the truth of $\forall$-statements we examine only the elements of D_{w_1}, the falsity of $E(\varepsilon)$ does not count. This means that in our models the classical axiom schema of *universal* instantiation,

$$\forall b \varphi(b) \rightarrow \varphi(t/b) \qquad \text{(UI)}$$

fails. Therefore, we do the standard abstraction present in the first-order modal literature (see [11], [16]): we replace this by the *actual instantiation* schema:

$$\mathrm{E}(t) \to (\forall b\varphi(b) \to \varphi(t/b)). \tag{AI}$$

3.3 Logical axioms

The logical axioms are

- the usual axioms and derivation rules of classical propositional logic.
- the usual axioms and derivation rules of classical first-order logic *for mathematics*.
- the usual axioms and derivation rules of classical first-order logic for physics, except the law (UI). We use (AI) instead.
- [13] showed that this system still not proves that the quantifiers of the same sort commute. We postulate these commutativities and we let commutate the quantifications of different sorts too:

$$\forall b\forall c\varphi \longleftrightarrow \forall c\forall b\varphi \qquad \forall b\forall x\varphi \longleftrightarrow \forall x\forall b\varphi.$$

- the usual axioms of identity for both sorts, and a new modal axiom about identity expressing that identity is invariant under thought experiments. During the axiomatization of special relativity we do not use such a radical thought experiment which could split one object into two different ones.

$$t = t, \qquad t = s \to (\varphi(t/x) \to \varphi(s/x)), \qquad t = s \to (\varphi(t/b) \to \varphi(s/b))$$

$$t \neq s \to \Box(s \neq t).$$

- the axiom and the derivation rule of the most general normal modal propositional logic **K**:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi), \qquad \frac{\varphi}{\Box\varphi}.$$

  For us, these express that the modal logical tautologies are invariant under thought experiments, and that invariance under thought experiments is closed to modus ponens.

- For simplicity, we assume that every world counts as a transformed version of itself, i.e., R is reflexive.[6] This can be ensured by the axiom:

$$\varphi \to \Diamond\varphi.$$

This proof system is strongly complete with respect to the semantics we use, see [11, Thm. 2.9 (i), p. 1502.].[7]

3.4 Mathematical Axioms

For the mathematical part, we use the theory of Euclidean fields.

Axiom 1 (Axioms of Euclidean Fields).
AxEField: The mathematical part of the model is a Euclidean field, i.e., an ordered field[8] in which every positive number has a square root.[9]

3.5 Axiom for characterizing the Framework

Here we specify some minimal requirements on the thought experimentation we will use.

Axiom 2 (Axioms of Modal Framework).
AxMFrame: Mathematics is invariant under thought experiments, and every (existing) observer remains an existing observer, i.e., the observers and their ability to coordinatize cannot vanish after a thought experiment:

$$(\forall k \in \mathrm{IOb})\Box(\mathrm{E}(k) \wedge \mathrm{IOb}(k)),$$

$$\begin{aligned}
(\forall x, y, z) \quad & x + y = z & \leftrightarrow \quad & \Box x + y = z, \\
(\forall x, y, z) \quad & x \cdot y = z & \leftrightarrow \quad & \Box x \cdot y = z, \\
(\forall x, y) \quad & x \leq y & \leftrightarrow \quad & \Box x \leq y, \\
(\forall x, y) \quad & x = y & \leftrightarrow \quad & \Box x = y.
\end{aligned}$$

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[6]However, this assumption can be evaded by replacing $\Box$ and $\Diamond$ with $\varphi \wedge \Box\varphi$ and $\varphi \vee \Diamond\varphi$ in all our axioms.

[7] [11] proved strong completeness for only one-sorted modal languages, but our language can be interpreted into it in the usual way, i.e., we introduce a D and a Q predicate to distinguish the sorts. To construct a one-sorted model for our system, we only have to stipulate that the mathematical part of the R-connected worlds are the same, i.e., it is invariant under R.

[8]For the axioms of ordered fields, see e.g., [10, p.41].

[9]That is, $(\forall x > 0)(\exists y)\ x = y^2$.

Note that AxMFrame allows an object to be an observer in a world w and a non-observer in an other world w'. This axiom ensures only that w' cannot be a transformed version of w, i.e., the relation R cannot connect them in this order.

The postulates about atomic statements of the mathematical sort implies that $\mu \longleftrightarrow \Box\mu$ whenever μ is a "purely" mathematical formula. Practically these axioms say that we do not consider thought experimentations according to which $2+2$ *can be* 5.

3.6 Physical axioms

In our first physical axiom, we use the following notations:

$$\bar{x} \in \mathrm{wline}_k(b) \stackrel{def.}{\iff} \mathrm{W}(k, b, \bar{x}),$$

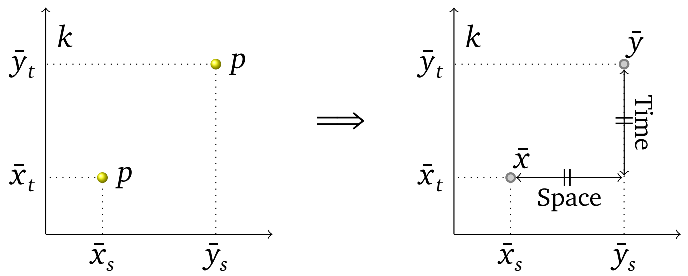
$$\bar{x}_t \stackrel{def.}{=} x_1, \qquad \bar{x}_s \stackrel{def.}{=} (x_2, \dots, x_d),$$

$$\mathrm{Time}(\bar{x}, \bar{y}) \stackrel{def.}{=} |\bar{x}_t - \bar{x}_t|, \qquad \mathrm{Space}(\bar{x}, \bar{y}) \stackrel{def.}{=} |\bar{x}_s - \bar{x}_s|.$$

Axiom 3 (Axiom of Observation of Light Signals.).
AxPhObs: Every observer sees the world-lines of photons as of slope 1. See Fig. 3:

$$(\forall k \in \mathrm{IOb})(\forall \bar{x}, \bar{y}) \left((\exists p \in \mathrm{Ph}) \bar{x}, \bar{y} \in \mathrm{wline}_k(p) \rightarrow \frac{\mathrm{Space}(\bar{x}, \bar{y})}{\mathrm{Time}(\bar{x}, \bar{y})} = 1 \right).$$

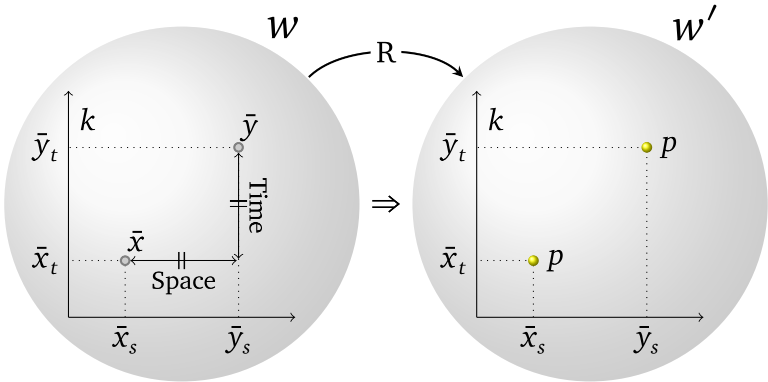
Figure 3: Axiom of Observation of Light Signals



Axiom 4 (Axiom of Light Signal Sending).
AxPhExp: Every observer can send a photon through coordinate points of slope 1. See Fig. 4:

$$(\forall k \in \mathrm{IOb})(\forall \bar{x}, \bar{y}) \left(\frac{\mathrm{Space}(\bar{x}, \bar{y})}{\mathrm{Time}(\bar{x}, \bar{y})} = 1 \rightarrow \Diamond(\exists p \in \mathrm{Ph}) \bar{x}, \bar{y} \in \mathrm{wline}_k(p) \right).$$

Figure 4: Axiom of the Light Signal-Sending



This axiom practically says that if there are two space-time locations where, according to AxPhObs, *there could be* a photon, (i.e., their slope is 1) then there is a thought experiment which transforms the actual world into a world in which *there is* a photon crossing through these space-time locations. That was the axiom we used in the example in section 2.

The most important message of the special theory of relativity is that relatively moving observers coordinatize the world differently *even with respect to time and simultaneity*. So the most interesting relation of this theory must be the relation which connects the "corresponding" coordinate points of different observers, because if we want to say something about relativistic effects, such as time dilatation, length contraction, etc., we have to compare different observers' "corresponding" coordinates. The usual way to achieve this is to introduce the notion of *events*. Intuitively an event is a meeting, an encountering, a collision etc. which itself is observer-independent. What is observer-dependent, is the space-time location of these events in the observers' coordinate-systems. We can introduce an observer-dependent formal counterpart for the notion of event:

Definition 1. An event at a coordinate point $\bar{x}$ according to k in a world w is the set of *existing (actual) bodies* occurring there:

$$\mathrm{ev}_k(\bar{x}) = \{b \in \mathrm{E} : \mathrm{W}(k, b, \bar{x})\}.$$

Let w and w', respectively, be the worlds before and after the thought experiment in the story of Alice and Bob in section 2. Then we had the following events:

$$\text{in } w: \quad \emptyset, \{\text{Alice}\}, \{\text{Bob}\}, \{\text{Alice, Bob}\}.$$
$$\text{in } w': \quad \emptyset, \{\text{Alice}\}, \{\text{Bob}\}, \{\text{Alice, Bob}\}, \{p_1\}, \{p_2\}, \{\text{Alice}, p_1\}, \{\text{Alice}, p_2\}.$$

This is not the usual concept of events. Since usually events are interpreted as *possible events*, usually different events correspond to different coordinate points. For *possible* events, this holds true even in our modal framework because, by AxPhExp and AxPhObs, the possible events are different in different coordinate points. For example, if Alice observes the same sets of bodies in two different coordinate points, say {Bob} at $\bar{x} = \langle 0, 1, 0, 0\rangle$ and $\bar{y} = \langle 1, 1.1, 0, 0\rangle$, then *there could be* a photon (there is a possible photon) moving in such direction that it is in $\bar{x}$ but cannot be in $\bar{y}$. Such a photon can distinguish the sets of possible bodies in $\bar{x}$ and $\bar{y}$.

So if events $\mathrm{ev}_{\mathrm{Alice}}(\bar{x})$ and $\mathrm{ev}_{\mathrm{Alice}}(\bar{y})$ contained not only actual, but *possible* bodies, there would be a photon telling apart the two sets. In this case, the following relation for connecting the "corresponding" coordinate points would be perfect:

$$\text{“}\mathrm{w}_{kh}(\bar{x}, \bar{y}) \overset{def.}{\iff} \mathrm{ev}_k(\bar{x}) = \mathrm{ev}_h(\bar{y}).\text{”}$$

However, such a notion of *possible* event is inexpressible since we have only *actualist quantifications* — we cannot quantify over possible bodies. Anyway, we can define appropriate worldview transformations using thought experiments. If two coordinate points are corresponding, i.e., the same set of possible bodies occur there, then we cannot tell apart them with thought experiments we have, and this can be formulated as:

$$\neg\Diamond \mathrm{ev}_k(\bar{x}) \neq \mathrm{ev}_h(\bar{y}).$$

Definition 2 (Worldview transformation)**.** We say that k sees at $\bar{x}$ what h sees at $\bar{y}$ iff in all transformed worlds the event in $\bar{x}$ for k is the same as the event in $\bar{y}$ for h. In other words, k sees at $\bar{x}$ what h sees at $\bar{y}$ iff it is *impossible* to tell apart these two events by thought experiments:

$$\mathrm{w}_{kh}(\bar{x}, \bar{y}) \overset{def.}{\iff} \Box \mathrm{ev}_k(\bar{x}) = \mathrm{ev}_h(\bar{y}).$$

Prop. 1 shows that AxPhExp provides enough thought experiments to prove that worldview transformations give a one-to-one correspondence between coordinate points.

Sometimes, to simplify our formulas, we list the conjuncts in a column:

$$\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \stackrel{def.}{\iff} (\varphi_1 \wedge \cdots \wedge \varphi_n).$$

Proposition 1. Worldview transformations are injective functions.

$$\{\mathsf{AxEField}, \mathsf{AxMFrame}, \mathsf{AxPhExp}, \mathsf{AxPhObs}\} \vdash$$

$$(\forall k, h \in \mathrm{IOb})(\forall \bar{x}, \bar{y}, \bar{z}) \begin{pmatrix} (\mathrm{w}_{kh}(\bar{x}, \bar{y}) \wedge \mathrm{w}_{kh}(\bar{x}, \bar{z})) \to \bar{y} = \bar{z} \\ (\mathrm{w}_{kh}(\bar{y}, \bar{x}) \wedge \mathrm{w}_{kh}(\bar{z}, \bar{x})) \to \bar{y} = \bar{z} \end{pmatrix}$$

Proof. By the definition of worldview transformation, it is clear that $\mathrm{w}_{kh}(\bar{x}, \bar{y}) = \mathrm{w}_{hk}(\bar{y}, \bar{x})$. Therefore, w_{kh} is injective iff w_{hk} is a function. So by the symmetry of h and k in the statement, it is enough to prove that w_{kh} is a function. To do so, let us assume towards contradiction that $\mathrm{w}_{kh}(\bar{x}, \bar{y})$, $\mathrm{w}_{kh}(\bar{x}, \bar{z})$, but $\bar{y} \neq \bar{z}$ in a world w. In this case, by AxPhExp and AxPhObs, h could send out a light signal from $\bar{y}$ in such a direction that it avoids $\bar{z}$, i.e., there is a possible world $w' Я w$, where there is a photon p such that

$$p \in \mathrm{ev}_h(\bar{y}) \quad \text{but} \quad p \notin \mathrm{ev}_h(\bar{z}). \tag{2}$$

However, since it is true in w that

$$\begin{aligned} \mathrm{w}_{kh}(\bar{x}, \bar{y}) &\iff \Box \mathrm{ev}_k(\bar{x}) = \mathrm{ev}_h(\bar{y}), \\ \mathrm{w}_{kh}(\bar{x}, \bar{z}) &\iff \Box \mathrm{ev}_k(\bar{x}) = \mathrm{ev}_h(\bar{z}), \end{aligned}$$

By the definition of $\Box$, it is true also in w' that

$$\mathrm{ev}_k(\bar{x}) = \mathrm{ev}_h(\bar{y}) \text{ and } \mathrm{ev}_k(\bar{x}) = \mathrm{ev}_h(\bar{z}), \quad \text{hence} \quad \mathrm{ev}_h(\bar{y}) = \mathrm{ev}_h(\bar{z}).$$

This contradicts (2), which proves that w_{kh} is a function. □

By Prop. 1, we can use the following notation for worldview transformations:

$$\mathrm{w}_{kh}(\bar{x}) = \bar{y} \stackrel{def.}{\iff} \mathrm{w}_{kh}(\bar{x}, \bar{y}).$$

So far we have not assumed that there is *at least* one corresponding coordinate point in the worldviews of observers. That is, in some sense we have not assumed that observers coordinatize the same physical reality. This is an important statement, so we take it as an axiom.

Axiom 5 (Axiom of Events).

AxMEv: The possible events are the same for every observer, i.e., there is no possible world in which there is an event for an observer, which is not observed by every other observers:

$$(\forall k, h \in \mathrm{IOb})(\forall \bar{x})(\exists \bar{y}) \quad \mathrm{w}_{kh}(\bar{x}) = \bar{y}.$$

Proposition 2. If we assume AxMEv, AxPhObs and AxPhExp, then worldview transformations are bijections from Q^4 to Q^4.

Now we introduce two more axioms to standardize coordinatizations:

Axiom 6 (Axiom of Self-Coordinatization).

AxSelf: Every observer coordinatizes itself stationary in the origin:

$$(\forall k \in \mathrm{IOb})(\forall \bar{x} \in \mathrm{wline}_k(k)) \quad \bar{x}_s = \bar{0}.$$

Axiom 7 (Axiom of Symmetry).

AxMSym: All observers use the same system of measurements:

$$(\forall k, h \in \mathrm{IOb})(\forall \bar{x}, \bar{x}', \bar{y}, \bar{y}')(\mathrm{Time}(\bar{x}, \bar{y}) = 0 \wedge \mathrm{Time}(\bar{x}', \bar{y}') = 0 \wedge \\ \wedge\, \mathrm{w}_{kh}(\bar{x}) = \bar{x}' \wedge \mathrm{w}_{kh}(\bar{y}) = \bar{y}') \rightarrow \mathrm{Space}(\bar{x}, \bar{y}) = \mathrm{Space}(\bar{x}', \bar{y}').$$

Within this axiomatic framework, we are able to introduce the axiomatization of modal kinematics of special relativity:

$$\mathsf{MSpecRel} \stackrel{def.}{=} \\ \{\mathsf{AxEField}, \mathsf{AxMFrame}, \mathsf{AxPhExp}, \mathsf{AxPhObs}, \mathsf{AxMEv}, \mathsf{AxSelf}, \mathsf{AxMSym}\}$$

Within this axiom system we can prove all the special relativistic kinematical effects such as time dilation and length contraction. See [3] for a direct proof for these effects in a classical axiomatic framework. Here instead of proving these effects directly, we prove that worldview transformations are Poincaré transformations, which imply all these effects.

Theorem 3.

$$\mathsf{MSpecRel} \vdash (\forall k, h \in \mathrm{IOb})\text{“}\mathrm{w}_{kh} \text{ is a Poincaré transformation.”}$$

The proof is in Appendix 5.1.

4 Dynamics

From now on we will assume MSpecRel without further mentioning.

4.1 Definition of Mass

In this section we introduce the special relativistic dynamics based on kinematical notions. We base our definition of mass on possible collisions with the mass-standard. So first we have to give a definition for inertial bodies and collisions. Instead of giving a definition generally for all type of collisions, we restrict ourselves to the inelastic collisions involving only two bodies. However, this does not mean that our dynamics is applicable only these types of collisions. The method can easily be generalized, see [31]. The reason why we choose these simple collisions is that they give a sufficient basis to define the relativistic mass explicitly.

Definition 3 (Inertial bodies and their speed)**.** A body is *inertial* iff its worldline can be covered by a line:

$$\mathrm{IB}(b) \stackrel{def.}{\iff} (\exists k \in \mathrm{IOb})(\forall \bar{x}, \bar{y}, \bar{z} \in \mathrm{wline}_k(b)) \\ (\bar{x}_t \leq \bar{y}_t \leq \bar{z}_t \rightarrow |\bar{x} - \bar{y}| + |\bar{y} - \bar{z}| = |\bar{x} - \bar{z}|).$$

If a body b is inertial and exists in at least two coordinate points, the following definition of speed is well-defined:

$$\mathrm{v}_k(b) = v \stackrel{def.}{\iff} (\exists x, y \in \mathrm{wline}_k(b)) \left(x \neq y \wedge v = \frac{\mathrm{Space}(x,y)}{\mathrm{Time}(x,y)} \right).$$

Two trivial examples for inertial bodies are the inertial observers (by AxSelf), and the photons (by AxPhObs). However, our intention with the definition of inertial bodies is to introduce the type of bodies to which we would like to assign mass. So first, inertial observers (i.e., coordinate-systems) are not such entities. Second, for simplicity, in this paper we will not consider the mass of photons. Therefore, we introduce the following notion for *other* inertial bodies.

Definition 4 (Ordinary body)**.** We call a body *ordinary* iff it is an inertial body which is not a photon nor an inertial observer:

$$\mathrm{OIB} \stackrel{def.}{=} (\mathrm{IB} - \mathrm{IOb}) - \mathrm{Ph}.$$

For ordinary bodies *other than the mass-standard* we introduce the following notation:

$$\mathrm{OIB}^{-} \stackrel{def.}{=} \mathrm{OIB} - \{\varepsilon\}$$

Definition 5 (Collision)**.** See Fig. 5. We say that b and c collide inelastically resulting a body d according to an observer k at the space-time-location $\bar{x}$, in formula $\mathrm{inecoll}_{k,\bar{x}}(b,c:d)$, iff b and c are different existing inertial bodies and their worldlines end in $\bar{x}$, and the worldline of the existing inertial body d begins also in $\bar{x}$ according to k.

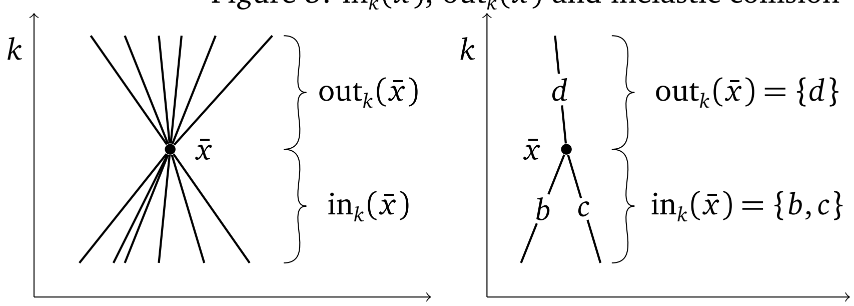
$$\begin{aligned}
\mathrm{in}_k(\bar{x}) &\stackrel{def.}{=} \{b \in \mathrm{IB} : b \in \mathrm{ev}_k(\bar{x}) \wedge (\forall \bar{y} \in \mathrm{wline}_k(b))\, \bar{y}_t < \bar{x}_t \vee \bar{y} = \bar{x}\},\\
\mathrm{out}_k(\bar{x}) &\stackrel{def.}{=} \{b \in \mathrm{IB} : b \in \mathrm{ev}_k(\bar{x}) \wedge (\forall \bar{y} \in \mathrm{wline}_k(b))\, \bar{y}_t > \bar{x}_t \vee \bar{y} = \bar{x}\},
\end{aligned}$$

$$\mathrm{inecoll}_{k,\bar{x}}(b,c:d) \stackrel{def.}{\Longleftrightarrow} \big(b,c,d \in \mathrm{E} \wedge b \neq c \wedge \mathrm{in}_k(\bar{x}) = \{b,c\} \wedge \mathrm{out}_k(\bar{x}) = \{d\}\big).$$

The omitted variables intended to be quantified over existentially:

$$\begin{aligned}
\mathrm{inecoll}_{k,\bar{x}}(b,c) &\stackrel{def.}{\Longleftrightarrow} (\exists d \in \mathrm{IB})\mathrm{inecoll}_{k,\bar{x}}(b,c:d),\\
\mathrm{inecoll}_{k}(b,c) &\stackrel{def.}{\Longleftrightarrow} (\exists \bar{x})\mathrm{inecoll}_{k,\bar{x}}(b,c),\\
\mathrm{inecoll}(b,c) &\stackrel{def.}{\Longleftrightarrow} (\exists k \in \mathrm{IOb})\mathrm{inecoll}_{k}(b,c).
\end{aligned}$$

Figure 5: $\mathrm{in}_k(\bar{x})$, $\mathrm{out}_k(\bar{x})$ and inelastic collision



We also introduce a notation for the space-time location of collisions:

$$\mathrm{locinecoll}_k(b,c) = \bar{x} \stackrel{def.}{\Longleftrightarrow} \mathrm{inecoll}_{k,\bar{x}}(b,c).$$

Let us note that, by the definition of inecoll, $\mathrm{locinecoll}_k(b,c)$ is well-defined.

Definition 6 (Covering line of inertial bodies)**.** The covering line $\overline{\mathrm{wline}}_k(d)$ of inertial body d according to observer k is the line which contains the world-line of d.

$$\overline{\mathrm{wline}}_k(d) \stackrel{\text{def.}}{=} \left\{ \bar{z} \,:\, (\forall \bar{x}, \bar{y} \in \mathrm{wline}_k(d)) \begin{pmatrix} |\bar{x}-\bar{y}| + |\bar{y}-\bar{z}| = |\bar{x}-\bar{z}| \lor \\ |\bar{x}-\bar{z}| + |\bar{z}-\bar{y}| = |\bar{x}-\bar{y}| \lor \\ |\bar{z}-\bar{x}| + |\bar{x}-\bar{y}| = |\bar{z}-\bar{y}| \end{pmatrix} \right\}$$

For inertial bodies participating in inelastic collisions, we can use the following notation since the covering line of these bodies cannot be horizontal:

$$\mathrm{wline}_k(d,t) = \bar{s} \stackrel{\text{def.}}{\iff} \langle t, \bar{s} \rangle \in \mathrm{wline}_k(d),$$

$$\overline{\mathrm{wline}}_k(d,t) = \bar{s} \stackrel{\text{def.}}{\iff} \langle t, \bar{s} \rangle \in \overline{\mathrm{wline}}_k(d).$$

How could we decide which of two colliding bodies, say b and c, is more massive? We can observe the resulting body d of the collision: if d is stationary, then the masses of b and c are equal; if d moves towards where from c have arrived, then b is more massive than c. So we can examine the ratio of the covering lines of the bodies b, c and d intersected with the simultaneity of an observer k, see Fig. 6. If this ratio is greater than 1, then b is more massive; and if this ratio is say 2.7, then b is 2.7 times more massive than c.

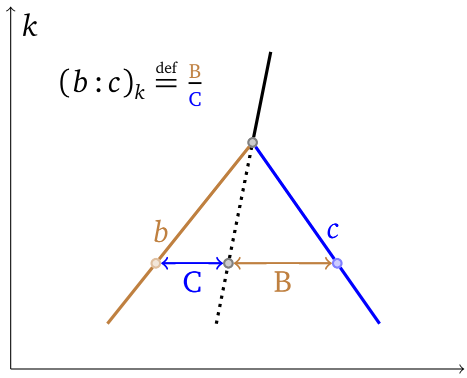
We will define the ratio of collision only for those collisions in which the resulting body's worldline is *between* the two colliding ordinary bodies, like in Fig. 6. Formally:

$$\begin{aligned} &\mathrm{Between}_k(b,d,c) \stackrel{\text{def.}}{\iff} \\ &\qquad (\forall \bar{x} \in \overline{\mathrm{wline}}_k(b))(\forall \bar{y} \in \overline{\mathrm{wline}}_k(c))(\forall \bar{z} \in \overline{\mathrm{wline}}_k(d))(\exists t) \\ &\qquad\qquad [0 < t < 1 \land \bar{z} = t\bar{x} + (1-t)\bar{y}] \lor \bar{x} = \bar{y} = \bar{z}. \end{aligned}$$

Definition 7 (Ratio of Collision). We say that *b is r times more massive than c according to k*, and we denote this by $(b:c)_k = r$, iff the covering line of the resulting body of the collision produced by b and c divides the simultaneity of k between the body b and c in the ratio of r:

$$\begin{aligned} (b:c)_k = r \stackrel{\text{def.}}{\iff} &\ \mathrm{inecoll}(b,c) \land \mathrm{Between}_k(b,d,c) \land \\ &\land (\exists t < \mathrm{locinecoll}_k(b,c)) \quad r = \frac{|\overline{\mathrm{wline}}_k(c,t) - \overline{\mathrm{wline}}_k(d,t)|}{|\overline{\mathrm{wline}}_k(b,t) - \overline{\mathrm{wline}}_k(d,t)|}. \end{aligned}$$

Figure 6: The Collision Ratio



Having the notion of ratio of collision we are close to give a mass function. Let us consider the following situation: Let c be the mass-standard ε. Then $(b:\varepsilon)_k = r$ should be read as "b is r times more massive than the mass-standard according to k." And what else would we like to understand by that "b has the relativistic mass r according to k" if not this?[10]

However, such a definition for mass seems to be too strict. There are three problems with this definition:

(1) ***Problem of the Non-interacting.*** How could we say anything about bodies that do *not collide* with something? Do they lack mass? Even if we do not know about the mass of such a body, it should *have* mass or at least it should be *meaningful* to speak about its mass.

(2) ***Problem of Reusability.*** If the mass-standard collides with a body, how could it collide again if we use only *inelastic* collisions?

(3) ***Problem of the Stationary.*** The mass-standard should have the mass 1 only if it is *stationary* since relativistic mass, similarly to length and time, de-

[10]Suppose that we already have a mass function m having the usual properties. So $m_k(b)$ denotes the mass of b according to k. Let b and c be two colliding bodies, and k be the inertial observer according to which the center of mass of b and c is stationary. Then $(b:c)_k$ is the ratio $\mathrm{v}_k(c)/\mathrm{v}_k(b)$. Therefore, the collision ratio $(b:c)_k$ corresponds to the ratio $m_k(b)/m_k(c)$ by the conservation of linear momentum. And since Poincaré transformations preserve the ratio of points on a line, the ratio of collision means the ratio of masses even if we choose a different observer than k. Practically, the ratio of collision is a formal implementation of Weyl's definition for ratios of masses, see [21, (1.4) on p.10.], implemented to special relativity.

pends on speed in relativity theory. What if b is at rest, too? How could such a stationary b be collided with the mass-standard such that the mass-standard is also stationary?

We solve these problems using possible world semantics and thought experiments. (1) can be solved by speaking about collisions in *alternative possible worlds where it collides with the mass-standard* instead of the actual one where it does not. This also solves (2): the actual world can be counted as the "first use" of the mass-standard, and the alternative world can be the "second use." And similarly every other measurement (collision with the mass-standard) can be done in another alternative world of the actual one.

So shortly: to define relativistic mass we will use collisions in *alternative possible worlds*. We can summarize the answer to the first two problems in a sketch of a definition of mass for *moving* bodies:

"Definition" 1 (Mass of the Moving)**.** The relativistic mass of a *moving* ordinary body b according to an observer k is r, iff it *could be* r times more massive than the mass-standard: there is a "very similar" alternative world in which b collides with the mass-standard with the collision ratio of $(b:\varepsilon)_k = r$.

We can also solve (3) using this "definition," i.e., we can define rest mass based on the mass of moving bodies by using a "transmitting body" between the stationary mass-standard and the stationary body which is going to be measured:

"Definition" 2 (Mass of the Stationary)**.** The relativistic mass of a *stationary* body b is $r_1 \cdot r_2$ according to k iff it *could be* r_1 times more massive than a body which *could be* r_2 times more massive than the mass-standard: There is a "very similar" alternative possible world in which b collides with a (moving) body c with the collision ratio $(b:c)_k = r_1$ and the relativistic mass of c is r_2.

There is only one problem with these two definitions: What does it mean that the alternative world is "very similar" to the actual one? Obviously not any kind of world is *relevant* if we want to collide the mass-standard to a body b. We are interested only in those worlds where b *has the same speed*. These considerations motivate the following two semantical definitions.

Since in modal logic, the predicates can vary in different worlds, when it is not straightforwardly determined by the context, we label the predicates by worlds. So here and from now on, superscript w in predicates P^w and terms t^w denote the worlds from which we took them.

Definition 8 (Collision Thought Experiments and their Relevance)**.** We say that in world w body c is *collidable* to b according to k iff b is an existing ordinary body and k is an existing observer in w, and there is an alternative world w' where these are still existing, inertial, the observer is still an observer, and there c is an existing ordinary body colliding with b. Formally, in w a body c is collidable to b according to k iff

$$
\begin{aligned}
&(\exists w' \in \mathrm{S}) w \mathrm{R} w',\\
&b \in \mathrm{D}_w \cap \mathrm{OIB}^{\mathfrak{M},w} \cap \mathrm{D}_{w'} \cap \mathrm{OIB}^{\mathfrak{M},w'},\\
&c \in \mathrm{OIB}^{\mathfrak{M},w'} \cap \mathrm{D}_{w'},\\
&k \in \mathrm{IOb}^{\mathfrak{M},w} \cap \mathrm{D}_w \cap \mathrm{IOb}^{\mathfrak{M},w} \cap \mathrm{D}_{w'},\\
&(\exists \bar{x} \in \mathrm{wline}_k(b)_w) \mathrm{inecoll}_{k,\bar{x}}(b,c)_{w'}.
\end{aligned}
$$

We call such a $\langle w, w', k, b, c\rangle$ tuple a *collision thought experiment*.

We call a collision thought experiment $\langle w, w', k, b, c\rangle$ *relevant* iff the worldline of b before the collision is the same in both worlds according to k.

$$
(\forall t \leq \mathrm{locinecoll}_k(b:c)_t^w) \quad \mathrm{wline}_k(b,t)^w = \mathrm{wline}_k(b,t)^{w'}.
$$

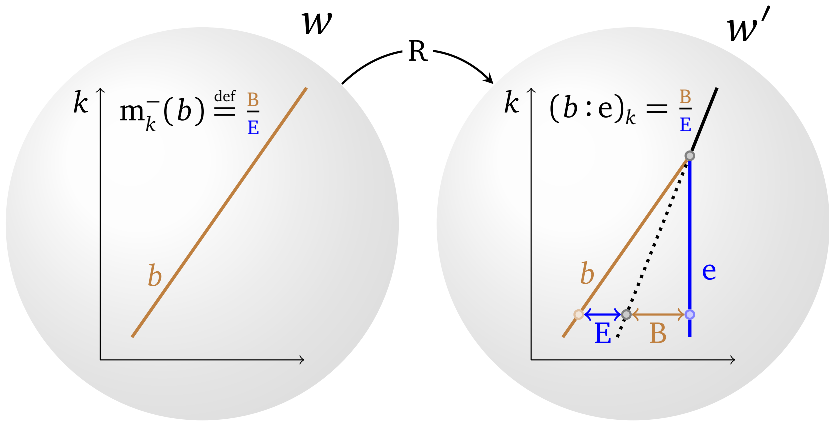
The following axiom ensure that all collision thought experiments are relevant:

Axiom 8 (Axiom of Relevant Collisions)**.**
AxCollRel: Every collision thought experiment is relevant:

$$
\begin{aligned}
&(\forall k \in \mathrm{IOb})(\forall b \in \mathrm{OIB})(\forall \bar{x}, \bar{y})\\
&\quad \Big(\big[\bar{y}_t \leq \bar{x}_t \wedge \mathrm{W}(k,b,\bar{y})\big] \rightarrow \Box \Big[(\exists c \in \mathrm{OIB}) \mathrm{inecoll}_{k,\bar{x}}(b,c) \rightarrow \mathrm{W}(k,b,\bar{y}) \Big] \Big).
\end{aligned}
$$

This axiom is the "engine" of our Dynamics. If we would like to *collide* an ordinary body c to an ordinary body b, then we have two expectations: The worldline of b *changes after* the collision and *remains unchanged before* the collision (otherwise its speed changes and that would ruin the whole experiment). So this axiom erases the worldline after a certain point to make room for a collision, but keep the rest of the worldline to preserve the speed. This axiom also

Figure 7: Direct Measurement



ensures a very important fact: the relative speed of two observers remains the same in collision thought experiments, see Item 1. of Prop. 4.[11]

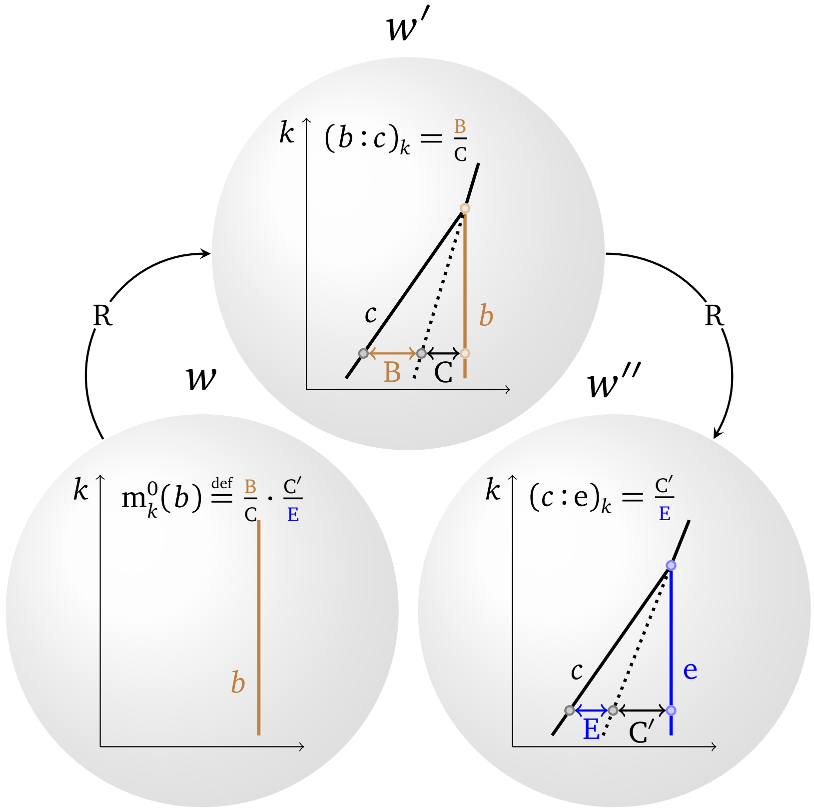
Now that we are able to filter out the relevant collisions, we can introduce collision experiments designed to determine the masses of the moving and stationary bodies.

Definition 9 (Measurements)**.** A collision experiment $\langle w, w', k, b, c\rangle$ is a *direct measurement* iff it is relevant, c is the mass-standard and it is stationary according to k. See Fig. 7. A collision experiment $\langle w, w', k, b, c\rangle$ is an *indirect measurement* iff it is relevant, b is stationary, and there exists a direct measurement $\langle w', w'', k, c, \varepsilon\rangle$. See Fig. 8.

The following two axioms will ensure that the measurements described above

[11]Note that the expressive power of first-order modal logic is not as strong as it seems. For example, it is hopeless to show a formula expressing exactly the following: "There is an alternative world w' in which every object from w having property P_w, has a property $\mathrm{Q}_{w'}$ in w'." The main reason for this is that we cannot 'quantify back' into the previous world after we used a $\Diamond$ operator. For a summary of expressivity problems of first-order modal logic, see [17], [19]. Now the dynamical statement like "Only b's worldline changes" is also such a statement. So this control is beyond the expressibility power of first-order modal logic. In the conference LR12 [26] and in [27], we sketched a solution which used a trick to enforce this kind of thought experiments, but it cost a lot: it used two modal operators such that one of them was a transitive closure of the other. A strong completeness theorem for such a logic is impossible, see [9, §4.8 Finitary Methods I.]. So AxCollRel seems to be the appropriate axiom which makes collision thought experiments possible, and is still *expressible*.

Figure 8: Indirect Measurement



are determined *uniquely*.

Axiom 9 (Axiom of Direct Measurements).

AxDir: According to any observer, every relatively moving ordinary body, other than the mass-standard, is *uniquely* collidable with the mass-standard such that the mass-standard is stationary for that observer:

$$(\forall k \in \mathrm{IOb})(\forall b \in \mathrm{OIB}^-)$$
$$\mathrm{v}_k(b) \neq 0 \rightarrow (\exists! r)\begin{pmatrix} \Diamond[\mathrm{v}_k(\varepsilon) = 0 \wedge (b:\varepsilon)_k = r] \\ \Box[\mathrm{v}_k(\varepsilon) = 0 \rightarrow (b:\varepsilon)_k = r] \end{pmatrix}.$$

If AxDir is assumed, we can define the mass of *relatively moving* ordinary bodies (except the mass-standard) as it was illustrated on Fig. 7:[12]

$$\mathrm{m}_k^-(b) = r \stackrel{def.}{\Longleftrightarrow} \Diamond[\mathrm{v}_k(\varepsilon) = 0 \wedge (b:\varepsilon)_k = r]. \tag{3}$$

Axiom 10 (Axiom of Indirect Measurements).

AxIndir: For every observer, every stationary ordinary body is involved in an indirect measurement, and the results of indirect measurements are *unique*, i.e., do not depend on the choices of the transmitting body:[13]

$$(\forall k \in \mathrm{IOb})(\forall b \in \mathrm{OIB})\mathrm{v}_k(b) = 0 \rightarrow$$
$$\rightarrow (\exists! r)\big[\Diamond(\exists c \in \mathrm{OIB}^-) r = (b:c)_k \cdot \mathrm{m}_k^-(c) \wedge$$
$$\wedge \Box(\forall c' \in \mathrm{OIB}^-)\mathrm{inecoll}(b,c) \rightarrow r = (b:c')_k \cdot \mathrm{m}_k^-(c')\big].$$

If AxDir and AxIndir are assumed, we can define the mass of *stationary* ordinary bodies as it was illustrated on Fig. 8:

$$\mathrm{m}_k^0(b) = r \stackrel{def.}{\Longleftrightarrow} \mathrm{v}_k(b) = 0 \wedge \Diamond(\exists c \in \mathrm{OIB})[r = (b:c)_k \cdot \mathrm{m}_k^-(c)]. \tag{4}$$

We can define an observer independent concept of rest mass as well:

$$\mathrm{m}_0(b) = r \stackrel{def.}{\Longleftrightarrow} (\exists k \in \mathrm{IOb}) \quad \mathrm{m}_k^0(b) = r.$$

To show that $\mathrm{m}_k^0(b)$ is a well-defined quantity, we have to prove that $\mathrm{m}_k^0(b)$ does not depend on k, i.e., co-moving observers get the same results from indirect measurements. We prove this in four steps:

[12] Note that the definitions (3), (4) and (5) express their intended meanings only if we assume AxCollRel as well.

[13] It is a question for further research to find natural and more elementary axioms implying that the results of indirect measurements do not depend on the choices of transmitting body.

Proposition 4.

1. The relative speed of observers remains the same in collision experiments, i.e., thought experiments described in Def. 8.

$$\mathsf{MSpecRel} \cup \{\mathsf{AxCollRel}\} \vdash$$
$$(\forall k,h \in \mathrm{IOb})(\exists r)(\exists b \in \mathrm{OIB})[\mathrm{v}_k(h) = r \rightarrow$$
$$\rightarrow \Box\big((\exists c \in \mathrm{OIB})\mathrm{inecoll}(b,c) \rightarrow \mathrm{v}_k(h) = r\big)].$$

2. In collision experiments, ordinary bodies have the same collision ratio for every two inertial observers co-moving with each other.

$$\mathsf{MSpecRel} \cup \{\mathsf{AxCollRel}\} \vdash$$
$$(\forall k,h \in \mathrm{IOb})\mathrm{v}_k(h) = 0 \rightarrow (\forall b \in \mathrm{OIB})\Box(\forall c \in \mathrm{OIB})(b:c)_k = (b:c)_h.$$

3. Inertial observers co-moving with each other get the same results in direct measurements.

$$\mathsf{MSpecRel} \cup \{\mathsf{AxCollRel}, \mathsf{AxDir}\} \vdash$$
$$(\forall k,h \in \mathrm{IOb})\mathrm{v}_k(h) = 0 \rightarrow (\forall b \in \mathrm{OIB}^-)(\mathrm{v}_k(b) \neq 0 \rightarrow \mathrm{m}^-_k(b) = \mathrm{m}^-_h(b)).$$

4. Inertial observers co-moving with each other get the same results in indirect measurements.

$$\mathsf{MSpecRel} \cup \{\mathsf{AxCollRel}, \mathsf{AxDir}, \mathsf{AxIndir}\} \vdash$$
$$(\forall k,h \in \mathrm{IOb})(\forall b \in \mathrm{OIB})\mathrm{m}^0_k(b) = \mathrm{m}^0_h(b).$$

Proof.

1.: Let w be an arbitrary but fixed world in which k and h are inertial observers moving with the speed of $\mathrm{v}_k(h) = \mathrm{v}_h(k) = r$. Let b an ordinary body from w. Let w' an arbitrary but fixed transformed version of w in which b collides inelastically with an ordinary body c. From $\mathsf{AxMFrame}$ we know that both k and h exist as observers in w', too. From now on we omit the details concerning $\mathsf{AxMFrame}$ in this proof.

We have to prove that $\mathrm{v}_k(h) = r$ in w', too. To prove that, by Thm. 3, it is enough to show that the transformations $\mathrm{w}_{kh}{}^{w}$ and $\mathrm{w}_{kh}{}^{w'}$, the worldview transformation *in* w and the worldview transformation *in* w', takes one timelike line to the same line, i.e.,

$$\mathrm{w}_{kh}{}^{w'}[\ell] = \mathrm{w}_{kh}{}^{w}[\ell].$$

For such a line, the covering line of b for k is a perfect choice, since $\langle w, w', k, b, c\rangle$ is a collision thought experiment, and by AxCollRel, it is relevant:

$$\mathrm{w}_{kh}^{w'}\left[\overline{\mathrm{wline}}_k(b)^{w'}\right] = \overline{\mathrm{wline}}_h(b)^{w'} \overset{\substack{\text{AxCollRel}\\ \downarrow}}{=} \overline{\mathrm{wline}}_h(b)^{w} = \mathrm{w}_{kh}^{w}\left[\overline{\mathrm{wline}}_k(b)^{w}\right] \overset{\substack{\text{AxCollRel}\\ \downarrow}}{=} \mathrm{w}_{kh}^{w}\left[\overline{\mathrm{wline}}_k(b)^{w'}\right].$$

2.: Let w be an arbitrary but fixed world in which k and h are co-moving existing observers coordinatizing an ordinary body b. Let w' an arbitrary but fixed transformed version of w, in which b collides inelastically with an ordinary body c.

Let d be the resulting body of the collision. To prove $(b:c)_k = (b:c)_h$, it is enough to show in w', that the covering lines of b, c and d according to k are parallel to the covering lines of b, c and d according to h, respectively. By Thm. 3 and AxSelf, assumption $\mathrm{v}_k(h) = 0$ ensures this in w. So we need that $\mathrm{v}_k(h) = 0$ in w' as well. But this follows from Item 1.

3.: Let w be a world in which k and h are co-moving existing inertial observers cooordinatizing an ordinary body b moving.

By AxDir and AxCollRel, "k can measure" the mass of b directly, i.e., there is a world $w'\text{Я}w$ in which the stationary mass-standard ε collides with b with a unique collision ratio $r = (b:\varepsilon)_k$. Because r is unique by AxDir, using Item 2 we have:

$$\mathrm{m}_k^-(b)^{w} = (b:\varepsilon)_k{}^{w'} \overset{\substack{\text{Item 2}\\ \downarrow}}{=} (b:\varepsilon)_h{}^{w'} = \mathrm{m}_h^-(b)^{w}$$

4.: Let w be a world in which k and h are co-moving observers cooordinatizing an ordinary body b stationary.

By AxIndir and AxCollRel, k can measure the mass of b indirectly, i.e., there is a world $w'Яw$ in which a transmitting ordinary body c collides with b with a collision ratio $r = (b:c)_k$. By Item 2, $r = (b:c)_h$ in w' as well. By Item 3, $\mathrm{m}_k^-(c) = \mathrm{m}_h^-(c)$ in w'. Since the result of the indirect measurement is unique because of AxIndir, we have the equations

$$\mathrm{m}_k^0(b)_w = (b:c)_k \cdot \mathrm{m}_k^-(c)^{w'} \overset{\substack{\text{Item 2,3} \\ \downarrow}}{=} (b:c)_h \cdot \mathrm{m}_h^-(c)^{w'} = \mathrm{m}_h^0(b)^w.$$

□

Definition 10 (Relativistic Mass). Assume AxDir and AxIndir. We define the relativistic mass of ordinary body b according to observer k by putting definitions (3) and (4) together:

$$\mathrm{m}_k(b) \overset{def.}{=} \begin{cases} \mathrm{m}_k^-(b), \text{ if } \mathrm{v}_k(b) \neq 0, \\ \mathrm{m}_k^0(b) \text{ otherwise.} \end{cases} \tag{5}$$

Because we defined the mass with the ratios of collision, we have a restricted "built-in" conservation-theorem: a restricted conservation of the *centerline of* mass:

Proposition 5. Assume AxDir and AxIndir. If c is at rest according to k, and b and c are colliding ordinary bodies, such that b is different from the mass-standard, then the collision ratio of this collision is the ratio of their masses.

$$\mathsf{MSpecRel} \cup \{\mathsf{AxDir}, \mathsf{AxIndir}\} \vdash$$
$$(\forall k \in \mathrm{IOb})(\forall b \in \mathrm{OIB}^-)(\forall c \in \mathrm{OIB})\mathrm{v}_k(c) = 0 \rightarrow \left((b:c)_k = \frac{\mathrm{m}_k(b)}{\mathrm{m}_k(c)}\right)$$

Proof. By the definition of collision ratio, for all inertial observer k and inertial bodies b and c:

$$(b:c)_k = \frac{1}{(c:b)_k}. \tag{6}$$

By equation (6), we have the following chain of equations:

$$(b:c)_k = \frac{1}{(c:b)_k} = \frac{\mathrm{m}_k^-(b)}{(c:b)_k \cdot \mathrm{m}_k^-(b)} = \frac{\mathrm{m}_k^-(b)}{\mathrm{m}_k^0(c)} = \frac{\mathrm{m}_k(b)}{\mathrm{m}_k(c)}.$$

□

So the collision ratio is determined by the masses of b and c, or in other words, the worldline of the resulting body is the continuation of the center of masses of the colliding bodies. So the *centerline* is conserved in the following sense: the covering line of the center of masses is the same as the covering line of the resulting body.

However, this conservation theorem is restricted: we proved only with the premise that one of the colliding bodies is at rest according to the observer. The more general statement in which the colliding bodies can both be moving according to the observer is the key axiom in [4], [23], [31, §5].[14] It is an interesting fact that in the modal framework we can prove the Mass Increase Theorem even without this assumption.

4.2 Equivalents of the mass-standard

We need one more "tool for measurement": the experiments with mass-standard-equivalents. These are, as their name says, introduced with the purpose to *replace* or substitute the mass-standard. We need such tools usually when we need more measuring tools than the only mass-standard, e.g., when two relatively moving observers try to compare their measuring results.

We have two main expectations about mass-standard-equivalents: they should be able to substitute the mass-standard ε in an equivalent way, and they should have the rest masses 1. Either property could be a good definition, but we start from a more basic level. We define mass-standard-equivalents by the following expectation: if a mass-standard-equivalent (whatever it is) collides with the mass-standard itself, then the resulting body should be stationary according to any "median" observer according to who these two bodies have opposite velocities.

Definition 11 (Median Observer)**.** An inertial observer m is median of the col-

[14] Proving that this key axiom does not follow even from our axiom system MSpecRelDyn (see below on p.34) is out of the scope of this paper, since it involves a complex model construction (because AxDir and AxIndir forces the (nontrivial) models of dynamics to have infinite number of worlds). However, the key idea of the construction is clear: we need worlds w, w_1 and w_2 such that $w_1 Я w \mathrm{R} w_2$, and the measurements between w_1 and w_2 are not 'harmonized', i.e., the ratios of $\mathrm{m}^-_k(b)$ and $\mathrm{m}^-_k(c)$ determined respectively by w_1 and w_2 do not correspond to the ratio of collision of b and c in w.

lision consisting body b and c iff the velocity of b is the opposite of that of c according to m:

$$\mathrm{MedianOb}_{b,c}(m) \overset{def.}{\iff} \mathrm{inecoll}(b,c) \land \bar{\mathrm{v}}_m(b) + \bar{\mathrm{v}}_m(c) = \bar{0}.$$

Definition 12 (Symmetric Collision). A collision is called symmetric iff there is a median observer of it coordinatizing the resulting body at rest:

$$\mathrm{SymColl}(b,c) \overset{def.}{\iff} (\exists m \in \mathrm{MedianOb}_{b,c})\ (b:c)_m = 1.$$

Definition 13 (Mass-standard-equivalents). A body b is a mass-standard-equivalent of k iff b is co-moving with k, different from the mass-standard, and whenever it collides with the mass-standard, it collides with it symmetrically.

$$\mathrm{Et}_k(b) \overset{def.}{\iff} \mathrm{OIB}(^{)} - b \land \mathrm{v}_k(b) = 0 \land \Box(\mathrm{inecoll}(b,\varepsilon) \to \mathrm{SymColl}(b,\varepsilon))$$

Theorem 6 (Symmetric Collision Theorem). Symmetric collisions have the collision ratio of $\sqrt{1-v^2}$ according to the co-moving observer of one of the colliding bodies, where v is the speed of the other body.

$$\mathsf{MSpecRel} \vdash (\forall b,c \in \mathrm{OIB})(\forall k,l \in \mathrm{IOb})\Big([\mathrm{SymColl}(b,c) \land \\ \land \mathrm{v}_k(b) = 0 \land \mathrm{v}_l(c) = 0] \to (b:c)_k = \sqrt{1-\mathrm{v}_l(k)^2}\Big)$$

For the proof of this theorem see Appendix 5.2.

Proposition 7 (Mass of moving equivalents). Assume AxDir and $\mathsf{AxCollRel}$. Every observer k measures every relatively moving l observer's equivalent(s) to be $\frac{1}{\sqrt{1-\mathrm{v}_k(\mathrm{e}_l)^2}}$.

$$\mathsf{MSpecRel} \cup \{\mathsf{AxDir}, \mathsf{AxCollRel}\} \vdash (\forall k,h \in \mathrm{IOb}) \\ \mathrm{v}_k(h) \neq 0 \to (\forall \mathrm{e}_h \in \mathrm{Et}_h)\ \mathrm{m}_k(\mathrm{e}_h) = \frac{1}{\sqrt{1-\mathrm{v}_k(h)^2}}$$

Proof. Let w a world in which there are two observers k and h and a mass-standard-equivalent e_h of h such that $\mathrm{v}_k(h) \neq 0$. Since e_h is stationary for h by definition, it moves for k. Therefore, by AxDir, there is a possible world w' where e_h collides with the mass-standard ε stationary for k. Since e_h exists in both worlds and collides in w', Item 1 of Prop. 4 ensures that the speed $\mathrm{v}_k(h)$

is the same in w and w'. Since e_h is an equivalent of the mass-standard, i.e., it collides symmetrically with the mass-standard, we know from the theorem of symmetric collisions (Thm. 6) that in this world the collision ratio according to k is

$$(\varepsilon : e_h)_k = \sqrt{1 - v_k(h)^2} \overset{(6)}{\Longleftrightarrow} (e_h : \varepsilon)_k = \frac{1}{\sqrt{1 - v_k(h)^2}}.$$

This is exactly the definition of the relativistic mass of e_h in w. □

To prove the substitutivity of the mass-standard, and that the equivalents' rest masses are 1 we use the following assumption:

Axiom 11 (Axiom of Symmetry of Equivalents).
AxEqSym: The equivalents of the mass-standard collide symmetrically with each other if there is a median observer of the collision:

$$(\forall k, l \in \mathrm{IOb})(\forall e_k \in \mathrm{Et}_k)(\forall e_l \in \mathrm{Et}_l)$$
$$[(\exists m)\mathrm{MedianOb}_{e_k, e_l}(m)] \rightarrow \mathrm{SymColl}(e_k, e_l).$$

We postulate a thought experimentation axiom very similar to the axiom of direct measurements AxDir. The *axiom of pseudo-direct measurement with comparison* enables to collide bodies with *equivalents* of the mass-standard (instead of the mass-standard, as AxDir does) in a way we can compare them with a median observer. This axiom comes handy, when we should collide "the mass-standard with itself."

Axiom 12 (Pseudo-Direct Experimentation with Comparison).
AxPDirComp: According to any observer, every body is collidable with one of the observer's mass-standard-equivalents. Moreover, if the body which is going to be collided is a mass-standard-equivalent as well, then a median observer stands ready to compare them:

$$(\forall k \in \mathrm{IOb})(\forall b \in \mathrm{OIB})\Diamond[(\exists e_k \in \mathrm{Et}_k)\mathrm{inecoll}(b, e_k)\wedge$$
$$\wedge [(\exists l)\mathrm{Et}_l(b)] \rightarrow (\exists m)\mathrm{MedianOb}_{b, e_k}(m)].$$

Now we can introduce axiom system MSpecRelDyn, which implies the Mass Increase Theorem.

$$\mathsf{MSpecRelDyn} \stackrel{def.}{=} \mathsf{MSpecRel} \cup \{\mathsf{AxCollRel}, \mathsf{AxDir}, \mathsf{AxIndir}, \mathsf{AxPDirComp}, \mathsf{AxEqSym}\}$$

Proposition 8 (Rest mass of mass-standard)**.**

$$\mathsf{MSpecRel} \cup \{\mathsf{AxCollRel}, \mathsf{AxDir}, \mathsf{AxIndir}\} \vdash \mathrm{m}_0(\varepsilon) = 1$$

Proof. Let c be the body consisted in the indirect measurement measuring the rest mass of ε. Since the collided body is the mass-standard, and the alternative relation is reflexive, we do not have to go further than one world away:

$$\mathrm{m}_0(\varepsilon) = (\varepsilon : c)_k \cdot (c : \varepsilon)_k \overset{(6)}{\underset{\downarrow}{=}} 1$$

□

In special relativity we are mostly interested in those situations, where there are at least two relatively moving observers. So from now on, we will refer to this assumption as $\exists 2\mathsf{IOb}$:

$$\exists 2\mathsf{IOb} \overset{def.}{\iff} (\exists k, l \in \mathrm{IOb}) \mathrm{v}_l(h) \neq 0.$$

Proposition 9 (Rest mass of the equivalents)**.** The rest mass of a mass-standard-equivalent is 1 if there are two relatively moving inertial observers.

$$\mathsf{MSpecRelDyn} \vdash \ (\forall k \in \mathrm{IOb})(\forall \mathrm{e}_k \in \mathrm{Et}_k) \ \exists 2\mathsf{IOb} \to \mathrm{m}_0(\mathrm{e}_k) = 1$$

Proof. By AxIndir and AxCollRel, the rest mass is well defined for every equivalent e_k of k. We determine the rest mass using a relatively moving observer's equivalent:

$$\mathrm{m}_0(\mathrm{e}_k) = \mathrm{m}_k^0(\mathrm{e}_k) = (\mathrm{e}_k : \mathrm{e}_l)_k \cdot \mathrm{m}_k^-(\mathrm{e}_l).$$

By AxEqSym and therefore the theorem of symmetric collisions (Thm. 6) and its corollary (Prop. 7), this is

$$\mathrm{m}_0(\mathrm{e}_k) = \sqrt{1 - \mathrm{v}_k(l)^2} \cdot \frac{1}{\sqrt{1 - \mathrm{v}_k(l)^2}} = 1.$$

□

Now we prove that the equivalents of the mass-standard are also equivalent in a formal sense: the mass-standard can be substituted by its co-moving equivalents.

Proposition 10 (Mass-standard-equivalence)**.** The mass-standard can be replaced by its co-moving equivalents if there are two relatively moving inertial observer.

$$\mathsf{MSpecRelDyn} \vdash (\forall k \in \mathrm{IOb})(\forall \mathrm{e}_k \in \mathrm{Et}_k)(\forall b \in \mathrm{OIB}^-)$$
$$[\exists 2\mathsf{IOb} \wedge \bar{\mathrm{v}}_k(\mathrm{e}_k) = \bar{\mathrm{v}}_k(\varepsilon)] \rightarrow (b : \mathrm{e}_k)_k = \mathrm{m}_k(b).$$

Proof. From Prop. 9 we know that the equivalents have exactly the same rest mass as the mass-standard. From Prop. 5, we also know that the mass of these bodies (since e_k is at rest for k) determines the collision ratio. So the collision ratios $(b : \mathrm{e}_k)_k$ and $(b : \varepsilon)_k$ cannot be different if the mass-standard is at rest. The latter is the definition of $\mathrm{m}_k(b)$. □

4.3 Mass Increase Theorem

Theorem 11 (Mass Increase Theorem)**.**

$$\mathsf{MSpecRelDyn} \vdash \ (\forall k \in \mathrm{IOb})(\forall b \in \mathrm{OIB}^-) \quad \mathrm{m}_0(b) = \sqrt{1 - \mathrm{v}_k(b)^2} \cdot \mathrm{m}_k(b)$$

Proof. If b is at rest according to k, then the statement is true by Def. 10. So let us assume that b is moving according to k.

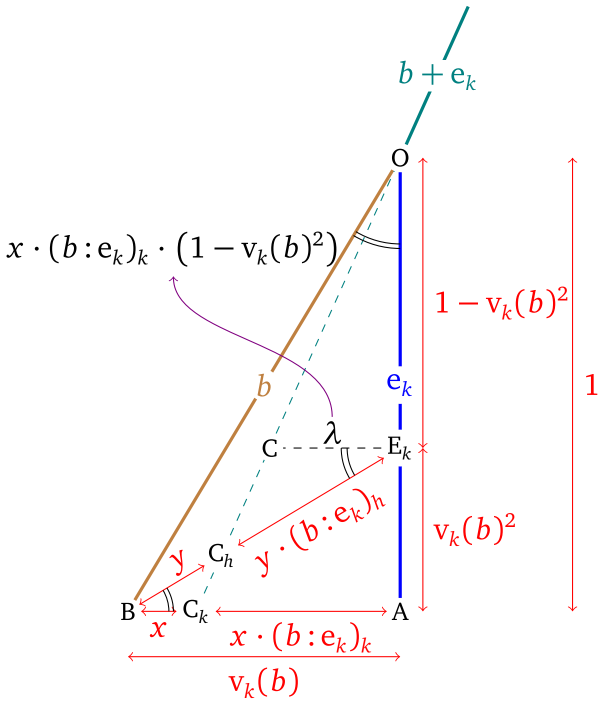
By AxIndir, AxDir and the definition of mass, there is an observer h according to whom b is at rest, and there is a "transmitting" body c such that there is an alternative possible world $w' \text{Я} w$, where

$$\mathrm{m}_0(b) = \mathrm{m}^0_h(b) = (b : c)_h \cdot \mathrm{m}_h(c),$$

see (4). Also by AxIndir, $\mathrm{m}_0(b)$ is independent from the choice of the 'transmitting' body c. Since k is moving in h's coordinate system, by AxPDirComp, we can use a mass-standard-equivalent of k for such a transmitting body c. Thus

$$\mathrm{m}_0(b) = (b : \mathrm{e}_k)_h \cdot \mathrm{m}_h(\mathrm{e}_k) \overset{\text{Cor. 7}}{\downarrow}{=} (b : \mathrm{e}_k)_h \cdot \frac{1}{\sqrt{1 - \mathrm{v}_k(h)^2}}.$$

Figure 9: Proof of Thm. 11: Transformation of the Collision Ratio



Since $v_h(b) = 0$, we have $v_k(b)^2 = v_k(h)^2$ and

$$m_0(b) = (b : e_k)_h \cdot \frac{1}{\sqrt{1 - v_k(b)^2}}. \tag{7}$$

To obtain the theorem from (7), we have to show that

$$(b : e_k)_h = (b : e_k)_k \cdot \left(1 - v_k(b)^2\right).$$

Let us now consider k's coordinate system on Fig. 9. Let O be the coordinate point of the collision, let A be a coordinate point on the very same spatial location but 'one second earlier,' i.e., $A_s = O_s$ and $A_t = O_t - 1$ (so A is on ε_k's covering line). Let B the coordinate point on b's covering line which is simultaneous to A according to k, i.e., $B_t = A_t$. Then $|BA| = \text{Space}(B, A) = v_h(b)$ since $\text{Time}(O, A) = 1$.

Let E_k be the coordinate point from the covering line of e_k which is 'simultaneous' for h, i.e., $w_{kh}(E_k)_t = w_{kh}(B)_t$. Then triangles AOB and AE_kB are similar, since $AOB\angle = ABE_k\angle$ and $BAO\angle = BAE_k\angle$ by Thm. 3. Therefore,

$$\frac{|AE_k|}{|AB|} = \frac{|AB|}{|AO|}, \text{ that is, } \frac{|AE_k|}{v_k(b)} = \frac{v_k(b)}{1}.$$

Hence

$$|AE_k| = v_k(b)^2 \text{ and } |OE_k| = 1 - v_k(b)^2. \tag{8}$$

Let C_h be the intersection of BE_k and the covering line of the resulting body, and C_k be the intersection of AB line of the resulting body. Then, by the definition of collision ratio, see Def. 7,

$$(b : e_k)_k = \tfrac{|C_k A|}{|BC_k|}, \quad \text{that is,} \quad |C_k A| = (b : e_k)_k \cdot |BC_k|, \tag{9}$$

Since w_{kh} is an affine transformation by Thm. 3, it preserves ratios of Euclidean distances between points on a line. Therefore,

$$(b : e_k)_h = \tfrac{|C_h E_k|}{|BC_h|}, \quad \text{that is,} \quad |C_h E_k| = (b : e_k)_h \cdot |BC_h|. \tag{10}$$

Let C be the coordinate point from the covering line of the resulting body which is simultaneous for k with E_k.

Triangles AOC_k and E_kOC are similar, and the ratio of the similarity is $1 - v_k(b)^2$ by (8). Therefore, using (9), we have

$$|CE_k| = |C_k A| \cdot \left(1 - v_k(b)^2\right) = (b : e_k)_k \cdot |BC_k| \cdot \left(1 - v_k(b)^2\right). \tag{11}$$

Now triangles $C_h E_k C$ and $C_h BC_k$ are also similar; therefore,

$$\frac{|BC_k|}{|BC_h|} = \frac{|CE_k|}{|C_h E_k|}. \tag{12}$$

Using (10) and (11) we can write the equation (12) in the following form:

$$\frac{|BC_k|}{|BC_h|} = \frac{(b : e_k)_k \cdot |BC_k| \cdot \left(1 - v_k(b)^2\right)}{(b : e_k)_h \cdot |BC_h|},$$

which simplifies to

$$(b : e_k)_h = (b : e_k)_k \cdot \left(1 - v_k(b)^2\right). \tag{13}$$

By (13), we can change observer h to k in (7):

$$m_0(b) = (b : e_k)_k \cdot \left(1 - v_k(b)^2\right) \cdot \frac{1}{\sqrt{1 - v_k(b)^2}} = (b : e_k)_k \cdot \sqrt{1 - v_k(b)^2}.$$

To turn $(b : e_k)_k$ into mass, we only have to replace tandard-mass-equivalent e_k to the real mass-standard ε. Prop. 10 enables this step, so in w' we have

$$m_0(b) = m_k(b) \cdot \sqrt{1 - v_k(b)^2}.$$

And since this is an equation of two *numbers*, by AxMFrame (the invariance of mathematics), this equation holds also in the starting point w. □

5 Appendix

5.1 Poincaré Transformation Theorem

Theorem 12.
$\mathsf{MSpecRel} \vdash (\forall k, h \in \mathrm{IOb})$ "w_{kh} is a Poincaré transformation."

To prove that w_{kh} is a Poincaré transformation, it is enough to show that it takes lines of slope 1 to lines of slope 1, since there is an Alexandrov-Zeeman type theorem which works only with these premises, see [24]. To prove this lemma, let us introduce the following notation for the speed corresponding to coordinate points $\bar{x}$ and $\bar{y}$:

$$\mathrm{v}(\bar{x}, \bar{y}) \stackrel{def.}{=} \frac{\mathrm{Space}(\bar{x}, \bar{y})}{\mathrm{Time}(\bar{x}, \bar{y})}.$$

Lemma 13 (Light-line)**.** Assume $\mathsf{MSpecRel}$. Then every worldview transformation is a bijection taking lines of slope 1 to lines of slope 1.

Proof. Worldview transformations are bijections by Prop. 2.

Now we prove that worldview transformations take lines of slope 1 to lines of slope 1. By $\mathsf{AxEField}$, $\mathrm{v}(\bar{x}, \bar{y}) = \mathrm{v}(\bar{y}, \bar{z}) = \mathrm{v}(\bar{z}, \bar{x}) = 1$ implies that $\bar{x}$, $\bar{y}$ and $\bar{z}$ are collinear. Therefore, to finish our proof, it is enough to derive the following formula:

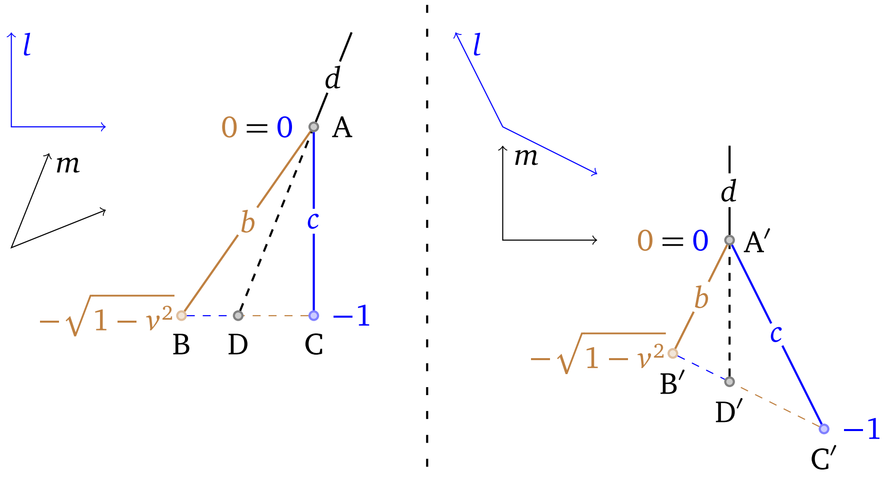
$$(\forall k, h \in \mathrm{IOb})(\forall \bar{x}, \bar{y})[\mathrm{v}(\bar{x}, \bar{y}) = 1 \rightarrow \mathrm{v}(\mathrm{w}_{kh}(\bar{x}), \mathrm{w}_{kh}(\bar{y})) = 1]. \tag{14}$$

Let k and h be arbitrary observers in a world w, and let $\bar{x}$ and $\bar{y}$ be coordinate points such that $\mathrm{v}(\bar{x}, \bar{y}) = 1$. By $\mathsf{AxPhExp}$, in every w world, there is an accessible world $w' Я w$ such that there is a light signal $p \in \mathrm{ev}_k(\bar{x}) \cap \mathrm{ev}_k(\bar{y})$ in w'. By $\mathsf{AxMFrame}$, k still exists as an observer in w'. So $p \in \mathrm{ev}_k(\mathrm{w}_{kh}(\bar{x})) \cap \mathrm{ev}_k(\mathrm{w}_{kh}(\bar{y}))$ by AxMEv. Consequently, by $\mathsf{AxPhObs}$:

$$\mathrm{v}(\mathrm{w}_{kh}(\bar{x}), \mathrm{w}_{kh}(\bar{y})) = 1;$$

and this is what we wanted to prove. □

Figure 10: The Symmetric Collision Theorem



5.2 Proof of Symmetric Collision Theorem

Here we are going to prove Thm. 6 stating that:

$$\mathsf{MSpecRel} \vdash (\forall b, c \in \mathrm{OIB})(\forall k, l \in \mathrm{IOb})\Big(\mathrm{SymColl}(b,c) \wedge \\ \wedge\, \mathrm{v}_k(b) = 0 \wedge \mathrm{v}_l(c) = 0 \rightarrow (b:c)_k = \sqrt{1 - \mathrm{v}_l(k)^2}\Big).$$

Proof. Let k and l be observers and let b and c be ordinary bodies in a world w such that $\mathrm{v}_k(b) = 0$, $\mathrm{v}_l(c) = 0$ and $\mathrm{SymColl}(b,c)$ holds for them. Since $\mathrm{SymColl}(b,c)$ holds, there is an observer m (the so called median observer) in w such that $\bar{\mathrm{v}}_m(b) + \bar{\mathrm{v}}_m(c) = \bar{0}$ and $(b:c)_m = 1$. See Fig. 10.

The time dilation effect, i.e.,

$$(\forall k, h \in \mathrm{IOb})(\forall \bar{x}, \bar{y} \in \mathrm{wline}_k(h)) \\ \mathrm{Time}(\bar{x}, \bar{y}) = \frac{\mathrm{Time}(\mathrm{w}_{kh}(\bar{x}), \mathrm{w}_{kh}(\bar{y}))}{\sqrt{1 - \mathrm{v}_k(h)^2}}, \quad (15)$$

is a consequence of Thm. 3, see [3, Thm. 2.4, (2)]:

We know from (15) that if the clocks of k and l show 0 at A, and the clock of l shows -1 at C, then the clock of k shows $-\sqrt{1-v^2}$ where v is $\mathrm{v}_l(k)$.

We are interested in $(c:b)_l$, which is now:

$$(c:b)_l = \frac{\mathrm{BD}}{\mathrm{DC}}. \quad (16)$$

Since worldview transformation are affine transformations, this ratio is the same in the worldview of the median observer m, i.e.,

$$\frac{\mathrm{BD}}{\mathrm{DC}} = \frac{\mathrm{B'D'}}{\mathrm{D'C'}}. \tag{17}$$

Since the worldline of d is an angle bisector of the triangle A′B′C′ in the worldview of m, by the angle bisector theorem,

$$\frac{\mathrm{B'D'}}{\mathrm{D'C'}} = \frac{\mathrm{B'A'}}{\mathrm{A'C'}}. \tag{18}$$

Since the clocks of k and l slow down with the same rate for the median observer m, we know that

$$\frac{\mathrm{B'A'}}{\mathrm{A'C'}} = \sqrt{1-v^2}. \tag{19}$$

Since the collision was symmetric, $(b:c)_k = (c:b)_l$. Therefore, from (16), (17), (18) and (19) we have

$$(c:b)_k = \sqrt{1-v^2},$$

and this is what we wanted to prove. □

6 Concluding Remarks

We have seen that the act of thought experimentation is a formalizable notion. The formal counterparts of thought experiments are model transformations of classical models. Modal logic express a large amount of these transformations. If we stay in the level of first-order logic, we still have a strong completeness theorem which is essential in foundational axiomatic approaches. With the aid of first-order modal logic, we axiomatized relativistic Kinematics and Dynamics with formal thought experiments.

An advantage of this approach is that we can distinguish *actual* and *theoretical* (or possible) objects. The philosophical importance of this is that the ontological statuses of actual and potential objects are clearly different. Theoretical objects are just non-existing objects we *need* to prove important statements. For example, the possible photons postulated by AxPhExp are needed to prove that the worldview transformations are Poincaré transformations. The transmitting bodies of AxIndir, the mass-standard and its equivalents of AxDir and AxPDirComp, respectively, are needed to give a well-defined concept of relativistic mass based on only kinematical terms.

Using this definition and AxEqSym, we proved the Mass Increase Formula without using the conservation postulates about linear momentum, mass, or even the centerline of relativistic mass (the key axiom of [4]). This result suggests that the presence of formal thought experiments make it possible to 'dig deeper' in the foundations of relativity theories.

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