THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 2: ELEMENTARY LOGARITHMIC INTEGRALS

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Abstract. We describe methods to evaluate elementary logarithmic integrals. The integrand is the product of a rational function and a linear polynomial in $\ln x$.

1. Introduction

The table of integrals by I. M. Gradshteyn and I. M. Rhyzik [3] contains a large selection of definite integrals of the form

$$\int_a^b R(x) \ln^m x \, dx, \tag{1.1}$$

where $R(x)$ is a rational function, $a, b \in \mathbb{R}^+$ and $m \in \mathbb{N}$. We call integrals of the form (1.1) *elementary logarithmic integrals*. The goal of this note is to present methods to evaluate them. We may assume that $a = 0$ using

$$\int_a^b R(x) \ln^m x \, dx = \int_0^b R(x) \ln^m x \, dx - \int_0^a R(x) \ln^m x \, dx. \tag{1.2}$$

Section 2 describes the situation when R is a polynomial. Section 3 presents the case in which the rational function has a single simple pole. Finally section 4 considers the case of multiple poles.

2. Polynomials examples

The first example considered here is

$$I(P; b, m) := \int_0^b P(x) \ln^m x \, dx, \tag{2.1}$$

where P is a polynomial. This can be evaluated in elementary terms. Indeed, $I(P; b, m)$ is a linear combination of

$$\int_0^b x^j \ln^m x \, dx, \tag{2.2}$$

and the change of variables $x = bt$ yields

$$\int_0^b x^j \ln^m x \, dx = b^{j+1} \sum_{k=0}^{m} \binom{m}{k} \ln^{m-k} b \int_0^1 t^j \ln^k t \, dt. \tag{2.3}$$

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The last integral evaluates to $(-1)^k k!/(j+1)^{k+1}$ either an easy induction argument or by the change of variables $t = e^{-s}$ that gives it as a value of the gamma function.

Theorem 2.1. Let $P(x)$ be a polynomial given by

$$P(x) = \sum_{j=0}^{p} a_j x^j. \tag{2.4}$$

Then

$$I(P;b,m) := \int_0^b P(x) \ln^m x \, dx = \sum_{k=0}^{m} (-1)^k k! \binom{m}{k} \ln^{m-k} b \sum_{j=0}^{p} a_j \frac{b^{j+1}}{(j+1)^{k+1}}. \tag{2.5}$$

This expression shows that $I(P;b,m)$ is a linear combination of $b^j \ln^k b$, with $1 \leq j \leq 1 + p(= 1 + \deg(P))$ and $0 \leq k \leq m$.

3. Linear denominators

We now consider the integral

$$f(b;r) := \int_0^b \frac{\ln x \, dx}{x+r} \tag{3.1}$$

for b, $r > 0$. This corresponds to the case in which the rational function in (1.1) has a single simple pole.

The change of variables $x = rt$ produces

$$\int_0^b \frac{\ln x \, dx}{x+r} = \ln r \, \ln(1 + b/r) + \int_0^{b/r} \frac{\ln t \, dt}{1+t}. \tag{3.2}$$

Therefore, it suffices to consider the function

$$g(b) := \int_0^b \frac{\ln t \, dt}{1+t}, \tag{3.3}$$

as we have

$$f(b;r) = \ln r \, \ln \left(1 + \frac{b}{r} \right) + g \left(\frac{b}{r} \right). \tag{3.4}$$

Before we present a discussion of the function g, we describe some elementary consequences of (3.2).

Elementary examples. The special case $r = b$ in (3.2) yields

$$\int_0^b \frac{dx}{x+b} = \ln 2 \, \ln b + \int_0^1 \frac{\ln t \, dt}{1+t}. \tag{3.5}$$

Expanding $1/(1+t)$ as a geometric series, we obtain

$$\int_0^1 \frac{\ln t \, dt}{1+t} = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}. \tag{3.6}$$

This appears as **4.231.1** in [3]. Differentiating (3.2) with respect to r produces

$$\int_0^b \frac{\ln x \, dx}{(x+r)^2} = -\frac{\ln(b+r)}{r} + \frac{\ln r}{r} + \frac{b \ln b}{r(r+b)}. \tag{3.7}$$

As b, $r \to 1$ we obtain

$$\int_0^1 \frac{\ln x\,dx}{(1+x)^2} = -\ln 2. \tag{3.8}$$

This appears as **4.231.6** in [3]. On the other hand, as $b \to \infty$ we recover **4.231.5** in [3]:

$$\int_0^\infty \frac{\ln x\,dx}{(x+r)^2} = \frac{\ln r}{r}. \tag{3.9}$$

The polylogarithm function. The evaluation of the integral

$$g(b) := \int_0^b \frac{\ln t\,dt}{1+t}, \tag{3.10}$$

requires the transcendental function

$$\text{Li}_n(x) := \sum_{k=1}^\infty \frac{x^k}{k^n}. \tag{3.11}$$

This is the *polylogarithm function* and it has also appeared in [5] in our discussion of the family

$$h_n(a) := \int_0^\infty \frac{\ln^n x\,dx}{(x-1)(x+a)}, \quad n \in \mathbb{R},\ a > 0. \tag{3.12}$$

In the current context we have $n = 2$ and we are dealing with the *dilogarithm function*: $\text{Li}_2(x)$.

Lemma 3.1. The function $g(b)$ is given by

$$g(b) = \ln b\,\ln(1+b) + \text{Li}_2(-b). \tag{3.13}$$

Proof. The change of variables $t = bs$ yields

$$g(b) = \ln b\,\ln(1+b) + \int_0^1 \frac{\ln s\,ds}{1+bs}. \tag{3.14}$$

Expanding the integrand in a geometric series yields the final identity. □

Theorem 3.2. Let b, $r > 0$. Then

$$\int_0^b \frac{\ln x\,dx}{x+r} = \ln b \ln\left(\frac{b+r}{r}\right) + \text{Li}_2\left(-\frac{b}{r}\right). \tag{3.15}$$

Corollary 3.3. Let $b > 0$. Then

$$\int_0^b \frac{\ln x\,dx}{x+b} = \ln 2\,\ln b - \frac{\pi^2}{12}. \tag{3.16}$$

Proof. Let $r \to b$ in Theorem 3.2 and use

$$\text{Li}_2(-1) = \sum_{n=1}^\infty \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}. \tag{3.17}$$

□

The expression in Theorem 3.2 and the method of partial fractions gives the explicit evaluation of elementary logarithmic integrals where the rational function has simple poles. For example:

Corollary 3.4. Let $0 < a < b$ and $r_1 \neq r_2 \in \mathbb{R}^+$. Then, with $r = r_2 - r_1$, we have

$$\begin{aligned} \int_a^b \frac{\ln x\, dx}{(x+r_1)(x+r_2)} &= \frac{1}{r}\left[\ln b \, \ln\left(\frac{r_2(b+r_1)}{r_1(b+r_2)}\right) + \ln a \, \ln\left(\frac{r_1(a+r_2)}{r_2(a+r_1)}\right)\right] + \\ &+ \frac{1}{r}\left[\text{Li}_2\left(-\frac{b}{r_1}\right) - \text{Li}_2\left(-\frac{a}{r_1}\right) - \text{Li}_2\left(-\frac{b}{r_2}\right) + \text{Li}_2\left(-\frac{a}{r_2}\right)\right]. \end{aligned}$$

The special case $a = r_1$ and $b = r_2$ is of interest:

Corollary 3.5. Let $0 < a < b$. Then

$$\begin{aligned} \int_a^b \frac{\ln x\, dx}{(x+a)(x+b)} &= \frac{1}{b-a}\left[\ln(ab)\ln(a+b) - \ln 2 \ln(ab) - 2\ln a \, \ln b\right] \\ &+ \frac{1}{b-a}\left[-2\text{Li}_2(-1) + \text{Li}_2\left(-\frac{b}{a}\right) + \text{Li}_2\left(-\frac{a}{b}\right)\right]. \end{aligned}$$

The integral in Corollary 3.5 appears as **4.232.1** in [3]. An interesting problem is to derive **4.232.2**

$$\int_0^\infty \frac{\ln x\, dx}{(x+u)(x+v)} = \frac{\ln^2 u - \ln^2 v}{2(u-v)} \tag{3.18}$$

directly from Corollary 3.5.

We now present an elementary evaluation of this integral and obtain from it an identity of Euler. We prove that

$$\int_a^b \frac{\ln x\, dx}{(x+a)(x+b)} = \frac{\ln ab}{2(b-a)} \ln \frac{(a+b)^2}{4ab}. \tag{3.19}$$

Proof. The partial fraction decomposition

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right).$$

reduces the problem to the evaluation of

$$I_1 = \int_a^b \frac{\ln x\; dx}{x+a} \text{ and } I_2 = \int_a^b \frac{\ln x\; dx}{x+b}.$$

The change of variables $x = at$ gives, with $c = b/a$,

$$\begin{aligned} I_1 &= \int_1^c \frac{\ln(at)\; dt}{1+t} \\ &= \ln a \int_1^c \frac{dt}{1+t} + \int_1^c \frac{\ln t}{1+t}\, dt \\ &= \ln a \ln(1+c) - \ln a \ln 2 + \int_1^c \frac{\ln t}{1+t}\, dt. \end{aligned}$$

Similarly,

$$I_2 = \ln b \ln 2 - \ln b \ln(1+1/c) + \int_1^{1/c} \frac{\ln t}{1+t}\, dt.$$

Therefore

$$\begin{aligned} I_1 - I_2 &= \ln a \ln(1+c) + \ln b \ln(1+1/c) - \ln 2 \ln a - \ln 2 \ln b + \\ &+ \int_1^c \frac{\ln t}{1+t}\,dt - \int_{1/c}^1 \frac{\ln t}{1+t}\,dt. \end{aligned}$$

Let $s = 1/t$ in the second integral to get

$$\int_{1/c}^1 \frac{\ln t}{1+t}\,dt = \int_c^1 \frac{\ln s}{s(1+s)}\,ds.$$

Replacing in the expression for $I_1 - I_2$ yields

$$\begin{aligned} I_1 - I_2 &= \ln a\,(\ln(a+b) - \ln a - \ln 2) - \ln b\,(\ln 2 - \ln(a+b) + \ln b) + \\ &+ \int_1^c \frac{\ln t}{t}\,dt. \end{aligned}$$

The last integral can now be evaluated by elementary means to produced the result. □

Now comparing the two evaluation of the integral in Corollary 3.5 produces an identity for the dilogarithm function.

Corollary 3.6. The dilogarithm function satisfies

$$\text{Li}_2(-z) + \text{Li}_2\left(-\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(z). \tag{3.20}$$

This is the first of many interesting functional equations satisfied by the polylogarithm functions. It was established by L. Euler in 1768. The reader will find in [4] a nice description of them.

4. A single multiple pole

In this section we consider the evaluation of

$$f_n(b,r) := \int_0^b \frac{\ln x\,dx}{(x+r)^n}. \tag{4.1}$$

This corresponds to the elementary rational integrals with a single pole (at $x = -r$). The change of variables $x = rt$ yields

$$f_n(b,r) = \frac{\ln r}{(n-1)r^{n-1}}\left[\frac{(b+r)^{n-1} - r^{n-1}}{(b+r)^{n-1}}\right] + \frac{1}{r^{n-1}}h_n(b/r),$$

where

$$h_n(b) := \int_0^b \frac{\ln t\,dt}{(1+t)^n}. \tag{4.2}$$

We first establish a recurrence for h_n.

Theorem 4.1. Let $n > 2$ and $b > 0$. Then h_n satisfies the recurrence

$$h_n(b) = \frac{n-2}{n-1}h_{n-1}(b) + \frac{b\,\ln b}{(n-1)(1+b)^{n-1}} + \frac{1-(1+b)^{n-2}}{(n-1)(n-2)(1+b)^{n-2}}. \tag{4.3}$$

Proof. Start with

$$h_n(b) = \int_0^b \frac{[(1+t)-t]\,\ln t\,dt}{(1+t)^n} = h_{n-1}(b) - \int_0^b \frac{t\,\ln t\,dt}{(1+t)^n}.$$

Integrate by parts in the last integral, with $u = t\ln t$ and $dv = dt/(1+t)^n$ to produce the result. □

The initial condition for this recurrence is obtained from the value

$$h_2(b) = \frac{b}{1+b}\ln b - \ln(1+b). \tag{4.4}$$

This expression follows by a direct integration by parts in

$$h_2(b) = -\lim_{\epsilon\to 0}\int_\epsilon^b \ln t\,\frac{d}{dt}(1+t)^{-1}\,dt. \tag{4.5}$$

The first few values of $h_n(b)$ suggest the introduction of the function

$$q_n(b) := (1+b)^{n-1}h_n(b), \tag{4.6}$$

for $n \geq 2$. For example,

$$q_2(b) = b\ln b - (1+b)\,\ln(1+b). \tag{4.7}$$

The recurrence for h_n yields one for q_n.

Corollary 4.2. The recurrence

$$q_n(b) = \frac{(n-2)}{(n-1)}(1+b)q_{n-1}(b) + \frac{b\,\ln b}{n-1} - \frac{(1+b)\left[(1+b)^{n-2}-1\right]}{(n-1)(n-2)}, \tag{4.8}$$

holds for $n \geq 2$.

Corollary 4.2 establishes the existence of functions $X_n(b)$, $Y_n(b)$ and $Z_n(b)$, such that

$$q_n(b) = X_n(b)\,\ln b + Y_n(b)\,\ln(1+b) + Z_n(b). \tag{4.9}$$

The recurrence (4.8) produces explicit expression for each of these parts.

Proposition 4.3. Let $n \geq 2$ and $b > 0$. Then

$$X_n(b) = \frac{(1+b)^{n-1}-1}{n-1}. \tag{4.10}$$

Proof. The function X_n satisfies the recurrence

$$X_n(b) = \frac{n-2}{n-1}(1+b)X_{n-1}(b) + \frac{b}{n-1}. \tag{4.11}$$

The initial condition is $X_2(b) = b$. The result is now easily established by induction. □

Proposition 4.4. Let $n \geq 2$ and $b > 0$. Then

$$Y_n(b) = -\frac{(1+b)^{n-1}}{n-1}. \tag{4.12}$$

Proof. The function Y_n satisfies the recurrence

$$Y_n(b) = \frac{n-2}{n-1}(1+b)Y_{n-1}(b). \tag{4.13}$$

This recurrence and the initial condition $Y_2(b) = -(1+b)$, yield the result. □

It remains to identify the function $Z_n(b)$. It satisfies the recurrence

$$Z_n(b) = \frac{n-2}{n-1}(1+b)Z_{n-1}(b) - \frac{(1+b)\left[(1+b)^{n-2}-1\right]}{(n-2)(n-1)}. \tag{4.14}$$

This recurrence and the initial condition $Z_2(b) = 0$ suggest the definition

$$T_n(b) := -\frac{(n-1)!\, Z_n(b)}{b(1+b)}. \tag{4.15}$$

Lemma 4.5. The function $T_n(b)$ is a polynomial of degree $n-3$ with positive integer coefficients.

Proof. The function $T_n(b)$ satisfies the recurrence

$$T_n(b) = (n-2)(1+b)T_{n-1}(b) + (n-3)!\left[\frac{(1+b)^{n-2}-1}{b}\right]. \tag{4.16}$$

Now simply observe that the right hand side is a polynomial in b. □

Properties of the polynomial $T_n(b)$ will be described in future publications. We now simply observe that its coefficients are *unimodal*. Recall that a polynomial

$$P_n(b) = \sum_{k=0}^{n} c_k b^k \tag{4.17}$$

is called *unimodal* if there is an index n^*, such that $c_k \leq c_{k+1}$ for $0 \leq k \leq n^*$ and $c_k \geq c_{k+1}$ for $n^* < k \leq n$. That is, the sequence of coefficients of P_n has a single peak. Unimodal polynomials appear in many different branches of Mathematics. The reader will find in [2] and [6] information about this property. We now use the result of [1] to establish the unimodality of T_n.

Theorem 4.6. Suppose $c_k > 0$ is a nondecreasing sequence. Then $P(x+1)$ is unimodal.

Therefore we consider the polynomial $S_n(b) := T_n(b-1)$. It satisfies the recurrence

$$S_n(b) = b(n-2)S_{n-1}(b) + (n-3)!\sum_{r=0}^{n-3} b^r. \tag{4.18}$$

Now write

$$S_n(b) = \sum_{k=0}^{n-3} c_{k,n} b^k, \tag{4.19}$$

and conclude that $c_{0,n} = (n-3)!$ and

$$c_{k,n} = (n-2)c_{k-1,n-1} + (n-3)!, \tag{4.20}$$

from which it follows that

$$c_{k+1,n} - c_{k,n} = (n-2)\left[c_{k,n-1} - c_{k-1,n-1}\right]. \tag{4.21}$$

We conclude that $c_{k,n}$ is a nondecreasing sequence.

Theorem 4.7. The polynomial $T_n(b)$ is unimodal.

Conclusions. We have given explicit formulas for integrals of the form

$$\int_a^b R(x) \, \ln x \, dx, \tag{4.22}$$

where R is a rational function with real poles. Future reports will describe the case of higher powers

$$\int_a^b R(x) \, \ln^m x \, dx, \tag{4.23}$$

as well as the case of complex poles, based on integrals of the form

$$C_n(a,r) := \int_0^b \frac{\ln x \, dx}{(x^2+r^2)^n}. \tag{4.24}$$

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