Corrigendum to: Weak Arithmetics and Kripke Models

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Abstract

We give a corrected proof of the main result in the paper mentioned in the title.

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By $W\neg\neg LNP$, we mean the scheme $\forall\overline{y}\neg\neg(\exists x\varphi(x,\overline{y}) \to \exists x(\varphi(x,\overline{y})\wedge\forall z < x\neg\varphi(z,\overline{y})))$. In [M], it is observed that $i\Pi_1 \equiv W\neg\neg l\Pi_1 \equiv W\neg\neg l\neg\Pi_1$. Here, $i\Pi_1$ is defined as $I\Pi_1$ but over intuitionistic logic. Also, $W\neg\neg l\Pi_1$ is the intuitionistic theory obtained by adding the scheme $W\neg\neg LNP$ for Π_1 formulas, to the intuitionistic version of $I\Delta_0$ (i.e. $i\Delta_0$). The theory $W\neg\neg l\neg\Pi_1$ is the intuitionistic theory obtained by adding the scheme $W\neg\neg LNP$ for negated Π_1 formulas to $i\Delta_0$. The above result was proved using a Lemma concerning Kripke models of the mentioned theories (Proposition 1.2 in [M]). The proof of this Lemma is not correct. Here, using the same idea, we give a more direct proof for the above mentioned equivalences. I should also note that Corollary 1.6 (ii) in [M], based on the mentioned Proposition, is wrong. One can construct a Kripke model of $i\Pi_1$ with a path such that the union of the worlds in it does not satisfy $I\Pi_1$. To see this, let M be a model of $I\Delta_0$ which is not a model of $I\Pi_1$ but is embeddable in a model $M' \vDash I\Pi_1$, see [W, Lemma 9] for existence of such models. Now let $\mathcal{K}$ be the Kripke model which is obtained by putting a world M' over each M in an ω-chain consisting of M's. This Kripke model clearly forces $i\Pi_1$ since double negation of any instance of induction on a Π_1-formula is forced in its root and is equivalent (in $\mathcal{K}$) to the same instance, see [W, Lemma 10] and [MM, Lemma 4.4].

We first recall the following Fact mentioned in [M].

Fact 1.1 Suppose $\mathcal{K} \Vdash i\Delta_0$ and $\alpha \in K$.

(i) $\alpha \Vdash \mathrm{PEM}_{\Delta_0}$.

(ii) If $\varphi \in \Sigma_1$ is a L_α-sentence then: $\alpha \Vdash \varphi \Leftrightarrow M_\alpha \vDash \varphi$.

(iii) If $\psi \in \Pi_1$ is a L_α-sentence then: $\alpha \Vdash \psi \Leftrightarrow \forall \beta \geq \alpha M_\beta \vDash \psi$.

Proposition 1.2 If a fragment $i\Gamma$ of HA is m-closed under the negative translation and $I\Gamma \vdash L\Gamma$, then for any formula $\varphi(x, \overline{y}) \in \Gamma$, $i\Gamma \vdash \forall \overline{y} \neg\neg(\exists x \varphi(x, \overline{y}) \rightarrow \exists x(\varphi(x, \overline{y}) \wedge \forall z < x \neg\varphi(z, \overline{y})))$.

Proof The second proof in [TD, p.131] for $HA \vdash W\neg\neg LNP$ actually proves the Proposition. For details see [MM]. □

Note that by the above Proposition, $i\Pi_1 \vdash W\neg\neg l\Pi_1$. Also, using $i\Pi_1 \equiv i\neg\Pi_1$ (see [W, Cor. 6]) and 1.2, we get $i\Pi_1 \vdash W\neg\neg l\neg\Pi_1$.

Proposition 1.3 $W\neg\neg l\neg\Pi_1 \vdash i\Pi_1$.

Proof Assume $\mathcal{K} \Vdash W\neg\neg l\neg\Pi_1$. Let $\alpha \in \mathcal{K}$ does not force $I_x\varphi(x, \overline{y})$, for some Π_1-formula φ. Therefore, by Fact 1.1, there will exist a node $\gamma \geqslant \alpha$ with $a, \overline{b} \in M_\gamma$ ($\overline{b}$ of the same arity as $\overline{y}$), such that

(i) $\gamma \Vdash \varphi(0, \overline{b}) \wedge \neg\varphi(a, \overline{b})$,

(ii) $\gamma \Vdash \forall x(\varphi(x, \overline{b}) \rightarrow \varphi(x+1, \overline{b}))$.

By $\mathcal{K} \Vdash W\neg\neg l\neg\Pi_1$, we get $\gamma \Vdash \neg\neg\exists x(\neg\varphi(x, \overline{b}) \wedge \forall z < x\varphi(z, \overline{b}))$. Therefore, for some $\delta \geq \gamma$ and some (necessarily nonzero) $d \in M_\delta$, $\delta \Vdash \neg\varphi(d, \overline{b}) \wedge (\forall z < d)\varphi(z, \overline{b})$. This is a contradiction to the fact that γ (and therefore, δ) forces $\forall x(\varphi(x, \overline{b}) \rightarrow \varphi(x+1, \overline{b}))$. □

Proposition 1.4 $W\neg\neg l\Pi_1 \vdash i\neg\Pi_1$.

Proof Let α be a node of a Kripke model $\mathcal{K} \Vdash W\neg\neg l\Pi_1$, $\varphi(x, \overline{y})$ negation of a Π_1-formula, and $\overline{a} \in M_\alpha$ of the same arity as $\overline{y}$. To prove $\alpha \Vdash I_x\varphi(x, \overline{a})$, assume without loss of generality that $\alpha \Vdash \varphi(0, \overline{a})$. It is enough to show that for every $\beta \geq \alpha$, there exists $\delta \geq \beta$ such that, $\delta \Vdash I_x\varphi(x, \overline{a})$, since in $i\Delta_0$ we have $\neg\neg I_x\varphi(x, \overline{a}) \vdash I_x\varphi(x, \overline{a})$. Fix $\beta \geq \alpha$. If $\beta \Vdash \forall x\varphi(x, \overline{a})$, then we may take $\delta = \beta$. Otherwise, by $\beta \Vdash W\neg\neg l\Pi_1$, one can see that there will exist $\gamma \geq \beta$ such that $\gamma \Vdash \neg\varphi(d, \overline{a}) \wedge (\forall z < d)\varphi(z, \overline{a})$ for some non-zero $d \in M_\gamma$. Clearly, such a node δ has the desired property. □

Corollary 1.5 $i\Pi_1 \equiv W\neg\neg l\Pi_1 \equiv W\neg\neg l\neg\Pi_1$.

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