Independence results for weak systems of intuitionistic arithmetic

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Abstract

This paper proves some independence results for weak fragments of Heyting arithmetic by using Kripke models. We present a necessary condition for linear Kripke models of arithmetical theories which are closed under the negative translation and use it to show that the union of the worlds in any linear Kripke model of HA satisfies PA. We construct a two-node PA-normal Kripke structure which does not force $i\Sigma_2$. We prove $i\forall_1 \nvdash i\exists_1$, $i\exists_1 \nvdash i\forall_1$, $i\Pi_2 \nvdash i\Sigma_2$ and $i\Sigma_2 \nvdash i\Pi_2$. We use Smorynski's operation $\Sigma^{'}$ to show $HA \nvdash l\Pi_1$.

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0. Introduction

In this note we use Kripke models to prove some independence results for weak fragments of Heyting arithmetic. Kripke semantics for intuitionistic logic was introduced by Kripke in 1965. The meta-logic of Kripke model theory is classical logic. Any intuitionistic theory is complete with respect to its Kripke models. The first comprehensive study of Kripke models of Heyting arithmetic HA (the intuitionistic counterpart of first order Peano arithmetic PA) was done by Smorynski in his PhD thesis which appeared as [S]. In [S], Smorynski introduced his method for constructing Kripke models of HA (Smorynski's operations $\Sigma^{'}$ and Σ^{*}) and used them to prove several independence results for HA. Even now, Smorynski's method for constructing non-trivial Kripke models of HA is the only method which is known.

Kripke models of weak fragments of HA are usually more accessible. In [W1], [MM], [M1] and [M2], Kripke models are used to prove results on certain weak fragments of HA. Here, we continue the line.

1. Preliminaries

Let PA^- be the finite set of usual axioms (including Trichotomy) for the nonnegative parts of discretely ordered commutative rings with 1 in the language $L = \{+, \cdot, <, 0, 1\}$ of arithmetic. Peano arithmetic PA (resp. Heyting arithmetic HA) is the classical (resp. intuitionistic, obtained by dropping the principle PEM of excluded middle) first order theory axiomatized by PA^- together with the induction scheme whose instance with respect to a distinguished free variable x on a formula $\varphi(x, \overline{y})$ is

$$I_x\varphi = I_x\varphi(x, \overline{y}) : \forall \overline{y}(\varphi(0, \overline{y}) \wedge \forall x(\varphi(x, \overline{y}) \rightarrow \varphi(x + 1, \overline{y})) \rightarrow \forall x\varphi(x, \overline{y})).$$

The instance of the least number principle LNP with respect to a distinguished free variable x on a formula $\varphi(x, \overline{y})$ is the sentence

$$L_x\varphi = L_x\varphi(x, \overline{y}) : \forall \overline{y}(\exists x\varphi(x, \overline{y}) \rightarrow \exists x(\varphi(x, \overline{y}) \wedge \forall z < x\neg\varphi(z, \overline{y})))$$

.

By *open* formulas we mean quantifier-free formulas. A formula is bounded if all quantifiers occurring in it are bounded. Δ_0-formulas are bounded formulas. Let $\Sigma_0 = \Pi_0 = \Delta_0$. For $n \geq 0, \Sigma_{n+1}$-formulas have the form $(\exists \overline{x})\varphi$ where φ is in Π_n, Π_{n+1}-formulas have the form $(\forall \overline{x})\varphi$ where φ is in Σ_n. The hierarchy of $\forall_n$-formulas and of $\exists_n$-formulas are defined similarly by changing bounded formulas to open formulas. To get the hierarchy of bounded formulas, U_n and E_n for $n \geqslant 0$, we start with open formulas and then we add bounded quantifiers in the above style.

For any set Γ of formulas we will use notations such as $i\Gamma$ and $l\Gamma$ to denote the intuitionistic theories obtained by PA^- plus the scheme of induction or LNP restricted to formulas in Γ respectively. $I\Gamma$ and $L\Gamma$ show the classical closures of them respectively. It is known that, for $n \geq 0$, $I\Sigma_n \equiv I\Pi_n \equiv L\Sigma_n \equiv L\Pi_n$, $I\exists_n \equiv I\forall_n \equiv L\exists_n$ and $IE_n \equiv IU_n \equiv LE_n$ (see, e.g. [K1, Result 1.1].

We adopt the usual Kripke semantics for intuitionistic theories based on L as in [TD]. A T-normal Kripke structure is one whose worlds are classical models of T. Here, we mention a few easy facts which will be freely used in this paper, see [AM] and [Ma]. Recall that $(PA^-)^i$ is the intuitionistic closure of PA^-.

Fact 1.1 We have

i) $(PA^-)^i \vdash PEM_{open}$,

ii) $i\Delta_0 \vdash PEM_{\Delta_0}$.

In other words, open formulas are decidable in $(PA^-)^i$ and bounded formulas are decidable in $i\Delta_0$.

Fact 1.2 Suppose that $\mathcal{K} \Vdash (PA^-)^i$ (resp. $\mathcal{K} \Vdash i\Delta_0$) and $\varphi \in \exists_1$ (resp. $\varphi \in \Sigma_1$). Then for each $\alpha \in K$, we have: $\alpha \Vdash \varphi \Leftrightarrow M_\alpha \vDash \varphi$.

If $\psi \in \forall_2$ (resp. $\psi \in \Pi_2$) then: $\alpha \Vdash \psi \Leftrightarrow \forall \beta \geq \alpha\ M_\beta \vDash \psi$.

Fact 1.3 Suppose that $\mathcal{K}$ is a Kripke structure.

i) $\mathcal{K} \Vdash (PA^-)^i$ iff $\mathcal{K}$ is PA^--normal.

ii) $\mathcal{K} \Vdash i\Delta_0$ iff $\mathcal{K}$ is $I\Delta_0$-normal and for each $\alpha, \beta \in K$, if $\alpha \leq \beta$ then $M_\alpha \prec_{\Delta_0} M_\beta$.

Fact 1.4 For a node α in a Kripke model deciding atomic and so open (resp. Δ_0-) formulas to force $I_x\varphi$, where φ is an $\exists_1$ (resp. Σ_1)-formula, it is enough that for each $\beta \geq \alpha, M_\beta \vDash I_x\varphi$.

2. Union of worlds in linear Kripke models

In this section we make some general observations about Kripke models of arithmetical theories. We will use them later.

In [M1], it is shown that for a Kripke structure deciding bounded formulas to force $i\Pi_1$ it is necessary and sufficient that the union of the worlds in any (maximal) path in it satisfies $I\Pi_1$. This provides us with a complete classical description of Kripke models of $i\Pi_1$, see Fact 1.3(ii). In the same time it shows some limitations on constructing infinite Kripke models of the theory.

Proposition 2.1 Suppose that T is any consistent extension of PA. There is a T-normal Kripke structure over the frame ω which does not force $i\Pi_1$.

Proof For each consistent extension T of PA, $I\Pi_1$ is not axiomatizable by Π_2-formulas in T (see Exercise 10.2(b) and Theorem 10.4 in [K2, P. 133-134]). So there is an ω- chain of models of T whose union does not model $I\Pi_1$, see e.g., [H, Th. 5.4.9]. Therefore, by considering this chain as an ω-framed Kripke structure, we get a T-normal Kripke structure which does not force $i\Pi_1$ (use the above mentioned criterion). $\Box$

Note that, if T is any consistent recursively axiomatized extension of PA, then one can construct an ω-framed T-normal Kripke structure which does not force $i\Pi_1$ exactly as Buss' model [Bus, Page 172]: in the Buss' proof replace $I\Sigma_n$ by the first n axioms of T.

Here, in the case of linear Kripke models, we prove the necessary part of the above mentioned criterion for a wide range of theories.

Lemma 2.2 Let $\mathcal{K} \Vdash PA^-$ be linear. For an $\exists$-free formula φ, $\mathcal{K} \Vdash \varphi$ iff the union of the worlds in $\mathcal{K}$ satisfies φ.

Proof Induction on the complexity of φ. $\Box$

Corollary 2.3 Let T^i be any intuitionistic theory containing PA^- and closed under the negative translation. The union of the worlds in any linear Kripke model of T^i satisfies the classical closure of T^i, i.e. T^c.

Proof Let $\mathcal{K}$ be a linear Kripke model of T^i. Suppose that $T^c \vdash \varphi$. This yields $T^i \vdash \varphi^-$ (here φ^- is the negative translation of φ (see [TD, page 57]). Now, using the above lemma, to prove the corollary, it is enough to note that the negative formulas are $\exists$-free and $\varphi^- \equiv_c \varphi$.□

Corollary 2.4 (i) The union of the worlds in any linear Kripke model of HA satisfies PA.

(ii) Linear Kripke models of $i\forall_1$ (resp. iU_1) are exactly those PA^--normal Kripke structures which the union of their worlds satisfy $I\forall_1$ (resp. IU_1).

Proof Closure of HA under the negative translation is well-known. It was observed in [AM, Example 2.4] that $i\forall_1$ is closed under the negative translation. Similarly, one can show the same for iU_1. Moreover, the scheme of induction on $\forall_1$ and U_1 formulas are $\exists$-free.□

Indeed, union of the worlds in linear Kripke models of $W\neg\neg HA$ are models of PA as well. By $W\neg\neg HA$ we mean the intuitionistic theory axiomatized by $i\Delta_0$ plus the scheme

$$\forall \overline{y}(\varphi(0,\overline{y}) \wedge \forall x(\varphi(x,\overline{y}) \rightarrow \varphi(x+1,\overline{y})) \rightarrow \forall x\neg\neg\varphi(x,\overline{y}))$$

for each formula $\varphi(x,\overline{y})$.

Here we give a counter example for the sufficient part. It is a variation of [Bus, Page 172].

Proposition 2.5 There is a two-node PA-normal Kripke structure which does not force $i\Sigma_2$.

Proof Consider a model $M \models PA + \neg Con(I\Sigma_a)$, where a is non-standard and the least solution of the formula $\neg Con(I\Sigma_x)$ in M. Embed M in a model $N \models PA + \neg Con(I\Sigma_b)$, where $N \models b < a$ and b is nonstandard and fresh and the least element with these properties. For existence of such models, see [Bus, Page 172]. It is not difficult to see that, the obvious two-node Kripke structure does not force $I_x(\varphi(x))$, where $\varphi(x)$ is the obvious classical Σ_2-equivalent of the formula $Con(I\Sigma_x) \vee \exists y < a \neg Con(I\Sigma_y)$. □

3. Independence results

It is known that HA proves the least number principle for bounded formulas. Indeed, $i\Delta_0 \vdash l\Delta_0$. This is a consequence of the decidability of bounded formulas in $i\Delta_0$. In [M2], it is proved that $HA \nvdash l\Sigma_1$. Here, we show the same for $l\Pi_1$.

Proposition 3.1 $HA \nvdash l\Pi_1$

Proof Let $M \vDash PA + Con(PA)$ and $M' \vDash PA + \neg Con(PA)$. Let $\mathcal{K}$ be the Kripke model of HA obtained by putting $\mathbb{N}$ below M and M' (the result of applying Smorynski's operation Σ' to M and M', see [S]). Let $\varphi(x)$ be the Π_1-formula $x = 1 \vee Con(PA)$. We have $\mathbb{N} \nVdash \varphi(0)$, $\mathbb{N} \Vdash \varphi(1)$ and $\mathbb{N} \nVdash \neg\varphi(0)$. Therefore $\mathcal{K} \nVdash L_x\varphi(x)$. $\square$

In [W1], it is shown that $i\Sigma_1 \nvdash i\Pi_1$ and $i\Pi_1 \nvdash i\Sigma_1$. Here, we prove similar results for $i\exists_1$ and $i\forall_1$.

proposition 3.2 $i\exists_1 \nvdash i\forall_1$

Proof By Corollary 2.4, linear Kripke models of $i\forall_1$ are exactly PA^--normal Kripke structures that the union of their worlds satisfy $I\forall_1$. Also by Fact 1.4, we know each $I\exists_1$-normal Kripke structure forces $i\exists_1$. So, to prove $i\exists_1 \nvdash i\forall_1$, it is enough to show that $I\exists_1$ is not $\forall_2$-axiomatizable. Since in this case there will exist an ω-chain of models of $I\exists_1$ such that its union does not satisfy $I\exists_1$(see [H, Th. 5.4.9]).

It is known that $I\exists_1$ is $\forall_2$-conservative over $I\exists_1^-$, where $I\exists_1^-$ is the theory of induction on parameter-free existential formulas [K1, Page 4]. So if $I\exists_1$ is $\forall_2$-axiomatizable, then we would have $I\exists_1^- \equiv I\exists_1$. This is impossible, since $I\exists_1^- \equiv I\Sigma_1^-$ (see [K1, page 4]) and $I\Sigma_1^-$ is not Π_2-axiomatizable by [KPD, Th. 06(ii)] (Σ_1^- is the set of parameter-free Σ_1-formulas). $\square$

Let us mention the two pruning lemmas in [DMKV]. The first says that if β is a node of a Kripke model $\mathcal{K}$, φ and ψ are formulas in L_β such that no free variables of ψ are bound in φ and $\beta \nVdash \psi$, then $\beta \Vdash \varphi^\psi$ iff $\beta \Vdash^\psi \varphi$. Here φ^ψ is the Friedman translation of φ by ψ and $\Vdash^\psi$ denotes forcing in the Kripke structure $\mathcal{K}^\psi$ obtained from the original one by pruning away nodes forcing ψ.

Let T^i be a fragment of HA which decides atomic formulas. The second pruning lemma essentially says if T^i is closed under Friedman's translation then whenever β is a node of a Kripke model of T^i, $\psi \in L_\beta$ and $\beta \nVdash \psi$, then $\beta \Vdash^\psi T^i$. Note that this is indeed true formula by formula (for pruning or translating by).

proposition 3.3 $i\forall_1 \nvdash i\exists_1$.

Proof Construct a Kripke model of $i\forall_1$ by putting a nonstandard model $M \models I\forall_1$ above $\mathbb{Z}[t]^{\geq 0}$. We show that this Kripke model is not closed under pruning with respect to $\exists_1$-formulas. Consider an element $s \in \mathbb{Z}[t]^{\geq 0}$ which is not even in this world. Let s become even in M (t or $t+1$ should work). So pruning this Kripke model by the formula $\exists y 2y = s$, will prune away just M. On the other hand, it is easy to see that $i\exists_1$ is closed under Friedman's translation by $\exists_1$-formulas. Using pruning lemmas, this shows that the Kripke structure does not model $i\exists_1$. $\square$

It is not known if $I\forall_1 \vdash L\forall_1$. For the intuitionistic version of this question we already have a negative answer. By [M2], we know that even $\neg\neg lop$ is not provable in $i\forall_1$. The proof is based on constructing a chain of submodels of an appropriate nonstandard model of $I\forall_1$ such that its union models $I\forall_1$ but non of its worlds models $Iopen$.

Let $W\neg\neg LNP$ be the scheme $\forall \overline{y} \neg\neg(\exists x \varphi(x, \overline{y}) \rightarrow \exists x(\varphi(x, \overline{y}) \wedge \forall z < x \neg\varphi(z, \overline{y})))$.

$W\neg\neg l\exists_1$ is the intuitionistic theory axiomatized by (PA^-) plus $W\neg\neg LNP$ on $\exists_1$-formulas.

proposition 3.4 $W\neg\neg l\exists_1 \vdash i\forall_1$.

Proof Let α be a node of a Kripke model $\mathcal{K} \Vdash W\neg\neg l\exists_1$, $\varphi(x, \overline{y})$ a $\forall_1$- formula, and $\overline{a} \in M_\alpha$ of the same arity as $\overline{y}$. We have to prove $\alpha \Vdash I_x\varphi(x, \overline{a})$. It is easy to see that $\neg\neg I_x\varphi(x, \overline{a}) \vdash I_x\varphi(x, \overline{a})$, so it is enough to show for every $\beta \geq \alpha$, there exists $\delta \geq \beta$ such that, $M_\delta \Vdash I_x\varphi(x, \overline{a})$. Assume without loss of generality $\alpha \Vdash \varphi(0, \overline{a})$. Fix β. If $\beta \Vdash \forall x\varphi(x, \overline{a})$, then we may take $\delta = \beta$. Otherwise, by Facts 1.1 and 1.2, and the assumption $\beta \Vdash W\neg\neg l\exists_1$, there will exist $\gamma \geq \beta$ such that, for some non-zero $d \in M_\gamma$, $\gamma \Vdash \neg\varphi(d, \overline{a}) \wedge \forall z < d\varphi(z, \overline{a})$. Clearly, one can take this node as δ. □

Corollary 3.5 $i\exists_1 \nvdash l\exists_1$

proposition 3.6 $i\exists_1 \nvdash l\forall_1$

Proof A similar proof as the one for Proposition 3.4, proves $W\neg\neg l\forall_1 \vdash i\neg\forall_1$. Also, in a theory containing PA^-, each $\forall_1$-formula is equivalent to its double negation. Therefore, $i\neg\forall_1 \vdash i\forall_1$. Hence, by Proposition 3.2, $i\exists_1 \nvdash l\forall_1$. □

Proposition 3.7 1) $i\Pi_2 \nvdash i\Sigma_2$.

2) $i\Sigma_2 \nvdash i\Pi_2$.

Proof (i) This is a consequence of Proposition 2.5 and this fact: every conversely well-founded $I\Pi_2$-normal Kripke structure forces $i\Pi_2$ (see [W2]).

(ii) By [Bur, Coro. 2.27], the provably recursive functions of $i\Sigma_2$ are exactly the primitive-recursive functions. But, by [Bur, Coro. 2.6], $I\Pi_2$ is Π_2-conservative over $i\Pi_2$ and so, for example, Ackerman's function is provably recursive in $i\Pi_2$. □

Remark It is known that $IE_1 \equiv IU_1$ [Wi, Lemma 2.1]. Proposition 3.3 actually shows that $i\forall_1 \nvdash iE_1$ (consider the formula $\exists y < s(2y = s)$ in its proof). Therefore, $iU_1 \nvdash iE_1$. The converse remains open: Is it true that $iE_1 \nvdash iU_1$?

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