PATRICK SUPPES, 1994

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PATRICK SUPPES: SCIENTIFIC PHILOSOPHER

Volume 1. Probability and Probabilistic Causality

Edited by

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*These volumes are dedicated
to Pat Suppes in return for
the inspiration that his
teaching and writings have
provided to generations of
students and colleagues.*

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PART II: PROBABILISTIC CAUSALITY

INTRODUCTION

The breadth and depth of Patrick Suppes's work and intellectual interests are extraordinary.[1] There are two easy ways by which one can convey this. To gain a grasp on the breadth of Suppes's work, a careful perusal of his bibliography contained at the end of Volume 3 of this collection will suffice. It reveals papers on topics ranging from historical work on Descartes and Aristotle through work in empirical psychology, computer assisted instruction, and robotics, to foundational work in measurement theory, probability, and theory structure, with a host of other subjects in between. Mere breadth is not, of course, in itself an indicator of a first-class intellect. It is the extent to which Suppes's work has influenced the direction of research in so many of these areas that is the truest measure of its depth and permanent influence. The best way to demonstrate the extent of that influence is to provide a (non-exhaustive) list of honors and awards he has received: Professor of Philosophy, of Statistics, of Psychology, and of Education at Stanford University; President of the American Philosophical Association, of the International Union of History and Philosophy of Science (twice), of the National Academy of Education, and of the American Educational Research Association; Member of the National Academy of Sciences, of the Finnish Academy of Sciences and Letters, of the Academie Internationale de Philosophie des Sciences, of the National Academy of Education, of the Société Francaise de Psychologie, of the Yugoslav Academy of Sciences and Arts, and of the USSR Academy of Pedagogical Sciences; Fellow of the American Academy of Arts and Sciences, of the American Psychological Association and of the American Association for the Advancement of Science. Other awards and honors have been plentiful, culminating in 1990 when Suppes was awarded the National Medal of Science by President Bush. This level of intellectual achievement and scholarly productivity is all the more impressive when one realizes that, since 1959, Suppes has been director of the Institute for Mathematical Studies in the Social Sciences (an institute that he co-founded with the economist Kenneth Arrow) and for a period of approximately twenty years, of a highly successful com-

PHILIPPE MONGIN

SOME CONNECTIONS BETWEEN EPISTEMIC LOGIC AND THE THEORY OF NONADDITIVE PROBABILITY

ABSTRACT. This paper is concerned with representations of belief by means of nonadditive probabilities of the Dempster–Shafer (D.S.) type. After surveying some foundational issues and results in the D.S. theory, including Suppes's related contributions, the paper proceeds to analyze the connection of the D.S. theory with some of the work currently pursued in epistemic logic. A preliminary investigation of the modal logic of belief functions à la Shafer is made. Then it is shown that the Alchourrón–Gärdenfors–Makinson (A.G.M.) logic of belief change is closely related to the D.S. theory. The final section compares the critique of Bayesianism which underlies the present paper with some important objections raised by Suppes against this doctrine.

1. INTRODUCTION AND OVERVIEW

The modelling of belief by means of nonadditive probability has over the years become increasingly familiar to researchers in the field of statistics, decision theory, and artificial intelligence. Still, for all its growing popularity, this remains a controversial approach. It seems fair to acknowledge the fact that the most systematic body of doctrine in the field, the Dempster–Shafer (henceforth D.S.) theory, cannot yet compare favorably with its formidable rival, Bayesianism. However, things are changing quickly for the better.

The scope of the D.S. theory was initially much more limited than that of Bayesianism. The latter is a full-fledged theory of rationality in human affairs, whereas the former was concerned with the rational formation of belief only. It is a deep methodological issue whether cognitive rationality should be investigated in and for itself or in connection with criteria of sound decision making. Some (especially in AI) would find the limitation in scope of Dempster's and Shafer's work a definite conceptual advantage. Others (especially in decision theory) are influenced by a view which was early promoted in theoretical economics and shared by de Finetti and Savage: to ascertain an individual's beliefs requires us to know his acts, most typically his betting behaviour. At any rate the D.S. theory is no longer isolated from utility theory. Betting

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schemes were devised at an early stage to rationalize nonadditive concepts (Smith, 1961; see also Jaffray, 1989a). More recently, important connections have been found with the independently developed models and axiomatics of nonlinear (or nonexpected) utility (Gilboa, 1987; Jaffray, 1989b; Schmeidler, 1989; Wakker, 1989).

A strength of the D.S. theory is that it included a dynamic component right from its beginning. Some of the most severe objections raised by Shafer against Bayesianism have to do with its exclusive reliance on Bayes's rule to model revision (1976, Ch. 1). Even abstracting from its dynamic component and its various connections with utility theory, the foundations of the D.S. theory have recently been subjected to useful investigations. I shall report below on a result by Wong *et al.* (1991), which provides Shafer's concept of a belief function with a qualitative axiomatization in the style of measurement theory. Only particular cases of, or concepts related to, Shafer's belief functions had been axiomatized before this result (see Dubois and Prade, 1991 for a survey). It is but one example of the current improvement in the axiomatic standing of the theory.

There seem to be at least as many interpretations of, e.g., Shafer's axioms as there are philosophical interpretations of Kolmogoroff's — a point which might rejuvenate the time-honoured discussion of subjectivism versus frequentism, but which does not seem to have been taken up yet. Although the literature on semantic issues is allusive and scattered, especially in A.I, it is replete with interesting insights. It contributes to disentangling the various features of the exceedingly compact notion of subjective uncertainty. If one looks at the post-War social sciences with hindsight, especially at the so-called revolution of uncertainty of the 70s and 80s in economics, one is struck by the fact that so much theoretical and empirical work was done around a largely unanalyzed concept (despite Shackle's warnings as early as 1949 and 1961).

Part of this paper is devoted to surveying a few foundational issues in D.S. theory. Section 2 recalls the basic formal definitions and axioms and then proceeds to a rough-and-ready classification of interpretations. For better or worse the decision-theoretic extensions have entirely been left out of this paper and even dynamics will hardly be touched upon (the reader is referred to the more extensive surveys of Smets, 1988, Dubois and Prade, 1991, or Walliser, 1991). Thus, the focus is on

such basic notions as: belief functions, basic probability assignments, plausibilities, lower and upper probabilities.

I hope to contribute towards the foundations of these concepts by stressing their connection with some of the work pursued in epistemic logic. It seems clear that the D.S. theory has gained acquiescence more easily from researchers who have a background in logic (such as computer scientists) than from those whose background is in probability theory (such as theoretical economists). The conviction that the logical approaches to belief and the nonadditive probability modelling connect well with each other pervades much current work in A.I. (see the papers collected in Shafer and Pearl, 1990). This view is supported in Sections 3 and 4 below. Section 3 is a preliminary investigation of the modal logic of belief functions. It is shown that a well-known system of modal epistemic logic has an interpretation in terms of classes of subsets having value 1 for some belief function. Presumably, this observation had not been made earlier because this system is standardly interpreted in terms of Kripke structures, a semantics that is perhaps not very well-suited for epistemic purposes. Section 4 discusses the concept of epistemic entrenchment, which was introduced by Gärdenfors and Makinson (1988) and belongs to the so-called A.G.M. (Alchourrón–Gärdenfors–Makinson) theory of belief change, one of the few truly dynamic accounts of belief available in today's epistemic logic (along with Spohn's work, 1987). I shall record the observations made in Dubois and Prade (1991) and Mongin (1992) to the effect that the epistemic entrenchment relation is a particular case of the D.S. theory, and ask how more general axioms for epistemic entrenchment could lead to recovery of the whole of Shafer's axioms.

It does not seem incongruous to connect a discussion of the D.S. theory with the work of Patrick Suppes. He made a significant contribution to the foundations of this theory in a 1974 paper, 'The Measurement of Belief'. His work with Zanotti (1977, 1989) on upper and lower probabilities is related to Dempster's (1967) seminal paper on the topic. Suppes's stand towards Bayesianism, as explained in *Probabilistic Metaphysics* (1974, 1984) and *La logique du probable* (1981), is a complex one. Although Suppes is obviously impressed by the scope and cogency of the Bayesian doctrine, he has always played down the normative claims made by its proponents. I suspect that his reservations have become more serious after the emergence of the theories on which I report here. Section 5 aims at connecting the interpretations

of this paper with some important objections raised by Suppes against Bayesianism.

2. BASIC DEFINITIONS AND SOME INTERPRETATIONS OF THE DEMPSTER–SHAFER THEORY

2.1. As it turned out, the D.S. theory is but a particular case of the earlier, mathematically much deeper 'Theory of Capacities' of Choquet (1953–1954). By assuming discrete (and even usually finite) measurable spaces, Dempster (1967) and Shafer (1976) cleared the ground for a technically accessible, primarily conceptual discussion of the modelling of belief. Schmeidler (1986), whose work closely parallels Choquet's, contributed importantly to the analysis of the non-discrete case and the related measure-theoretic issues.

Given a finite nonempty set Θ and $\mathcal{A} = 2^\theta$, Shafer (1976) defines a *belief function* to be an $f : \mathcal{A} \to \mathbb{R}$ such that:

(i) $f(\emptyset) = 0;$ $f(\Theta) = 1;$ $f(A) \leq f(B),$ $\forall A, B \in \mathcal{A}$ s.t. $A \subseteq B;$

(ii) $\forall n \in \mathbb{N}^*,$ $\forall A_1, \ldots, A_n \in \mathcal{A},$

$$f\left(\bigcup_{1 \leq i \leq n} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} f\left(\bigcap_{i \in I} A_i\right).$$

Following Chateauneuf and Jaffray's (1989) terminology, I shall define a *capacity* to be an f which satisfies (i) only and an *n-monotone capacity* to be an f which satisfies (i) as well as (ii) up to the order n. In particular, a *2-monotone* (or *convex*) capacity is defined by (i) and:

(ii') $f(A_1 \cup A_2) + f(A_1 \cap A_2) \geq f(A_1) + f(A_2).$

Many interesting results can be proved without using the full force of (ii). In particular there is a well-developed theory of convex capacities.

Given any function $f : \mathcal{A} \to \mathbb{R}$ another function $m : \mathcal{A} \to \mathbb{R}$ can be associated to it by defining:

(1) $m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} f(B),$ $\forall A \in \mathcal{A}.$

Shafer calls this mapping a *Möbius inversion* and proves that it is one-to-one (1976, Appendix). The inverse mapping of Möbius inversion is given by:

(2) $f(A) = \sum_{B \subseteq A} m(B),$ $\forall A \in \mathcal{A}.$

Now, a very important result is that f is a belief function if and only if the Möbius inverse m of f satisfies:

(iii) $m(\emptyset) = 0;$ $\sum_{A \in \mathcal{A}} m(A) = 1;$ $m(A) \geq 0,$ $\forall A \in \mathcal{A}.$

(Shafer, 1976, p. 51). Shafer calls a *basic probability assignment* (b.p.a.) any function $m : \mathcal{A} \to \mathbb{R}$ that satisfies (iii). Thus, basic probability assignments and belief functions are interdefinable. Which one is taken as a primitive of the theory will depend on the epistemic interpretation aimed at. In particular, it may happen that the seemingly unintuitive condition (ii) can be defended indirectly, in view of a plausible interpretation of the b.p.a.

To any capacity f on $\mathcal{A}$ one can associate its *dual* $g : \mathcal{A} \to \mathbb{R}$ as defined by:

(3) $g(A) = 1 - f(A^c),$ $\forall A \in \mathcal{A}.$

Shafer shows that g can be recovered from the Möbius inverse m of f as:

(4) $g(A) = \sum_{A \cap B \neq \emptyset} m(B).$

Whenever f is a belief function I shall follow the current practice (e.g. Smets, 1988) and call its dual g the associated *plausibility function*. If the convexity inequality (ii) holds, then

(5) $f(A) \leq g(A),$ $\forall A \in \mathcal{A}.$

The interdefinability of f and g means that there are once again two modes of presentation of the theory between which the choice is a matter of epistemic interpretation.

The concepts of capacity and k-mononotone capacity were mentioned above as generalizations of Shafer's belief function, but they have a life of their own. Given any nonempty set $\mathcal{P}$ of probability

measures on $\mathcal{A}$, the *lower probability* f and the *upper probability* g, as defined by:

$$(6) \qquad f(A) = \inf_{P \in \mathcal{P}} P(A) \quad \text{and} \quad g(A) = \sup_{P \in \mathcal{P}} P(A), \quad \forall A \in \mathcal{A}$$

are clearly capacities. Not every $\mathcal{P}$ (even convex) gives rise to a convex capacity, let alone a belief function. There is an elegant characterization of sets $\mathcal{P} \subseteq \Delta(\Theta, \mathcal{A})$ the lower probability of which satisfies the convexity inequality (ii'); see Jaffray (1989b, p. 243) for an exact statement. This result was first used in the altogether different context of cooperative games (Shapley, 1971). It encompasses the existence result proved in Suppes and Zanotti (1977, Theorem 3) and already underlies the construction in Dempster's (1967) seminal paper.

Dempster (1967) was concerned with providing a generalization of the standard notion of image probability in measure theory when the mapping is multivalued. In essence, he claims that the image of a probability measure μ by a multivalued mapping (correspondence) Γ is captured by two functions P_* and P^* that are the lower and upper probabilities, respectively, of a relevant probability set $\mathcal{P}$. Dempster studies the properties of $\mathcal{P}$, f and g. He implicitly shows that P_* and P^* satisfy Shafer's axioms for a belief function and a plausibility, respectively.

To be more specific, suppose there is $(\Theta, \mathcal{A})$ as above, another nonempty set $(T, \mathcal{B})$, with $\mathcal{B} = 2^T$, and a correspondence $\Gamma : T \rightrightarrows \Theta$. Given any $A \subseteq \Theta$ define its lower and upper preimages A_* and A^* as follows:

$$A_* = \{t \in T \mid \Gamma(t) \subseteq A\}$$
$$A^* = \{t \in T \mid \Gamma(t) \cap A \neq \emptyset\}.$$

(There are some minor difficulties involved with correspondences having $\Gamma(t) = \emptyset$ for some t, so that it is best to assume this case away.) Now if μ is a probability on $(T, \mathcal{B})$, what would be a reasonable definition of the image of μ by Γ? Two relevant notions are P_* and P^* on $(\Theta, \mathcal{A})$ as defined by:

$$(7) \qquad P_*(A) = \mu(A_*) \quad \text{and} \quad P^*(A) = \mu(A^*), \quad \forall A \in \mathcal{A}.$$

Dempster argues as follows for these functions (which he calls *lower* and *upper probabilities*, respectively):

Since A^* consists of those $t \in T$ which *can possibly* correspond under Γ to a $\theta \in \Theta$, one may naturally regard $\mu(A^*)$ as the largest possible amount of probability from the measure μ which can be transferred to outcomes $\theta \in \Theta$. Similarly A_* consists of those $t \in T$ which *must lead* to a $\theta \in \Theta$, so that $\mu(A_*)$ represents the minimum amount of probability which can be transferred to outcomes $\theta \in \Theta$. (1967, p. 326).

I have emphasized two expressions in the quoted sentence because they are suggestive of modalities: there is a connection between P_*, P^* and the dual operators of modal logic $\square$, $\diamond$. Dempster studies the properties of the set $\mathcal{P}$ of those probabilities which are *compatible* with P_* and P^*:

$$(8) \qquad \mathcal{P} = \{P \mid P \in \Delta(\Theta, \mathcal{A}) \quad \text{and}$$

$$\forall A \in \mathcal{B}, P_*(A) \leq P(A) \leq P^*(A)\}.$$

We may as well leave Dempster's construction at that, but record the important fact (which is only implicit in the original article): P_* satisfies Shafer's axioms (i) and (ii) for a belief function. (Proofs of this statement are given in Suppes and Zanotti, 1977, Section 4, and in Wong *et al.*, 1991.)

2.2. Conceptually, the important point is that Dempster had started his own theory with none of the concepts that can be used (interchangeably) as primitives in Shafer's. This appears to express a cleavage of interpretations among the users of the D.S. theory. The rest of this section elaborates on this and related philosophical issues. At the risk of oversimplification I distinguish between three broad classes of interpretations, depending on which concept is taken to be the true primitive: belief function or plausibility (type I), basic probability assignment (type II), probability (type III).

2.2.1. A characteristic feature of type I interpretations is that they emphasize allegedly natural properties of subjective uncertainty that they claim are better captured by the chosen primitive than by probability. A standard argument for belief functions goes in terms of *extreme uncertainty*. If one is very uncertain about whether A or A^c will take place, probability theory would leave no other choice than putting: $\text{Prob}\, A = \text{Prob}\, A^c = 1/2$. This 'principle of indifference', as Keynes (1921) named it (some writers refer to it as to the 'principle of insufficient reason'), has a long and notorious history of paradoxes. One

classic problem is that there are many seemingly natural, though eventually inconsistent ways of defining equiprobable distributions. Mathematicians such as Jaynes have explored this problem in the context of geometrical probability. Another source of paradoxes is that the Prob function deals with extreme uncertainty as if it were the certainty of equiprobability. This last difficulty disappears from the theory of belief functions because the convexity inequality (ii') implies that:

$$(9) \qquad f(A) + f(A^c) \leq 1.$$

[The dual inequality for plausibilities is of course:

$$(9') \qquad g(A) + g(A^c) \geq 1.]$$

That is, the theory is compatible with the individual's attributing a small fractional amount of belief to both A and A^c.

A relevant particular case of belief functions is the *necessity* function defined as Nec : $\mathcal{A} \to [0, 1]$, Nec $(\emptyset) = 0$, Nec $(\Theta) = 1$ and

$$(10) \qquad \text{Nec } (A \cap B) = \min(\text{Nec } A, \text{Nec } B).$$

Its dual is called the *possibility* function and has the defining feature that Pos $(\emptyset) = 0$, Pos $(\Theta) = 1$ and

$$(10') \qquad \text{Pos } (A \cup B) = \max(\text{Pos } A, \text{Pos } B).$$

The fact that Nec and Pos are indeed belief functions and plausibilities, respectively, is proved in Shafer (1976) using the convenient tool of Möbius inverses. The subsets that are given strictly positive value by the Möbius inverse m of Nec or Pos turn out to satisfy a remarkable property: they are a nested family of sets. Besides convenience there are several reasons for becoming interested in the particular cases of Nec or Pos. These reasons are spelled out in great detail by Dubois and Prade (1985). An important connection was stressed by Zadeh (1978) at an early stage: the basic notions and results of 'fuzzy set theory' can be expressed in the language of Pos functions. This fact motivates the claim that Pos captures the class of uncertainty situations in which 'uncertainty' does not so much refer to the absence of relevant information as to the use of *intrinsically vague* predicates ('John is tall'). A no less important connection on which I elaborate in Section 4 is that the A.G.M. logic of belief change makes implicit use of Nec functions.

Leaving aside the special cases of Nec and Pos there is a serious problem with the intuitive justification of belief functions and plausibilities by (allegedly natural) properties of subjective uncertainty. These properties usually depend on (ii') rather than on the whole set of conditions (ii). The above argument of 'extreme uncertainty' can be taken care of by imposing a weaker principle than (ii'):

$$(ii'') \quad f(A_1 \cup A_2) \geq f(A_1) + f(A_2) \quad \text{whenever} \quad A_1 \cap A_2 = \emptyset.$$

(Interestingly, only this weaker principle is required by Suppes (1974), although he motivates it differently. It had already been selected as being of particular relevance in Good, 1962.) These difficulties lead one to the next class of interpretations. An alternative solution is to concentrate on sets of belief function 1 as in Section 3 below. The issue of extreme uncertainty can be pursued usefully in this simplified context.

2.2.2. Type II interpretations take the Möbius function (or basic probability assignment) m as being the relevant primitive. As already suggested by Shafer's terminology, this is a semantically ubiquitous concept. The defining conditions (iii) are in general incompatible with m being a probability on $(\Theta, \mathcal{A})$. But it is trivial to introduce *another* space $(T, \mathcal{B})$ on which m can be regarded as a probability. Whether or not this formal step is justified depends on the purported interpretation. Hence there will be two species of type II interpretations.

Here is an example due to Smets (1988) in the style most favored by Shafer. Mrs Jones was murdered. The investigator believes that the murderer belongs to the set $\Theta = \{$ Peter, Paul, Mary $\}$. Some evidence points to the fact that the murderer is a small man, which fits the description of Peter and Paul; another piece of evidence suggests that Peter was at home, which would leave Paul and Mary as the only suspects. The uncertainty situation is well captured – or so it is argued – by *allocating* a unit total mass of nonnegative belief to each proposition having evidential support. From the m allocation a belief function and a plausibility are then constructed by simple summations as in (2) and (4), respectively. The formula (2) expresses more caution than does (4). The investigator *believes* to a positive degree only those propositions which are logical consequences of some evidentially supported proposition. He regards as *plausible* to a positive degree the wider class of those propositions which are not contradicted by all evidentially sup-

ported propositions. (This intuitive distinction parallels Dempster's in the quotation above.) There is an important feature of the above analysis of belief functions and plausibilities: it can be generalized to the case where there are several kinds of evidence. Suppose that there is another investigation and it gives support to the view that the murderers were a couple. Hence another m' is constructed with $m'(\{Peter, Mary\} > 0$, $m'(\{Paul, Mary\}) > 0$ and the further problem is that of *mixing* m and m' (Shafer, 1976, Ch. 1). The relevant conclusion, however, that I wish to draw from the murderer's example is simply this: the b.p.a. can be given an epistemic interpretation as weight of evidence and there is nothing in the intuitive discussion above that would *compel* a probabilistic view of the b.p.a.

I now move to an example of a very different sort borrowed from Jaffray (1991). A medical researcher investigates the relation between the presence or the absence of a symptom S and the outbreak or otherwise of a disease D. The investigation is incomplete. As a result not all of the patients studied by the researcher have cards of one of the following types: D^+S^+, D^+S^-, D^-S^+, D^-S^-. Some patients have incompletely filled cards D^+ (= just the disease was evidenced) and S^- (= just the lack of symptom was evidenced). Let us denote by m the function that assigns to each of the eight possible cases of completely or incompletely filled cards its proportion in the sample. How could the researcher summarize his view of the relation between D and S, given his knowledge of m? Once again, it is argued that the belief function equation and plausibility equations (2) and (3) are two reasonable ways of doing this. There are several interesting features in this example that I will not pursue here. I just stress that the b.p.a. has again been given the epistemic interpretation of weight of evidence, but it is now a probability – and unproblematically so. In the medical example m is simply a *proportion* function.

2.2.3. Type III interpretations start with either a probability or a set of probabilities and construct the special functions of the D.S. theory using such data. From what I have just said some of the interpretations in terms of b.p.a. are indistinguishable from type III interpretations.

A common example of a type III interpretation occurs when an individual is faced with a well-defined probability space $(T, 2^T, \mu)$ and then required to express beliefs on a *different though related* measurable space $(\Theta, 2^\theta)$. More formally suppose as in Wong *et al.* (1991, Sec-

tion 4) that there is a compatibility relation C on T. It is symmetric, does not have to be complete, and can be assumed to satisfy the property that for all $t \in T$, there is $\theta \in \Theta$ such that $tC\theta$. The intended interpretation is that knowledge of state t would not contradict the individual's belief that state θ occurs. Now, C gives rise to the multivalued mapping $\Gamma : T \rightrightarrows \Theta$,

$$\Gamma(t) = \{\theta \in \Theta \mid tC\theta\}$$

and Dempster's 1967 construction (see Section 2.1.) can be repeated literally. Dempster indeed had in mind the case in which the statistician's sample space is different from the space on which he is asked to make statistical statements. Two features of the Dempster-Wong interpretation should be noted. One, it assumes a given probability μ at the start but is agnostic about its origin: μ may be either a frequency or a subjective prior in the Bayesian sense. Two, the Dempster–Wong interpretation is general enough to encompass the medical example of Section 2.2.2. A closer look at the latter shows that it is best formalized in terms of two distinct sets: the set of patients Θ and the set of all possible (i.e. either completely or incompletely filled) cards T. (If the m function is to be construed as a frequency, it *must* be defined on a sample space T different from the population space Θ. Specifically, $m(D^+S^-)$ does *not* measure the proportion of D^+S^- individuals in Θ.) I also understand Shafer and Tversky's (1985) examples of 'randomly coded messages' to be particular applications of the Dempster–Wong interpretation.

An altogether different example of type III interpretation is Suppes's (1974). In a vein typical of measurement theory he assumes a set X, an algebra of subsets $\mathcal{F}$ on X, and a binary relation $\succeq$ on $\mathcal{F}$ that satisfies de Finetti's qualitative axioms for probability: $\succeq$ is a weak ordering, it is monotonic in a suitably strong ('additive') sense, $A \succeq \emptyset$ and $X \succ \emptyset$. Suppes also assumes a subalgebra $\mathcal{S}$ which is *finite* and satisfies the properties that:

(a) if $S \in \mathcal{S}$ and $S \neq \emptyset$, then $S \succ \emptyset$.
(b) if $S, T \in \mathcal{S}$ and $S \succeq T$, then there is $V \in \mathcal{S}$ such that $S \sim T \cup V$.

Condition (b) is of course a 'resolution principle' or 'structural axiom' in the sense made clear by measurement theory. Suppes's first result is that there is a unique probability P on $\mathcal{S}$ that represents $\succeq$. Because of (a) P has to assign the same value to every minimal event in $\mathcal{S}$. This

first result (which goes back to an earlier paper in Suppes, 1969) needs emphasizing. Since the counterexample by Kraft, Pratt and Seidenberg (1959), it is well known that the de Finetti axioms of qualitative probability are not sufficient for a probability representation even in the finite case. The structural condition (b) is a simple and elegant way of circumventing this problem in the case of $\mathcal{S}$.

Suppes's second result is directed towards the problem of representing $\succeq$ on the larger algebra $\mathcal{F}$. He introduces two functions P_*, P^* on $\mathcal{F}$ defined as follows: if A is equivalent to some $S \in \mathcal{S}$, one sets $P_*(A) = P^*(A) = P(S)$. If not, there are $S, S' \in \mathcal{S}$ such that $S' \succeq A \succeq S$, and $S' \sim S \cup V$ where V is a minimal element of $\mathcal{S}$; one then sets $P_*(A) = P(S)$ and $P^*(A) = P(S')$. The notation is suggested by the fact, which is proved by Suppes, that P_*, P^* obey several properties of lower and upper probabilities in the sense of 2.1, above. In particular they satisfy $P_* \leq P^*$, that is inequality (5), and P_* satisfies the weaker convexity inequality (ii"). Now, one half of the representation theorem states that

$$P_*(A) \geq P^*(B) \Rightarrow A \succeq B;$$

and another half that if one defines $^*\!\succ$ on $\mathcal{F}$ as

$$A \, ^*\!\succ B \quad \text{iff} \quad \exists S \in \mathcal{S}, \quad A \succ S \succ B,$$

then $^*\!\succ$ is a semiorder and $A \, ^*\!\succ B \Rightarrow P_*(A) \geq P^*(B)$. The two halves of the representation theorem are only loosely connected with each other but, as Suppes explains, this is in the nature of things. Given the relatively weak axioms it is not surprising that the weak semiorder structure emerges. Attempts to strengthen it while still using the P_*, P^* representations will be frustrated; for instance, $A \succ B \Rightarrow P_*(A) > P^*(B)$ is clearly false.

Suppes's 1974 theory is highly polemical against Bayesianism because it stresses the *inexact* nature of 'the measurement of belief'. A unique probability representation, as is usually required by Bayesians, can be derived on a special subset of $\mathcal{S}$ only. The ('standard') events in $\mathcal{S}$ serve a measurement purpose elsewhere. Specifically, any minimal event of $\mathcal{S}$ can be used as a measuring rod for any $A \in \mathcal{F}$. This leads to approximations, exactly as in the measurement of solid bodies, which has been the heuristic model for Suppes's theory (see 1981, pp. 55–57). From the point of view of this survey the latter theory definitely leans towards type III interpretations. It relies on (qualitative)

probability as the only primitive. Suitably modified concepts of belief functions and plausibilities are *then* introduced derivatively. There is some remote analogy between Dempster's construction and Suppes's. As in Dempster, a probability function on a special set is used to generate lower and upper probabilities (here in a weaker than the usual sense) on the set of primary interest. It would be interesting to clarify the mathematical connection between the two constructions.

Dempster's and Suppes's constructions have the common feature that they start with *one* probability function or relation. Other type III interpretations such as Kyburg's (1987) rely on the assumption that the individual has availed himself of a whole probability set $\mathcal{P}$. This assumption (with $\mathcal{P}$ being convex) also underlies some of Levi's work (e.g. 1984). A standard application occurs when the sample space T and the space of interest Θ are the same, and the statistician assumes that the observations in the sample may be drawn from any distribution in a given set. For instance, he samples results of dice-throwing, knowing only that the thrown dice belong to a certain collection of biased dice. This is a case where the primitive concept should clearly be taken to be a probability set rather than a single probability. There are seemingly related examples that are less convincing, i.e. where the probability set is better seen as the collection $\mathcal{P}$ of probabilities compatible with some antecedently constructed belief function and plausibility.

3. ON THE MODAL LOGIC OF BELIEF FUNCTIONS

This section is a preliminary investigation of the links between epistemic modal logics and the Dempster–Shafer theory. It shows that the system *KD* of propositional modal logic has an interpretation in terms of sets of possible worlds which are given value 1 by some belief function. The same applies to some relevant systems containing *KD*. This is an easy observation to make once *KD* is interpreted in terms of *neighborhood* structures rather than Kripke structures. To work out this intuition properly and develop a completely autonomous semantics of *KD* systems in terms of belief functions involves some further, nontrivial steps which will only be alluded to here.

The vocabulary of our systems is built upon the set $VP = (p_k)_{k \in K}$ of propositional variables, the usual propositional connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, as well as the unary operator B. $\top$, $\perp$ denote the tautological

and contradictory propositions respectively. The syntax will include any axiomatization of the propositional calculus as well as part or the whole of the following rule and axiom schemata:

$$(RM) \quad \frac{\varphi \to \psi}{B\varphi \to B\psi}$$

$$(N) \quad B\top$$

$$(C) \quad B\varphi \wedge B\psi \to B(\varphi \wedge \psi)$$

$$(P) \quad \neg B\bot$$

$$(D_c) \quad \neg B\varphi \to B\neg\varphi.$$

The monotonicity rule (RM) together with (N) and (C) are an axiomatization of the standard system K. Adding (P) one gets KD. An alternative way of defining KD is to replace (P) with

$$(D) \quad B\varphi \to \neg B\neg\varphi$$

which is seen to be equivalent to (P) in any system containing K (Chellas, 1980, p. 133). (D_c) is the converse of (D). This is a rather strong and unusual axiom. For (D) and (D_c) taken together imply that one of the operators B and $\neg B\neg$ (or $\square$ and $\diamond$ in the more usual notation) is redundant given the other. I shall follow Chellas's conventions as closely as possible and then denote the complete system by $KD!$ (see 1980, p. 143). As usual, the rules and axioms make it possible to define a notion of formal inference $\varphi \vdash \psi$ and a notion of theorem $\vdash \varphi$.

The (informal) epistemic interpretation of each item is plain. (RM) means that the individual's belief system can reproduce the inferences warranted by the logic, and (N) that the former includes any theorem of the latter. Both requirements have been criticized in the AI and philosophical literatures under the name of *logical omniscience* (Vardi, 1986; Dubucs, 1992). To get the full force of the epistemic demands made by the K system it is useful to recall that it has the theorem:

$$(K) \quad B(\varphi \to \psi) \to (B\varphi \to B\psi).$$

Perhaps more obviously than (RM), this wording makes it clear that the individual draws any inference that is licensed by his own system

of beliefs. The most obvious interpretation of (C) goes in terms of the preservation of conjunctive belief, a property that can be seen to be violated by probabilistic reasoning in the following case: supposing that an individual has a high but still uncertain degree of probabilistic belief in φ and ψ, it surely does not follow that he has a similarly high degree of probabilistic belief in $\varphi \wedge \psi$. However, if an individual has a *maximal* degree of probabilistic belief in φ and ψ, he also has a *maximal* degree of probabilistic belief in $\varphi \wedge \psi$. This latter observation will be used to derive the proposition stated below.

A *neighborhood* (or *Scott*) *structure* is defined to be any triple:

$$m = \langle \Theta, N, v \rangle$$

where Θ is a nonempty set, N a function $\Theta \to 2^{2^\Theta}$ (which is said to associate a *neighborhood*, or set of subsets of Θ to each θ), v a valuation function $\Theta \times VP \to \{0, 1\}$. $\mathcal{N}$ will denote the class of all neighborhood structures. I shall concentrate on particular subsets of $\mathcal{N}$ determined by the properties of the N functions, to be reviewed below. Given $m \in \mathcal{N}$ and $\theta \in \Theta$ the semantic validity relation $m, \theta \models \varphi$ is defined recursively in the following way:

- if $\varphi \in VP$ $m, \theta \models \varphi$ iff $v(\theta, \varphi) = 1$
- if $\varphi = \neg\psi$ $m, \theta \models \varphi$ iff [not $m, \theta \models \psi$]
- if $\varphi = \varphi_1 \wedge \varphi_2$ $m, \theta \models \varphi$ iff $m, \theta \models \varphi_1$ and $m, \theta \models \varphi_2$ and analogously for the remaining propositional clauses
- if $\varphi = B\psi$ $m, \theta \models \varphi$ iff $[\![\psi]\!]_m \overset{\text{def}}{=} \{\theta' \mid m, \theta' \models \psi\} \in N(\theta)$.

As usual $m \models \varphi$ means that $m, \theta \models \varphi$ for all $\theta \in \Theta$ and $\mathcal{N} \models \varphi$ that $m \models \varphi$ for all m in the relevant subclass $\mathcal{N}' \subseteq \mathcal{N}$.

The point is worth stressing that neighborhood structures are a very natural semantics when it comes to the epistemic interpretations of propositional modal logic. For the last (and most crucial) clause of the above definition indicates that $B\psi$ holds at θ if and only if the truth set of ψ belongs to $N(\theta)$; hence $N(\theta)$ is simply the list of propositions (viewed extensionally) to which the individual adheres. The informal epistemic account of the Kripke semantics is not as plain as that. The basic epistemic intuition underlying the Kripke relation $\omega R\omega'$ is, 'the individual regards world ω' as accessible from world ω'. This meaning is *prima facie* obscure. Not only is the concept of a possible world

open to a variety of philosophical interpretations – a well recognized problem that neighborhood semantics also has to face – but the notion of a *subjective* accessibility relation clearly needs analyzing.

To specify the relevant subclasses of $\mathcal{N}$ the following properties of the N function will be used: for all $\theta \in \Theta$,

1. $N(\theta)$ is supplemented, i.e. $X \in N(\theta)$ and $X \subseteq X' \Rightarrow X' \in N(\theta)$
2. $\Theta \in N(\theta)$
3. $N(\theta)$ is closed under intersections
4. $\emptyset \notin N(\theta)$
5. $N(\theta)$ is maximal, i.e. $\forall X \in 2^\Theta$, either $X \in N(\theta)$ or $X^c \in N(\theta)$.

In order to connect the neighborhood semantics with the Dempster–Shafer theory one should restate these properties in terms of functions $f : 2^\Theta \to [0, 1]$. Given such a function define its *essential preimage* as the set of those $X \subseteq \Theta$ such that $f(X) = 1$. Now, condition 1, above is equivalent to stating that $N(\theta)$ is the essential preimage of some monotone function f. The remaining conditions can similarly be rephrased. Owing to this crude translation device the classic results proved on neighborhood semantics become available to Dempster–Shafer theorists, provided some care is taken of the definitional restriction of the D.S. theory to finite spaces. I shall define $\mathcal{N}_{BF}(N_{\text{Prob}})$ to be the class of those structures $m = \langle \Theta, N, v \rangle$ such that Θ is finite and for all θ, $N(\theta)$ is the essential preimage of a belief function (respectively a probability); and $\mathcal{N}^2_{BF}(\mathcal{N}^2_{\text{Prob}})$ to be the class of those m in which Θ is finite and for all θ, $N(\theta)$ is the essential preimage of a *two-valued* belief function (respectively a two-valued probability).

PROPOSITION. *The system KD is a sound and complete axiomatization of $\mathcal{N}_{BF} = \mathcal{N}_{\text{Prob}} = \mathcal{N}^2_{BF}$), KD! a sound and complete axiomatization of $\mathcal{N}^2_{\text{Prob}}$. KD! is false of $\mathcal{N}^2_{BF}$.*

Here is a sketch of the proof. (i) If f is a belief function on 2^Θ, its essential preimage e satisfies conditions 1, 2, 3, 4. (Note the role of the inequality of convex capacities in deriving 3.) (ii) Given a finite Θ and $e \subseteq 2^\Theta$ satisfying 1–4 it is easy to construct a probability, (hence a belief function), P such that e is the essential preimage of P. (Take $X = \cap e$ and set $P(\{\theta\}) = 1/|X|$ if $\theta \in X$ and $P(\{\theta\}) = 0$ if $\theta \notin X$.) (iii) Taken together (i) and (ii) imply that $\mathcal{N}_{BF}$ is exactly the

subclass of those m in which Θ is finite and for all θ, $N(\theta)$ satisfies 1–4. Then, using theorems on the determination of modal logic systems by classes of finite neighborhood structures (Chellas, 1980, pp. 264–267), one concludes that KD is a sound and complete axiomatization of $\mathcal{N}_{BF}$. (iv) Point (ii) above shows that $\mathcal{N}_{BF} \subseteq \mathcal{N}_{\text{Prob}}$. (v) To show that $\mathcal{N}_{BF} \subseteq \mathcal{N}^2_{BF}$ one starts with a finite Θ and $e \subseteq 2^\Theta$ satisfying 1–4, and finds a two-valued necessity (hence a two-valued belief function), Nec, such that e is the essential preimage of Nec. (Set Nec = 1 if $X \in e$ and Nec $X = 0$ otherwise.)

Regarding the second part. (vi) If P is a two-valued probability on any measurable space, its essential preimage satisfies 1–5. Conversely, given a finite Θ and $e \subseteq 2^\Theta$ satisfying 1–5, there is a two-valued probability, P, such that e is the essential preimage of P. (Take $X = \cap e$. It has to be a singleton $\{\theta\}$. Set $P = \delta_\theta$, the Dirac mass located at θ.) (vii) Hence $\mathcal{N}^2_{\text{Prob}}$ is exactly the subclass of those m in which Θ is finite and for all θ, $N(\theta)$ satisfies 1–5. Using determination results for finite neighborhood structures it is seen that $KD!$ is a sound and complete axiomatization of $\mathcal{N}^2_{\text{Prob}}$. (viii) Clearly, $\mathcal{N}^2_{\text{Prob}} \neq \mathcal{N}^2_{BF}$ and the (D_c) schema is violated in some elements of $\mathcal{N}^2_{BF}$. (Take a two-element set $\{\theta_1, \theta_2\}$ with Bel $\theta_1 = $ Bel $\theta_2 = 0$.)

In sum one cannot logically distinguish between probabilities and belief functions if one considers validity in $\mathcal{N}_{BF}$ and $\mathcal{N}_{\text{Prob}}$, but one can if the validity concept is modified by considering $\mathcal{N}^2_{BF}$ and $\mathcal{N}^2_{\text{Prob}}$. The distinguishing axiom is (D_c), which is semantically related to the strengthening of the convex capacity inequality (ii') into the familiar equation $\text{Prob}(A \cup B) + \text{Prob}(A \cap B) = \text{Prob} A + \text{Prob} B$. The proposition above is nothing but a convenient rephrasing of known results on finite neighborhood structures. To go further into the logical differences between probabilities and belief functions the following two strategies seem to be commendable. For one, a deeper logical analysis would result from enriching the language with operators B_r that are indexed by the degree of belief r. In order to keep a finite number of axiom schemata r would have to vary over a *finite* subset of $\{0\} \cup {]}0, 1{[} \cup \{1\}$ as in some recently developed probabilistic logics (Bacchus, 1990), and in a system of Lismont (1992). For another, even if one accepts the restriction of r to $\{0, 1\}$, the proposition stated above is somewhat unsatisfactory. For it still relies on the basic tool of neighborhood semantics

– the $N(\theta)$ function. To develop a completely autonomous semantics of modal propositional logic in terms of belief functions would involve one with considering *infinite hierarchies* of belief functions. In essence, statements of 'depth' equal to that of $B\varphi$ where φ is a purely propositional formula would then be interpreted by means of a belief function, statements of 'depth' equal to that of $BB\varphi$ by means of a belief function over a set of belief functions, and so on. The pioneering work of Fagin, Halpern and Vardi (1984) or Vardi (1986) becomes relevant at this juncture, provided that one forgets about the added generality of their multi-individual framework. For instance, Mongin (1991) analyzes infinite hierarchies of two-valued probabilities and connects them with Vardi's concept of belief structures.

4. EPISTEMIC ENTRENCHMENT AND THE DEMPSTER-SHAFER THEORY

This section elaborates upon another significant connection between epistemic logic and nonadditive probability. The Alchourrón–Gärdenfors–Makinson (A.G.M.) logic of belief change has recently attracted much attention, especially in the field of AI, for (at least) the following two reasons. On the one hand, it axiomatizes belief change in a *prima facie* highly natural way; on the other hand, the axioms have elegant and nontrivial semantic counterparts. (The *locus classicus* of the A.G.M. logic is Alchourrón, Gärdenfors and Makinson, 1985, to be complemented by the recent results in Gärdenfors, 1988, and Gärdenfors and Makinson, 1988.) Broadly speaking, I show that the A.G.M. axioms have a semantic counterpart in the Dempster–Shafer theory. This observation was already made in Dubois and Prade (1991) and Mongin (1992). Section 4.1. provides the background definitions, Section 4.2. states the main relevant facts, while 4.3. investigates a very tentative generalization of the A.G.M. concept of epistemic entrenchment.

4.1. In contradistinction with the well-established Bayesian approach to belief revision, the A.G.M. approach never explicitly refers to the individual's decisions. Nor does it formalize the individual's beliefs in measure-theoretic – let alone probabilistic – terms. The building blocks of the theory are propositions. The major mathematical constraint is that these propositions are expressed in a language which in an appropriate sense includes the sentential calculus. *Epistemic states*, or

states of belief, are captured by deductively closed sets of propositions. *Epistemic attitudes* – belief, disbelief, and indeterminacy – are then described by means of the membership relation. The *epistemic input*, that is the incoming information, is normally restricted to be propositional. *Epistemic changes* are axiomatized in terms of the following items: the input which bring them about, the initial epistemic state and the resulting epistemic state. There are three such operations: contraction, expansion and revision. In the principal case at least, *contraction* may be viewed as a move from belief or disbelief to indeterminacy, *expansion* as a move from indeterminacy to either belief or disbelief, and finally *revision* as a move from either determinate attitude to the opposite one. When it comes to technical definitions, contraction and revision come to the forefront and turn out to be closely related to each other. An important heuristic principle of the theory is that the final result of either contractions or revisions should as much as possible (i) preserve logical consistency, and (ii) avoid unnecessary changes in the initial epistemic state (minimality principle).

In a significant development of the A.G.M. framework, Gärdenfors (1988) and Gärdenfors and Makinson (1988) introduced the novel concept of *epistemic entrenchment*. This concept is meant to capture the relative priority of one proposition over another in the initial epistemic state. It then has a bearing on what in the initial state is given up, and what is retained, when the contraction operation takes place. More precisely, the enlarged A.G.M. approach axiomatizes epistemic entrenchment as a binary relation on propositions subject to an ordering and further special constraints. It is then shown that the epistemic entrenchment axioms can be recovered from those already defined on contraction, and conversely. This result is described as a 'grande finale' in Gärdenfors (1988, p. 96). It is a significant achievement because the axioms of epistemic entrenchment, on the one hand, and contraction, on the other, have much to say for themselves, each in a seemingly different sphere of epistemic intuition. Furthermore, epistemic entrenchment relations are more concrete objects than contractions, the definition of which is natural and plausible, but *nonconstructive*. Broadly speaking, the former relations play a semantic role, in a way somewhat analogous to the so-called partial meet contraction functions, which provided the constructive counterpart of the latter operation in the 1985 version of the A.G.M. theory. The primary aim of this section is to improve on the 1988 theorem and gain a better understanding of epistemic entrenchment.

For simplicity reasons I shall deal with the A.G.M. theory as if it had been stated in the propositional calculus strictly speaking. Define $VP = \{p_1, \ldots, p_n, \ldots\}$ to be a denumerable set of propositional variables. $L(VP)$ stands for the set of all propositions, $\vdash$ for the inference relation, and $\mathcal{A}$ for $L(VP)/\leftrightarrow$, i.e. the quotient of the proposition set by the logical equivalence relation. Following A.G.M., the individual's epistemic states are closed under $\vdash$, that is they are *systems* in Tarski's sense. We denote by $\mathcal{S}$ the set of all systems and use for them the letters $S, S', \ldots, K, K', \ldots$. Of special interest is the subset of *axiomatizable* systems, i.e., of those $S^\varphi \in \mathcal{S}$ such that $S^\varphi = Cn(\{\varphi\})$ for some $\varphi \in L(VP)$. By assumption, the individual's initial state of belief K may be any system whatsoever, including the tautology set $S^\top$ of $L(VP)$ and the inconsistent system $S^\perp = L(VP)$, two limiting cases which are *not* deprived of epistemic relevance. Note emphatically that K does not have to be complete.

If $K \in \mathcal{S}$ is the individual's current epistemic state, his epistemic attitudes – acceptance, rejection, and indeterminacy with respect to a proposition φ – are clearly rendered as: $\varphi \in K$, $\neg\varphi \in K$, and $[\varphi \notin K$ and $\neg\varphi \notin K]$. A.G.M. define the belief change operations as functions $\mathcal{S} \times L(VP) \to \mathcal{S}: K_\varphi^+$ ('the expansion of K by φ'), K_φ^- ('the contraction of K by φ'), and K_φ^* ('the revision of K by φ'). There is a definite loss of generality in the restriction of the domain to $\mathcal{S} \times L(VP)$ instead of $\mathcal{S} \times \mathcal{S}$ (but attempts have recently been made to generalize). K_φ^+ is simply $Cn(K \cup \{\varphi\})$. I shall now state the A.G.M. axioms for K_φ^- and K_φ^* in the current version (Gärdenfors, 1988, Ch. 3, and Gärdenfors and Makinson, 1988), making a few comments at the right and below.

A.G.M. CONTRACTION AXIOMS

$(K\text{-}1)$ $K_\varphi^- \in \mathcal{S}$ *(deductive closure)*

$(K\text{-}2)$ $K_\varphi^- \subseteq K$

$(K\text{-}3)$ If $\varphi \notin K$, then $K_\varphi^- = K$ *(an application of the minimality principle)*

$(K\text{-}4)$ If $\nvdash \varphi$, then $\varphi \notin K_\varphi^-$

$(K\text{-}5)$ $K \subseteq (K_\varphi^-)_\varphi^+$ *('recovery postulate', see below)*

$(K\text{-}6)$ If $\vdash \varphi \leftrightarrow \psi$, then $K_\varphi^- = K_\psi^-$ *(a form of logical omniscience)*

Rather than stating the remaining two axioms $(K\text{-}7)$ and $(K\text{-}8)$, I shall use a convenient 'factoring' condition to which these two axioms can be seen to be equivalent in the presence of $(K\text{-}1)$–$(K\text{-}6)$:

$(K\text{-}V)$ Either $K_{\varphi \wedge \psi}^- = K_\varphi^-$ or $K_{\varphi \wedge \psi}^- = K_\psi^-$ or $K_{\varphi \wedge \psi}^- = K_\varphi^- \cap K_\psi^-$.

Some of the contraction axioms strike one as definitional (given the informal account of this operation as resulting in belief indeterminacy with respect to φ). Others are primarily intended to provide mathematical structure but they may have the epistemic flavor of logical omniscience. Several axioms, among them the somewhat questionable 'recovery postulate', are clearly influenced and possibly justified by the minimality principle (ii) above: K_φ^- should depart as little as possible from K. Regarding (i) (preservation of logical consistency) note that $K_\varphi^- \in S^\perp$ except in one limiting case (when $K = S^\perp$ and $\varphi = \top$).

A.G.M. REVISION AXIOMS

$(K*1)$ $K_\varphi^* \in \mathcal{S}$ *(deductive closure)*

$(K*2)$ $\varphi \in K_\varphi^*$

$(K*3)$ $K_\varphi^* \subseteq K_\varphi^+$

$(K*4)$ If $\neg\varphi \notin K$, then $K_\varphi^+ \subseteq K_\varphi^*$ *(the consequence operation is all right if there is no inconsistency)*

$(K*5)$ $K_\varphi^* = S^\perp$ only if $\vdash \neg\varphi$ *(inconsistent K_φ^* are exceptional)*

$(K*6)$ If $\vdash \varphi \leftrightarrow \psi$, then $K_\varphi^* = K_\psi^*$ *(a form of logical omniscience).*

Rather than stating the remaining two axioms $(K*7)$ and $(K*8)$ I shall complete the list with the following 'factoring' condition to which these two axioms are equivalent in the presence of $(K*1)$–$(K*6)$ (for a proof, see Gärdenfors, 1988, p. 57):

$(K*V)$ Either $K_{\varphi \vee \psi}^* = K_\varphi^*$ or $K_{\varphi \vee \psi}^* = K_\psi^*$ or $K_{\varphi \vee \psi}^* = K_\varphi^* \cap K_\psi^*$.

Again, the revision axioms reflect an admixture of definitional considerations, mathematical constraints with a flavor of logical omniscience, as well as heuristic motivations along the lines of (i) and (ii). Granting the informal account of K_φ^* as resulting in acceptance of φ, the definitional component of this axiom set strikes one as large indeed. There is no such problematic axiom as the 'recovery postulate' of contraction. Finally, the comparison of $(K\text{-}V)$ and $(K*V)$ would suggest exploring the algebraic duality of $-$ and $*$. This line of research is

pursued in Mongin (1992).

4.2. I now move to a set of definitions of epistemic entrenchment (e.e.). There are two accounts of epistemic entrenchment (e.e.) in the current A.G.M. theory. The first account is an axiom set which does not involve contraction as a primitive term.

A.G.M. AXIOMS FOR E.E. *Define an e.e. relation to be any binary relation $\leq$ on $L(VP)$ that satisfies the following properties:*

(*EE*1) *Transitivity*
(*EE*2) *Monotonicity: if* $\vdash \varphi \rightarrow \psi$*, then* $\varphi \leq \psi$
(*EE*3) *Conjunctiveness:* $\forall \varphi, \psi$*, either* $\varphi \leq \varphi \wedge \psi$ *or* $\psi \leq \varphi \wedge \psi$
(*EE*4) *Minimality: when* $K \neq S^\perp$*,* $\varphi \notin K$ *iff* $\varphi \leq \psi$ *for all* ψ
(*EE*5) $\psi \leq \varphi$ *for all* ψ*, then* $\vdash \varphi$.

The relation $\leq$ implicitly depends on the initial epistemic state K. The five axioms are meant to capture normatively desirable properties, granting the basic intuition: ψ is more entrenched than φ if the individual has more confidence in ψ than in φ, *ceteris paribus*. (*EE*1) and (*EE*2) are easy to justify along this line, as is the condition of connectedness (which follows from (*EE*2) and (*EE*3)). (*EE*4) means that the set of propositions with the lowest e.e. degree is exactly the set of those propositions which are not believed in the initial epistemic state. (*EE*5) says that only tautologies have maximal e.e. (*EE*3) is clearly the most problematic of the lot. Along with (*EE*2), it implies the rather strong condition:

(*N*) $\forall \varphi, \psi$ either $\varphi \wedge \psi \sim \varphi$ or $\varphi \wedge \psi \sim \psi$.

The second A.G.M. account of e.e. consists in introducing a link between e.e. and contraction. The guiding idea is now: ψ is more entrenched than φ if the individual is more willing to give up φ than he is to give up ψ, when he is given the choice.

DEFINITION ($C \leq$) OF E.E. *Given a function K_φ^- define $\leq^-$ as the following binary relation on $L(VP)$:*

$\varphi \leq^- \psi$ *if either* $\varphi \notin K_{\varphi \wedge \psi}^-$ *or* $\vdash \varphi \wedge \psi$.

As was mentioned in Section 4.1 and is formally restated in the Proposition below, the current A.G.M. theory connects the two accounts of e.e. with each other. That is, the axiom set (*EE*1) to (*EE*5) is shown to bear an indirect relationship – through the ($C \leq$) definition – to the *contraction* operation. This fact suggests to devise an alternative axiom set (*EE*1′) to (*EE*5′) that would connect – through a suitably defined binary relation ($R \leq$) – with the *revision operation*.

ALTERNATIVE AXIOMS FOR E.E. *Define an e.e. relation to be any binary relation $\leq'$ on $L(VP)$ that satisfies:*

(*EE*1′) *Transitivity*
(*EE*2′) *Monotonicity*
(*EE*3′) *Disjunctiveness:* $\forall \varphi, \psi$*, either* $\varphi \vee \psi \leq' \varphi$ *or* $\varphi \vee \psi \leq' \psi$
(*EE*4′) *Alternative maximality: when* $K \neq S^\perp$*,* $\varphi \notin K$ *iff* $\neg \psi \leq' \neg \varphi$ *for all* ψ
(*EE*5′) *Alternative minimality: if* $\varphi \leq' \psi$ *for all* ψ*, then* $\vdash \neg \varphi$.

DEFINITION ($R \leq$) OF E.E. *Given a function K_φ^*, define $\leq^*$ as the following binary relation on $L(VP)$:*

$$\varphi \leq^* \psi \text{ iff either } \neg \psi \notin K_{\varphi \vee \psi}^* \text{ or } [\vdash \neg \varphi \text{ and } \vdash \neg \psi].$$

To motivate these new concepts: (*EE*4′) means that the set of propositions whose *negations* are maximally entrenched is exactly the set of those propositions which are not believed in the initial epistemic state. One should expect this liberal criterion to allow for *more* beliefs than the corresponding (*EE*4). (*EE*5′) says that only contradictions have minimal e.e. In the presence of (*EE*2′), (*EE*3′) implies that

(*P*) $\forall \varphi, \psi$ either $\varphi \vee \psi \sim \varphi$ or $\varphi \vee \psi \sim \psi$.

On the face of it, this rather demanding condition is no more and no less problematic than (*N*) above. As far as ($R \leq$) is concerned, the guiding idea is: ψ is more entrenched than φ if the individual is more willing to give up $\neg \psi$ than he is to give up $\neg \varphi$, when he is given the choice.

It is time to state the main facts of this section.

PROPOSITION (Gärdenfors, 1988, p. 96; Gärdenfors and Makinson, 1988, Theorem 5; Mongin, 1992, Observation 5). *If K_φ^- and K_φ^* satisfy the A.G.M. axioms for contraction and revision respectively, then*

(C ≤) and (R ≤) lead to epistemic entrenchment relations of types (EE1)–(EE5) and (EE1′)–(EE5′), respectively. Furthermore these epistemic entrenchment relations can be represented by numerical functions of the necessity and possibility types, respectively.

This proposition encapsulates a major result of the current A.G.M. theory: if K_φ^- satisfies the A.G.M. axioms for contraction, the $(C \leq)$ relation satisfies $(EE1)$–$(EE5)$. The most difficult part in this theorem is to establish $(EE1)$, which involves using the 'recovery axiom' of contraction. Once the theorem is granted it is not difficult to move to the statement that if K_φ^* satisfies the A.G.M. axioms for revision, the $(R \leq)$ relation satisfies $(EE1′)$–$(EE5′)$. This follows from some further facts of the A.G.M. theory and a duality argument. The remaining part of the proposition is derived routinely, using especially conditions (N), (P) and the fact that VP was taken to be denumerable.

The Nec and Pos functions that represent $\leq^-$ and $\leq^*$ have special features. $(EE4)$–$(EE5)$ and $(EE4′)$–$(EE5′)$ imply that:

(4)–(4′)   If $K \neq S^\perp$, φ is in K if and only if φ has positive necessity [respectively its negation $\neg\varphi$ has less than maximal possibility].

(5)–(5′)   Only tautologies have maximal necessity [respectively only contradictions have minimal possibility].

Now, using further results of the current A.G.M. theory it is possible to complement the above propositions with a converse statement:

PROPOSITION (Gärdenfors, 1988, p. 96; Gärdenfors and Makinson, 1988, Theorem 4; Mongin, 1992, Observation 6). *Suppose that* Nec *is a necessity function on* $L(VP)/\leftrightarrow$ *and satifies (4) and (5). Thus, one can construct a function* K_φ^- *on* $S \times L(VP)$ *that satisfies the A.G.M. axioms for contraction and is such that* Nec *represents* $\leq^-$. *Similarly, if* Pos *is a possibility function on* $L(VP)/\leftrightarrow$ *and satisfies (4′) and (5′), one can construct a function* K_φ^* *on* $S \times L(VP)$ *that satisfies the A.G.M. axioms for revision and is such that* Pos *represents* $\leq^*$.

4.3. The semantic exercise of this section is complete. Relying on the two-way theorem that Gärdenfors called his 'grande finale' I have stressed the close connection between the conformity with the contraction (revision) axioms and necessity (resp. possibility) reasoning. Taken together, the two propositions above have some remote analogy with a soundness and completeness theorem. They may be used in a variety of discursive strategies. One of them is to argue from the reasonableness of the A.G.M. system to the reasonableness of the necessity/possibility modelling of belief. This anti-Bayesian point seems to underlie some of Dubois and Prade's work (especially their 1991 paper). Here I shall explore the alternative strategy of arguing from the defects of Nec and Pos to an unnoticed problem in the A.G.M. theory.

One could indeed object to e.e. being representable by such highly specific non-additive probabilities as are Nec and Pos. Some properties of these functions lead to puzzling consequences. Take an individual who believes with certainty that it is equally likely that the coin will fall heads (H) and that it will fall tails (T). If we are to model this individual's belief by means of one of the functions under review, we *must* put Nec $(H) = $ Nec $(T) = 0$ or Pos $(H) = $ Pos $(T) = 1$. That is to say, the Nec and Pos functions deal with the certainty of equilikelihood as if it were *also* something else: complete ignorance of the result H or T of the tossing of the coin (in the Nec case), and complete confidence in either result (in the Pos case). Nec and Pos raise an interpretative issue which is (informally) opposite to the classic problem of equiprobability, as briefly reported in Section 2.

In brief one might well render the A.G.M. theory a bad service when stressing its connection with 'the theory of possibilistic reasoning'. But note emphatically that even if it were successful, the objection sketched above would lead to an ambiguous result. One would have to decide against *either* the A.G.M. axioms of belief change *or* against $(C \leq)$ and $(R \leq)$. The latter solution is much more attractive than the former, given the strong direct and indirect warrant of the contraction and revision axioms. Hence the research program: how should one modify the definition of contraction- and revision-induced e.e. relations in order to generate other functional representations than Nec and Pos?

A recent result by Wong *et al.* (1991) is to the point here. They solve the problem of axiomatizing *qualitative* belief structures. Specifically, they show that if Θ is finite and $\succ$ is an asymmetric, negatively transitive

binary relation on 2^{Θ}, there is a belief function representing $\succ$ if and only if

(a) $\succ$ is nondecreasing, i.e. $B \subseteq A \rightarrow$ Not $[B \succ A]$
(b) $\succ$ satisfies 'partial monotonicity', i.e. $[B \subset A, A \cap C = \emptyset, A \succ B] \Rightarrow [A \cup C \succ B \cup C]$.

[Thus far only limiting cases of belief functions had been given qualitative axiomatizations: probabilities of course (see Krantz, Luce, Suppes and Tversky, 1971, as well as the results by Suppes, 1969, and Suppes and Zanotti, 1976), necessities or possibilities (Dubois, 1988)].

This result would suggest trying the following *generalized* definition of e.e. ((*VP*) is henceforth taken to be finite.)

GENERALIZED AXIOMS FOR E.E. *Define an e.e. relation to be any binary relation* $\leq''$ *on* $L(VP)$ *that satisfies:*

(*EE1''*) *Transitivity.*
(*EE2''*) *Monotonicity.*
(*EE3''*) *Belief Structure: if* $\vdash \varphi \rightarrow \psi$, $\vdash \neg(\psi \wedge \chi)$ *and* $\varphi \vee \chi \sim \psi \vee \chi$, *then* $\varphi \sim \psi$.
(*EE4''*) *Minimality (as in the A.G.M. axioms).*
(*EE5''*) *Maximality (as in the A.G.M. axioms).*
(*EE6''*) *Completeness.*

The following observations are readily made. (i) Define $\psi \,''\!> \varphi$ as [not $\psi \leq'' \varphi$]. Then it is equivalent to assume the generalized axioms on $\leq''$ or to assume that $''\!>$ has a qualitative belief structure in the sense of Wong *et al.* (ii) The A.G.M. axioms for e.e. imply the generalized axioms. Note that the weakening of (*EE2*)–(*EE3*) into (*EE2''*)–(*EE3''*) involves the loss of a useful consequence – the completeness property; this explains the added axiom (*EE6''*). Now, in view of observation (i) we can avail ourselves of Wong's theorem and conclude that the generalized axioms imply a representation of e.e. $\leq''$ by a belief function Bel. Exactly as Nec before, such a Bel should satisfy the numerical properties corresponding to (*EE4''*) and (*EE5''*).

The remaining, much more difficult step would then be to devise a novel definition of e.e. in terms of contraction that would generalize $(C \leq)$ and imply axioms (*EE1''*) to (*EE6''*). If this succeeded, one would have carried out the project of keeping the A.G.M. theory of contraction while modifying their notion of epistemic entrenchment in

order to derive a semantic counterpart of the A.G.M. logic in terms of Shafer's belief functions.

5. CONCLUSIONS

From the survey of Section 2 some interpretations of the D.S. theory have emerged as particularly relevant. I repeat them here while slightly modifying the initial taxonomic principle:

α: possibility functions can be used to formalize the individual's reasoning with intrinsically imprecise concepts ('John is tall');
β: the basic probability assignment formalizes natural properties of the weight of evidence and this derivatively provides belief functions and plausibilities with psychological significance (the Mrs. Jones' example);
γ: generalized D.S. functions should be used to solve the problem of the measurement of belief (Suppes's 1974 interpretation);
δ: belief functions and plausibilities naturally occur in the following statistical context: the individual has evidential data of a probabilistic sort and *either* these data are incomplete (the medical example) *or* he is asked to make statistical statements on a space different from the sample space (the typical application of Dempster's construction) *or* the data are known to be drawn from several probability distributions (the dice example).

Roughly speaking points α, β and γ relate to supposedly natural properties of the individual's belief system, hence are of crucial relevance to epistemology and AI; while δ underlies the statistician's case for the D.S. theory. But there is of course an overlap: the AI literature is *also* interested in the statistical point of view (as evidenced by some of the discussions of Dempster's construction in this literature). An important feature of the above list is that not one of its items really supports the view that the D.S. theory is a universally applicable theory of belief in the sense in which Bayesianism is. This comment would even apply to the list taken as whole. It describes *areas of application* at the same time as ways of understanding the formal concepts. It is not implied that these areas taken together exhaust the application domain of the theory of uncertain belief; so that there could well remain a significant scope for Bayesianism. In sum there seems to be a built-in eclecticism in the D.S. theory. The surveys by Smets (1988) and Dubois and Prade

(1991) go a long way in this direction. I also think that this conclusion should be congenial to Suppes. A recurrent complaint in his work is that the current rationality theories, especially Bayesianism, are exceedingly 'simple and general' (e.g. 1981, p. 60).

An application of this point is Suppes's critique of the indiscriminate use of Bayes's rule to explain learning. In 1966 he argued that there is no satisfactory account of concept formation along this line. At the other extreme, as it were, some learning processes involve acquiring coordinations of movements and perceptions and it strikes one as rather artificial to explain these processes in terms of Bayesian revisions. Suppes's interest in Markov models of learning was no doubt prompted by his view that the domain of application of Bayesianism is much more limited than is usually recognized by the proponents of this doctrine. I am claiming the benefit of this critique also in the *static* context of Bayesianism and praising in contrast the lack of arrogance of D.S. theorists.

There is much more to say about the Bayesian's exclusive reliance on probabilities. In Bogdan (1979, p. 21) Suppes writes that there are

three fundamental problems underlying probability theory. One is the addition of probability for mutually exclusive events, the second is the concept of independence of events and random variables, and the third is the concept of randomness.

Each of these three issues can be discussed *per se*, irrespective of what Bayesians claim, but it is also clear that each could uncover a possible objection against them. To start with the third point, Suppes's *Probabilistic Metaphysics* (1984) argues from examples in physics, and even in quantum mechanics, that subjective ignorance is not the only source of randomness. The second point implicitly refers to de Finetti's analysis of independence by means of the exchangeability concept (1937). The problem mentioned in the first instance was of course at the center of the present paper. Suppes's own critique of additivity may now be summarized.

For one, he has always been concerned with the problem of finding a suitable *qualitative* equivalent to the additivity axiom of probability theory, after de Finetti's all too simple solution was declared a failure. The problem is important in Suppes's eyes because of the methodological conviction underlying his measurement theory: any numerical concept used for measurement purposes – then probabilistic belief as well as weight or mass – should be explained in terms of an n-ary relation and, if possible, of *finite* structures. (This methodology, of course, pervades

the jointly authored volume by Krantz *et al.*, 1971. It is usefully surveyed in Luce, 1979.) As I mentioned in Section 2, Suppes's 1969 theorem offers a solution, but it depends on a strong structural axiom. An alternative approach was to change the initial object of interest and to axiomatize probability in terms of extended indicator functions, as in the joint work with Zanotti.

For another, the additivity issue is – conceptually at least – connected with the *uniqueness* problem of probability. As is well known, Savage (1954) derives a unique probability from his (structurally-laden) axioms. For Suppes, this result virtually counts as a *reductio ad absurdum* of Bayesianism, at least in Savage's version. For belief is like any quantity to be measured – it can only be assigned an approximate value:

Dans la théorie de Savage il y a quelque chose d'effectivement paradoxal dans l'accent mis sur le caractère d'exactitude de la distribution de probabilité . . . En comparant cette situation avec celle qui prévaut dans le cas de la mesure physique, nous pouvons voir l'absurdité de cette exigence de résultats exacts. Il nous paraîtrait impensable d'exiger d'une théorie physique qu'elle soit fondée sur des mesures exactes de la masse ou de la position (1981, p. 73).

This strongly expressed methodological view seems the main heuristic underlying Suppes's weakening of additivity in his 1974 construction.

Having argued that there is much common ground between the D.S. critique of Bayesianism and Suppes's, I should hasten to add that there is another tenet of the doctrine to which he appears to adhere wholeheartedly: rationality is best understood in terms of rational *actions* (e.g. 1981, p. 11). This view is of course typical of 20th-century Bayesianism and it may even underlie several passages of Thomas Bayes's *magnum opus* (see the quotations discussed in 1981). It is by no means shared by all D.S. theorists, especially in the field of AI. To avoid any misunderstanding it should be clear that the point of the dissent is not ontological. That is, there is no question of whether or not "belief can exist outside any decision or betting context", as a distinguished member of the AI community has it (Smets, 1988, p. 254). The question is rather the methodological one of the significance and indispensability of an account of belief functions in terms of decision schemes. Writing this, I hope to capture the antireductionist, primarily methodological essence of Suppes's behaviorism.

There is a related, but distinctive theme in Suppes's analysis of probability to which I cannot do full justice here: the concept of mathematical expectation is in some sense more important than that of probability

(e.g. 1981, pp. 16, 25). This is a related theme, of course, if only because it underlies much of the subjectivist literature (for instance, de Finetti's 1937 definition of probability as an exchange rate between two sums of money). But this theme has also a distinctive epistemological flavor. Even if one does not believe in the analytical importance of betting schemes, one may ask, as Suppes does boldly, whether or not probability should be regarded as a primitive concept of the theory of knowledge. He goes as far as to suggest that the repeated attempts in measurement theory to axiomatize qualitative probability by reference to *events* are misdirected (in Bogdan, 1979, p. 21). By contrast, his work with Zanotti (1976, 1977, 1989) starts with various relations defined on *extended indicator functions*, i.e. sums of indicator functions. The question arises, then, of whether or not Suppes regards this axiomatic work as more fundamental than his altogether different axiomatization of the 'measurement of belief'. Importantly, Suppes extends the former line of research in the direction of the D.S. theory, too.

There are two further comments that I would like to excerpt from Suppes's discussion of Bayesianism because they connect with the logical exercise of this paper. At one place (1981, p. 42) he complains about the Bayesian's inability to make room for unknown probabilities. This objection seems to be the same as that of the D.S. theorist when he claims that probabilities cannot accommodate situations of extreme uncertainty whereas belief functions can. I have tried to clarify the logical basis of this claim by comparing the modal logics *KD* and *KD!* in Section 3. At another point (1981, p. 44) Suppes objects to Bayes's rule on the grounds that the likelihood of the data is not always easy to assess. From this point of view it is a definite advantage of the A.G.M. logic to replace the Bayesian duality of prior probability and likelihood with a *single* concept of a priori belief: epistemic entrenchment. This may be another motivation for pursuing much further the exploration of Section 4.

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COMMENTS BY PATRICK SUPPES

Philippe Mongin's long and detailed paper is full of more things than I can comment on with care but he raises so many issues that it is easy to select a few fundamental topics to discuss.

Bayesianism. He catches very well my own skepticism toward many aspects of Bayesianism, in spite of my generally sympathetic and appreciative recognition of what a Bayesian viewpoint toward belief offers. I do want to make one positive Bayesian point that I do not think Philippe emphasizes enough. This is that the Bayesian view of statistics is now playing a conceptual and increasingly practical role, but the same cannot be said for the Dempster–Shafer theory which he labels the theory of nonadditive probability. The fundamental critique of a purely objective view in statistics has been carried out almost entirely by Bayesians. They have gone on to make detailed proposals as to how standard statistical methods of analysis can be modified so as to fall within a Bayesian framework. Many different contributions taken together have shown how natural it is to look upon a belief viewpoint as the proper way to think about statistics in general. The awkwardness of computing any complicated statistics in the Dempster-Shafer theory, on the other hand, suggests that without substantial modification there is little hope of it coming to play the same role in statistics that the modern Bayesian viewpoint does.

Skepticism about the Dempster–Shafer Theory. In making the criticism I just did of the Dempster–Shafer theory I want to emphasize that I see a natural place for two kinds of theory of belief, or perhaps I should say *at least* two kinds of theory of belief. It is natural in scientific context to have a theory very much like the modern Bayesian one, which fits in naturally with detailed scientific experimentation or analysis of data which may be nonexperimental but is substantial and naturally quantitative in character. On the other hand, most of our personal ruminations about belief are not of this character. They are qualitative, ill-formed, and often hastily put together on the spot to meet the needs of a particular real-time problem at hand. For this latter purpose, the direction taken by the Dempster–Shafer theory is a natural one, and one that I heartily support. The awkwardness of computations, the difficulties of making connections with methods of statistical analysis that continue to work well, or the need to fit into detailed scientific programs of experimentation are not at all fundamental criticisms.

My skepticism about the Dempster–Shafer theory runs in the opposite direction. It is too conservative and too close to the kind of thing needed for scientific practice. I mention in my comments on Isaac Levi's paper that I would favor at the least a weakening of the Dempster–Shafer

theory to the existence of only an upper probability. I also want to mention *en passant* that for many reasons it seems better to use an upper probability than a lower probability just because conditionalization will work much more smoothly.

In any case, I urge nonmonotonic upper probabilities as a better general framework because they permit the holding of incoherent beliefs, which still seem natural and hard to avoid once any large number of concepts or variables are considered. For example, most of us probably hold incoherent beliefs about the pairwise correlations of large numbers of concepts, and certainly most physicians probably hold incoherent beliefs about the correlations of various symptoms with various diseases and we need to have an apparatus that naturally expresses these beliefs for a starting point. Of course, I do not think that even a weak nonmonotonic upper probability is a Holy Grail of belief theory, and something weaker still may be needed to express widespread aberrations that characterize our natural belief structures from the standpoint of a stricter Bayesian or even Dempster–Shafer viewpoint.

Epistemic Logic. Mongin gives a clear and detailed survey of current results in epistemic logic. I would only comment here that considering their weakness as logical systems, I find that the epistemic logics tend to be too normative in spirit. As Philippe remarks and as I agree, they seem to require deductive closure, as the A.G.M. revision axioms do, and the A.G.M. axioms for epistemic entrenchment, in particular axiom *EE*3 on conjunctiveness, seem much too strong for fallible human entrenchment. In other words, it seems to me desirable to explore psychologically realistic rather than strong normative axioms as the appropriate qualitative approach to epistemic entrenchment.

My more skeptical and more serious criticism, however, is that the apparatus of propositional logic is too weak to catch anything like the full sense of what we want from a notion of epistemic entrenchment. It is rather like asking us to do physics with only propositional logic as our resource. We hold subtle and complicated views about many different phenomena. The many puzzling and complex results about human perception which have been so thoroughly explored experimentally by psychologists throughout this century alone provide data for a much more complicated and rich theory of entrenchment than has yet been proposed.

Apart from perception, the second aspect of entrenchment that seems fundamental to our own psychological experience is the entrenchment that comes from learning, broadly conceived. Surely a theory of learning is required in any theory of entrenchment and revision of entrenchment. Again, no serious theory of learning can be expressed purely within a propositional logic, even one enriched in the ways that Mongin describes.

Finally, I want to make a remark about the axiomatization of qualitative belief structures by Wong *et al.* (1991), which Mongin mentions. The positive result is very nice. Using axioms very similar to de Finetti's original qualitative axioms for probability, they are able to prove that their closely-related five axioms are necessary and sufficient for the existence of a lower probability, more particularly, a lower probability that satisfies the capacity restriction for a belief function in the sense of Shafer. However, it also needs to be pointed out that the result of Wong *et al.* is essentially that to be found in Walley and Fine (1979) and discussed in some detail in Fine's article in the present volume.

What neither Wong *et al.* nor Mongin makes clear is the fact that the results are still not complete, for although a representation theorem is proved, there is no corresponding result about the uniqueness of the representation. As I have said on several occasions and also in remarks and comments on various papers in these volumes, this is of course the difficulty with the Kraft, Pratt and Seidenberg (1959) or Scott (1964) necessary and sufficient conditions for a qualitative comparative relation on a finite set to have a strictly agreeing probability measure. Uniqueness is, to put it bluntly, a mess. For real quantitative work one needs a very restricted form of uniqueness that is for probability, most desirably, absolute uniqueness, and if not that, something like a ratio scale.

For me, an interesting problem is to try to extend the Dempster–Shafer axiomatization by the use of elementary random variables, as in Suppes and Zanotti (1976), to obtain a unique representation or to understand why such an approach cannot be made to work. In this respect, it is also to be noted that Mongin does not discuss the complex problems of having an adequate theory of expectation to go with Shafer belief functions, a matter discussed with some care again in Terry's article.

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