Social Preference Under Twofold Uncertainty*

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Abstract

We investigate the conflict between the *ex ante* and *ex post* criteria of social welfare in a new framework of individual and social decisions, which distinguishes between two sources of uncertainty, here interpreted as being objective and subjective respectively. This framework makes it possible to endow the individuals and society not only with *ex ante* and *ex post* preferences, as is usually done, but also with *interim* preferences of two kinds, and correspondingly, to introduce *interim* forms of the Pareto principle. After characterizing the two social welfare criteria, we present two compromises between them, one based on the *ex ante* criterion and absorbing as much as possible of the *ex post* criterion (Theorem 1), the other based on the *ex post* criterion and absorbing as much as possible of the *ex ante* criterion (Theorem 2). Both solutions translate the assumed Pareto conditions into weighted additive utility representations, as in Harsanyi's Aggregation Theorem, and both attribute to the individuals common probability values on the objective source of uncertainty, and different probability values on the subjective source. We discuss these solutions in terms of the by now classic *spurious unanimity* argument and a novel informational argument labelled *complementary ignorance*. The paper complies with the standard economic methodology of basing probability and utility representations on preference axioms.

Keywords: *Ex ante* social welfare; *ex post* social welfare; objective *versus* subjective uncertainty; objective *versus* subjective probability; Pareto principle; separability; Harsanyi social aggregation theorem; spurious unanimity; complementary ignorance.
JEL classification: D70; D81.

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1 Introduction

Any normative analysis of collective decisions under uncertainty must confront an old and unresolved problem: the conflict between the *ex ante* and *ex post* criteria of social welfare. This paper proposes new solutions to this problem, which are based on a distinction between two sources of uncertainty. In our framework, agents may hold different beliefs about one source while holding the same beliefs about the other. Before explaining what difference this twofold uncertainty makes, we restate the conflict in its classical form.

The *ex ante social welfare criterion* assumes that the individuals form preferences over social uncertain prospects according to some decision theory – typically that of subjective expected utility (SEU) – and it applies the Pareto principle to these *ex ante* individual preferences, thus following an *ex ante* version of the principle. In contrast, the *ex post social welfare criterion* assumes that society itself forms preferences over social prospects according to the normative decision theory under consideration, while it endows the individuals only with state-by-state preferences. It then applies the Pareto principle statewise to these *ex post* individual preferences, thus following an *ex post* version of the principle.[1] If all agents satisfy the axioms of subjective expected utility, or even weaker axioms, then the *ex ante* and *ex post* criteria are generally incompatible.

This conflict has long been recognized, although the problem has been formulated in several different ways. The early statements by Starr (1973) and Hammond (1981, 1983) belonged to traditional welfare economics, and envisaged only two extreme solutions, i.e., endorsing one of the two criteria and rejecting the other. Mongin's (1995) abstract formulation in terms of Savage's (1972) SEU postulates avoided the domain-specific assumptions made by the welfare economists, thus sharpening the conflict, while also showing that probability or utility dependencies between the individuals could alleviate it. This axiomatic approach also facilitated comparison with Harsanyi's (1955) Social Aggregation Theorem, which famously says that, if both individuals and society form their preferences over social lotteries according to von Neumann-Morgenstern (VNM) theory, and the social preferences satisfy the Pareto principle, then society's preferences can be represented by a weighted ("utilitarian") sum of individual utility representations.[2] As the Pareto principle applies here both *ex ante* (to lotteries) and *ex post* (to final outcomes as a degenerate case of lotteries), Harsanyi's assumptions contain all the ingredients of the two welfare criteria, and his weighted sum formula seems to contradict the claim that the two criteria are incompatible. However, the assumption of a common lottery set amounts to imposing identical probabilities on the individuals and society, an extreme case of those probabilistic dependencies which make the criteria compatible.

This paper will also exploit the fact that the conflict between them vanishes when probabilities are identical, but in a much more general fashion than Harsanyi. We also derive weighted sum representations of social utility, but depart from his simplifying probabilistic

[1]Note the difference between a social welfare *criterion* and the corresponding *Pareto principle.* There is more to the the *ex ante* (*ex post*) social welfare criterion than just the *ex ante* (*ex post*) Pareto principle, because a criterion also decides where rationality assumptions apply (to the individuals or society).

[2]The utilitarian interpretation of Harsanyi remains controversial; see Sen's (1986) objections and the discussion in Weymark (1991). Fleurbaey and Mongin (2016) propose a new defense of Harsanyi's position.

treatment. Our individuals have probabilistic beliefs that differ on one source of uncertainty and are identical on the other, while society, if it entertains probabilistic beliefs at all, adopts those of the individuals when they are identical. This contrast emerges only at the level of representation theorems, because the paper follows the standard methodology of taking preferences to be the only axiomatic primitives. Thus, our treatment generalizes Harsanyi's in two ways, i.e., both by reducing the scope of his common probability assumption and by endogenizing it.

Semantically, we distinguish the two sources of uncertainty as being *objective* and *subjective* according to whether the endogenous probabilities are identical or not - hence the labels $\mathcal{O}$ and $\mathcal{S}$ we later use for their respective state spaces. This is but one interpretation of the results, but we can defend it as follows. If preferences are the only primitives, it is impossible to differentiate between the forms of uncertainty in terms of exogeneous probabilitic information, and what the endogeneous probabilistic information reveals is only whether probabilities are identical or not. Then, either one should drop the distinction between objective and subjective uncertainty, or one should base it on what is available. Since the distinction seems well established in the ordinary language of economics, we opt for the former solution and thus say that uncertainty is *objective* when both the individuals and society assign identical probabilities, and *subjective* when the individuals assign different probabilities and society leaves this diversity unregulated.

This endogenous distinction is actually foreshadowed at the level of preference axioms. A key step is the introduction of *conditional preferences*: for both society and the individuals, we posit preferences conditional on $\mathcal{O}$ and preferences conditional on $\mathcal{S}$, each obeying distinctive properties. These properties will eventually determine whether or not probabilistic beliefs exist for society, the individuals or both, and whether or not such probabilistic beliefs are shared between those who entertain them. By introducing conditional preferences, we are also able to consider new, *interim* forms of the Pareto principle, in addition to the classic *ex ante* and *ex post* forms. By simultaneously varying the forms of the Pareto principle and the properties of conditionals, we obtain a rich set of possibilities.

The paper explores these possibilities to discover new compromises between the *ex ante* and the *ex post* social welfare criteria. The most interesting ones are those which capitalize on one of the two criteria and absorb as much of the contents of the other criterion as is possible without returning the classic conflict between them. There are two such optimal compromises in the paper, which are characterized in our two main results.

Theorem 1 combines the *ex ante* criterion in full with a partial version of the *ex post* criterion, which is limited to the objective source of uncertainty. At the level of representations, all individuals and society assign identical probabilities to $\mathcal{O}$, but regarding $\mathcal{S}$, individuals can assign different probabilities, while society contents itself with recording preferences without forming probabilities of its own. This solution makes society's *ex post* preferences state-dependent – i.e., the *ex post* social solution varies according to which state is realized – a flexibility that several writers have already judged attractive in the standard framework of a single source of uncertainty. By contrast, Theorem 2 encapsulates the *ex post* criterion in full and a partial version of the *ex ante* criterion, which replaces the *ex ante* Pareto principle by the *interim* form relative to the objective source of uncertainty. At the level of representations, it is still the case that all individuals and society assign identical probabilities on $\mathcal{O}$, while individuals can assign different probabilities to $\mathcal{S}$. However, this time, society forms probabilities on $\mathcal{S}$; importantly, these are unrelated to those of the individuals. This solution calls for a comparison with the influential one defined by Gilboa et al. (2004) for the standard framework of uncertainty.

Besides axiomatizing these two compromises, the paper attempts to assess them conceptually. Two main critical arguments are involved, i.e., the *spurious unanimity objection*, which was first developed by Mongin (1997) and has become fairly well accepted today, and the *complementary ignorance objection*, which we introduce in this paper. Our *ex ante*-based compromise (Theorem 1) avoids the complementary ignorance objection, while falling prey to the spurious unanimity objection, as does any solution that retains the *ex ante* Pareto principle; however, this defect can be traded off against the advantage of having a state-dependent *ex post* social welfare function at one's disposal. The compromise of Gilboa et al. (2004) was devised to eschew the spurious unanimity objection, but turns out to be open to the complementary ignorance objection, and this also holds of various recent proposals. By contrast, our *ex post*-based compromise (Theorem 2) avoids *both* objections at the same time, which is a reason for preferring it to its predecessors. More specifically, Gilboa, Samet and Schmeidler's representation theorem turns society's probabilities into a weighted sum of the individual probabilities, whereas, in ours, the two kinds of probabilities remain unconnected. The "linear pooling rule", as it is called in the statistics and management literatures, is the implicit target of our complementary ignorance objection.[3]

The paper is divided as follows. Section 2 introduces the framework, and the various decision-theoretic and Pareto conditions. As a preliminary for our more original results, Section 3 axiomatizes the *ex ante* social welfare criterion (Proposition 1), the *ex post* social welfare criterion (Proposition 2), and illustrates their conflict (Proposition 3). Section 4 proposes our first optimal compromise (Theorem 1), based on the *ex ante* criterion, and Section 5 our second optimal compromise (Theorem 2), based on the *ex post* criterion. Corollaries 1 and 2 show that reinforcing the assumptions of either theorem would reproduce the conflict of Proposition 3. Section 6 describes the spurious unanimity and complementary ignorance problems, and discusses our solutions as well as that of Gilboa et al. (2004) in this light. Section 7 reviews recent literature also in this light. The more technical material, including the proofs of the results, appear in two appendices. The results of this paper primarily depend on the mathematics of separable preference orders. Appendix A restates the concepts and general theorems from this theory (some well-known, others our own work); Appendix B uses these tools to prove our results.

2 The framework

Uncertain prospects. We assume that states of the world are pairs (s, o), where $s \in \mathcal{S}$ and $o \in \mathcal{O}$ represent two distinctive sources of uncertainty. We often refer to them as the

[3]The earlier literature on the linear pooling rule is vast; see among others Genest and Zidek (1986) and Clemen and Winkler (2007).

subjective and *objective* source —but this is merely a terminological convention until we reach our main results (Theorems 1 and 2). We often identify the events $\{s\} \times \mathcal{O}$ and $\mathcal{S} \times \{o\}$ with the states s and o; similar routine identifications will occur later in the paper.

We assume that $\mathcal{S}$ and $\mathcal{O}$ are finite with $|\mathcal{S}|, |\mathcal{O}| \geq 2$. Let $\Delta_{\mathcal{S}}$ and $\Delta_{\mathcal{O}}$ be the sets of probability vectors on $\mathcal{S}$ and $\mathcal{O}$, respectively. We assume that the individuals i belong to a finite set $\mathcal{I}$ with $|\mathcal{I}| \geq 2$, and that each individual i and society face uncertain prospects. These can be *completely uncertain* (when both s and o are unknown), *subjectively uncertain* (o is fixed and s is unknown), or *objectively uncertain* (s is fixed and o is unknown). We refer to prospects in the last two classes as *interim* prospects. "Learning s" means observing the event $\{s\} \times \mathcal{O}$. Likewise, "learning o" means observing the event $\mathcal{S} \times \{o\}$.

We think of prospects as mappings from states of the world to consequences, and express consequences directly in terms of payoff numbers x^i_{so} for the individuals. For all $i \in I$, $s \in S$, $o \in O$, we assume that x^i_{so} varies across $\mathbb{R}$; these domain assumptions are discussed at the end of the section. We leave it for the interpretation to decide whether the x^i_{so} numbers represent physical payoffs (levels of consumption in a good) or subjective payoffs (utility values in some metric). We define a *completely uncertain* prospect:

- in the case of an individual $i \in \mathcal{I}$, as a matrix $\mathbf{X}^i = (x^i_{so})_{s \in \mathcal{S}, o \in \mathcal{O}} \in \mathbb{R}^{\mathcal{S} \times \mathcal{O}}$,
- in the case of society, as a three-dimensional array, $\mathbb{X} = (x^i_{so})^{i \in \mathcal{I}}_{s \in \mathcal{S}, o \in \mathcal{O}} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}}$.

We define an *objectively uncertain* (respectively, *subjectively uncertain*) prospect:

- in the case of an individual $i \in \mathcal{I}$, as any vector $\mathbf{x}^i_s = (x^i_{so})_{o \in \mathcal{O}} \in \mathbb{R}^{\mathcal{O}}$ for some fixed $s \in \mathcal{S}$ (respectively, $\mathbf{x}^i_o = (x^i_{so})_{s \in \mathcal{S}} \in \mathbb{R}^{\mathcal{S}}$ for some fixed $o \in \mathcal{O}$),
- in the case of society, as any matrix $\mathbf{X}_s = (x^i_{so})^{i \in \mathcal{I}}_{o \in \mathcal{O}} \in \mathbb{R}^{\mathcal{I} \times \mathcal{O}}$ for some fixed $s \in \mathcal{S}$ (respectively, $\mathbf{X}_o = (x^i_{so})^{i \in \mathcal{I}}_{s \in \mathcal{S}} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S}}$for some fixed $o \in \mathcal{O}$).

When uncertainty is completely resolved, an individual i faces a scalar x^i_{so}, while society faces a vector $\mathbf{x}_{so} = (x^i_{so})^{i \in \mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$.

Preferences. We assume that the individuals and society assess completely uncertain prospects in terms of *ex ante* preference relations, denoted by $\succsim^i$ for $i \in \mathcal{I}$ and $\succsim$ for society; these are our only preference primitives. *Throughout the paper, we take $\succsim^i$ and $\succsim$ to be continuous weak orders*, thus representable by continuous real-valued utility functions.

The other preference relations of this paper are *conditional relations* induced by either the $\succsim^i$ or $\succsim$. There are six of them to consider: $\succsim^i_s$, $\succsim^i_o$ and $\succsim^i_{so}$ for individual i, and $\succsim_s$, $\succsim_o$ and $\succsim_{so}$ for society. While $\succsim^i_{so}$ and $\succsim_{so}$ make *ex post* comparisons, $\succsim^i_s$, $\succsim_s$, $\succsim^i_o$ and $\succsim_o$ make *interim* comparisons, which are specific to the present framework. As usual, conditional preferences are defined by restricting unconditional preferences to prospects that vary only along the component of interest. It suffices to illustrate this definition by two cases (see Appendix A for more formalism). For all $\mathbf{x}^i_s, \mathbf{y}^i_s \in \mathbb{R}^{\mathcal{O}}$, the conditional $\succsim^i_s$ is defined by:

> $\mathbf{x}_s^i \succsim_s^i \mathbf{y}_s^i$ iff $\mathbf{X}^i \succsim^i \mathbf{Y}^i$ for some $\mathbf{X}^i, \mathbf{Y}^i \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ such that $\mathbf{x}_s^i$ and $\mathbf{y}_s^i$ are the (vector-valued) s-components of $\mathbf{X}^i$ and $\mathbf{Y}^i$, while $\mathbf{X}^i$ and $\mathbf{Y}^i$ coincide on all other components.

(Definitions for $\succsim_o^i$, $\succsim_o$ and $\succsim_s$ follow *mutatis mutandis.*) For all $\mathbf{x}_{so}, \mathbf{y}_{so} \in \mathbb{R}^{\mathcal{I}}$, the conditional $\succsim_{so}$ is defined by:

> $\mathbf{x}_{so} \succsim_{so} \mathbf{y}_{so}$ iff $\mathbb{X} \succsim \mathbb{Y}$ for some $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$ such that $\mathbf{x}_{so}$ and $\mathbf{y}_{so}$ are the (vector-valued) (s, o)-components of $\mathbb{X}$ and $\mathbb{Y}$, while $\mathbb{X}$ and $\mathbb{Y}$ coincide on all other components.

Properties of conditionals. The conditional relations defined above are complete, but not necessarily transitive. To make them so, a *separability* condition must be added. This says in effect that the postulated comparisons between the conditional and the source relation do not depend on which alternatives are chosen, provided these alternatives coincide outside the components of the conditional. To illustrate with the first example, $\succsim^i$ is said to be *separable in s* if, for all $\mathbf{X}, \mathbf{Y}, \widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$,

if $\mathbf{x}_s = \widetilde{\mathbf{x}}_s$ and $\mathbf{y}_s = \widetilde{\mathbf{y}}_s$, while $\mathbf{X}$ coincides with $\mathbf{Y}$, and $\widetilde{\mathbf{X}}$ with $\widetilde{\mathbf{Y}}$, on the components other than s, then the following holds:

$$\mathbf{X} \succsim^i \mathbf{Y} \text{ iff } \widetilde{\mathbf{X}} \succsim^i \widetilde{\mathbf{Y}}.$$

If $\succsim^i$ is separable in s, then $\succsim_s^i$ is transitive, hence a *bona fide* ordering. The converse also holds. Appendix A restates this basic fact in general form.

The conditional relations $\succsim_{so}^i$ compare real numbers, and we will assume that they agree with the natural ordering of these numbers. That is, for all $(s, o) \in \mathcal{S} \times \mathcal{O}$, all $i \in \mathcal{I}$ and all $x_{so}^i, y_{so}^i \in \mathbb{R}$,

$$x_{so}^i \succsim_{so}^i y_{so}^i \quad \text{if and only if} \quad x_{so}^i \geq y_{so}^i. \tag{1}$$

This is consistent with the payoff interpretation of these numbers, and it automatically makes the $\succsim_{so}^i$ transitive.

For all other conditional relations, we do *not* generally assume transitivity (or the equivalent property of separability). Our results crucially depend on selecting which conditionals are transitive. For ease of expression, when a conditional has this property, whether by way of assumption or by way of conclusion, we say that its source relation *induces a preference ordering*. Thus, "$\succsim^i$ induces an interim preference ordering $\succsim_s^i$" means that $\succsim_s^i$ is transitive, or equivalently, that $\succsim^i$ is separable in s; and similarly with the other cases. We always make such transitivity assumptions uniformly across the uncertainty type. That is, we take the ordering property of conditionals to hold either either for all s or for none, either for all o or for none, either for all (s, o) or for none. For example, we will simply say, "$\succsim^i$ induces interim preference orderings $\succsim_s^i$" without adding the implied "for all s in $\mathcal{S}$".[4]

When $\succsim^i$ or $\succsim$ induces conditional preference orderings of some uncertainty type, we may, by a separate decision, require that these preferences be identical across the given

[4] The requirement that separability conditions hold across the uncertainty type means that they are equivalent to *dominance conditions* for the given type.

type, in which case we say that they are *invariant*. That is, we may assume not only that $\succsim^i$ induces interim preference orderings $\succsim^i_s$, but also that $\succsim^i_s = \succsim^i_{s'}$ for all $s, s' \in \mathcal{S}$; not only that $\succsim$ induces *ex post* preference orderings $\succsim_{so}$, but also that $\succsim_{so} = \succsim_{s'o'}$ for all $s, s' \in \mathcal{S}$, and all $o, o' \in \mathcal{O}$, etc. For any i in $\mathcal{I}$, $\{\succsim^i_{so}\}_{(s,o)\in\mathcal{S}\times\mathcal{O}}$ are invariant, by statement (1).

The meaning of these invariance assumptions should be clear. For instance, if $\succsim^i$ induces invariant interim $\succsim^i_s$ preferences, then the resolution of uncertainty about s has no influence on i's preferences over interim prospects that depend on o, hence for all decision purposes, i regards s as being *uninformative* about o. If $\succsim$ induces invariant *ex post* preferences $\succsim_{so}$, then society has state-independent *ex post* preferences.

For suitable ordering and invariance assumptions put on the conditionals, our framework give rise to SEU representations of individual preferences, social preferences, or both of them. The theorems and propositions below make the existence and uniqueness properties of these representations part of a wider set of conclusions, but we have extracted the underlying representation theorem as Proposition A2 in Appendix A. It could be compared with the founding results of Savage (1972) and Anscombe and Aumann (1963), but we do not pursue this comparison here; see Mongin and Pivato (2015) and Mongin (2017).[5]

As some assumptions relative to these conditionals recur in the formal statements below, it will be convenient to have labels for them.

- Assumption (A1) (resp. (B1)) says that for each individual i (resp. for society), the preference ordering $\succeq^i$ (resp. $\succeq$) induces interim preference orderings $\succeq^i_s$ (resp. $\succeq_s$) for all $s \in \mathcal{S}$, and also interim preference orderings $\succeq^i_o$ (resp. $\succeq_o$) for all $o \in \mathcal{O}$.

- When (A1) (resp. (B1)) holds, Assumption (A2) (resp. B2)) says that these interim preferences are moreover *invariant* —i.e. independent of the s and o indices.

Pareto conditions. In the standard framework, social preference under uncertainty is subjected to Pareto conditions defined either *ex ante* or *ex post*. But our framework yields more options. Here, the *ex ante* condition applies to completely uncertain prospects, the *ex post* condition applies to fully resolved prospects, and two newly defined *interim* conditions apply to objectively uncertain prospects and subjectively uncertain prospects. These are formally defined as follows.

- *Ex ante* Pareto principle: for all $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$: if $\mathbf{X}^i \succeq^i \mathbf{Y}^i$ for all $i \in \mathcal{I}$, then $\mathbb{X} \succeq \mathbb{Y}$; if, in addition, $\mathbf{X}^i \succ^i \mathbf{Y}^i$ for some $i \in \mathcal{I}$, then $\mathbb{X} \succ \mathbb{Y}$.

- *Ex post* Pareto principle: for all $(s, o) \in \mathcal{S} \times \mathcal{O}$, and all $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^{\mathcal{I}}$: if $x^i \geq y^i$ for all $i \in \mathcal{I}$, then $\mathbf{x} \succeq_{so} \mathbf{y}$; if, in addition, $x^i > y^i$ for some $i \in \mathcal{I}$, then $\mathbf{x} \succ_{so} \mathbf{y}$;

- *Objective interim* Pareto principle: for all $s \in \mathcal{S}$, and all $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}$: if $\mathbf{x}^i \succeq^i_s \mathbf{y}^i$ for all $i \in \mathcal{I}$, then $\mathbf{X} \succeq_s \mathbf{Y}$; if, in addition, $\mathbf{x}^i \succ^i_s \mathbf{y}^i$ for some $i \in \mathcal{I}$, then $\mathbf{X} \succ_s \mathbf{Y}$;

[5] Briefly: despite finiteness assumptions that resemble Anscombe and Aumann's, our theorems follow Savage by relying on a pure uncertainty framework, and eschewing any probabilistic primitives.

- *Subjective interim* Pareto principle: for all $o \in \mathcal{O}$, and all $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S}}$: if $\mathbf{x}^i \succeq_o^i \mathbf{y}^i$ for all $i \in \mathcal{I}$, then $\mathbf{X} \succeq_o \mathbf{Y}$; if, in addition, $\mathbf{x}^i \succ_o^i \mathbf{y}^i$ for some $i \in \mathcal{I}$, then $\mathbf{X} \succ_o \mathbf{Y}$.

The *interim* Pareto principles are called *objective* or *subjective*, depending on which uncertainty remains to be resolved (they draw their denomination from the the prospects they handle, not the conditioning variable).[6] Since these are recurring assumptions, we will label the *ex ante* Pareto and *ex post* Pareto principles by (A3) and (B3), respectively.

Domain assumptions. In this paper, we assume maximal domains of objects; all are indeed of the form $\mathbb{R}^L$ for some L. This neglects feasibility considerations. It may well be that not every array, matrix or vector of payoff values can be obtained by a prospect that is feasible for the individuals or society. But our domain assumptions are just for mathematical simplicity, and it would be possible to replace them by more realistic ones. The proofs of this paper use a mathematical theory of separability, developed in Mongin and Pivato (2015), which allows for domains that are not Cartesian products, but only satisfy certain connectedness properties (convex sets being a particular case). We refrained from applying this theory in full generality in order not to add further complexity.

3 The *ex ante* and *ex post* criteria of social welfare

The first result of this section axiomatically characterizes the *ex ante* social welfare criterion in the twofold uncertainty framework. Assumptions (A1) and (A2) indirectly endow individual preferences with SEU representations. In other words, to obtain such a representation, it suffices that individuals have well-defined interim preferences for both types of uncertainty, and that these preferences be invariant. This is the SEU representation theorem mentioned in Section 2; it will recur in Proposition 2 as well as Theorems 1 and 2 below. Assumption (A3) (the *ex ante* Pareto principle) then allows the *ex ante* social preference to be represented by a function that is increasing with the individual SEU representations.

Proposition 1 *Suppose that* (A1) *for all $i \in \mathcal{I}$, the individual preferences $\succeq^i$ induce interim preference orderings $\succeq_s^i$ and $\succeq_o^i$, and* (A2) *both families of orderings are invariant. Suppose also that* (A3) *$\succeq$ satisfies the* ex ante *Pareto principle.*

Then, for all $i \in \mathcal{I}$, there are strictly positive probability vectors $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ and $\mathbf{q}^i \in \Delta_{\mathcal{O}}$, and an increasing continuous utility function u^i on $\mathbb{R}$, such that the preference $\succeq^i$ admits the following SEU representation:

$$U^i(\mathbf{X}) \quad := \quad \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} q_o^i p_s^i u^i(x_{so}), \quad \textit{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{S} \times \mathcal{O}}. \qquad (2)$$

[6] Implicitly, we define all forms of the Pareto principle except for the *ex ante* one in terms of binary relations rather than *bona fide* preference orderings. This makes these Pareto conditions logically independent of the decision-theoretic conditions discussed above.

Moreover, there is a continuous increasing function F on the range of the vector-valued function $(U^i)_{i\in\mathcal{I}}$ such that $\succeq$ is represented by the ex ante social welfare function

$$W_{\mathrm{xa}}(\mathbb{X}) \quad := \quad F\big(\big[U^i(\mathbf{X}^i)\big]_{i\in\mathcal{I}}\big), \quad \textit{for all } \mathbb{X}\in\mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}. \tag{3}$$

In these representations, for all $i\in\mathcal{I}$, the probability vectors $\mathbf{p}^i$ and $\mathbf{q}^i$ are unique, and u_i is unique up to positive affine transformations, while F is unique up to continuous increasing transformations.

Each SEU representation U^i builds upon two probability functions $\mathbf{p}^i$ and $\mathbf{q}^i$, which represent i's beliefs about $\mathcal{S}$ and $\mathcal{O}$, respectively. Given the multiplicative form $q^i_o\,p^i_s$, the events in $\mathcal{S}\times\mathcal{O}$ associated with s values and those associated with o values are *stochastically independent* according to i. Accordingly, i would not revise the probability values for one class of events upon learning which event of the other class of events occurs. Foreshadowing this informational property, the conditional preferences $\succeq^i_s$ and $\succeq^i_o$ have SEU representations independent of s and o, respectively. What is missing is a semantic distinction between the two sources of uncertainty, as their symmetric treatment does not yet permit interpreting one as being objective and the other as being subjective.

Paralleling Proposition 1, the second result of this section axiomatically characterizes the *ex post* social welfare criterion in the twofold uncertainty framework. We apply to society the same decision-theoretic constraints as we earlier put on the individuals: assumptions (B1) and (B2) require that society have well-defined interim preferences for both types of uncertainty, and that these preferences be invariant. This endows society with an SEU representation. Assumption (B3) (the *ex post* Pareto principle) ensures that the utility function in this representation (the *ex post* social welfare function) is increasing in the individuals' utility functions.

Proposition 2 *Suppose that* (B1) *the social preference $\succeq$ induces interim preference orderings $\succeq_s$ and $\succeq_o$, and* (B2) *both families of orderings are invariant. Suppose also that* (B3) *$\succeq$ satisfies the* ex post *Pareto principle.*

Then, the ex post *social preference relations $\succeq_{so}$ are orderings and they are invariant, and there is a continuous and increasing representation W_{xp} for them. There also exist strictly positive probability vectors $\mathbf{p}\in\Delta_{\mathcal{S}}$ and $\mathbf{q}\in\Delta_{\mathcal{O}}$ such that $\succeq$ has the following SEU representation:*

$$W_{\mathrm{xa}}(\mathbb{X}) \quad := \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} p_s q_o W_{\mathrm{xp}}(\mathbf{x}_{so}), \quad \textit{for all } \mathbb{X}\in\mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}. \tag{4}$$

In this representation, $\mathbf{p}$ and $\mathbf{q}$ are unique, and the ex post *social welfare function W_{xp} is unique up to positive affine transformations.*

The probabilities $\mathbf{p}$ and $\mathbf{q}$ that appear here, again in multiplicative form, belong to society exclusively. Like the individuals in Proposition 1, society regards the two sources of uncertainty as being informationally unrelated, and as in this proposition, it is not yet possible to distinguish between objective and subjective uncertainty.

We close this section by restating in the twofold uncertainty framework a theorem from Mongin and Pivato (2015) that formalizes the conflict between the *ex ante* and *ex post* welfare criteria. This result improves on the classic ones by deriving the same undesirable conclusion — that the individuals' and society's probabilities are all identical — under weaker axiomatic conditions. These consist solely of the *ex ante* Pareto principle, which is just one part of the *ex ante* criterion, and the decision-theoretic requirement that society have state-independent *ex post* social preferences, which is only one logical implication of the *ex post* criterion. In the present framework, the undesirable conclusion says that all individuals and society have the same beliefs on the whole state space $\mathcal{S} \times \mathcal{O}$.

Proposition 3 *Suppose that the social preference $\succeq$ induces invariant* ex post *preference orderings $\succeq_{so}$ and the* ex ante *Pareto principle holds.*

Then, there are a strictly positive probability vector $\boldsymbol{\pi} \in \Delta_{\mathcal{S}\times\mathcal{O}}$, and for all $i \in \mathcal{I}$, continuous and increasing utility functions u^i on $\mathbb{R}$ such that the ex ante *social preference $\succeq$ admits the following expected utility representation:*

$$W_{\mathrm{xa}}(\mathbb{X}) \quad = \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} \pi_{so}\, W_{\mathrm{xp}}(\mathbf{x}_{so}), \quad \text{for all } \mathbb{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}},$$

with

$$W_{\mathrm{xp}}(\mathbf{x}_{so}) = \sum_{i\in\mathcal{I}} u^i(x^i_{so}), \text{ for all } \mathbf{x}_{so} \in \mathbb{R}^{\mathcal{I}}.$$

As a result, the ex post Pareto principle holds, and for all $i \in \mathcal{I}$, $\succeq^i$ admits a SEU representation:

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} \pi_{so}\, u^i(x_{so}), \quad \text{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}.$$

In these representations, $\boldsymbol{\pi}$ is unique, and the u^i are unique up to a positive affine transformation with a common multiplier.

Our strategy will now consist in enriching each social welfare criterion with as much as possible of the other that can be added without triggering a conclusion similar to that of 3. Section 4 proceeds from the *ex ante* criterion, and Section 5 from the *ex post* criterion.

4 An *ex ante*-oriented reconciliation

Our first main result, Theorem 1, enriches the *ex ante* social welfare criterion —or equivalently assumptions (A1), (A2) and (A3) of Proposition 1 —with part of the decision-theoretic content of the *ex post* social welfare criterion. Regarding the social interim preferences on o, we require that they be both well-defined and invariant, but regarding the social interim preferences on s, we only require that they be well-defined. In terms of Proposition 2, this amounts to postulating the full force of (B1) while weakening (B2). We also want (B3) (the *ex post* Pareto principle) to hold, but this turns out to be implied by the aforementioned assumptions. Indeed, besides delivering a full *ex ante* representation for society, Theorem 1 endows it with a state-dependent *ex post* representation satisfying the *ex post* Pareto principle. More comments follow the formal statement, regarding the status of the probabilities and the additive form of the representations. The ensuing Corollary 1 shows the compromise of Theorem 1 to be optimal: adding any more of the *ex post* criterion yields an undesirable conclusion.

Theorem 1 *Take the full set of assumptions for the ex ante criterion in Proposition 1), i.e.,* (A1), (A2) *and* (A3). *Among the assumptions for the ex post criterion in Proposition 2, take* (B1), *i.e., that* $\succeq$ *induces interim social preference orderings* $\succeq_s$ *and* $\succeq_o$, *but suppose only that the interim preferences* $\succeq_o$ *are invariant, a weakening of* (B2).

Then, for all $i \in \mathcal{I}$, *the SEU representations of Proposition 1 for* ex ante *individual preferences hold with* $\mathbf{q}^1 = ... = \mathbf{q}^n = \mathbf{q}$, *i.e., for all* $i \in \mathcal{I}$,

$$U^i(\mathbf{X}) \quad = \quad \sum_{o\in\mathcal{O}} \sum_{s\in\mathcal{S}} q_o\, p^i_s\, u^i(x_{so}), \quad \textit{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}.$$

Furthermore, the ex ante *social preference* $\succeq$ *is now represented by the additive* ex ante *social welfare function*

$$W_{\text{xa}}(\mathbb{X}) \quad := \quad \sum_{i\in\mathcal{I}} U_i \quad = \quad \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} q_o\, p^i_s\, u^i(x^i_{so}), \quad \textit{for all } \mathbb{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}, \tag{5}$$

and for all states $(s, o) \in \mathcal{S} \times \mathcal{O}$, $\succeq$ *induces a state-dependent* ex post *social preference ordering* $\succeq_{so}$ *represented by*

$$W_{\text{xp},s}(\mathbf{x}) \quad := \quad \sum_{i\in\mathcal{I}} p^i_s\, u^i(x^i_{so}), \quad \textit{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{I}}. \tag{6}$$

The interim social preferences $\succeq_s$ *and* $\succeq_o$ *are represented by the relevant sums in the formula for* W_{xa}. *As a consequence,* (B3) *holds, i.e., the* ex post *Pareto principle holds, and the objective interim Pareto principle also holds.*

In these representations, $\mathbf{q}$ *and each* $\mathbf{p}^i$ *are unique, while the utility functions* u^i *are unique up to a positive affine transformation with a common multiplier.*

The conclusions of Theorem 1 strengthens those of the baseline Proposition 1 in several ways. First of all, the individuals can entertain idiosyncratic probabilistic beliefs $\mathbf{p}^i$ on $\mathcal{S}$, but must adopt common probabilistic beliefs $\mathbf{q}$ on $\mathcal{O}$; on its part, society entertains probabilistic beliefs on $\mathcal{O}$, which are also given by $\mathbf{q}$, but no such beliefs about $\mathcal{S}$. The asymmetric treatment now justifies our interpretation of $\mathcal{S}$ as a *subjective* and $\mathcal{O}$ as an *objective* source of uncertainty.

Second, Theorem 1 turns the unspecified social welfare function W_{xa} of Proposition 1 into a weighted sum of individual expected utilities, as in Harsanyi's (1955) Social Aggregation Theorem. However, the identity of probabilistic beliefs is here *derived*, not merely assumed, and moreover restricted to the objective part of the uncertainty. From the W_{xa}

function, weighted sum representations follow for conditional social preferences. We now comment on these representations. The representation of the $\succsim_o$ conditionals:

$$\sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}} p_s^i u^i(x_{so}^i)$$

is independent of o, but the representation of the $\succeq_s$ conditionals:

$$\sum_{i\in\mathcal{I}}\sum_{o\in\mathcal{O}} q_o p_s^i u^i(x_{so}^i)$$

depends on s, as does the representation (6) of the state-dependent *ex post* social preference relation. $\succeq_{so}$. State-dependence of social preferences is one way to reconcile the *ex ante* and *ex post* criteria of social welfare.[7] An important feature of the last two utility sums is that state-dependence occurs through the factors p_s^i. This means that the individuals who assigned s the highest probability are those who are most favoured by the *ex post* social welfare function if s realizes. Although this is an uncommon arrangement, it can be defended on the ground that these individuals were better predictors than the others, and presumably made more suitable provisions, which somehow deserves rewarding. Also, perhaps the individuals themselves would endorse these weights, since their subjective likelihood assessments express the subjective importance they put on each state. (We thank a referee for this suggestion.) However, these normative intuitions must be balanced against another, which is that poor predictors presumably made less adequate provisions, which may call for compensation. Both intuitions clash with a third: that questions of social ethics should not be decided by comparing the correctness (or luck) of the individuals' beliefs. We must leave the interpretational debate at this stage.

Theorem 1 adds as much as possible of the *ex post* criterion to the *ex ante* criterion. By assuming the full force of (B2) —i.e., making the social interim preferences on s not only well-defined, but also also invariant —one derives a social probability $\mathbf{p}$ also on the subjective source, and this forces the individuals to align their probabilities $\mathbf{p}_i$ with this $\mathbf{p}$.

Corollary 1 *The assumptions are those of Theorem 1 except that* (B2) *now holds in full, i.e., both the* $\succsim_o$ *and the* $\succsim_s$ *are invariant. Then, the representations of Theorem 1 hold with a common probability vector* $\mathbf{p} \in \Delta_{\mathcal{S}}$ *such that* $\mathbf{p}_1 = ... = \mathbf{p}_n = \mathbf{p}$.

All in all, Theorem 1 appears to be a welcome improvement on the position of those writers, in the early welfare economics controversy, who bluntly adopted the *ex ante* social welfare criterion and rejected the *ex post* one. Among the more recent participants, Hild et al. (2003) and Risse (2003) have made a sophisticated case for the former against the

[7]In the standard framework, with a single source of uncertainty, Mongin (1998), Chambers and Hayashi (2006) and Keeney and Nau (2011) have explored this solution. It is not merely formal, but normatively defensible, as it seems desirable that society could change the individuals' weights in the utility sum according to which state of the world is realized. In the standard framework, this solution might leave society without any probabilistic beliefs at all. But Theorem 1 does away with social probability only for the subjective part of uncertainty.

latter. In effect, they argue that social and individual preferences are always *ex ante*. For them, the distinction between final consequences and uncertain prospects is a matter of convention; a more refined analysis of these consequences would reveal that they define yet another class of uncertain prospects. By focusing on this particular class, the *ex post* Pareto principle makes an arbitrary restriction to the *ex ante* principle, while being open to the same difficulties; hence it should be avoided, and so should the *ex post* social welfare criterion, which includes it.[8] However, this argument does not have the same implications here as in a framework with a single source of uncertainty. While it encourages the adoption of the *ex ante* criterion when the only other choice is the *ex post* criterion, it may push in favour of compromise solutions when these become available, as is the case here.

5 An *ex post*-oriented reconciliation

Our second main result, Theorem 2, enriches the *ex post* social welfare criterion —or equivalently assumptions (B1), (B2) and (B3) of Proposition 2 —with the decision-theoretic part of the *ex ante* social welfare criterion, but only some of its Paretian content. Specifically, we require the objective interim Pareto principle to hold instead of the full *ex ante* principle. In terms of Proposition 1, this amounts to invoking assumptions (A1) and (A2), but replacing (A3) by a weaker version of the Pareto principle. Theorem 2 delivers SEU representations for both the individuals and society, with a utilitarian *ex post* SWF, and common beliefs about $\mathcal{O}$ (but not $\mathcal{S}$). More comments follow the formal statement, regarding the status of the probabilities and the additive form of the representations. The ensuing Corollary 2 shows the compromise of Theorem 2 to be optimal: adding any more of the *ex ante* criterion yields an undesirable conclusion.

Theorem 2 *Take the full set of assumptions for the* ex post *criterion in Proposition 2, i.e.,* (B1), (B2) *and* (B3). *Among the assumptions for the* ex ante *criterion in Proposition 1, take* (A1) *and* (A2). *Suppose also that the objective interim Pareto principle holds.*

Then, there exist strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ *and* $\mathbf{q} \in \Delta_{\mathcal{O}}$, *and for all* $i \in \mathcal{I}$, *there exist strictly positive probability vectors* $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ *and continuous and increasing utility functions* u^i *on* $\mathbb{R}$, *with the following properties. For all* $i \in \mathcal{I}$, *the* ex ante *individual preferences* $\succeq^i$ *have the SEU representation:*

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} p^i_s \, q_o \, u^i(x_{so}), \quad \text{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}, \tag{7}$$

while the ex ante *social preference* $\succeq$ *has the SEU representation:*

$$W_{\text{xa}}(\mathbb{X}) \quad = \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} p_s \, q_o \, W_{\text{xp}}(\mathbf{x}_{so}), \quad \text{for all } \mathbb{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}. \tag{8}$$

[8] This troubling argument indirectly connects with worries that Savage (1972) once expressed concerning "small worlds" and the problem they raise for his SEU theory.

Furthermore, there is a vector of positive weights $\mathbf{r} = (r^i)_{i \in \mathcal{I}}$ *such that the* ex post *social welfare function* W_{xp} *has the additive form*

$$W_{\text{xp}}(\mathbf{x}) \quad := \quad \sum_{i \in \mathcal{I}} r^i \, u^i(x^i), \quad \text{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{I}}. \tag{9}$$

In these representations, the vectors $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$ *are unique, and the utility functions* u^i *are unique up to positive affine representations with a common multiplier.*

These conclusions make two major additions to those of the base-line Proposition 2. First, while that result said nothing of the decision theory satisfied by the individuals, they now have SEU representations for their preferences $\succeq^i$. By endowing the individuals with identical probabilities on $\mathcal{O}$ and idiosyncratic probabilities on $\mathcal{S}$, these representations justify our objective/subjective interpretation of the two sources of uncertainty. Observe that Theorem 1 and 2 involve different epistemic attitudes on the part of society. While it previously did not form probabilistic beliefs on s, it now entertains such beliefs. A common feature, however, society never regulates individual beliefs on the subjective source.

Second, all Pareto conditions that are being assumed are translated into weighted sums of individual utilities. Indeed, the *ex post* social utility W_{xp}, which Proposition 2 did not determine, turns out to be a weighted sum of *ex post* individual utilities. Note that this function is state-independent, unlike the corresponding one in Theorem 1. Also, for each given s, the objective *interim* social welfare function is a weighted sum of the individuals' objective *interim* expected utilities:

$$\sum_{o \in \mathcal{O}} q_o \, W_{\text{xp}}(\mathbf{x}_o) = \sum_{i \in \mathcal{I}} r^i \sum_{o \in \mathcal{O}} q_o \, u^i(x_{so}).$$

No such decomposition exists for either the subjective interim social welfare function $\sum_{s \in \mathcal{S}} p_s \, W_{\text{xp}}(\mathbf{x}_s)$ or the *ex ante* social utility W_{xa}, which reflects the absence of the subjective interim and *ex ante* Pareto principles.[9]

At this juncture, we can finalize the comparison of our results with Harsanyi's (1955). In his von Neumann-Morgenstern framework, the conclusion that the *ex post* SWF is a weighted sum of *ex post* individual utilities automatically follows from the earlier conclusion that *ex ante* SWF is a weighted sum of *ex ante* individual utilities. By contrast, we obtain the two weighted sum conclusions from different assumptions in separate results, i.e., Theorem 1 for the *ex ante* conclusion, and the present Theorem 2 for the *ex post* one.

Lastly, we show that any further attempt at reconciling the two social welfare criteria would precipitate the classic impossibility. Indeed, adding the *ex ante* Pareto principle collapses the individual probabilities p^i_s onto the social ones p_s. We actually prove this in slightly stronger form.

Corollary 2 *Under the assumptions of Theorem 2, the following three conditions are equivalent:* (a) *for all* $s \in \mathcal{S}$, $\succsim_s$ *satisfies the subjective interim Pareto principle;* (b) *for all* $i \in \mathcal{I}$, $\mathbf{p}^i = \mathbf{p}$; (c) *the* ex ante *Pareto principle holds.*

[9]By substituting (9) into (8), we can express W_{xa} as a weighted sum of u^i values. But since society uses its own probability $\mathbf{p}$, this expression is *not* a sum of the individual SEU representations from (7).

The solution offered by Theorem 2 should be compared with that of Gilboa et al. (2004) (hereafter GSS), which has recently attracted significant attention. In the same Savage framework as Mongin (1995), GSS assume the *ex post* criterion in full and the *ex ante* criterion in part. They limit the *ex ante* Pareto principle to comparisons of social prospects which do not involve probabilistic disagreements between the individuals. From this, they are able to conclude that (i) society's *ex post* preference is represented by a weighted sum of *ex post* individual utility functions, i.e., Harsanyi's *ex post* conclusion, and that (ii) society's probability equals a weighted average of individual probabilities, i.e., the *linear pooling rule* of the statistics and the management literature:

$$\mathbf{p} = \sum_{i \in \mathcal{I}} a^i \mathbf{p}^i \qquad \text{for some constants } a^i \geq 0.$$

Importantly, their assumptions do *not* entail that (iii) society's SEU representation can be represented in terms of the individuals' SEU representations, let alone by a weighted sum of them, as in Harsanyi's *ex ante* conclusion. Adding (iii) would simply reinstate one of the impossibilies proved in Mongin (1995).

Despite the dissimilar frameworks, a comparison is possible between the GSS conclusions and those of Theorem 2. We also derive (i), also refrain from deriving (iii), but most importantly, do *not* derive (ii). Whether (ii) is desirable is a central concern of Section 6.

6 Spurious unanimity and complementary ignorance

Let us first consider the *ex ante*-oriented compromise embodied in Theorem 1. As it retains the *ex ante* social welfare criterion in full, it is open to the objection of *spurious unanimity* that is now often raised against this criterion, and more specifically, against the *ex ante* Pareto principle. Mongin (1997) introduced this objection and coined the expression for it. To make our argument self-contained, we briefly restate an example from his initial paper.

Imagine the members of society are spread out in two areas, an island and a mainland, and the Government is concerned with the lack of communication between them. Thus, it considers connecting one area to the other by a bridge, and being democratically inspired, organizes a public hearing. At this stage, it is not clear whether the traffic flow will mostly go from the island to the mainland, or in the other direction.[10] As it happens, the Islanders think that the former consequence is more probable than the latter, while the Mainlanders have the opposite belief. It is also the case that the Islanders value the former consequence more than the latter, while the Mainlanders have the opposite ranking. For relevant probability and utility values fitting this situation, SEU and even more general decision theories predict that both groups will express support for the project. Thus, if the Government adheres to the *ex ante* Pareto principle, it will push it forward. However, this would be a dubious decision to make. The two groups are unanimous in preferring the bridge, but spuriously, since they are in fact *twice opposed* - i.e., in their utility and probability comparisons – and their disagreements just offset in the SEU or

[10]We may assume that the bridge is financially feasible, whichever the main direction of the flow is.

related calculation. Arguably, the Government should refrain from endorsing unanimous preferences in those cases in which individuals so strongly differ from each other.

The bridge example takes the direction of the flow to be subjectively, not objectively uncertain; in the present notation, this would be evaluated in terms of the p^i, not the q. Thus, the example is damaging for the solution proposed in Theorem 1, and even for any solution that would just retain the subjective *interim* Pareto principle instead of the *ex ante* Pareto principle. But this defect must be weighed against the advantage we stressed earlier of endowing society with a state-dependent representation of its *ex post* preferences, an advantage that the solution proposed in Theorem 2 does not enjoy.

Gilboa et al. (2004) have taken up the spurious unanimity objection, using the different example of a duel. If two people agree to fight a duel, this is presumably because they have both opposite beliefs (on who will win) and opposite preferences (on who should survive). Applying the *ex ante* Pareto principle would make duels socially permissible, a seemingly undesirable consequence. While the bridge example questions the optimality of procuring a unanimously desired public good, this even simpler example questions the optimality of legally endorsing mutually agreed individual transactions. The GSS theorem mentioned in last section defines a compromise between the *ex ante* and *ex post* criteria that takes care of the spurious unanimity objection. Let us now examine this theorem in more detail.

Formally, given a set of states Ω, GSS introduce the family $\mathcal{F}$ of all events on whose probabilities all individuals agree, and they require society to respect only those unanimous comparisons which take place between social prospects on Ω that are *measurable* with respect to $\mathcal{F}$. For instance, if a social prospect is constant on the cells of a partition, and each cell receives the same probabilities from all individuals, then it satisfies the measurability restriction. This restricted form of the *ex ante* Pareto principle makes the bridge and duel examples inoperative, since it precludes probabilistic disagreement from cancelling utility disagreement. Remarkably, no more than this condition is needed to entail conclusions (i) and (ii) of last section when it is conjoined with the assumption that the individuals satisfy Savage's SEU postulates (which correspond to the decision-theoretic part of the *ex ante* criterion) and the assumption that society also does (which corresponds to the decision-theoretic part of the *ex post* criterion[11]).

However, as we now show, this elegant construction can go wrong if the individuals have private information and revise their probabilities accordingly. Consider a society made out of two individuals, Alice and Bob, a partitition of the set of states Ω into three events E_1, E_2, E_3, and two prospects, f and g, which we describe in terms of the utility values that each of the three events brings to Alice and Bob. We assume that they share the same utility function, and initially have the same probabilistic beliefs, as in the next table.

	E_1	E_2	E_3
Alice's and Bob's utilities for f	1	0	1
Alice's and Bob's utilities for g	0	1	0
Alice's and Bob's probabilities	0.49	0.02	0.49

[11] In the Savage framework, the remaining part of the *ex post* criterion, i.e., the *ex post* Pareto principle, follows from either the *ex ante* principle or its GSS restricted form.

Initially, Alice and Bob both assign a SEU of 0.98 to f and an SEU of 0.02 to g, so that they both prefer f over g. Thus, GSS's restricted *ex ante* Pareto principle says that society should also prefer f over g.

To make the example more concrete, suppose a card has been drawn at random from a well-shuffled deck containing only one Joker. Let E_1 denote the event "Red Suit"(i.e., Hearts or Diamonds), E_3 denote the event "Black Suit" (i.e., Clubs or Spades), and E_2 denote the event "Joker". Thus, f is a bet on "Suit Card", whereas g is a bet on "Joker". Alice and Bob will equally split the winnings of either bet. Given the (unanimous) probabilities and utility assignments in the table, both Alice and Bob prefer to bet on "Suit Card" rather than "Joker". Now, suppose Alice privately receives information that the card is not from a black suit; i.e., she observes $E_1 \cup E_2$. Meanwhile Bob privately learns that the card is not from a red suit; i.e., he observes $E_2 \cup E_3$. Granting that Alice and Bob do not communicate with each other, they will reach the following probabilities and SEU values from Bayesian updating:

	$P(E_1)$	$P(E_2)$	$P(E_3)$	$SEU(f)$	$SEU(g)$
Alice	0.96	0.04	0	0.96	0.04
Bob	0	0.04	0.96	0.96	0.04

Let us construct the family $\mathcal{F}$ of events on the probabilities of which the Alice and Bob agree. Both put $P(E_1 \cup E_3) = 0.96$ and $P(E_2) = 0.04$, so $\mathcal{F} = \{\varnothing, E_1 \cup E_3, E_2, \mathcal{S}\}$. The prospects f and g are constant on $E_1 \cup E_3$ and constant on E_2, hence measurable with respect to $\mathcal{F}$, so from GSS's restricted *ex ante* Pareto principle, f should be socially preferred to g. However, if Alice knows the card is not black, and Bob knows the card is not red, then the card must be the Joker. But then, society should prefer g to f, contrary to what GSS's restricted *ex ante* Pareto principle says. If Alice and Bob unanimously prefer f over g, this is only because each one has information the other one lacks; let us say they are in a state of *complementary ignorance.*

This counterexample also hits the linear pooling rule, which is not surprising since GSS recover it as a consequence of their restricted *ex ante* Pareto principle. If the social observer averages Alice and Bob's posterior probabilities, an incorrect value $P(E_2) = 0.04$ follows. One may object that this is not the proper interpretation of the linear pooling rule in a context where individuals have private information. Rather, the social observer should first average Alice and Bob's prior probabilities, and then update this average after deducing from their posteriors that E_2 occurred; this would deliver the correct value $P(E_2) = 1$. But this conclusion could be reached by simply observing the posteriors, without any averaging. So this argument does not save the linear pooling rule.[12]

At this point, another objection will perhaps surface: neither the linear pooling rule nor the GSS restriction of the *ex ante* Pareto principle are intended to cover the case of private information. In other words, both are stated under the implicit proviso that the individuals' probabilities are *priors*, unlike the *posteriors* of the last table. However, the distinction

[12]We may also note that this two-step process of revising and averaging does not give the same result as the direct averaging of posteriors. In technical jargon, the linear pooling rule is not "externally Bayesian", an old observation of the literature (see, e.g., Genest and Zidek (1986) and Clemen and Winkler (2007)).

between these two cases is partly a matter of convention. Depending on what one considers to be the underlying knowledge and what one considers to be information, a prior in one context becomes a posterior in another, and *vice versa*. Leaving aside the hypothetical constructs of the "original position" or the "veil of ignorance", which are usually understood in terms of pure priors, the individual probabilities relevant to normative economics are typically posteriors. Instead of the prior *versus* posterior distinction, we would privilege that between those posteriors which can be analyzed so as to reveal information to the social observer, as in the Alice and Bob example, and those which cannot.

All this is to say that to derive the linear pooling rule, as GSS do, is a mixed blessing. In the end, one might view as a advantage that Theorem 2 involves no connection between the social probability and the individual ones. This does not give a full solution to the complementary ignorance problem, but at least it does not give an incorrect answer.[13]

7 Comparisons with the literature

The conflict between the *ex ante* and *ex post* social welfare criteria has attracted much attention recently. The common strategies are to weaken either the *ex ante* Pareto principle or the SEU assumptions, whether on society or the individuals, while preserving the *ex post* Pareto principle. Gilboa et al. (2004) (GSS) epitomize the first strategy. An original variant can be found in Nehring (2004) and Chambers and Hayashi (2014), who assume that agents have private information and restrict the *ex ante* Pareto principle to situations where it is common knowledge that one prospect *ex ante* Pareto-dominates another. If society satisfies statewise dominance, even this weak assumption delivers the undesirably strong conclusion that the agents share the same prior on the common knowledge events. This impossibility theorem refines the classic one.[14]

Gilboa et al. (2014) also explore the first strategy. Working with utility representations rather than axiomatically, they define a new form of Paretian comparison by saying that prospect f *No-Betting Pareto* (NBP) dominates prospect g if there exists some probability measure p such that f yields at least as high a p-expected utility as g for every individual. They show that NBP-dominance holds if and only if, for any weighted sum of *ex post* utilities, g does not statewise dominate f. They explore the consequences for financial markets of restricting the *ex ante* Pareto principle to NBP-dominance comparisons, an analysis continued by Gayer et al. (2014).

Other writers use the second strategy, possibly in conjunction with the first. Three analyses in this category lead to related conclusions despite having been developed independently. Thus, Qu (2017) assumes that both society and the individuals conform to Gilboa and Schmeidler (1989)'s *maximin expected utility* (MEU) theory, a by now classic generalization of Anscombe and Aumann's SEU theory that endows decision makers with *sets* of probabilistic beliefs rather than single probabilistic beliefs. Qu also constructs a

[13]To check formally that there is no connection, suppose that, for any choice of u^i, the $\mathbf{p}^i$ are the same and differ from $\mathbf{p}$. The assumptions of the theorem can be satisfied under this supposition.

[14]Chambers and Hayashi further show that the *ex ante* social welfare function is a weighted sum of individual expected utilities, whereas Nehring assumes this.

variant of the *ex ante* Pareto principle (RCEPO) that extends it in one direction and restricts it in another. Given his decision-theoretic assumptions, the conjunction of the *ex post* Pareto principle and this *ex ante* variant holds if and only if the *ex post* social welfare function is a weighted sum of individual utilities and the social set of probabilities $\mathcal{P}$ is a convex combination of the individuals' sets $\mathcal{P}_i$. In terms of GSS's two conclusions described in Section 5, Qu recovers (i) and generalizes (ii) in accordance with MEU decision theory.

Alon and Gayer (2014) assume Savage's SEU theory for the individuals, and put axioms on society that endow it with a MEU representation, thus with a probability set $\mathcal{P}$. They strengthen GSS's restricted *ex ante* Pareto principle to a *Consensus Pareto* version, which says that if all individuals (according to their own probabilistic beliefs) deem that prospect f yields a higher SEU than prospect g *for every individual*, then society should prefer f over g. This excludes the spurious unanimity implication of the *ex ante* Pareto principle, while accommodating some situations where individuals have different probabilistic beliefs. Given the decision-theoretic assumptions, society satisfies Consensus Pareto if and only if the *ex post* SWF is a weighted sum of individual utilities and the social probability set $\mathcal{P}$ is included in the convex hull of the individual probability measures. Then, Alon and Gayer recover (i) and generalize (ii), although this generalization differs from Qu's.[15]

In an Anscombe-Aumann framework, Danan et al. (2016) assume that society and the individuals have partial orders $\succeq$ and $\succeq_i$ that admit representations in the sense of Bewley (2002), i.e., there are sets $\mathcal{P}$ and $\mathcal{P}_i$ of probability distributions such that $f \succeq g$ (resp. $f \succeq_i g$) if and only if f yields at least as high an expected utility as g according to *all* elements in $\mathcal{P}$ (resp. in $\mathcal{P}_i$). Given these decision-theoretic assumptions and the further condition that individual utility functions are affinely independent, society satisfies an *ex ante* Pareto principle if and only if the *ex post* social welfare function is a weighted sum of individual utilities and $\mathcal{P}$ is included in the intersection of the $\mathcal{P}_i$. This amounts to a conjunction of (i) and a substitute for - not a generalization of - (ii). In the particular case of SEU theory, the individuals' unique probability measure must be the same for all individuals and society, which recovers the classic impossibility. The authors also consider a *Common Taste Pareto* version of the *ex ante* Pareto principle; this amounts to restricting the principle to pairs of prospects such that all individuals agree on the ranking of the consequences obtained from these prospects (and convex combinations thereof). Given the decision-theoretic assumptions, Common Taste Pareto holds if and only if and $\mathcal{P}$ is included in the convex hull of the union of the $\mathcal{P}_i$. This is again (i) conjoined with a generalization of (ii), which is specific to the paper.

The positive results in the three aforementioned papers have attractive features, but they are still vulnerable to the "complementary ignorance" example from Section 6, because SEU is a special case of both the MEU and Bewley theories. In that example, Alice and Bob have the *same* utility function, so Alon and Gayer's (2014) Consensus Pareto and Danan et al.'s (2016) Common Taste Pareto both reduce to *ex ante* Pareto, which as we have seen, forces the social preferences to wrongly ignore private information. Likewise,

[15]Here we focus on Alon and Gayer's (2014) preprint rather than their final (2016) version because it better connects with the other work reviewed here. See Billot and Qu (2017) for more on the same line.

for SEU preferences, Qu's (2017) RCEPO axiom reduces to *ex ante* Pareto.[16]

For the sake of completeness, we briefly mention other investigations of the *ex ante* and *ex post* social welfare criteria that do not fit in the previous discussion. Thus, Billot and Vergopoulos (2016) endow each individual with a personalized state space and a personalized consequence set, and society with a state space and a consequence set that are Cartesian products of these spaces. Assuming SEU theory for individuals and society, Billot and Vergopoulos show that the latter satisfies a set of Pareto conditions if and only if the usual conclusion (i) holds, and the social probability measure is the *product* of the individual ones, an interesting alternative to (ii). However, their framework presupposes that individuals face *independent risks*, as in standard insurance markets. This severely constrains its applications to economic policy.[17]

In a framework with a finite state space and given *ex post* individual utilities, Hayashi and Lombardi (2018) further explore the consequences of assuming MEU theory for either the individuals or society. Unlike the earlier writers, they do not attempt to reconcile the *ex ante* and *ex post* criteria, but rather to compare their normative consequences, focusing on maximin "egalitarianism" rather than weighted sum "utilitarianism". This paper mostly belongs to the separate and growing literature on egalitarianism under uncertainty; see the surveys by Mongin and Pivato (2016) and Fleurbaey (2018).

Some other papers have explored the problems the *ex ante* criterion in relation to financial markets. Standard economic theory generally endorses transactions on these markets, thus implicitly assuming the *ex ante* Pareto principle, but uncertainty raises spurious unanimity objections here as it does elsewhere. Thus, Posner and Weyl (2013), Blume et al. (2015) and others identify *purely speculative* transactions with those driven by different beliefs, and argue for public regulation in this case. Instead of Pareto dominance, Brunnermeier et al. (2014) strengthen the concept of Pareto *inefficiency*, by defining a prospect f to be *belief-neutral inefficient* if, for every probability measure p arising from a convex combination of the individual ones, there is some prospect yielding a higher p-expected utility for every agent than f. By this criterion, they identify speculative transactions for which they recommend regulatory scrutiny. Going against the tide, Crès and Tvede (2018) construct a paradox of regulation, to the effect that if it applies to *multiple related* non-speculative transactions, it must consistently authorize *all* transactions, including speculative ones.

8 Conclusion

By complexifying the underlying model of uncertainty, we have identified intermediate positions between the classical *ex ante* and *ex post* social welfare criteria. Whereas some of the recent literature has departed from SEU theory, we have instead chosen to enrich this theory with a distinction it does not normally explore: that between two sources of

[16]Indeed, applying the results of Qu or Danan et al. to this example yields SEU social preferences with social beliefs satisfying (ii). Applying Alon and Gayer's result yields MEU social preferences in which all priors agree that $P(E_1 \cup E_3) = 0.96$ and $P(E_2) = 0.04$, so the social ranking of f and g is still incorrect.

[17] In the bridge example, the risk faced by the Islanders is not independent of the risk faced by the Mainlanders, and in the duel example, the risks faced by the duellists are not independent either.

uncertainty, here interpreted as being objective and subjective respectively. This semantic distinction emerges in the representation theorems through the difference between multiple probability assessments for one source and a single probability assessment for the other.

The intermediate positions we have uncovered do not fully resolve the underlying trade-off: one must still choose between a primarily *ex ante* and a primarily *ex post* solution. But, as we have shown, the former can reap the benefits of an *ex post* state-dependent social welfare function without depriving society of all probabilistic beliefs, and the latter can extend some way in the direction of *ex ante* Paretianism, while avoiding both the classic spurious unanimity objection and the novel complementary ignorance objection. To introduce the latter was another contribution of this paper.

We feel that the theoretical potential of the twofold uncertainty framework is not exhausted. Elsewhere, it has served to introduce a preference axiomatization, in a Bayesian theorist's style, of the property of stochastic independence (see Mongin, 2017). It may guide other foundational explorations in probability theory, in particular a more thorough discussion of objective probability than that provided here. We also feel that the complementary ignorance objection calls for more attention, as it suggests that collective beliefs should take into account not merely individual beliefs, but also their origin and justification, and this opens new avenues for the theory of social preference under uncertainty.

Appendices

A Technical background

We begin by restating the definition of a conditional relation in terms of its source relation, and the separability property that turns a conditional relation into an ordering.

Suppose that a weak preference ordering R is defined on a product set $\mathcal{X} = \prod_{\ell \in \mathcal{L}} \mathcal{X}_\ell$, where $\mathcal{L}$ is a finite set of indexes. Take a subset of indexes $\mathcal{J} \subseteq \mathcal{L}$ and its complement $\mathcal{K} := \mathcal{L} \setminus \mathcal{J}$. Denote the subproduct sets $\prod_{\ell \in \mathcal{J}} \mathcal{X}_\ell$ and $\prod_{\ell \in \mathcal{K}} \mathcal{X}_\ell$ by $\mathcal{X}_\mathcal{J}$ and $\mathcal{X}_\mathcal{K}$, respectively. By definition, the *conditional induced by* R *on* $\mathcal{J}$ is the relation $\mathsf{R}_\mathcal{J}$ on $\mathcal{X}_\mathcal{J}$ thus defined: for all $\xi_\mathcal{J}, \xi'_\mathcal{J} \in \mathcal{X}_\mathcal{J}$,

$$\xi_\mathcal{J} \; \mathsf{R}_\mathcal{J} \; \xi'_\mathcal{J} \;\text{ if and only if }\; \text{for some } \xi_\mathcal{K} \in \mathcal{X}_\mathcal{K}, (\xi_\mathcal{J}, \xi_\mathcal{K}) \; \mathsf{R} \; (\xi'_\mathcal{J}, \xi_\mathcal{K}).$$

By a well-known fact, the conditional $\mathsf{R}_\mathcal{J}$ is an ordering if and only if R is *separable in* $\mathcal{J}$, that is: for all $\xi_\mathcal{J}, \xi'_\mathcal{J} \in \mathcal{X}_\mathcal{J}$ and $\xi_\mathcal{K}, \xi'_\mathcal{K} \in \mathcal{X}_\mathcal{K}$,

$$(\xi_\mathcal{J}, \xi_\mathcal{K}) \; \mathsf{R} \; (\xi'_\mathcal{J}, \xi_\mathcal{K}) \text{ if and only if } (\xi_\mathcal{J}, \xi'_\mathcal{K}) \; \mathsf{R} \; (\xi'_\mathcal{J}, \xi'_\mathcal{K}).$$

In this case, we may also say that $\mathcal{J}$ is a R-*separable*. Clearly, separability in $\mathcal{J}$ entails that R is *increasing* with $\mathsf{R}_\mathcal{J}$, i.e., that for all $\xi_\mathcal{J}, \xi'_\mathcal{J} \in \mathcal{X}_\mathcal{J}$ and $\xi_\mathcal{K} \in \mathcal{X}_\mathcal{K}$,

$$\text{if } \xi_\mathcal{J} \; \mathsf{R}_\mathcal{J} \; \xi'_\mathcal{J}, \text{ then } (\xi_\mathcal{J}, \xi_\mathcal{K}) \; \mathsf{R} \; (\xi'_\mathcal{J}, \xi_\mathcal{K}),$$

and if the $\mathsf{R}_{\mathcal{J}}$-comparison is in fact strict, so is the resulting R-comparison. Conversely, if $\mathsf{R}_{\mathcal{J}}$ is some ordering on $\mathcal{X}_{\mathcal{J}}$, the property that R on $\mathcal{X}$ is increasing with $\mathsf{R}_{\mathcal{J}}$ entails that R is weakly separable in $\mathcal{J}$.[18]

Section 2 considers several cases of this construction; in each of them, $\mathcal{X}_\ell = \mathbb{R}$ for all $\ell \in \mathcal{L}$. If $\mathcal{L} = \mathcal{I}\times\mathcal{S}\times\mathcal{O}$ and R is the social preference $\succeq$, then we can let $\mathcal{J} = \mathcal{I}\times\{s\}\times\mathcal{O}$ to obtain s-conditional social preference relations, let $\mathcal{J} = \mathcal{I}\times\mathcal{S}\times\{o\}$ to obtain o-conditional social preference relations, and let $\mathcal{J} = \{i\}\times\mathcal{S}\times\mathcal{O}$ for the *ex ante* Pareto principle. If $\mathcal{L} = \mathcal{S}\times\mathcal{O}$ and R is an individual preference $\succeq^i$, then we let $\mathcal{J} = \{s\}\times\mathcal{O}$ to obtain s-conditional individual preference relations, and let $\mathcal{J} = \mathcal{S}\times\{o\}$ to obtain o-conditional individual preference relations. If $\mathcal{L} = \mathcal{I}\times\mathcal{O}$ and R is the s-conditional social preference $\succeq_s$, then we let $\mathcal{J} = \{i\}\times\mathcal{O}$ to obtain the subjective interim Pareto principle. Other cases are similar.[19] These relations may or may not be orderings, i.e., from the first equivalence stated above, their source relation may or may be separable in the associated index set. As we assume that conditional preference orderings exist across the uncertainty type or not at all, it follows from the second equivalence that the ordering assumptions could be restated in terms of *dominance*.[20]

Let us now take $\mathcal{X}$ to be an open box in $\mathbb{R}^{\mathcal{L}}$, i.e., $\mathcal{X} = \prod_{\ell\in\mathcal{L}}\mathcal{X}_\ell$, where the $\mathcal{X}_\ell$ are open intervals. An ordering $\succeq$ on $\mathcal{X}$ has an *additive representation* if it is represented by a function $U : \mathcal{X}\longrightarrow\mathbb{R}$ of the form

$$U(\mathbf{x}) \quad := \quad \sum_{\ell\in\mathcal{L}} u_\ell(x_\ell), \tag{A1}$$

where $u_\ell :\ \mathcal{X}_\ell\longrightarrow\mathbb{R}$, $\ell\in\mathcal{L}$. We now state a proposition that will be useful later. A proof can be devised from the formal arguments in Mongin and Pivato (2015).

Proposition A1 *Take $\mathcal{L} = \mathcal{I}\times\mathcal{S}\times\mathcal{O}$, with $|\mathcal{I}|, |\mathcal{S}|, |\mathcal{O}| \geq 2$, and view elements $\mathbb{X}\in\mathbb{R}^{\mathcal{L}}$ as arrays $[x^i_{so}]^{i\in\mathcal{I}}_{s\in\mathcal{S},o\in\mathcal{O}}$. If a continuous order $\succeq$ on $\mathbb{R}^{\mathcal{L}}$ is increasing in every coordinate, and is separable in each $i\in\mathcal{I}$, each $s\in\mathcal{S}$, and each $o\in\mathcal{O}$, then it has an additive utility representation $U : \mathcal{X}\longrightarrow\mathbb{R}$ of the form $U(\mathbf{x}) := \displaystyle\sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} u^i_{so}(x^i_{so})$, where each $u^i_{so} : \mathbb{R}\longrightarrow\mathbb{R}$ is a continuous, increasing function. Furthermore, the functions $\{u^i_{so}\}^{i\in\mathcal{I}}_{s\in\mathcal{S},o\in\mathcal{O}}$ are unique up to positive affine transformation (PAT) with a common multiplier.*

We now specialize the basic sets still differently. Take $\mathcal{L} = \mathcal{J}\times\mathcal{K}$ with $|\mathcal{J}|, |\mathcal{K}| \geq 2$, and $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$, viewing elements $\mathbf{X}\in\mathcal{X}$ as matrices $[x^j_k]^{j\in\mathcal{J}}_{k\in\mathcal{K}}$, with $j\in\mathcal{J}$ indexing the rows and $k\in\mathcal{K}$ indexing the columns. Alternatively, think of $\mathbf{X}$ as a $\mathcal{J}$-indexed array of row vectors $\mathbf{x}^j := [x^j_k]_{k\in\ \mathcal{K}}\in\mathbb{R}^{\mathcal{K}}$, or as a $\mathcal{K}$-indexed array of columns vectors $\mathbf{x}_k := [x^j_k]^{j\in\mathcal{J}}\in\mathbb{R}^{\mathcal{J}}$. Now consider a continuous ordering $\succeq$ on $\mathcal{X}$. Here are three axioms that $\succeq$ might satisfy.

[18]For these definitions and basic facts, see Fishburn (1970), Keeney and Raiffa (1976), and Wakker (1989). What is called *separable* here is sometimes called *weakly separable* elsewhere.

[19]Here we make the obvious identifications of $\{s\}\times\mathcal{O}$ with $\mathcal{O}$, $\mathcal{S}\times\{o\}$ with $\mathcal{S}$, etc. Similar routine identifications will occur below without being mentioned.

[20]Thus, our statement "$\succsim^i$ induces conditional preferences $\succsim^i_s$", which definitionally says that the $\succsim^i_s$ are orderings, equivalently says that $\succsim^i$ satisfies dominance with respect to the $\succsim^i_s$.

Coordinate Monotonicity: For all $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$, if $x_k^j \geq y_k^j$ for all $(j,k) \in \mathcal{J} \times \mathcal{K}$, then $\mathbf{X} \succeq \mathbf{Y}$. If, in addition, $x_k^j > y_k^j$ for some $(j,k) \in \mathcal{J} \times \mathcal{K}$, then $\mathbf{X} \succ \mathbf{Y}$.

Row Preferences: For each column $j \in \mathcal{J}$, $\succeq$ is separable in $\{j\} \times \mathcal{K}$.

Column Preferences: For all rows $k \in \mathcal{K}$, $\succeq$ is separable in $\mathcal{J} \times \{k\}$.

Define $\succeq^j$ and $\succeq_k$ to be the conditional relations of $\succeq$ on j and k, respectively. It follows from Row Preferences that the $\succeq^j$ are orderings on $\mathbb{R}^{\mathcal{K}}$, and from Column Preferences that the $\succeq_k$ are orderings on $\mathbb{R}^{\mathcal{J}}$. Moreover, $\succeq$ is increasing with respect to each of these conditional relations. The next two axioms force the conditional orders to be invariant.

Invariant Row Preferences: Row Preferences holds, and there is an ordering $\succeq^{\mathcal{J}}$ on $\mathcal{Y}^{\mathcal{K}}$ such that $\succeq^j = \succeq^{\mathcal{J}}$ for all $j \in \mathcal{J}$.

Invariant Column Preferences: Column Preferences holds, and there is an ordering $\succeq_{\mathcal{K}}$ on $\mathcal{Y}^{\mathcal{J}}$ such that $\succeq_k = \succeq_{\mathcal{K}}$ for all $k \in \mathcal{K}$.

These five axioms draw their use from the following proposition, which the proofs in Appendix B will repeatedly use. (Each of these proofs will involve two of the sets $\mathcal{I}$, $\mathcal{S}$, $\mathcal{O}$ taking the place of the abstract indexing sets $\mathcal{J}$ and $\mathcal{K}$.)

Proposition A2 (a) *Suppose a continuous preference order $\succeq$ on $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$ satisfies Coordinate Monotonicity, Row Preferences and Column Preferences. Then for all $j \in \mathcal{J}$ and $k \in \mathcal{K}$, there is an increasing, continuous function $v_k^j : \mathbb{R} \longrightarrow \mathbb{R}$, such that $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by: $W(\mathbf{X}) := \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} v_k^j(x_k^j)$. In this representation, the functions v_k^j are unique up to PAT with a common multiplier.*

(b) *Assume Invariant Column Preferences instead of Column Preferences, holding the other conditions the same as in part (a). Then there is a strictly positive probability vector $\mathbf{p} \in \Delta_{\mathcal{K}}$, and for all $j \in \mathcal{J}$, there is an increasing, continuous function $u^j : \mathbb{R} \longrightarrow \mathbb{R}$, such that $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by: $W(\mathbf{X}) := \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} p_k \, u^j(x_k^j)$. In this representation, the probability vector $\mathbf{p}$ is unique, and the functions u^j are unique up to PAT with a common multiplier.*

(c) *Assume Invariant Row Preferences instead of Row Preferences, holding the other conditions the same as in part (b). Then there is an increasing, continuous function $u : \mathbb{R} \longrightarrow \mathbb{R}$ and strictly positive probability vectors $\mathbf{q} \in \Delta_{\mathcal{J}}$ and $\mathbf{p} \in \Delta_{\mathcal{K}}$ such that $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by $W(\mathbf{X}) := \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} q^j \, p_k \, u(x_k^j)$. In this representation, $\mathbf{q}$ and $\mathbf{p}$ are unique, and u is unique up to a PAT.*

Proof. See Mongin and Pivato (2015). Part (a) follows from Proposition 1(b). Part (b) follows from Theorem 1(c,d), and part (c) from Corollary 1(c,d). The axioms of that paper are stated differently, because the domains considered there are not necessarily Cartesian products. □

B Proofs of the results of the paper

Our framework may seem to raise the possibility that conditional orderings depend on how they are induced; e.g., that $\succsim_{so}$, as directly induced by $\succsim$, differs from $\succsim_{so}$, as induced by the ordering $\succsim_s$ induced by $\succsim$, or from $\succsim_{so}$, as induced by the ordering $\succsim_o$. But such a discrepancy cannot occur, as the different forms of conditionalization commute with one another (we skip the proof). In what follows, we use this property without saying.

Lemma B1 *Let $\succeq$ be a continuous order on $\mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$.*

(a) *If $\succeq$ induces interim preferences $\succeq_s$ and $\succeq_o$, then it induces* ex post *preferences $\succeq_{so}$.*

(b) *If, moreover, the interim preferences $\succeq_o$ are invariant, then for any given s, $\succeq_s$ induces invariant* ex post *preferences $\succeq_{so}$.*

(c) *If, moreover, the interim preferences $\succeq_o$ and $\succeq_s$ are both invariant, then the* ex post *preferences $\succeq_{so}$ are invariant.*

Proof. Let $(s,o) \in \mathcal{S}\times\mathcal{O}$. For all $o \in \mathcal{O}$, let $\mathcal{J}_o := \{(i',s',o);\ i' \in \mathcal{I} \text{ and } s' \in \mathcal{S}\}$. Then $\mathcal{J}_o$ is a $\succeq$-separable subset of $\mathcal{I}\times\mathcal{S}\times\mathcal{O}$, because, by hypothesis, $\succeq$ induces interim preferences $\succeq_o$. Similarly, for all $s \in \mathcal{S}$, let $\mathcal{K}_s := \{(i',s,o');\ i' \in \mathcal{I} \text{ and } o' \in \mathcal{O}\}$; this is a $\succeq$-separable subset of $\mathcal{I}\times\mathcal{S}\times\mathcal{O}$, because $\succeq$ induces interim preferences $\succeq_s$. The nonempty intersection $\mathcal{I}_{so} := \mathcal{J}_o \cap \mathcal{K}_s$ is $\succeq$-separable by a classic result of Gorman (1968). Thus, $\succeq$ induces *ex post* preferences $\succeq_{so}$.

Adding the assumption that the interim preferences $\succeq_o$ induced by $\succeq$ are invariant, we fix s and consider any pair $o \neq o'$. By commutativity of conditonalization , we can regard the *ex post* preferences $\succeq_{so}$ and $\succeq_{so'}$ as being induced by $\succeq_o$ and $\succeq_{o'}$, respectively. But $\succeq_o = \succeq_{o'}$, so that $\succeq_{so} = \succeq_{so'}$, and now regarding these *ex post* preferences as being induced by $\succeq_s$, we conclude that this ordering induces invariant *ex post* preferences.

Now we add the assumption that the interim preferences $\succeq_s$ induced by $\succeq$ are invariant, fix o and consider any pair $s \neq s'$. By symmetric reasoning, we conclude that $\succeq_{so} = \succeq_{s'o}$. The two paragraphs together prove that, for all $o, o' \in \mathcal{O}$ and $s, s' \in \mathcal{S}$, $\succeq_{so} = \succeq_{s'o'}$, meaning that $\succeq$ induces invariant *ex post* preferences. □

Proof of Proposition 1. Let $\mathcal{J} := \mathcal{S}$ and $\mathcal{K} := \mathcal{O}$. We will check which of the axioms of Appendix A apply to the ordering $\succeq^i$, for any $i \in \mathcal{I}$. Coordinate Monotonicity holds because $\succeq^i$ induces preference orderings $\succeq^i_{so}$ that coincide with the natural ordering of real numbers, by statement (1). As the $\succeq^i_s$ (resp. the $\succeq^i_o$) are invariant, Invariant Row Preferences (resp. Invariant Column Preferences) holds. Thus, Proposition A2(c) yields the expected utility representation (2) for $\succeq_i$. Since $\succeq$ has a numerical representation that is increasing with the $\succeq^i$ by the *ex ante* Pareto principle, the social representation (3) follows. The uniqueness condition for F is obvious, and the other uniqueness statements follow from Proposition A2(c). □

Proof of Proposition 2. By Lemma B1(c), the assumption that $\succeq$ induces invariant interim preferences of both kinds guarantees that $\succeq$ also induces invariant *ex post* preferences $\succeq_{\text{xp}}$ on $\mathbb{R}^{\mathcal{I}}$. These preferences inherit the continuity of $\succeq$ and the *ex post* Pareto principle makes them increasing in every coordinate. Thus, each of them is represented by a continuous and increasing function $v : \mathbb{R}^{\mathcal{I}} \longrightarrow \mathbb{R}$.

To any $\mathbb{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$, we associate the element $\widetilde{\mathbf{X}} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ whose (s, o) component is $\widetilde{x}_{so} := v(\mathbf{x}_{so})$. The function $V : \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}} \to \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ defined by $V(\mathbb{X}) := \widetilde{\mathbf{X}}$ is continuous and increasing in each component. By these two properties, the image set $\widetilde{\mathcal{X}} := V(\mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}})$ is a set of the form $\mathcal{Y}^{\mathcal{S}\times\mathcal{O}}$, where $\mathcal{Y} := v(\mathbb{R}^{\mathcal{I}})$ is an open interval.

Define an ordering $\widetilde{\succeq}$ on $\widetilde{\mathcal{X}}$ by the condition that for all $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} \in \widetilde{\mathcal{X}}$, if $\widetilde{\mathbf{X}} = V(\mathbb{X})$ and $\widetilde{\mathbf{Y}} = V(\mathbb{Y})$, then

$$\widetilde{\mathbf{X}} \mathrel{\widetilde{\succeq}} \widetilde{\mathbf{Y}} \text{ if and only if } \mathbb{X} \succeq \mathbb{Y}. \tag{B1}$$

(To see that $\widetilde{\succeq}$ is mathematically well-defined by (B1), suppose $V(\mathbb{X}) = \widetilde{\mathbf{X}} = V(\mathbb{X}')$ for some $\mathbb{X}, \mathbb{X}' \in \mathcal{X}$. Then for all $(s, o) \in \mathcal{S} \times \mathcal{O}$, we have $v(\mathbf{x}_{so}) = v(\mathbf{x}'_{so})$, and hence $\mathbf{x}_{so} \approx_{\text{xp}} \mathbf{x}'_{so}$. Thus $\mathbb{X} \approx \mathbb{X}'$, because $\succeq$ is increasing relative to $\succeq_{\text{xp}}$.) In terms of the Appendix A, putting $\mathcal{J} := \mathcal{S}$ and $\mathcal{K} := \mathcal{O}$, we conclude that $\widetilde{\succeq}$ is continuous and satisfies Invariant Row Preferences and Invariant Column Preferences, and Coordinate Monotonicity, by using the respective properties that $\succeq$ is continuous, induces invariant interim orderings $\succeq_s$, and induces invariant interim orderings $\succeq_o$, and induces invariant *ex post* orderings $\succeq_{\text{xp}}$. Thus, Proposition A2(c) yields strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ and $\mathbf{q} \in \Delta_{\mathcal{O}}$, and a continuous increasing function $u : \mathbb{R} \longrightarrow \mathbb{R}$, such that $\widetilde{\succeq}$ is represented by the function $\widetilde{W} : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ defined by $\widetilde{W}(\widetilde{\mathbf{X}}) := \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} q_o\, p_s\, u(\widetilde{x}_{so})$. Now set $W_{\text{xa}}(\mathbb{X}) := \widetilde{W} \circ V(\mathbf{X})$ for all $\mathbf{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$, and $W_{\mathbf{xp}}(\mathbf{x}) := u \circ v(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{\mathcal{I}}$ to get the desired representations. The uniqueness properties are those of Proposition A2(c).[21] □

Proof of Proposition 3. First we show that $\succeq$ is increasing in every coordinate. Let $(i, s, o) \in \mathcal{I} \times \mathcal{S} \times \mathcal{O}$. Statement (1) implies that $\succeq^i$ is increasing with respect to the coordinate $x^i_{s,o}$. By the *ex ante* Pareto principle, $\succeq$ is also increasing with respect to $x^i_{s,o}$.

The result now follows from Theorem A2(b), by setting $\mathcal{J} := \mathcal{I}$ and $\mathcal{K} := \mathcal{S}\times\mathcal{O}$. *Ex ante* Pareto then becomes Row Preferences, while the existence of invariant *ex post* preferences yields Invariant Column Preferences. Meanwhile, $\succeq$ satisfies Coordinate Monotonicity by the previous paragraph. □

Proof of Theorem 1. First note that $\succeq$ is increasing in every coordinate, by exactly the same argument as the first paragraph in the proof of Proposition 3. Next, since the $\succeq^i$ relations are orderings and the *ex ante* Pareto principle makes $\succeq$ increasing with them, $\succeq$ is separable in each $i \in \mathcal{I}$. As $\succeq$ induces interim preferences of both types, $\succeq$ is also separable in each $s \in \mathcal{S}$ and $o \in \mathcal{O}$. It then follows from Proposition A1 that, for all $(i, s, o) \in \mathcal{I} \times \mathcal{S} \times \mathcal{O}$, there exist continuous and increasing functions $u^i_{so} : \mathbb{R} \longrightarrow \mathbb{R}$ such

[21]Proposition A2(c) is stated for $\mathbb{R}^{\mathcal{J}\times\mathcal{K}}$, but it carries through to subsets $Y^{\mathcal{J}\times\mathcal{K}} \subseteq \mathbb{R}^{\mathcal{J}\times\mathcal{K}}$, when these are open and take the form of a product of intervals.

that $\succeq$ is represented by the function $W_{\text{xa}} : \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$W_{\text{xa}}\left(\mathbb{X}\right) \quad := \quad \sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} u^i_{so}(x^i_{so}). \tag{B2}$$

Furthermore, the u^i_{so} are unique up to positive affine transformations (PAT) with a common multiplier. We can fix any $\mathbb{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$ and add constants to these functions so as to ensure that $u^i_{so}(y^i_{so}) = 0$ for all $(i,s,o) \in \mathcal{I}\times\mathcal{S}\times\mathcal{O}$.[22] For convenience, fix some $\overline{y} \in \mathbb{R}$, and suppose that $y^i_{so} = \overline{y}$ for all $(i,s,o) \in \mathcal{I}\times\mathcal{S}\times\mathcal{O}$.

For all $i \in \mathcal{I}$, equation (B2) implies that the preference ordering $\succeq^i$ can be represented by the function $U^i : \mathbb{R}^{\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} u^i_{so}(x_{so}). \tag{B3}$$

From Proposition 1 , there are continuous increasing utility functions $\tilde{u}^i : \mathbb{R} \longrightarrow \mathbb{R}$, and two strictly positive probability vectors $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ and $\mathbf{q}^i \in \Delta_{\mathcal{O}}$, such that $\succeq^i$ is represented by the function $U^i : \mathbb{R}^{\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} q^i_o\, p^i_s\, \tilde{u}^i(x_{so}). \tag{B4}$$

Furthermore, in this representation, $\mathbf{p}^i$ and $\mathbf{q}^i$ are unique, and $\tilde{u}^i$ is unique up to PAT. By adding a constant, we ensure that $\tilde{u}^i(\overline{y}) = 0$.

From the uniqueness property applied to (B3) and (B4), there exist constants $\alpha^i > 0$ and $\beta^i \in \mathbb{R}$ such that :

$$u^i_{so}(x) = \alpha^i\, q^i_o\, p^i_s\, \tilde{u}^i(x) + \beta^i, \text{ for all } (s,o) \in \mathcal{S}\times\mathcal{O}. \tag{B5}$$

Substituting $x = \overline{y}$ into (B5) leads to $\beta^i = 0$. Then substituting (B5) (for all $i \in \mathcal{I}$) into the representation (B2) yields:

$$W_{\text{xa}}\left(\mathbb{X}\right) \quad = \quad \sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} \alpha^i\, q^i_o\, p^i_s\, \tilde{u}^i(x^i_{so}). \tag{B6}$$

For given $s \in \mathcal{S}$ in this representation, we obtain a representation $V_s : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ of the interim preference $\succeq_s$ on $\mathbb{R}^{\mathcal{I}\times\mathcal{O}}$:

$$V_s(\mathbf{X}) \quad := \quad \sum_{i\in\mathcal{I}}\sum_{o\in\mathcal{O}} \alpha^i\, q^i_o\, p^i_s\, \tilde{u}^i(x^i_o). \tag{B7}$$

Let $\mathbf{Y}_s := (\overline{y}, \ldots, \overline{y}) \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}$; then $V_s(\mathbf{Y}_s) \; = 0$.

Let us now put $\mathcal{J} := \mathcal{I}$ and $\mathcal{K} := \mathcal{O}$, and check which axioms in Appendix A the interim preference $\succeq_s$ satisfies. This is a continuous ordering by the continuity of $\succeq$. By

[22] To avoid burdening notation, we refer to the original and translated functions by the same symbol. This convention is applied throughout the proofs.

the representation (B7), $\succeq_s$ is separable in each $\{i\} \times \mathcal{O}$ and each $\mathcal{I} \times \{o\}$, and increasing in every coordinate , and thus satisfies Row Preferences, Column Preferences, and Coordinate Monotonicity. As $\succeq$ induces invariant $\succeq_o$, Lemma B1(b) entails that the induced preferences $\succeq_{so}$ are invariant, meaning that the stronger axiom of Invariant Column Preferences holds. Hence, Proposition A2(b) yields a strictly positive probability vector $\mathbf{r}_s \in \Delta_{\mathcal{O}}$, and for all $i \in \mathcal{I}$, continuous, increasing utility functions $\widehat{u}_s^i : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\succeq_s$ is represented by the function $\widehat{V}_s : \mathbb{R}^{\mathcal{I} \times \mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$\widehat{V}_s(\mathbf{X}) \quad := \quad \sum_{i \in \mathcal{I}} \sum_{o \in \mathcal{O}} r_{so} \, \widehat{u}_s^i(x_o^i). \tag{B8}$$

In this representation, $\mathbf{r}_s$ is unique and $\{\widehat{u}_s^i\}_{i \in \mathcal{I}}$ are unique up to PAT with a common multiplier. Add constants to ensure that $\widehat{u}_s^i(\overline{y}) = 0$ for all $i \in \mathcal{I}$. It follows that $\widehat{V}_s(\mathbf{Y}_s) = 0$.

From the uniqueness property applied to (B7) and (B8), there exist $\gamma_s > 0$ and $\delta_s \in \mathbb{R}$ such that $\widehat{V}_s = \gamma_s \, V_s + \delta_s$. Substituting $\mathbf{Y}_s$ leads to $\delta_s = 0$. Since this holds for all $s \in \mathcal{S}$, we can conclude that

$$\gamma_s \, r_{so} \, \widehat{u}_s^i \quad = \quad \alpha^i \, q_o^i \, p_s^i \, \tilde{u}^i, \text{ for all } (i, s, o) \in \mathcal{I} \times \mathcal{S} \times \mathcal{O} \tag{B9}$$

Let us now fix i and s in these equations. All the coefficients are positive and the increasing functions $\widehat{u}_s^i$ and $\tilde{u}^i$ are nonzero for some $y^* \in \mathbb{R}$. Thus we can derive the relations:

$$\frac{r_{so}}{q_o^i} \quad = \quad \frac{\alpha^i \, p_s^i \, \tilde{u}^i(y^*)}{\gamma_s \, \widehat{u}_s^i(y^*)}, \quad \text{for all } o \in \mathcal{O}. \tag{B10}$$

The right-hand side of (B10) does not depend on o. Thus, the left-hand side must also be independent of o, which means that the vectors $\mathbf{q}^i$ and $\mathbf{r}_s$ are scalar multiples of one another. Thus, since they are probability vectors, we have $\mathbf{q}^i = \mathbf{r}_s$. Since this holds for all i and s, we can drop the indexes. Denote the common probability vector by $\mathbf{q}$. Substituting $\mathbf{q}$ into (B6) and defining $u^i := \alpha^i \, \tilde{u}^i$, we get the formula (5) of the theorem. The other parts readily follow. □

Proof of Theorem 2. For each $i \in \mathcal{I}$, $\succeq^i$ satisfies the assumptions of Proposition 1. Thus, by the argument used to prove this proposition, we conclude that there exist a continuous increasing utility function $u^i : \mathbb{R} \longrightarrow \mathbb{R}$, and strictly positive probability vectors $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ and $\mathbf{q}^i \in \Delta_{\mathcal{O}}$, such that $\succeq^i$ is represented by the function $U^i : \mathbb{R}^{\mathcal{S} \times \mathcal{O}} \longrightarrow \mathbb{R}$ defined by $U^i(\mathbf{X}) := \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} q_o^i \, p_s^i \, u^i(x_{so})$, and the u^i are unique up to PAT. This establishes the SEU representation (7). Fix $\overline{x} \in \mathbb{R}$. By adding constants, we ensure that $u^i(\overline{x}) = 0$ for all $i \in \mathcal{I}$.

Meanwhile, Proposition 2 yields strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ and $\mathbf{q} \in \Delta_{\mathcal{O}}$, and a continuous increasing function $W_{\text{xp}} : \mathbb{R}^{\mathcal{I}} \longrightarrow \mathbb{R}$, such that $\succeq$ is represented by the function $W_{\text{xa}} : \mathcal{X} \longrightarrow \mathbb{R}$ defined by $W_{\text{xa}}(\mathbb{X}) := \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} p_s \, q_o \, W_{\text{xp}}(\mathbf{x}_{so})$, where $\mathbf{p}$ and $\mathbf{q}$ are unique, and W_{xp} is unique up to PAT. This establishes the SEU representation (8). Let $\overline{\mathbf{x}} := (\overline{x}, \ldots, \overline{x})$. By adding a constant, we ensure that $W_{\text{xp}}(\overline{\mathbf{x}}) = 0$.

Now let $\mathcal{J} = \mathcal{I}$ and $\mathcal{K} = \mathcal{O}$ and consider how the axioms of Appendix A apply to $\succeq_s$ for any given $s \in \mathcal{S}$, recalling that these interim social preferences are well-defined and invariant (i.e. independent of s) . The objective interim Pareto principle makes $\succeq_s$ separable in each $i \in \mathcal{I}$, so that Row Preferences holds. By Proposition 2, the *ex post* social preferences $\succeq_{so}$ are well-defined and invariant, so that Invariant Column Preferences holds. Then, by Proposition A2(b), there exist a probability vector $\widetilde{\mathbf{q}} \in \Delta_{\mathcal{O}}$, and for all $i \in \mathcal{I}$, continuous increasing functions v^i such that $\succeq_s$ is represented by the function $W : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by $W(\mathbf{X}) := \sum_{i\in\mathcal{I}} \sum_{o\in\mathcal{O}} \widetilde{q}_o\, v^i(x^i_{so})$, where $\widetilde{\mathbf{q}}$ is unique and the v^i are unique up to PAT with a common multiplier. The same representation holds for all $s \in \mathcal{S}$. Adding a constant, we ensure that $v^i(\overline{x}) = 0$ for all $i \in \mathcal{I}$.

We now show that $\mathbf{q} = \widetilde{\mathbf{q}}$. By fixing $s \in \mathcal{S}$ and applying the representation W_{xa} to elements $\mathbb{X}$ whose components for $s' \neq s$ are fixed at some values, we obtain a new representation for $\succeq_s$ and reduce it to the representation just obtained in terms of W by the standard uniqueness property. That is, there exist constants $\alpha > 0$ and β such that $\sum_{o\in\mathcal{O}} q_o\, W_{\text{xp}}(\mathbf{x}_o) = \alpha \sum_{i\in\mathcal{I}} \sum_{o\in\mathcal{O}} \widetilde{q}_o\, v^i(x^i_o) + \beta$, for all $\mathbf{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}$. Substituting $x^i_o = \overline{x}$ for all $i \in \mathcal{I}$ and $o \in \mathcal{O}$ leads to $\beta = 0$. Now fixing o and putting $x^i_{o'} = \overline{x}$ for all $o' \neq o$ leads to the equation: $W_{\text{xp}}(\mathbf{x}_o) = \frac{\widetilde{q}_o}{q_o} \sum_{i\in\mathcal{I}} \alpha\, v^i(x^i_{so})$, for all $\mathbf{x}_o \in \mathbb{R}^{\mathcal{I}}$. Since this holds for all $o \in \mathcal{O}$, the two probability vectors $\mathbf{q}$ and $\widetilde{\mathbf{q}}$ are proportional, hence equal. Hence

$$W_{\text{xp}}(\mathbf{x}_o) \quad = \quad \sum_{i\in\mathcal{I}} \alpha\, v^i(x^i_{so}), \qquad \text{for all } \mathbf{x}_o \in \mathbb{R}^{\mathcal{I}}. \tag{B11}$$

and the invariant conditional preference $\succeq_s$ is represented by the function $\widetilde{W} : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by $\widetilde{W}(\mathbf{X}) := \sum_{i\in\mathcal{I}} \sum_{o\in\mathcal{O}} q_o\, \alpha\, v^i(x^i_{so})$. We now use a similar argument to show that $\mathbf{q} = \mathbf{q}^i$ for all $i \in \mathcal{I}$. Fixing $i \in \mathcal{I}$ and $s \in \mathcal{S}$, we can obtain a representation for the invariant interim preferences $\succeq^i_s$ in two ways: first, from $\widetilde{W}$ by applying this representation to elements of $\mathbb{R}^{\mathcal{I}\times\mathcal{O}}$ whose components for $i' \neq i$ are fixed at some values (because $\succeq_s$ satisfies the objective interim Pareto principle), and second, from U^i by applying this representation to elements of $\mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ whose components for $s' \neq s$ are fixed at some values. By the standard uniqueness property, there exist $\gamma^i_s > 0$ and δ^i_s such that

$$\sum_{o\in\mathcal{O}} q_o \alpha v^i(x_o) \quad = \quad \gamma^i_s \sum_{o\in\mathcal{O}} q^i_o\, p^i_s\, u^i(x_o) + \delta^i_s, \quad \text{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{O}}. \tag{B12}$$

Substituting $x_o = \overline{x}$ into (B12) leads to $\delta^i_s = 0$. Fix $o \in \mathcal{O}$. Put $x_{o'} = \overline{x}$ for all $o' \neq o$ yields:

$$\frac{q_o}{q^i_o} \alpha v^i(x) \quad = \quad \gamma^i_s\, p^i_s\, u^i(x) \text{ for all } x \in \mathbb{R}. \tag{B13}$$

The right-hand side of (B13) is independent of o. Thus, the probability vectors $\mathbf{q}$ and $\mathbf{q}^i$ are proportional, hence equal, and thus

$$\alpha v^i(x) \quad = \quad \gamma^i_s\, p^i_s\, u^i(x) \text{ for all } x \in \mathbb{R}. \tag{B14}$$

Equation (B14) holds for all $s \in \mathcal{S}$. Hence, for all $i \in \mathcal{I}$, the product $r^i := \gamma^i_s p^i_s$ is independent of s; note that $r^i > 0$. Equation (B14) now says $\alpha v^i = r^i u^i$. Substituting this into the representation (B11) yields the representation (9) for $\succeq_{\text{xp}}$. □

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