MODEL THEORY OF FIELDS WITH FREE OPERATORS IN CHARACTERISTIC ZERO

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Abstract. Generalising and unifying the known theorems for difference and differential fields, it is shown that for every finite free $\mathbb{S}$-algebra $\mathcal{D}$ over a field A of characteristic zero the theory of $\mathcal{D}$-fields has a model companion $\mathcal{D}\text{-CF}_0$ which is simple and satisfies the Zilber dichotomy for finite-dimensional minimal types.

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1. Introduction

The theories of differential and difference fields instantiate some of the most sophisticated ideas and theorems in model theoretic stability theory. For example, the theory of differentially closed fields of characteristic zero, DCF_0, is an ω-stable theory for which the full panoply of geometric stability theory applies from the existence and uniqueness of prime models (and, hence, of differential closures) to the theory of liaison groups (and, thus, a very general differential Galois theory) to the Zilber trichotomy for strongly minimal sets (from which strong theorems about function field arithmetic have been deduced). Likewise, the model companion of the theory of difference fields, ACFA, is supersimple and admits an analogous theory of internal automorphism groups and satisfies a version of the Zilber dichotomy for its minimal types. Beyond the formal analogies and parallel theorems, the proofs of the basic results in the model theory of differential and difference fields follow similar though not identical lines. In this paper we formalise the sense in which these theories are specialisations of a common theory of fields with operators and how the theories may be developed in one fell swoop. On the other hand, features which emerge from the general theory explain how the theories of differential and difference fields diverge.

By definition a derivation on a commutative ring R is an additive map $\partial : R \to R$ which satisfies the Leibniz rule $\partial(xy) = x\partial(y) + y\partial(x)$. Equivalently, the function $e : R \to R[\epsilon]/(\epsilon^2)$ given by $x \mapsto x + \partial(x)\epsilon$ is a homomorphism of rings. An endomorphism $\sigma : R \to R$ of a ring is simply a ring homomorphism from the ring R back to itself, but at the risk of complicating the definition, we may also say that a function $\sigma : R \to R$ is an endomorphism if the function $e : R \to R \times R$ given by $x \mapsto (x, \sigma(x))$ is a homomorphism of rings. With each of the latter presentations we see differential (respectively, difference) ring as a $\mathcal{D}$-ring in the sense introduced in [11].

As the details of the $\mathcal{D}$-ring formalism along with many examples are presented in Section 3, we limit ourselves to a loose discussion here. For each fixed ring scheme $\mathcal{D}$ (possibly over some base ring A) satisfying some additional requirements we have a theory of $\mathcal{D}$-fields. In particular, we require that the underlying additive group scheme of $\mathcal{D}$ be some power of the additive group scheme so that for any A-algebra R, $\mathcal{D}(R) = (R^n, +, \boxtimes)$ where the multiplication $\boxtimes$ is given by some bilinear form defined over A. We require that $\mathcal{D}$ comes equipped with a functorial projection map to the standard ring scheme and that for the sake of concreteness, read relative to coordinates this projection map be given by projection onto the first coordinate. A $\mathcal{D}$-ring is then a pair (R, e) consisting of an A-algebra R and a map of A-algebras $e : R \to \mathcal{D}(R)$ which is a section of the projection. In the motivating examples, $\mathcal{D}(R) = R[\epsilon]/(\epsilon^2) = (R^2, +, \boxtimes)$ where $(x_1, x_2) \boxtimes (y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ ($\mathcal{D}(R) = R \times R$ with coordinatewise ring operations, respectively).

In general, using the coordinatization of $\mathcal{D}(R)$, the data of a $\mathcal{D}$-ring (R, e) is equivalent to that of a ring R given together with a sequence $\partial_0, \ldots, \partial_{n-1}$ of operators $\partial_i : R \to R$ for which the map $e : R \to \mathcal{D}(R)$ is given in coordinates by $x \mapsto (\partial_0(x), \ldots, \partial_{n-1}(x))$. The requirements on such a sequence of operators that they define a $\mathcal{D}$-ring structure may be expressed by certain universal axioms. For example, to say that the map $e : R \to \mathcal{D}(R)$ is a section of the projection is just to say that $(\forall x \in R)\partial_0(x) = x$ and the requirement that e convert multiplication in R to the multiplication of $\mathcal{D}(R)$ may be expressed by insisting that certain polynomial relations hold amongst $\partial_0(x), \ldots, \partial_{n-1}(x); \partial_0(y), \ldots, \partial_{n-1}(y); \partial_0(xy), \ldots, \partial_{n-1}(xy)$. In this way, the class of $\mathcal{D}$-fields is easily seen to be first order in the language of rings augmented by unary function symbols for the operators $\partial_0, \ldots, \partial_{n-1}$. On the other hand, the interpretation of the operators as components of a ring homomorphism permits us to apply ideas from commutative algebra and algebraic geometry to analyze these theories.

Our first main theorem is that for any ring scheme $\mathcal{D}$ (meeting the requirements set out in Section 3), the theory of $\mathcal{D}$-fields of characteristic zero has a model companion, which we denote by $\mathcal{D}\text{-CF}_0$ and call the theory of $\mathcal{D}$-closed fields. Our axiomatization of $\mathcal{D}\text{-CF}_0$ follows the geometric style which first appeared in the Chatzidakis-Hrushovski axioms for ACFA [2] and was then extended to differential fields by Pierce and Pillay [13]. Moreover, the proofs will be familiar to anyone who has worked through the corresponding results for difference and differential fields. Following the known proofs for difference and differential fields, we establish a quantifier simplification theorem and show that $\mathcal{D}\text{-CF}_0$ is always simple.

As noted above, the theory DCF_0 is the quintessential ω-stable theory, but ACFA, the model companion of the theory of difference fields is not even stable. At a technical level, the instability of ACFA may be traced to the failure of quantifier elimination which, algebraically, is due to the non-uniqueness (up to isomorphism) of the extension of an automorphism of a field to an automorphism of its algebraic closure. We show that this phenomenon, namely that instability is tied to the nonuniqueness of extensions of automorphisms, pervades the theory of $\mathcal{D}$-fields. That is, for each $\mathcal{D}$ there is a finite list of associated endomorphisms expressible as linear combinations of the basic operators. Since we require that ∂_0 is the identity map, one of these associated endomorphisms is always the identity map. If there are any others, then the theory of $\mathcal{D}\text{-CF}_0$ suffers from instability and the failure of quantifier elimination just as does ACFA. On the other hand, if there are no other associated endomorphisms, then $\mathcal{D}\text{-CF}_0$ is stable.

The deepest of the fine structural theorems for types in DCF_0 and in ACFA is the Zilber dichotomy for minimal types, first established by Sokolović and Hrushovski for DCF_0 using Zariski geometries [6], for ACFA_0 by Chatzidakis and Hrushovski through a study of ramification [2], and by Chatzidakis, Hrushovski and Peterzil for ACFA in all characteristics using the theory of limit types and a refined form of the theory of Zariski geometries [3]. Subsequently, Pillay and Ziegler established a stronger form of the trichotomy theorem in characteristic zero [14] by adapting jet space arguments Campana and Fujiki used to study complex manifolds [1, 4]. Here we implement the Pillay-Ziegler strategy for $\mathcal{D}\text{-CF}_0$ by using the theory of $\mathcal{D}$-jet spaces from [12]. In particular, we show that $\mathcal{D}\text{-CF}_0$ has the canonical base property.

The model companion of the theory of difference fields of characteristic zero with n automorphisms appears as $\mathcal{D}\text{-CF}_0$ where $\mathcal{D}(R) := R^{1+n}$ in contradistinction to the well-known fact that the theory of difference fields with n (> 1) commuting automorphisms does not have a model companion. On the other hand, if $(\mathbb{U}, \partial_0, \partial_1, \ldots, \partial_n) \models \mathcal{D}\text{-CF}_0$ is sufficiently saturated, then the type definable field obtained as the intersection of the fixed fields of all the elements of the commutator group of the group generated by $\partial_1, \ldots, \partial_n$ has Lascar rank ω^n and may be regarded as a universal domain for difference fields with n commuting automorphisms. (See Section 1.2 of [7] for a discussion of these issue.) Likewise, models of the theory $\mathrm{DCF}_{0,n}$ of differentially closed fields with n commuting derivations may be realized as type definable fields in models of $\mathcal{D}$-CF_0 where $\mathcal{D}(R) = R[\epsilon_1, \ldots, \epsilon_n]/(\epsilon_1, \ldots, \epsilon_n)^2$.

While omitting commutation allows for model companions in characteristic zero, it complicates matters in positive characteristic. Under a natural algebraic hypothesis on p and $\mathcal{D}$, namely that there be some $\epsilon \in \mathcal{D}(A)$ which is nilpotent but for which $\epsilon^p \neq 0$, we observe with Proposition 7.2 that no model companion of the theory of $\mathcal{D}$-fields of characteristic p exists. This proposition is consonant with the known examples of ACFA_p and DCF_p where no such ϵ exists. However, it implies that the theory of (not necessarily iterative) Hasse-Schmidt differential fields of positive characteristic does not have a model companion, which is at odds with the iterative theory $\mathrm{SCH}_{p,e}$ considered by Ziegler [16]. While the theory of iterative $\mathcal{D}$-fields developed in [12] was intended as an abstraction of the theory of iterative Hasse-Schmidt differential fields, we have not yet understood the extent to which the theorems around $\mathrm{SCH}_{p,e}$ generalise to iterative $\mathcal{D}$-fields. As such, we leave open the problems of which theories of iterative $\mathcal{D}$-fields and which theories of positive characteristic $\mathcal{D}$-fields have model companions.

This paper is organized as follows. We begin in Section 2 with some remarks about our conventions. With Section 3 we recall the formalism of $\mathcal{D}$-rings in detail and present several examples. In Section 4 we give axioms for the theory $\mathcal{D}$-CF_0 and prove that it is in fact the model companion of the theory of $\mathcal{D}$-fields of characteristic zero. In Section 5 we establish the essential model theoretic properties of $\mathcal{D}$-closed fields. In Section 6 we give a proof of the Zilber dichotomy for minimal types of finite dimension. We conclude with an appendix in which we show that the theory of $\mathcal{D}$-fields does not have a model companion for most choices of $\mathcal{D}$ in positive characteristic, and also explain how a convenient set of assumptions made early in the paper can be removed.

2. Notation and conventions

As a general rule, we follow standard conventions in model theory and differential algebra and introduce unfamiliar notation as needed. We do move between scheme theory and Weil-style algebraic geometry. For the most part, the theory of prolongation and jet spaces must developed scheme theoretically as we make essential use of nonreduced bases. However, in the applications to the first order theories of fields, as is common in model theoretic algebra, we sometimes use Weil-style language. Let us note some of these conventions. If X is some variety over a field K, L is an extension field of K, and $a \in X(L)$ is an L-rational point, then $\mathrm{loc}(a/K)$, the locus of a over K, is the intersection of all closed K-subvarieties $Y \subseteq X$ with $a \in Y(K)$. If X is affine with coordinate ring $\mathcal{O}_X$, then we define $I(a/K) := \{f \in \mathcal{O}_X : f(a) = 0\}$ to be the ideal of a over K. Generalizing somewhat, for $Y \subseteq X_L$ a subvariety of the base change of X to L, we define $I(Y/K) := \{f \in \mathcal{O}_X : f \text{ vanishes on } Y\}$. Note that the locus of a over K is the variety defined by the ideal of a over K. We say that a is a generic point of X over K (or is K-generic in X) if $\mathrm{loc}(a/K) = X$. Scheme theoretically, one would say that $I(a/K)$ is the generic point of X, but these two points of view will not appear in the same section.

3. $\mathcal{D}$-RINGS

Throughout this paper we will fix the following data, sometimes making further assumptions about them:

A. a base ring A
B. a finite free A-algebra $\mathcal{D}(A)$,
C. an A-algebra homomorphism $\pi^A : \mathcal{D}(A) \to A$, and
D. an A-basis $(\epsilon_0, \dots, \epsilon_{\ell-1})$ for $\mathcal{D}(A)$ such that $\pi^A(\epsilon_0) = 1$ and $\pi^A(\epsilon_i) = 0$ for all $i = 1, \dots, \ell - 1$.

An equivalent scheme-theoretic way to describe these data is as a *finite free* $\mathbb{S}$-*algebra scheme over* A *with basis* in the sense of [11] and [12]. Here $\mathbb{S}$ denotes the ring scheme which when evaluated at any A-algebra R is just the ring R itself. That is, $\mathbb{S}$ is simply the affine line $\operatorname{Spec}\big(A[x]\big)$ endowed with the usual ring scheme structure. Instead of B, C, D as above we could consider the basic data as being

B′. an $\mathbb{S}$-algebra scheme $\mathcal{D}$ over A; that is, a ring scheme $\mathcal{D}$ over A together with a ring scheme morphism $s : \mathbb{S} \to \mathcal{D}$ over A,
C′. a morphism of $\mathbb{S}$-algebra schemes $\pi : \mathcal{D} \to \mathbb{S}$ over A, and
D′. an $\mathbb{S}$-linear isomorphism $\psi : \mathcal{D} \to \mathbb{S}^\ell$ over A such that π is ψ composed with the first co-ordinate projection on $\mathbb{S}^\ell$.

Indeed, as is explained on page 14 of [12], we obtain the second presentation from the first as follows: given any A-algebra R define $\mathcal{D}(R) = R \otimes_A \mathcal{D}(A)$, $\pi^R = \mathrm{id}_R \otimes \pi^A$, and $\psi^R = \mathrm{id}_R \otimes \psi^A$ where $\psi^A : \mathcal{D}(A) \to A^\ell$ is the A-linear isomorphism induced by the choice of basis $(\epsilon_0, \dots, \epsilon_{\ell-1})$. To go in the other direction is clear, one just evaluates all the scheme-theoretic data on the ring A. A key point is that given $(\mathcal{D}, \pi, \psi)$, for any A-algebra R there is a canonical isomorphism induced by ψ between $\mathcal{D}(R)$ and $R \otimes_A \mathcal{D}(A)$. While the first presentation of the data is more immediately accessible, it is the second scheme-theoretic one that is more efficient and that we will use.

Remark 3.1. The assumption in D′ that π is ψ composed with the first co-ordinate projection on $\mathbb{S}^\ell$ is new in that it was not made in Definition 2.2 of [12]. However, it can always be made to hold through a change of basis.

The multiplicative structure on $\mathcal{D}(A)$, and hence on $\mathcal{D}(R)$ for any A-algebra R, can be described in terms of the basis by writing

$$\epsilon_i \epsilon_j = \sum_{k=0}^{\ell-1} a_{i,j,k} \epsilon_k \tag{1}$$

$$1_{\mathcal{D}(A)} = \sum_{k=0}^{\ell-1} c_k \epsilon_k \tag{2}$$

where the $a_{i,j,k}$'s and c_k's are elements of A. Note that $a_{0,0,0} = 1$ and $c_0 = 1$.

Definition 3.2 ($\mathcal{D}$-rings)**.** By a $\mathcal{D}$-*ring* we will mean an A-algebra R together with a sequence of operators $\partial := (\partial_1, \dots, \partial_{\ell-1})$ on R such that the map $e : R \to \mathcal{D}(R)$ given by

$$e(r) := r\epsilon_0 + \partial_1(r)\epsilon_1 + \cdots + \partial_{\ell-1}(r)\epsilon_{\ell-1}$$

is an A-algebra homomorphism.

Via the above identity we can move back and forth between thinking of a $\mathcal{D}$-ring as (R, ∂) or as (R, e). It should be remarked that this is not exactly consistent with [11]. In that paper, a "$\mathcal{D}$-ring" was defined to be simply an A-algebra R together with an A-algebra homomorphism $e : R \to \mathcal{D}(R)$. Hence, under the correspondence $(R, \partial) \mapsto (R, e)$, the $\mathcal{D}$-rings of the current paper are precisely the "$\mathcal{D}$-rings" of [11] with the additional assumption that e is a section to $\pi^R : \mathcal{D}(R) \to R$. (Note that this latter assumption already appears in Definition 2.4 of [12].)

The class of $\mathcal{D}$-rings is axiomatisable in the language

$$L_{\mathcal{D}} := \mathcal{L}(0, 1, , +, -, \times, (\lambda_a)_{a \in A}, \partial_1, \dots, \partial_{\ell-1})$$

where λ_a is scalar multiplication by $a \in A$. Indeed, the class of A-algebras is cleary axiomatisable, and the A-linearity of $e : R \to \mathcal{D}(R)$, which is equivalent to $\partial_1, \dots, \partial_{\ell-1}$ being A-linear operators on R, is also axiomatisable. Finally, that e is in addition a ring homomorphism corresponds to the satisfaction of certain A-linear functional equations on the operators. Indeed, using (1) above, we see that the multiplicativity of e is equivalent to

$$\partial_k(xy) = \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} a_{i,j,k}\partial_i(x)\partial_j(y) \quad \text{for all } x, y, \text{ and} \tag{3}$$

$$\partial_k(1_R) = c_k \tag{4}$$

for all $k = 1, \dots, \ell - 1$.

Example 3.3 (Prime $\mathcal{D}$-ring)**.** For any $(A, \mathcal{D}, \pi, \psi)$ there is a unique $\mathcal{D}$-ring structure on A, namely where all the ∂_i are zero on A. This corresponds to $e = s^A$, and is called the *prime $\mathcal{D}$-ring*.

Example 3.4 (Fibred products)**.** We can always combine examples. Given $(\mathcal{D}, \pi, \psi)$ and $(\mathcal{D}', \pi', \psi')$ we can consider the fibred product $\mathcal{D} \times_{\mathbb{S}} \mathcal{D}'$ with $\pi \times \pi' : \mathcal{D} \times_{\mathbb{S}} \mathcal{D}' \to \mathbb{S} \times_{\mathbb{S}} \mathbb{S} = \mathbb{S}$, and $\psi \times \psi' : \mathbb{S}^\ell \times_{\mathbb{S}} \mathbb{S}^{\ell'} = \mathbb{S}^{\ell+\ell'-1}$. The $\mathcal{D} \times_{\mathbb{S}} \mathcal{D}'$-rings will be precisely those of the form (R, ∂, ∂') where (R, ∂) is a $\mathcal{D}$-ring and (R, ∂') is a $\mathcal{D}'$-ring.

Example 3.5. We list here some of the main motivating examples. Characteristic 0 or p specialisations of these examples are obtained by letting A be $\mathbb{Q}$ or $\mathbb{F}_p$, respectively. In what follows R ranges over all A-algebras.

(a) *Differential rings.* Let $\mathcal{D}(R) = R[\eta]/(\eta^2)$ with the natural R-algebra structure, $\pi^R : R[\eta]/(\eta^2) \to R$ be the quotient map, and $(1, \eta)$ the R-basis. Then a $\mathcal{D}$-ring is precisely an A-algebra equipped with a derivation over A.

(b) *Truncated higher derivations.* Generalising the above example, let

$$\mathcal{D}(R) = R[\eta]/(\eta^{n+1})$$

with the natural R-algebra structure, $\pi^R : R[\eta]/(\eta^{n+1}) \to R$ the quotient map, and take as an R-basis $(1, \eta, \dots, \eta^n)$. Then a $\mathcal{D}$-ring is precisely an A-algebra equipped with a *higher derivation of length n over A* in the sense of [9]; that is, a sequence of A-linear maps $(\partial_0 = \mathrm{id}, \partial_1, \dots, \partial_m)$ such that $\partial_i(xy) = \sum_{r+s=i} \partial_r(x)\partial_s(y)$.

It is worth pointing out here that even in characteristic zero (so when $A = \mathbb{Q}$ for example) this is a proper generalisation of differential rings. It is true that ∂_1 is a derivation, and *if* we had imposed the usual iterativity condition then we would have $\partial_i = \dfrac{\partial_1^i}{i!}$. But the point is that being a $\mathcal{D}$-ring does not impose iterativity, the operators are in this sense "free".

(c) *Difference rings.* Let $\mathcal{D}(R) = R^2$ with the product R-algebra structure, π^R the projection onto the first co-ordinate, and (ϵ_0, ϵ_1) the standard basis. Then a $\mathcal{D}$-ring is precisely an A-algebra equipped with an A-endomorphism.

(d) *Partial higher differential-difference rings.* Taking fibred products as in Example 3.4, we can combine the above examples. That is, suppose we are given positive integers $m_1, n_1, \dots, n_{m_1}$, and m_2. For an appropriate choice of $(\mathcal{D}, \pi, \psi)$ the $\mathcal{D}$-rings will be precisely the A-algebras equipped with m_1 higher A-derivations (of length $n_1, \dots, n_{m_1}$ respectively) and m_2 A-endomorphisms. Note that being a $\mathcal{D}$-ring will not impose that the various operations commute.

(e) *D-rings.* Fix $c \in A$ and let $\mathcal{D}(R) := R^2$ as an A-module and define multiplication by

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1x_2, x_1y_2 + y_1x_2 + y_1y_2c).$$

A $\mathcal{D}$-ring is then an A-algebra R equipped with an A-linear map $D : R \to R$ satisfying the twisted Leibniz rule $D(xy) = xD(y)+D(x)y+D(x)D(y)c$. If we define $\sigma : R \to R$ by $\sigma(x) := x + D(x)c$, then D is a ring endomorphism of R. If $c = 0$ then D is a derivation on R, if c is invertible in A then D may be computed from σ by the rule $D(x) = e^{-1}(\sigma(x) - x)$. Such structures were considered by the second author in [15].

(f) *Derivations of an endomorphism.* Consider the case when $\mathcal{D}(R) = R \times R[\eta]/(\eta^2)$ with basis $\{(1, 0), (0, 1), (1, \eta)\}$ and π the projection onto the first co-ordinate. A $\mathcal{D}$-ring is then an A-algebra equipped with an A-endomorphism σ and an A-linear map δ satisfying the σ-twisted Leibniz rule $\delta(xy) = \sigma(x)\delta(y) + \delta(x)\sigma(y)$.

In the next chapter we will show that the theory of $\mathcal{D}$-fields in characteristic zero has a model companion. In order to describe the axioms of this model companion we will make use of the *abstract prolongations* introduced and discussed in §4 of [11]. Let us briefly recall them here.

Given a $\mathcal{D}$-ring (R, ∂) and an algebraic scheme X over R, the prolongation[1] of X, denoted by $\tau(X, \mathcal{D}, e)$ or just τX for short, is itself a scheme over R with the characteristic property that its R-points can be canonically identified with $X(\mathcal{D}(R))$ where X is regarded as a scheme over $\mathcal{D}(R)$ via the base change coming from $e : R \to \mathcal{D}(R)$. Via this identification, note that e induces a map $\nabla : X(R) \to \tau(X, \mathcal{D}, e)(R)$.

In terms of equations, if $X \subset \mathbb{A}^n_R$ is the affine scheme $\operatorname{Spec}\big(R[x]/I\big)$ where $x = (x_1, \dots, x_n)$ is really an n-tuple of indeterminates then τX will be the closed subscheme of $\mathbb{A}^{n\ell}_R$ given by $\operatorname{Spec}\big(R[x^{(0)}, x^{(1)}, \dots, x^{(\ell-1)}]/I'\big)$ where I' is obtained as follows: For each $P(x) \in I$ let $P^e(x) \in \mathcal{D}(R)[x]$ be the polynomial obtained by applying e to the coefficients of P, and compute

$$P^e(\sum_{j=0}^{\ell-1} x^{(j)}\epsilon_j) \quad = \quad \sum_{j=0}^{\ell-1} P^{(j)}(x^{(0)}, x^{(1)}, \dots, x^{(\ell-1)})\epsilon_j$$

[1] The prolongation does not always exist; however it does exist for any quasi-projective scheme. See the discussion after Definition 4.1 of [11] for details.

in the polynomial ring $\mathcal{D}(R)[x^{(0)}, x^{(1)}, \ldots, x^{(\ell-1)}] = \bigoplus_{i=0}^{\ell-1} R[x^{(0)}, x^{(1)}, \ldots, x^{(\ell-1)}] \cdot \epsilon_i$.
Then I' is the ideal of $R[x^{(0)}, x^{(1)}, \ldots, x^{(\ell-1)}]$ generated by $P^{(0)}, \ldots, P^{(\ell-1)}$ as P ranges in I. Note that the $P^{(i)}$'s are computed using (1) above. With respect to these co-ordinates, the map $\nabla : X(R) \to \tau(X, \mathcal{D}, e)(R)$ is given by $\nabla(a) = \big(a, \partial_1(a), \ldots, \partial_{\ell-1}(a)\big)$.

4. Existentially closed $\mathcal{D}$-fields

We aim to show that the theory of $\mathcal{D}$-fields of characteristic zero admits a model companion when the base ring A enjoys some additional properties that are spelled out with the following assumptions. In fact, as is explained in the appendix, these assumptions are not necessary. Nevertheless, for the sake of significant ease of notation, and in order to better fix ideas, we will impose the following:

Assumptions 4.1. *The following assumptions will be in place throughout the rest of the paper, unless explicitly stated otherwise:*

(i) *The ring A is a field.*
(ii) *Writing $\mathcal{D}(A) = \prod_{i=0}^{t} B_i$, where the B_i are local finite A-algebras, the residue field of each B_i, which is necessarily a finite extension of A, is in fact A itself.*

Note that the B_i's are unique up to isomorphism and reordering of the indices. They can be obtained by running through the finitely many maximal ideals of $\mathcal{D}(A)$ and quotienting out by a sufficiently high power of them.

All of the motivating examples described in Example 3.5, specialised to the case when $A = \mathbb{Q}$ or $\mathbb{F}_p$, satisfy Assumptions 4.1.

4.1. The associated endomorphisms.

Our axioms for the model companion must take into account certain definable endomorphisms that are induced by the $\mathcal{D}$-operators given the above decomposition of $\mathcal{D}(A)$ into local artinian A-algebras.

First some notation. Fixing A-bases for $B_0, \ldots, B_t$, we get

- finite free local $\mathbb{S}$-algebra schemes with bases, $(\mathcal{D}_i, \psi_i)$ for $i = 0, \ldots, t$, such that $\mathcal{D}_i(A) = B_i$,
- $\mathbb{S}$-algebra homomorphisms $\theta_i : \mathcal{D} \to \mathcal{D}_i$ corresponding to $\mathcal{D} = \prod_{i=0}^{t} \mathcal{D}_i$,
- $\mathbb{S}$-algebra homomorphisms $\rho_i : \mathcal{D}_i \to \mathbb{S}$ which when evaluated at A are the residue maps $B_i \to A$, and
- $\pi_i := \rho_i \circ \theta_i : \mathcal{D} \to \mathbb{S}$.

Note that one of the maximal ideals of $\mathcal{D}(A)$, say the one corresponding to B_0, is the kernel of our A-algebra homomorphism $\pi^A : \mathcal{D}(A) \to A$. In particular, $\pi_0 = \pi$.

Now suppose (R, ∂) is a $\mathcal{D}$-ring. For each $i = 0, \ldots, t$, we have the A-algebra endomorphism $\sigma_i := \pi_i^R \circ e : R \to R$. Since $\pi_0 = \pi$, $\sigma_0 = \mathrm{id}$. The others, $\sigma_1, \ldots, \sigma_t$, will be possibly nontrivial endomorphisms of R. As the π_i are $\mathbb{S}$-linear morphisms over A, the σ_i are A-linear combinations of the operators $\partial_1, \ldots, \partial_{\ell-1}$. In particular, these are 0-definable in (R, ∂). We call them the *associated endomorphisms* and $(R, \sigma_1, \ldots, \sigma_t)$ the *associated difference ring*.

Example 4.2. In the partial difference-differential case of Example 3.5(c), that is, of (R, ∂, σ) where ∂ is a tuple of m_1 (higher truncated) derivations on R and σ is a tuple of m_2 endomorphisms of R, the associated difference ring is, as expected, (R, σ). In the D-rings of Example 3.5(e), the associated endomorphism is the map $\sigma(x) := x + D(x)c$.

Definition 4.3. An *inversive* $\mathcal{D}$-ring is one for which the associated endomorphisms are surjective. That is, a $\mathcal{D}$-ring (R, ∂) is inversive just in case the associated difference ring (R, σ) is inversive.

Suppose now that (R, ∂) is an inversive integral $\mathcal{D}$-ring. The associated endomorphisms are thus automorphisms of R. Given $B \subseteq R$, the *inversive closure of B in R*, denoted by $\langle B\rangle$, is the smallest inversive $\mathcal{D}$-subring of R containing B. That is, it is the intersection of all inversive $\mathcal{D}$-subrings containing B. If k is an inversive $\mathcal{D}$-subring of R and $a = (a_1, \dots, a_n)$ is a tuple from R, then $k\langle a\rangle$ denotes $\big\langle k \cup \{a_1, \dots, a_n\}\big\rangle$.

Remark 4.4.
(a) Because σ and ∂ do not necessarily commute, it is not the case that $\langle B\rangle$ is simply the inversive difference ring generated by the $\mathcal{D}$-subring generated by B. Rather, $\langle B\rangle = \bigcup_{i<\omega} R_i$ where $R_{-1} = B$ and R_i is the $\mathcal{D}$-subring generated by $\{\sigma_j^{-1}(a) : a \in R_{i-1}, j = 1, \dots, t\}$.
(b) We denote by Θa the (infinite) tuple whose co-ordinates are of the form θa_i as θ ranges over all finite words on the set $\{\partial_1, \dots, \partial_{\ell-1}, \sigma_1^{-1}, \dots, \sigma_t^{-1}\}$. So $k\langle a\rangle = k[\Theta a]$.

4.2. **The model companion.** Our model companion for $\mathcal{D}$-fields will include a "geometric" axiom in the spirit of Chatzidakis-Hrushovski [2] or Pierce-Pillay [13]. To state these we need some further notation. Suppose X is an algebraic scheme over a $\mathcal{D}$-ring R. For each $i = 0, \dots, t$, since $\sigma_i = \pi_i^R \circ e$, the morphism $\pi_i : \mathcal{D} \to \mathbb{S}$ induces[2] a surjective morphism on the prolongations $\widehat{\pi}_i : \tau(X, \mathcal{D}, e) \to \tau(X, \mathbb{S}, \sigma_i)$. For ease of notation we will set $X^{\sigma_i} := \tau(X, \mathbb{S}, \sigma_i)$. Note that X^{σ_i} is nothing other than X base changed via $\sigma_i : R \to R$, and that in terms of equations it is obtained by applying σ_i to the coefficients of the defining polynomials.

Theorem 4.5. *With Assumptions 4.1 in place, let $\mathcal{K}$ denote the class of $\mathcal{D}$-rings (R, ∂) such that R is an integral domain of characteristic zero and the associated endomorphisms are injective. Then $(K, \partial) \in \mathcal{K}$ is existentially closed if and only if*

I. *K is an algebraically closed field,*
II. *(K, ∂) is inversive, and*
III. *if X is an irreducible affine variety over K and $Y \subseteq \tau(X, \mathcal{D}, e)$ is an irreducible subvariety over K such that $\widehat{\pi}_i(Y)$ is Zariski dense in X^{σ_i} for all $i = 0, \dots, t$, then there exists $a \in X(K)$ with $\nabla(a) \in Y(K)$.*

Note that $\mathcal{K}$ is a universally axiomatisable class. (This uses the fact that the associated endomorphisms of a $\mathcal{D}$-ring are A-linear combinations of the operators, and hence definable.) Moreover, the characterisation of existentially closed models given in the theorem is also first-order. Indeed, the only one that is not obviously elementary is condition III, but since irreducibility and Zariski-density are parametrically definable in algebraically closed fields, the only thing to check is that if

[2] See §4.1 of [11] for the construction.

X varies in an algebraic family then so do the $\widehat{\pi}_i : \tau X \to X^{\sigma_i}$. That τX and X^{σ_i} vary uniformly in families follows from Proposition 4.7(b) of [11], but can also be verified directly by looking at the equations that define the prolongations. That $\widehat{\pi}_i$ also varies algebraically follows from the construction of these morphisms in §4.1 of [11]; see in particular Proposition 4.8(b) of that paper. The following is therefore an immediate corollary of Theorem 4.5:

Corollary 4.6. *Under assumptions 4.1 the theory of $\mathcal{D}$-fields of characteristic zero admits a model companion. We denote the model companion by $\mathcal{D}\text{-CF}_0$, and we call its models* $\mathcal{D}$-closed fields.

We now work toward a proof of Theorem 4.5.

To prove properties I through III of Theorem 4.5 for every existentially closed model in $\mathcal{K}$ is to prove various extension lemmas about $\mathcal{K}$. In order to facilitate this we introduce the following auxiliary class.

Definition 4.7 (The class $\mathcal{M}$)**.** The class $\mathcal{M}$ is defined to be the class of triples (R, S, ∂) where $R \subseteq S$ are integral A-algebras of characteristic zero and $\partial = (\partial_0, \dots, \partial_{\ell-1})$ is a sequence of maps from R to S such that $e : R \to \mathcal{D}(S)$ given by $e(r) := \partial_0(r)\epsilon_0 + \cdots + \partial_{\ell-1}(r)\epsilon_{\ell-1}$ has the following properties:

(i) e is an A-algebra homomorphism,
(ii) $\sigma_0 := \pi^S \circ e : R \to S$ is the identity on R, and
(iii) for each $i = 1, \dots, t$, $\sigma_i := \pi_i^S \circ e : R \to S$ is injective.

Note that $(R, \partial) \in \mathcal{K}$ if and only if $(R, R, \partial) \in \mathcal{M}$.

The following lemma will imply that every existentially closed member of $\mathcal{K}$ is a field. It is here that we require the associated endomorphisms to be injective.

Lemma 4.8. *Suppose $(R, L, \partial) \in \mathcal{M}$ with L a field. Then we can (uniquely) extend ∂ to the fraction field F of R so that $(F, L, \partial) \in \mathcal{M}$.*

Proof. Let us first extend $e : R \to \mathcal{D}(L)$ to an A-algebra homomorphism from F to $\mathcal{D}(L)$. By the universal property of localisation it suffices (and is necessary) to show that e takes nonzero elements of R to units in $\mathcal{D}(L)$. Note that an element $x \in \mathcal{D}(L)$ is a unit if and only if each of its projections $\theta_i^L(x)$ is a unit in the local L-algebra $\mathcal{D}_i(L)$, which in turn is equivalent to the residue of $\theta_i^L(x)$, namely $\pi_i^L(x) \in L$, being nonzero. But for all $i = 0, \dots, t$, $\pi_i^L \circ e$ is injective on R by assumption. So $e(a)$ is a unit in $\mathcal{D}(L)$ for nonzero $a \in R$.

We thus have an extension $e : F \to \mathcal{D}(L)$. The injectivity of $\sigma_1, \dots, \sigma_t$ is immediate as these are A-algebra homomorphisms between fields. Moreover, σ_0 is the identity on F as it is extends the identity on R. So, letting ∂ be the operators corresponding to $e : F \to \mathcal{D}(L)$, we have that $(F, L, \partial) \in \mathcal{M}$. □

The next lemma shows that existentially closed models are algebraically closed.

Lemma 4.9. *Suppose $(F, L, \partial) \in \mathcal{M}$ where F and L are fields and L is algebraically closed. Then we can extend ∂ to F^{alg} so that $(F^{\mathrm{alg}}, L, \partial) \in \mathcal{M}$.*

In fact, more is true. If σ is the tuple of embeddings $F \to L$ associated to ∂, and σ' is any extension of σ to F^{alg}, then there is exactly one extension ∂' of ∂ to F^{alg} with associated embeddings σ'.

Proof. By iteration it suffices to prove, for any given $a \in F^{\mathrm{alg}}$, that we can extend e to $F(a)$ in such a way that $\pi^L \circ e$ is still the identity on $F(a)$. Let $P(x) \in F[x]$ be the minimal polynomial of a over F. Fixing $i = 0, \dots, t$, we let $c_i \in L$ be a root for $P^{\sigma_i}(x) \in L[x]$, where $\sigma_i : F \to L$ is the field embedding $\pi_i^L \circ e : F \to L$. In the case when $i = 0$ note that $P^{\sigma_0}(x) = P(x)$, and so we can, and do, choose $c_0 = a$. Now, let $e_i : F \to \mathcal{D}_i(L)$ be $\theta_i^L \circ e$. Note that by construction $P^{\sigma_i}(x) \in L[x]$ is the reduction of $P^{e_i}(x) \in \mathcal{D}_i(L)[x]$ modulo the maximal ideal of the local artinian ring $\mathcal{D}_i(L)$. Since $P^{\sigma_i}(x)$ is separable (we are in characteristic zero), Hensel's Lemma allows us to lift c_i to a root b_i of $P^{e_i}(x)$ in $\mathcal{D}_i(L)$. In fact there is a unique such lifting as $F \hookrightarrow F(a)$ is étale. Then $b = (b_0, \dots, b_t) \in \mathcal{D}(L)$ is a root of $P^e(x) \in \mathcal{D}(L)[x]$, and we can extend e to $F(a)$ by sending a to b. By construction $\pi^L e(a) = \pi_0^L e(a) = c_0 = a$, so that $\pi^L \circ e = \mathrm{id}_{F(a)}$.

It is the fact that in the above argument any choice of roots $c_1, \dots, c_t$ work, and that once they are chosen there is a unique possibility for b, that leads to the "in fact" clause of the lemma. □

The next lemma will imply that existentially closed models are inversive.

Lemma 4.10. *Suppose $(F, L, \partial) \in \mathcal{M}$ where F and L are fields. Then there exists an extension L' of L and an extension of ∂ to an inversive $\mathcal{D}$-field structure on L'.*

Proof. First, we can extend $\sigma_1, \dots, \sigma_t$ to automorphisms $\sigma'_1, \dots, \sigma'_t$ of some algebraically closed $L' \supseteq L$. Now fix a transcendence basis B for L' over F. For each $b \in B$ let $b_0 \in \mathcal{D}_0(L')$ lift b, and let $b_i \in \mathcal{D}_i(L')$ lift $\sigma'_i(b)$ for $i = 1, \dots, t$. Then define $e(b)$ to be $(b_0, \dots, b_t) \in \mathcal{D}(L')$. This gives us an extension $F[B] \to \mathcal{D}\big(L'\big)$ of e such that $\pi^{L'} \circ e$, $\pi_1^{L'} \circ e, \dots, \pi_t^{L'} \circ e$ agree with $\mathrm{id}, \sigma'_1, \dots, \sigma'_t$, respectively. By Lemma 4.8 we can extend e to $F(B) \to \mathcal{D}(L')$ preserving this property. By Lemma 4.9 we can extend e to a $\mathcal{D}$-structure on $L' = F(B)^{\mathrm{alg}}$ in such a way that the associated endomorphisms remain the automorphisms $\sigma'_1, \dots, \sigma'_t$. □

Proof of Theorem 4.5. Suppose $(K, \partial) \in \mathcal{K}$ is existentially closed. By Lemmas 4.8, 4.9, and 4.10 we know that K is an algebraically closed field and that $\sigma_1, \dots, \sigma_t$ are automorphisms of K. It remains to check condition III. Let $X \subseteq \mathbb{A}^n_K$ and $Y \subseteq \tau X$ be as in that condition. Let L be an algebraically closed field extending K and let $b \in Y(L)$ be a K-generic point of Y. Let $a := \hat{\pi}(b) \in X(L)$. Our goal is to extend ∂ to a $\mathcal{D}$-field structure on some extension of L in such a way that $\nabla(a) = b$. This will suffice, because then by existential closedness there must exist a K-point of X with the property that its image under ∇ is a K-point of Y.

As described in §4 of [11], $\tau X(L)$ can be canonically identified with the $\mathcal{D}(L)$-points of the affine scheme over $\mathcal{D}(K)$ obtained from X by applying e to the coefficients of the defining polynomials. Let b' be the n-tuple from $\mathcal{D}(L)$ that corresponds to $b \in \tau X(L)$ under this identification. So $P^e(b') = 0$ for all $P(x) \in I(X/K)$. Since $\hat{\pi}(Y) = \hat{\pi}_0(Y)$ is Zariski dense in $X^{\sigma_0} = X$ and b is K-generic in Y, we have that $a = \hat{\pi}(b)$ is K-generic in X. So $I(X) = I(a/K)$. We thus have that $P^e(b') = 0$ for all $P(x) \in I(a/K)$. That is, we can extend $e : K \to \mathcal{D}(L)$ to an A-algebra homomorphism $e : K[a] \to \mathcal{D}(L)$ by $e(a) = b'$. The fact that $\hat{\pi}(b) = a$ implies that $\pi^L(b') = a$ so that $\pi^L \circ e = \mathrm{id}_{K[a]}$. For each $i = 1, \dots, t$, the fact that $\hat{\pi}_i(Y)$ is Zariski dense in X^{σ_i} implies that $\hat{\pi}_i(b)$ is K-generic in X^{σ_i}, and hence for any $P(x) \in K[x]$ on which a does not vanish, $\pi_i^L e\big(P(a)\big) = P^{\sigma_i}\big(\pi_i^L(b')\big) \neq 0$. That is, $\pi_i^L \circ e : K[a] \to L$ is injective for each $i = 1, \dots, t$. So, letting ∂ be the corresponding operators, we have $\big(K[a], L, \partial\big) \in \mathcal{M}$. By Lemma 4.8 this extends to $\big(K(a), L, \partial\big) \in \mathcal{M}$. By Lemma 4.10, there is an extension L' of L such that ∂

extends to a $\mathcal{D}$-field structure on L'. The fact that $e(a) = b'$ implies that $\nabla(a) = b$, as desired.

Now for the converse. Suppose (K, ∂) is a $\mathcal{D}$-field satisfying I through III. To show that (K, ∂) is existentially closed in $\mathcal{K}$ it suffices to consider a conjunction of atomic $\mathcal{L}_{\mathcal{D},K}$-formulae that is realised in some extension of (K, ∂) in $\mathcal{K}$, and show that it is already realised in (K, ∂). Indeed, all inequations of the form $t(x_1, \ldots, x_m) \neq 0$ that might appear can be replaced by $t(x_1, \ldots, x_m)y - 1 = 0$ where y is a new variable. We can also assume, by Lemmas 4.8 and 4.9, that the extension in which we have a realisation is an algebraically closed $\mathcal{D}$-field.

Let $\phi(x)$ be a conjunction of atomic $\mathcal{L}_{\mathcal{D},K}$-formulae where $x = (x_1, \ldots, x_m)$ is an m-tuple of variables, let (L, ∂) be an algebraically closed $\mathcal{D}$-field extension of (K, ∂), and let $c_0 \in L^m$ realise $\phi(x)$. Let Ξ be the set of all finite words on $\{\partial_1, \ldots, \partial_{\ell-1}\}$, and for each $r \geq 0$ let Ξ_r be those words of length at most r. Fix an enumeration of Ξ so that Ξ_r is an initial segement of Ξ_{r+1} for all $r \geq 0$. Define $\nabla_r : L \to L^{n_r}$ by $b \mapsto \big(\xi(b) : \xi \in \Xi_r\big)$. Then for some $r \geq 0$, $\phi(x)^L = \{b \in L^m : \nabla_r(b) \in Z\}$ where $Z \subseteq L^{mn_r}$ is a Zariski-closed set over K. Note that if $r = 0$ then $\phi(x)$ is equivalent to a formula over K in the language of rings with a realisation in an extension, and so, as K is algebraically closed, $\phi(x)$ is realised in K. We may thus assume that $r > 0$. Let

$$
\begin{aligned}
c &:= \nabla_{r-1}(c_0) \in L^{mn_{r-1}} \\
X &:= \operatorname{loc}(c/K) \subseteq L^{mn_{r-1}} \\
Y &:= \operatorname{loc}(\nabla c/K) \subseteq \tau X(L) \subseteq L^{\ell mn_{r-1}}
\end{aligned}
$$

Note that for each $i = 0, \ldots, t$, $\hat{\pi}_i(\nabla c) = \sigma_i(c) \in X^{\sigma_i}(L)$. Since ∇c is K-generic in Y and $\sigma_i(c)$ is K-generic in X^{σ_i} (as σ_i restricts to an automorphism of K by assumption), it follows that $\hat{\pi}_i(Y)$ is Zariski dense in X^{σ_i}. Hence, by III, there exists $a \in X(K)$ such that $\nabla a \in Y(K)$. Let a_0 be the first m co-ordinates of a. It remains to verify that a_0 satisfies $\phi(x)$; that is, that $\nabla_r(a_0) \in Z(K)$.

First of all, we note that $\nabla_{r-1}(a_0) = a$. Indeed, we show by induction on the length of $\xi \in \Xi_{r-1}$ that $\xi(a_0) = a_\xi$, where $a = (a_\xi : \xi \in \Xi_{r-1})$. For $\xi = \mathrm{id}$ this is clear by choice of a_0. Now suppose $\xi = \partial_i \xi'$. Since $\nabla_{r-1}(c_0) = c$, we know that $\partial_i c_{\xi'} = c_\xi$. Because ∇a is in the K-locus of ∇c, we have $\partial_i a_{\xi'} = a_\xi$ also. But by the inductive hypothesis, $\partial_i a_{\xi'} = \partial_i \xi'(a_0) = \xi(a_0)$, so that $\xi(a_0) = a_\xi$ as desired.

Finally, since c_0 is a realisation of $\phi(x)$, we know that $\nabla_r c_0 \in Z$. The latter can be seen as an algebraic fact about $\nabla\nabla_{r-1}c_0$. Since $\nabla\nabla_{r-1}a_0 = \nabla a$ is in the K-locus of $\nabla\nabla_{r-1}c_0 = \nabla c$, it follows that $\nabla_r a_0 \in Z$, as desired.

This completes the proof of Theorem 4.5. $\square$

5. Basic model theory of $\mathcal{D}$-CF$_0$

We begin now to investigate the model theory of $\mathcal{D}$-CF$_0$ using the study of existentially closed difference fields as it appears in §1 of [2] as a template. Assumptions 4.1 and the notation of the previous chapter remain in place.

5.1. **Completions.** We aim to describe the completions of $\mathcal{D}$-CF$_0$.

Lemma 5.1. *Suppose (K, ∂) and (L, γ) are inversive $\mathcal{D}$-fields extending an inversive $\mathcal{D}$-field (F, ∂) with K and L linearly disjoint over F (inside some fixed common* *field extension). Then we can simultaneously extend (K,∂) and (L,γ) uniquely to a $\mathcal{D}$-field structure on the compositum KL.*

Proof. It follows from linear disjointedness that $R := K\otimes_F L$ is an integral domain whose fraction field is the compositum KL. Here we identify K with $K\otimes 1\subset R$, L with $1\otimes L\subset R$ and F with $1\otimes F = F\otimes 1\subseteq K\cap L$. It suffices to find a common extension of ∂ and γ to R. Indeed, by Lemma 4.8 we can then further extend R uniquely to a $\mathcal{D}$-field structure on the fraction field KL.

Let $e_1 : K\to\mathcal{D}(K)\subseteq\mathcal{D}(R)$ and $e_2 : L\to\mathcal{D}(L)\subseteq\mathcal{D}(R)$ be the corresponding A-algebra homomorphisms. Since these agree on F we have the induced map $e : R\to\mathcal{D}(R)$ determined by $e(a\otimes b) := e_1(a)e_2(b)$, which is easily seen to be an A-algebra homomorphism that extends both e_1 and e_2. For $i = 0,\dots,t$,

$$\begin{aligned}\pi_i^R e(a\otimes b) &= \big(\pi_i^R e_1(a)\big)\big(\pi_i^R e_2(b)\big)\\ &= \big(\pi_i^K e_1(a)\big)\big(\pi_i^L e_2(b)\big)\\ &= \big(\sigma_i(a)\otimes 1\big)\big(1\otimes\tau_i(b)\big)\\ &= \sigma_i(a)\otimes\tau_i(b)\end{aligned}$$

where the $\sigma_i s$ and τ_i's are the associated automorphisms of K and L respectively. Applying this to $i=0$ we see that $\pi^R\circ e = \pi_0^R\circ e$ is the identity on R; hence (R,e) is a $\mathcal{D}$-ring. For $i\geq 1$, since σ_i and τ_i extend an automorphism of F (by the inversiveness assumption) and K is linearly disjoint from L over F, $a\otimes b\mapsto\sigma_i(a)\otimes\tau_i(b)$ determines an automorphism of $R = K\otimes_F L$. Hence $\pi_i^R\circ e$ is an automorphism of R for $i = 1,\dots,t$. So (R,e) is in the class $\mathcal{K}$, and extends (K,e_1) and (L,e_2), as desired. □

Proposition 5.2. *If (K,∂) and (L,γ) are models of $\mathcal{D}$-CF_0 with a common algebraically closed inversive $\mathcal{D}$-subfield F, then $(K,\partial)\equiv_F(L,\gamma)$.*

Proof. The fact that F is algebraically closed allows us to assume, after possibly replacing (L,γ) by an F-isomorphic copy, that as subfields of some common field extension, K and L are linearly disjoint over F. By Lemma 5.1 we can extend (K,∂) and (L,γ) simultaneously to a $\mathcal{D}$-field structure on KL, which we can then extend further to a model, say $(K',\partial)\models\mathcal{D}$-$\mathrm{CF}_0$. By model completeness, $(K,\partial)\preceq(K',\partial)$ and $(L,\gamma)\preceq(K',\partial)$. It follows that $(K,\partial)\equiv_F(L,\gamma)$. □

Lemma 5.3. *Suppose $(F,\partial)\subseteq(K,\partial)$ is a $\mathcal{D}$-field extension such that K is algebraically closed and F is inversive. Then $F^{\mathrm{alg}}\subseteq K$ is an inversive $\mathcal{D}$-subfield.*

Proof. Inversiveness comes for free once we see that F^{alg} is a $\mathcal{D}$-subfield. Let $a\in F^{\mathrm{alg}}$ and $P(x)\in F[x]$ be the minimal poynomial of a. Let $e : K\to\mathcal{D}(K)$ be the A-algebra homomorphism corresponding to ∂. We need to show that $e(a)\in\mathcal{D}(F^{\mathrm{alg}})$. Under the identification $\mathcal{D}(K) = \prod_{i=0}^t\mathcal{D}_i(K)$, we have $e(a) = \big(e_0(a),\dots,e_t(a)\big)$, where $e_i := \theta_i^K\circ e$, and it suffices to show that each $e_i(a)\in\mathcal{D}_i(F^{\mathrm{alg}})$. Now $\sigma_i(a)\in F^{\mathrm{alg}}$ and by the inversiveness assumption $P^{\sigma_i}(x)$ is the minimal polynomial of $\sigma_i(a)$ over F. So by Hensel's Lemma $\sigma_i(a)$ has a lifting to a root of $P^{e_i}(x)$ in $\mathcal{D}_i(F^{\mathrm{alg}})$. On the other hand, $e_i(a)$ also lifts $\sigma_i(a)$ to a root of $P^{e_i}(x)$ in $\mathcal{D}_i(K)$. As the extension is ètale these liftings agree, and so $e_i(a)\in\mathcal{D}_i(F^{\mathrm{alg}})$, as desired. □

Corollary 5.4 (Completions of $\mathcal{D}$-CF_0)**.** *The completions of $\mathcal{D}$-CF_0 are determined by the difference-field structure on the algebraic closure of the prime $\mathcal{D}$-field. That is, two models (K,∂) and (L,γ) of $\mathcal{D}$-CF_0 are elementarily equivalent if and* *only if* $(A^{\mathrm{alg}}, \sigma\restriction_{A^{\mathrm{alg}}}) \approx_A (A^{\mathrm{alg}}, \tau\restriction_{A^{\mathrm{alg}}})$*, where* σ *and* τ *are the sequences of automorphisms of* K *and* L *associated to* ∂ *and* γ*, respectively.*

Proof. First of all, both (K,∂) and (L,γ) extend the prime $\mathcal{D}$-field A, which is itself inversive (the difference-field structure on A is trivial). Hence, by Lemma 5.3, $(A^{\mathrm{alg}}, \partial\restriction_{A^{\mathrm{alg}}})$ and $(A^{\mathrm{alg}}, \gamma\restriction_{A^{\mathrm{alg}}})$ are inversive $\mathcal{D}$-field extensions of A. By Lemma 4.9 their $\mathcal{D}$-field structures are determined by the action of the corresponding automorphisms on A^{alg}. Hence, if $(A^{\mathrm{alg}}, \sigma\restriction_{A^{\mathrm{alg}}}))$ and $(A^{\mathrm{alg}}, \tau\restriction_{A^{\mathrm{alg}}}))$ are isomorphic then $(A^{\mathrm{alg}}, \partial\restriction_{A^{\mathrm{alg}}})$ and $(A^{\mathrm{alg}}, \gamma\restriction_{A^{\mathrm{alg}}})$ are isomorphic, and so by Proposition 5.2, (K,∂) and (L,γ) are elementarily equivalent. For the converse, if $(K,\partial) \equiv (L,\gamma)$ then there is an elementary embedding of (K,∂) into an elementary extension (L',γ) of (L,γ). This elementary embedding will restrict to an isomorphism from $(A^{\mathrm{alg}}, \partial\restriction_{A^{\mathrm{alg}}})$ to its image in (L',γ), which is $(A^{\mathrm{alg}}, \gamma\restriction_{A^{\mathrm{alg}}})$. In particular, $(A^{\mathrm{alg}}, \sigma\restriction_{A^{\mathrm{alg}}}) \approx_A (A^{\mathrm{alg}}, \tau\restriction_{A^{\mathrm{alg}}})$. □

5.2. **Algebraic closure.** We characterise model-theoretic algebraic closure.

Proposition 5.5. *Suppose* $(K,\partial) \models \mathcal{D}\text{-CF}_0$*. For all* $B \subseteq K$*,* $\mathrm{acl}(B) = \langle B\rangle^{\mathrm{alg}}$*.*

Proof. Recall that $\langle B\rangle$ is the inversive closure of B, the smallest inversive $\mathcal{D}$-subring of K containing B. As $\sigma_1,\dots,\sigma_t$ are $\mathcal{L}_{\mathcal{D}}$-definable, $\langle B\rangle \subseteq \mathrm{dcl}(B)$. Hence $F := \langle B\rangle^{\mathrm{alg}} \subseteq \mathrm{acl}(B)$. It remains to show that if $a \in K\setminus F$ then $\mathrm{tp}(a/F)$ is nonalgebraic.

Note that, by Lemma 5.3, F is an inversive $\mathcal{D}$-subfield of K. Since F is algebraically closed we can find, in some common field extension, an isomorphic copy of K over F, witnessed say by an F-isomorphism $\alpha : K \to K'$, and such that K is linearly disjoint from K' over F. Via α we can put a $\mathcal{D}$-field structure ∂' on K' that extends (F,∂) and so that α is an isomorphism of $\mathcal{D}$-fields. Now we extend (K,∂) and (K',∂') to a $\mathcal{D}$-field structure on KK' using Lemma 5.1, and then further to a model of $\mathcal{D}\text{-CF}_0$. We have thus found a common elementary extension of (K,∂) and (K',∂'). In this elementary extension, $\alpha(a)$ will be a realisation of $\mathrm{tp}(a/F)$ that is distinct from a. Iterating, we find infinitely many realisations of $\mathrm{tp}(a/F)$ in some elementary extension, proving that this type is nonalgebraic. □

5.3. **Types.** We characterise types and deduce a quantifier reduction theorem. Recall that Θa is the (infinite) tuple whose co-ordinates are of the form θa_i as θ ranges over all finite words on the set $\{\partial_1,\dots,\partial_{\ell-1},\sigma_1^{-1},\dots,\sigma_t^{-1}\}$.

Proposition 5.6. *Suppose* $(K,\partial) \models \mathcal{D}\text{-CF}_0$*,* $k \subseteq K$ *is an inversive* $\mathcal{D}$*-subfield, and* $a,b \in K^n$*. Then the following are equivalent:*

(i) $\mathrm{tp}(a/k) = \mathrm{tp}(b/k)$*,*
(ii) $\mathrm{tp}_\sigma\big(\Theta a/k\big) = \mathrm{tp}_\sigma\big(\Theta b/k\big)$ *(where* $\mathrm{tp}_\sigma(c/k)$ *denotes the type of* c *over* k *in the reduct to the language of difference fields),*
(iii) *there is an isomorphism from* $\big(k\langle a\rangle,\partial\big)$ *to* $\big(k\langle b\rangle,\partial\big)$ *sending* a *to* b *and fixing* k *that extends to an isomorphism from* $\big(k\langle a\rangle^{\mathrm{alg}},\sigma\big)$ *to* $\big(k\langle b\rangle^{\mathrm{alg}},\sigma\big)$*.*

Proof. (i) $\Longrightarrow$ (ii) is clear.

(ii) $\Longrightarrow$ (iii). Work in a sufficiently saturated elementary extension (L,∂) of (K,∂). Then (L,σ) is also saturated as a difference-field, and so $\mathrm{tp}_\sigma\big(\Theta a/k\big) = \mathrm{tp}_\sigma\big(\Theta b/k\big)$ is witnessed by a difference-field automorphism α of L over k, taking

Θa to Θb. Then $\beta := \alpha \restriction_{k[\Theta a]}$ is the desired $\mathcal{D}$-field isomorphism from $k\langle a\rangle = k[\Theta a]$ to $k\langle b\rangle = k[\Theta b]$, and $\alpha \restriction_{k\langle a\rangle^{\mathrm{alg}}}$ is the desired extension.

(iii) $\Longrightarrow$ (i). First note that the difference-field isomorphism, α, from $k\langle a\rangle^{\mathrm{alg}}$ to $k\langle b\rangle^{\mathrm{alg}}$ will necessarily be a $\mathcal{D}$-field isomorphism. Indeed, α will take $\partial \restriction_{k\langle a\rangle^{\mathrm{alg}}}$ to a $\mathcal{D}$-field structure on $k\langle b\rangle^{\mathrm{alg}}$ whose associated endomorphism is $\sigma \restriction_{k\langle a\rangle^{\mathrm{alg}}}$. But by the uniqueness part of Lemma 4.9, this new $\mathcal{D}$-structure must co-incide with $\partial \restriction_{k\langle b\rangle^{\mathrm{alg}}}$. The equality of types is now an immediate consequence of 5.2 and 5.3. □

The equivalence of parts (i) and (iii) above yields the following corollary:

Corollary 5.7 (Quantifier Reduction). *Every L-formula $\phi(x_1,\dots,x_n)$ is equivalent modulo $\mathcal{D}$-CF_0 to an L-formula of the form $\exists y\ \psi(x_1,\dots,x_n,y)$ where*

- *$\psi(x_1,\dots,x_n,y) = \xi(\bar{x},\bar{y})$ where ξ is a quantifier-free ring formula, the co-ordinates of $\bar{x}$ are of the form θx_i where $\theta \in \Theta$ and $\bar{y} = (y,\sigma_1(y),\dots,\sigma_t(y))$,*
- *each disjunct of ξ written in disjunctive normal form includes a conjunct of the form $t_N(\bar{x}) \neq 0$ & $\sum_{j=0}^{N} t_j(\bar{x})y^j = 0$ where each t_i is a polynomial.*

In particular, when the associated endomorphisms are all trivial the existential quantifier may be omitted and we have quantifier elimination.

5.4. **Independence and simplicity.** In this section we observe that $\mathcal{D}$-CF_0 is simple, and we give an algebraic characterisation of nonforking independence. The results here follow more or less axiomatically from the results of the previous sections, as established by Chatzidakis and Hrushovski in [2].

Let $(\mathbb{U},\partial)$ be a sufficiently saturated model of $\mathcal{D}$-CF_0.

Definition 5.8. Suppose A, B, C are (small) subsets of $\mathbb{U}$. Then *A is independent from B over C*, denoted by $A \mathop{\smile\!\!\!\!\!\!|}_C B$, if $\mathrm{acl}(A\cup C)$ is algebraically independent (equivalently linearly disjoint) from $\mathrm{acl}(A\cup C)$ over $\mathrm{acl}(C)$.

Theorem 5.9. *Independence in $(\mathbb{U},\partial)$ satisfies the following properties:*

(a) *Symmetry. $A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $B \mathop{\smile\!\!\!\!\!\!|}_C A$.*
(b) *Transitivity. Given $A \subseteq B \subseteq C$ and tuple a,*
$$a \mathop{\smile\!\!\!\!\!\!|}_A C \text{ if and only if } a \mathop{\smile\!\!\!\!\!\!|}_B C \text{ and } a \mathop{\smile\!\!\!\!\!\!|}_A B.$$
(c) *Invariance. If $\alpha \in \mathrm{Aut}(\mathbb{U},\partial)$ then $A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $\alpha(A) \mathop{\smile\!\!\!\!\!\!|}_{\alpha(C)} \alpha(B)$.*
(d) *Finite character. $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}_C B_0$ for all finite $B_0 \subset B$.*
(e) *Local character. Given a set B and a tuple a, there exists countable $B_0 \subset B$ such that $a \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.*
(f) *Extension. Given $A \subseteq B$ and tuple a, there exists a tuple a' such that $\mathrm{tp}(a/A) = \mathrm{tp}(a'/A)$ and $a' \mathop{\smile\!\!\!\!\!\!|}_A B$.*
(g) *Independence theorem. Suppose*
   – *F is an algebraically closed inversive $\mathcal{D}$-field,*
   – *A and B are supersets of F with $A \mathop{\smile\!\!\!\!\!\!|}_F B$,*
   – *$a \mathop{\smile\!\!\!\!\!\!|}_F A$ and $b \mathop{\smile\!\!\!\!\!\!|}_F B$*
   – *$\mathrm{tp}(a/F) = \mathrm{tp}(b/F)$.*

   Then there is $d \mathop{\smile\!\!\!\!\!\!|}_E AB$ with $\mathrm{tp}(d/A) = \mathrm{tp}(a/A)$ and $\mathrm{tp}(d/B) = \mathrm{tp}(b/B)$.

In particular, $\mathrm{Th}(\mathbb{U},\partial)$ is simple and $\mathop{\smile\!\!\!\!\!\!|}$ is nonforking independence.

Proof. (a) through (e) follow easily from the corresponding properties for algebraic independence; part (e) using also the fact that if K is an inversive $\mathcal{D}$-field then $K\langle a\rangle$ is countably generated as a field over K.

(f). Let $F = \operatorname{acl}(A)$, $K = \operatorname{acl}(B)$, and $K_1 := F\langle a\rangle^{\mathrm{alg}}$. Let K_1' be a field-isomorphic copy of K_1 over F – say with $\alpha : K_1 \to K_1'$ witnessing this – such that K_1' is linearly disjoint from K over F. We can put a $\mathcal{D}$-field structure ∂' on K_1' extending (F, ∂) such that α is a $\mathcal{D}$-field isomorphism. Now by Lemma 5.1 we can find a model of $\mathcal{D}$-CF_0 extending both (K_1', ∂') and (K, ∂). By Proposition 5.2 and saturation we may assume this model is an elementary substructure of $(\mathbb{U}, \partial)$. Hence $\operatorname{tp}(\alpha(a)/F) = \operatorname{tp}(a/F)$ by the equivalence of (i) and (iii) in Proposition 5.6, and $\alpha(a) \mathop{\smile\!\!\!\!\!\!|}_A B$ by linear disjointedness.

(g). We follow the spirit of the argument used for $ACFA$ in [2]. Fix $c \models p(x) := \operatorname{tp}(a/F) = \operatorname{tp}(b/F)$. It suffices to find A', B' such that

(i) $\{A', B', c\}$ is independent over F,
(ii) $A'c \models \operatorname{tp}(Aa/F)$,
(iii) $B'c \models \operatorname{tp}(Bb/F)$, and
(iv) $A'B' \models \operatorname{tp}(AB/F)$.

Indeed, if $\alpha \in \operatorname{Aut}_F(\mathbb{U}, \partial)$ with $\alpha(A'B') = AB$, then $d := \alpha(c)$ will witness the conclusion.

Since $\operatorname{tp}(c/F) = \operatorname{tp}(a/F) = \operatorname{tp}(b/F)$, there exists $A'B'$ satisfying (ii) and (iii). Moreover, by extension, we may also assume that $A' \mathop{\smile\!\!\!\!\!\!|}_{Fc} B'$. Hence, by transitivity, we have (i) as well. The only thing missing is (iv).

Let $K_0 := \operatorname{acl}(A') \cdot \operatorname{acl}(B')$, and $K_1 := \operatorname{acl}(A'c) \cdot \operatorname{acl}(B'c)$, and $K_2 := \operatorname{acl}(A'B')$. So K_1 and K_2 are field extensions of K_0. We wish to give K_2 a $\mathcal{D}$-field structure γ such that

$$(K_2, \gamma) \approx_F (\operatorname{acl}(AB), \partial \restriction_{\operatorname{acl}(AB)}). \tag{5}$$

To do so, denote by α and β the F-automorphisms of the universe taking A to A' and B to B', respectively. Then since $A \mathop{\smile\!\!\!\!\!\!|}_F B$ and $A' \mathop{\smile\!\!\!\!\!\!|}_F B'$, $\alpha \restriction_{\operatorname{acl}(A)} \otimes \beta \restriction_{\operatorname{acl}(B)}$ induces an isomorphism over F between the fields $\operatorname{acl}(A)\cdot\operatorname{acl}(B)$ and $\operatorname{acl}(A')\cdot\operatorname{acl}(B')$, and hence between their field-theoretic algebraic closures $\big(\operatorname{acl}(A) \cdot \operatorname{acl}(B)\big)^{\mathrm{alg}} = \operatorname{acl}(AB)$ and $\big(\operatorname{acl}(A')\cdot\operatorname{acl}(B')\big)^{\mathrm{alg}} = \operatorname{acl}(A'B') = K_2$. We use this field isomorphism to define the desired γ on K_2 such that (5) holds.

Since $\alpha \restriction_{\operatorname{acl}(A)}$ and $\beta \restriction_{\operatorname{acl}(B)}$ are $\mathcal{D}$-field ismorphisms, we have that γ agrees with ∂ on each of $\operatorname{acl}(A')$ and $\operatorname{acl}(B')$. Hence γ must agree with ∂ on the composite K_0. That is, $(K_1, \partial \restriction_{K_1})$ and (K_2, γ) are $\mathcal{D}$-field extensions of $(K_0, \partial \restriction_{K_0})$. If we can find a common extension τ of $\partial \restriction_{K_1}$ and γ to the composite $K_1 \cdot K_2$, then we could extend $(K_1 \cdot K_2, \tau)$ to a model of $\mathcal{D}$-CF_0 which will be elementarily embeddable in $(\mathbb{U}, \partial)$ over F by Proposition 5.2. We will thus have achieved (iv) because of (5), without ruining (i) through (iii), thereby proving the independence theorem.

To find such an extension, by Lemma 5.1, it suffices to show that K_1 and K_2 are linearly disjoint over K_0. This follows from the following field-theoretic fact proved by Chatzidakis and Hrushovski in [2]: *If A, B, C are algebraically closed fields extending an algebraically closed field F, with C algebraically independent from AB over F, then $(AC)^{\mathrm{alg}}(BC)^{\mathrm{alg}}$ is linearly disjoint from $(AB)^{\mathrm{alg}}$ over AB.*[3] □

[3]See Remark 2 following the proof of the Generalised Independence Theorem in [2].

Definition 5.10 (Dimension)**.** Suppose a is a tuple and k is an algebraically closed inversive $\mathcal{D}$-subfield. We let $\dim_{\mathcal{D}}(a/k) := (\operatorname{trdeg}(\Theta_r(a)/k) : r < \omega)$ where $\Theta_r(a) := \big(\theta a : \theta$ a word of length $\leq r$ on $\{\partial_1, \ldots, \partial_\ell, \sigma_1^{-1}, \ldots, \sigma_t^{-1}\}\big)$. We view $\dim_{\mathcal{D}}(a/k)$ as an element of ω^ω equipped with the lexicographic ordering.

Note that this dimension is not preserved under interdefinability, and that a more robust notion would depend only on the the eventual growth of the sequence of transcendence degrees. This dimension should be regarded as an analogue of the Kolchin function in differential algebra. In some sense, it is too fine, but it will measure nonforking.

Lemma 5.11. *Suppose a is a tuple and $k \subseteq L$ are algebraically closed inversive $\mathcal{D}$-subfields. Then $a \mathop{\smile\!\!\!\!\!\!|}_k L$ if and only if $\dim_{\mathcal{D}}(a/L) = \dim_{\mathcal{D}}(a/k)$.*

Proof.

$$
\begin{aligned}
a \mathop{\smile\!\!\!\!\!\!|}_{k} L \quad &\Longleftrightarrow \quad \operatorname{acl}(ka) \text{ is algebraically independent of } L \text{ over } k \quad \text{(by definition)}\\
&\Longleftrightarrow \quad k(\Theta a)^{\mathrm{alg}} \text{ is algebraically independent of } L \text{ over } k \quad \text{(by 5.5)}\\
&\Longleftrightarrow \quad k\big(\Theta_r(a)\big)^{\mathrm{alg}} \text{ is algebraically independent of } L \text{ over } k \text{ for all } r < \omega\\
&\Longleftrightarrow \quad \operatorname{trdeg}(\Theta_r(a)/L) = \operatorname{trdeg}(\Theta_r(a)/k) \text{ for all } r < \omega\\
&\Longleftrightarrow \quad \dim_{\mathcal{D}}(a/L) = \dim_{\mathcal{D}}(a/k)
\end{aligned}
$$

□

5.5. **Elimination of imaginaries.** We follow the same basic strategy for proving elimination of imaginaries as that of Chatzidakis and Hrushovski in [2].

Theorem 5.12. $\operatorname{Th}(\mathbb{U}, \partial)$ *eliminates imaginaries.*

Proof. The proof of elimination of imaginaries for ACFA given in §1.10 of [2] actually proves that a simple theory admits weak elimination of imaginaries if *given any imaginary element $e = f(a)$, where a is a tuple from the home sort and f is a definable function, there exists $c \models \operatorname{tp}\big(a/\operatorname{acl}^{\mathrm{eq}}(e) \cap \mathbb{U}\big)$ with $f(c) = e$ and $c \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}^{\mathrm{eq}}(e)\cap\mathbb{U}} a$.* Since in any theory of fields weak elimination of imaginaries implies full elimination of imaginaries, it suffices to prove the existence of such a c.

Let $E := \operatorname{acl}^{\mathrm{eq}}(e) \cap \mathbb{U}$. As pointed out in §1.10 of [2], Neumann's Lemma implies that there exists $b \models \operatorname{tp}(a/Ee)$ with $\operatorname{acl}^{\mathrm{eq}}(Ea) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} = E$. We first claim that such a b can be chosen of maximal $\dim_{\mathcal{D}}$ over $\operatorname{acl}(Ea)$ in the lexicographic ordering. First, for each r, choose b_r so that the above holds and $\big(\operatorname{trdeg}(\Theta_i(b_r/\operatorname{acl}(Ea)) : i \leq r\big)$ is maximal possible. Let $n_r := \operatorname{trdeg}(\Theta_r(b_r)/\operatorname{acl}(Ea))$. Note that for all $i \leq r$, $\operatorname{trdeg}(\Theta_i(b_r)/\operatorname{acl}(Ea)) = n_i$. Now let $\Phi(x)$ be the partial type over $\operatorname{acl}(Ea)$ saying that

$x \models \operatorname{tp}(a/Ee)$,
$\operatorname{acl}^{\mathrm{eq}}(Ea) \cap \operatorname{acl}^{\mathrm{eq}}(Ex) \cap \mathbb{U} = E$, and,
for each $r < \omega$, $\operatorname{trdeg}(\Theta_r(x)/\operatorname{acl}(Ea)) \geq n_r$.

The b_r's witness that $\Phi(x)$ is finitely satisfiable, and hence by compactness it is satisfiable. Letting b realise $\Phi(x)$ we have that $b \models \operatorname{tp}(a/Ee)$, $\operatorname{acl}^{\mathrm{eq}}(Ea) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} = E$, and $\dim_{\mathcal{D}}(b/\operatorname{acl}(Ea))$ is maximal.

Now we proceed as in §1.10 of [2]. Let $c \models \operatorname{tp}(b/\operatorname{acl}(Ea))$ with $c \mathop{\smile\!\!\!\!\!\!|}_{Ea} b$. Then $c \models \operatorname{tp}(a/Ee)$ and so $f(c) = e$. So it remains to show that $c \mathop{\smile\!\!\!\!\!\!|}_{E} a$.

We have that $\operatorname{acl}^{\mathrm{eq}}(Ec) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \subseteq \operatorname{acl}^{\mathrm{eq}}(Ea)$ by independence, and hence $\operatorname{acl}^{\mathrm{eq}}(Ec) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} \subseteq \operatorname{acl}^{\mathrm{eq}}(Ea) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} = E$. Letting c' be such that $\operatorname{tp}(bc/Ee) = \operatorname{tp}(ac'/Ee)$ we have that $c' \models \operatorname{tp}(a/Ee)$ and $\operatorname{acl}^{\mathrm{eq}}(Ec') \cap \operatorname{acl}^{\mathrm{eq}}(Ea) \cap \mathbb{U} = E$. So by maximality, $\dim_{\mathcal{D}}(c'/\operatorname{acl}(Ea)) \leq \dim_{\mathcal{D}}(b/\operatorname{acl}(Ea))$. Hence, as $\dim_{\mathcal{D}}$ is automorphism invariant, $\dim_{\mathcal{D}}(c/\operatorname{acl}(Eb)) \leq \dim_{\mathcal{D}}(b/\operatorname{acl}(Ea))$. But, on the other hand,

$$\dim_{\mathcal{D}}(c/\operatorname{acl}(Eb)) \geq \dim_{\mathcal{D}}(c/\operatorname{acl}(Eab)) = \dim_{\mathcal{D}}(c/\operatorname{acl}(Ea)) = \dim_{\mathcal{D}}(b/\operatorname{acl}(Ea))$$

where the first equality is by Lemma 5.11. Hence we have equality throughout, and $\dim_{\mathcal{D}}(c/\operatorname{acl}(Eb)) = \dim_{\mathcal{D}}(c/\operatorname{acl}(Eab))$ which, by Lemma 5.11 again, implies that $c \mathop{\smile\!\!\!\!\!\!|}_{Eb} a$. Since we also have $c \mathop{\smile\!\!\!\!\!\!|}_{Ea} b$, and $\operatorname{acl}^{\mathrm{eq}}(Ea) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} = E$, we get $c \mathop{\smile\!\!\!\!\!\!|}_{E} ab$. In particular $c \mathop{\smile\!\!\!\!\!\!|}_{E} a$, as desired. □

6. The Zilber dichotomy for finite-dimensional minimal types

In this final chapter we begin to study the fine structure of definable sets in $\mathcal{D}$-CF_0. As is by now a standard approach, the first step is to prove a Zilber dichotomy type theorem for the types of SU-rank one as these form the building blocks of the finite rank definable sets. A second step, which we do not carry out here, would be to consider regular types more generally. Our current methods only allow us to handle the "finite-dimensional" case.

Definition 6.1. Working in a saturated model of $\mathcal{D}$-CF_0 and over an inversive $\mathcal{D}$-subfield k, a type $p = \operatorname{tp}(a/k)$ is called *finite-dimensional* if $k\langle a\rangle$, the inversive $\mathcal{D}$-field generated by a over k, is of finite transcendence degree over k. Equivalently, $\dim_\theta(a/k)$ is an eventually constant sequence (see Definition 5.10).

Remark 6.2. Another possibility here would have been to ask simply that the $\mathcal{D}$-field (not necessarily inversive) generated by a realisation be of finite transcendence degree. Indeed, that would better suit the geometry of generalised Hasse-Schmidt varieties as studied in [12]. However, for the purposes of the model-theoretic analysis taking into account the inversive $\mathcal{D}$-field that the realisation generates is necessary.

The goal of this chapter then is to prove that if p is a finite-dimensional type of SU-rank one, then either p is one-based or it is almost internal to the field of constants $C := \{x : e(x) = s(x)\}$. We follow here the strategy of Pillay and Ziegler [14] by proving first a canonical base property (see 6.18 for a precise statement in our context) using an appropriate notion of *jet spaces*. One such notion well-suited to the present context was developed in [12], but some preliminaries are necessary to relate the formalisms of that paper and the current one.

Assumptions 4.1 and the notation of Chapter 4 remain in place.

6.1. **Iterativity.** We begin by describing how $\mathcal{D}$ gives rise to a generalised iterative Hasse-Schmidt system in the sense of [12]. The construction here is essentially the same as (though dual to) that of Kamensky (Proposition 2.3.2 of [8]).

First of all, one can always form a completely free iterative Hasse-Schmidt system by simply iterating $\mathcal{D}$ with itself. That is, one defines the projective system of finite free algebra schemes $(\mathcal{D}^{(n)}, s_n, \psi_n,)$ by

$$\begin{aligned}
&\mathcal{D}^{(n+1)}(R) = \mathcal{D}\big(\mathcal{D}^{(n)}(R)\big)\\
&s^R_{n+1} := s^{\mathcal{D}^n(R)} \circ s^R_n : R \to \mathcal{D}^{(n+1)}(R)\\
&\psi^R_{n+1} := (\psi^R_n)^\ell \circ \psi^{\mathcal{D}^{(n)}(R)} : \mathcal{D}^{(n+1)}(R) \to (R^\ell)^{n+1}\\
&f^R_{n+1} := \pi^{\mathcal{D}^{(n)}(R)} : \mathcal{D}^{(n+1)}(R) \to \mathcal{D}^{(n)}(R)
\end{aligned}$$

for any A-algebra R. For details on composing finite free $\mathbb{S}$-algebras see §4.2 of [11]. Equipped with the trivial iterativity maps (since $\mathcal{D}^{(n+m)} = \mathcal{D}^{(n)} \circ \mathcal{D}^{(m)}$), this becomes a generalised iterative Hasse-Schmidt system (see 2.2 and 2.17 of [12]).

However, the above construction does not take into account the fact that in our $\mathcal{D}$-rings (R, e), the coefficient of ϵ_0 in $e(a)$ is always a. In other words, the fact that e is a section to $\pi = \mathrm{pr}_1 \circ \psi$. We will thus need to define a sequence of subalgebra schemes $\mathcal{D}_n \subseteq \mathcal{D}^{(n)}$ by identifying the appropriate co-ordinates. This is done as follows. Given an A-algebra R, fix the R-basis $\{\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_n} : 0 \le i_j \le \ell - 1\}$ for $\mathcal{D}^{(n)}(R)$. Define $(i_1, \dots, i_n) \sim (j_1, \dots, j_n)$ if $(i_1, \dots, i_n)$ and $(j_1, \dots, j_n)$ yield the same ordered tuple when all the zeros are dropped. Then $\mathcal{D}_n(R)$ is the subalgebra of elements

$$\Big\{\sum r_{i_1,\dots,i_n}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_n}) \mid r_{i_1,\dots,i_n} = r_{j_1,\dots,j_n} \text{ whenever } (i_1, \dots, i_n) \sim (j_1, \dots, j_n)\Big\}$$

It follows from this that $\psi_n : \mathcal{D}^{(n)} \to (S^\ell)^n$ maps $\mathcal{D}_n$ onto the diagonal defined by equating the $(i_1, \dots, i_n)$th and $(j_1, \dots, j_n)$th co-ordinates whenever $(i_1, \dots, i_n) \sim (j_1, \dots, j_n)$. This diagonal is canonically identified with the free $\mathbb{S}$-module scheme $\mathbb{S}^{L_n}$ where $L_n := \{(i_1, \dots, i_m) : 0 \le m \le n, 0 < i_j \le \ell - 1\}$.

Remark 6.3. We can define $\mathcal{D}_n$ in a co-ordinate free manner as follows. Given $n > 0$ and $1 \le i \le n$, consider the morphism of algebra schemes

$$\lambda_{i,n} := \mathcal{D}^{i-1}(f_{n-i+1}) : \mathcal{D}^{(n)} \to \mathcal{D}^{(n-1)}$$

Then $\mathcal{D}_n$ is the equaliser of $\lambda_{1,n}, \dots, \lambda_{n,n}$.

The following properties follow:

- $s_n : \mathbb{S} \to \mathcal{D}^{(n)}$ maps $\mathbb{S}$ to $\mathcal{D}_n$, so the latter becomes an $\mathbb{S}$-algebra scheme.
- $\psi_n : \mathcal{D}^{(n)} \to (S^\ell)^n$ maps $\mathcal{D}_n$ to $\mathbb{S}^{L_n}$ isomorphically as an $\mathbb{S}$-module
- $f_n : \mathcal{D}^{(n)} \to \mathcal{D}^{(n-1)}$ restricts to a surjective morphism of $\mathbb{S}$-algebra schemes from $\mathcal{D}_n$ to $\mathcal{D}_{n-1}$.
- As subalgebra schemes of $\mathcal{D}^{(m+n)}$, $\mathcal{D}_{m+n} \subseteq \mathcal{D}_m \circ \mathcal{D}_n$.

Hence, $\underline{\mathcal{D}} := (\mathcal{D}_n)$ is a generalised iterative Hasse-Schmidt system.

Proposition 6.4. *Suppose (R, e) is a $\mathcal{D}$-ring. Then there is a unique iterative $\underline{\mathcal{D}}$-ring structure $E = \big(E_n : R \to \mathcal{D}_n(R) : n < \omega\big)$ on R with $E_1 = e$. In terms of co-ordinates, this $\underline{\mathcal{D}}$-ring structure is given by*

$$E_n(a) \quad = \sum_{(i_1,\dots,i_m)\in L_n} \partial_{i_1} \cdots \partial_{i_m}(a)\epsilon_{i_1,\dots,i_m} \tag{6}$$

where $L_n := \{(i_1, \dots, i_m) : 0 \leq m \leq n, 0 < i_j \leq \ell - 1\}$ *and* $\{\epsilon_{i_1,\dots,i_m}\}$ *is the* R*-basis for* $\mathcal{D}_n(R)$ *obtained from the standard basis for* R^{L_n} *via* ψ_n^R.

Proof. First of all, recall that $E = \big(E_n : R \to \mathcal{D}_n(R) : n < \omega\big)$ is an iterative $\underline{\mathcal{D}}$-ring structure on R, according to Definitions 2.2 and 2.17 of [12], if the maps are all ring homomorphisms and

(i) $E_0 = \mathrm{id}$
(ii) $f^R_{m,n} \circ E_m = E_n$ for all $m \geq n$, where $f_{m,n} : \mathcal{D}_m \to \mathcal{D}_n$ is $f_m \circ \cdots \circ f_{n+1}$,
(iii) $E_{m+n} = \mathcal{D}_m(E_n) \circ E_m$ for all m, n.

For existence, we define $E_n : R \to \mathcal{D}^{(n)}(R)$ by composing e with itself n-times. That is, recursively, $E_0 = \mathrm{id}$ and $E_{n+1} = \mathcal{D}(E_n) \circ e$. That E_n maps R to $\mathcal{D}_n(R)$, and that it has the form claimed in (6), is not difficult to check using the fact that $e(a) = a + \partial_1(a)\epsilon + \cdots + \partial_{\ell-1}(a)\epsilon_{\ell-1}$. Properties (i) through (iii) follow more or less immediately from (6).

For uniqueness, note that the assumption that $E_1 = e$ and property (iii) force $E_{n+1} = \mathcal{D}(E_n) \circ e$, so that the above construction is the only one possible. □

6.2. **$\underline{\mathcal{D}}$-varieties and generic points.** We return now to our saturated model $(\mathbb{U}, \partial)$ of $\mathcal{D}$-CF_0, and equip it with the definable iterative $\underline{\mathcal{D}}$-field structure E given by Proposition 6.4. We fix also an inversive $\mathcal{D}$-subfield k.

In Section 3 of [12] the rudiments of $\underline{\mathcal{D}}$-algebraic geometry are developed. Let us recap some of the notions introduced there. We work inside a fixed irreducible algebraic variety X over k. While the treatment in [12] is more general, for the sake of concreteness and also with an eye toward our model-theoretic intentions, we will assume that X is affine. By a *$\underline{\mathcal{D}}$-subvariety of X over k* is meant a sequence $\underline{Z} = (Z_n)_{n<\omega}$ of subvarieties $Z_n \subseteq \tau_n X$ over k such that for all $0 < n < \omega$,

(1) $\hat{f}_n : \tau_n X \to \tau_{n-1} X$ restricts to a morphism from $Z_n \to Z_{n-1}$, and
(2) $Z_n \subseteq \tau(Z_{n-1})$.

Here, $\tau_n X = \tau(X, \mathcal{D}_n, E_n)$ is the nth prolongation of X in the sense of §2.1 of [12]. In particular, $\tau_n X(\mathbb{U})$ is canonically identified with $X\big(\mathcal{D}_n(\mathbb{U})\big)$.

Remark 6.5. The iterativity condition of Definition 3.1 of [12] here simplifies to (2) above because the iterativity map here is just the containment $\mathcal{D}_n \subseteq \mathcal{D} \circ \mathcal{D}_{n-1}$ as subschemes of $\mathcal{D}^{(n)}$.

The map E_n induces a definable map $\nabla_n : X(\mathbb{U}) \to \tau_n X(\mathbb{U})$ which with respect to the standard bases is given by

$$\nabla_n(p) := \big(\partial_{i_1} \cdots \partial_{i_m}(p) : 0 \leq m \leq n, 0 < i_j \leq \ell - 1\big)$$

We have seen this map appear already in the proof of Theorem 4.5.

We say $\underline{Z}$ is (*absolutely*) *irreducible* if each Z_n is (absolutely) irreducible.

By the $\mathbb{U}$-*rational points* (or simply the *rational points*) of $\underline{Z}$ is meant the type-definable set

$$\underline{Z}(\mathbb{U}) := \{p \in X(\mathbb{U}) : \nabla_n(p) \in Z_n(\mathbb{U}), \text{ for all } n < \omega\}$$

A rational point $p \in \underline{Z}(\mathbb{U})$ is *k-generic* if $\nabla_n(p)$ is generic in Z_n over k in Weil's sense that there is no proper k-subvariety $Y \subsetneq Z_n$ with $\nabla_n(p) \in Y(\mathbb{U})$), for all n.

Of course nothing so far has guaranteed the existence of generic points, or even of rational points. In [12] this is dealt with by working in "rich" $\underline{\mathcal{D}}$-fields. Here we do not have richness of $(\mathbb{U},\partial)$, however, we can characterise precisely which $\underline{\mathcal{D}}$-varieties do have generic points. Indeed, what is required is the following higher-order analogue of the condition appearing in axiom III of $\mathcal{D}$-CF$_0$ (cf. Theorem 4.5).

Definition 6.6 (σ-dominance)**.** Suppose $\underline{Z} = (Z_n)_{n<\omega}$ is a $\underline{\mathcal{D}}$-subvariety of an algebraic variety X. We will say that $\underline{Z}$ is *σ-dominant* if for each $n>0$ and each $i=0,\dots,t$, the morphism $\hat{\pi}_i : \tau(Z_{n-1}) \to Z_{n-1}^{\sigma_i}$ restricts to a dominant morphism from Z_n to $Z_{n-1}^{\sigma_i}$. (See §4.2 to recall what $\pi=\pi_0,\dots,\pi_t$ are.)

Remark 6.7. It follows from σ-dominance that $\hat{f}_n\restriction_{Z_n}: Z_n \to Z_{n-1}$ is dominant. Indeed, this is because $\pi_0=\pi$ and f_n is just π applied to $\mathcal{D}^{(n)}$. Hence a σ-dominant $\underline{\mathcal{D}}$-variety is in particular *dominant* in the sense of [12].

Proposition 6.8. *Suppose $\underline{Z}$ is an absolutely irreducible $\underline{\mathcal{D}}$-subvariety of X over k. Then $\underline{Z}$ has a k-generic point if and only if $\underline{Z}$ is σ-dominant.*

Proof. First suppose that $\underline{Z}$ has a k-generic point $p \in \underline{Z}(\mathbb{U})$. Then $\nabla_n(p)$ is generic in $Z_n(\mathbb{U})$. To prove σ-dominance we will show that $\hat{\pi}_i\big(\nabla_n(p)\big)$ is generic in $Z_{n-1}^{\sigma_i}$ over k. Under the identification of $\tau Z_{n-1}(\mathbb{U})$ with $Z_{n-1}\big(\mathcal{D}(\mathbb{U})\big)$, $\nabla_n(p)$ corresponds $e\big(\nabla_{n-1}(p)\big)$, and we need to show that $\pi_i\big(e(\nabla_{n-1}(p))\big) = \sigma_i\big(\nabla_{n-1}(p)\big)$ is generic in $Z_{n-1}^{\sigma_i}$ over k. But this follows from the fact that $\nabla_{n-1}(p)$ is generic in Z_{n-1} and σ_i is an automorphism.

For the converse we assume that $\underline{Z}$ is σ-dominant and seek a generic rational point. Without loss of generality we may assume that k is algebraically closed. By saturation it suffices to fix $n<\omega$ and show that there exists $p \in X(\mathbb{U})$ such that $\nabla_n(p)$ is generic in Z_n over k. For this we follow the general strategy in the proof of Theorem 4.5. Let L be an algebraically closed field extension of k, $b \in Z_n(L)$ a generic point over k, and $a := \hat{f}_{n,0}(b) \in X(L)$. We will show how to extend ∂ from k to a $\mathcal{D}$-field structure on some extension of L, such that $\nabla_n(a)=b$. This $\mathcal{D}$-field structure could then be further extended to a model of $\mathcal{D}$-CF$_0$, which by Proposition 5.2 and saturation can then be embedded into $(\mathbb{U},\partial)$ over k; thus establishing the existence of the desired $p \in X(\mathbb{U})$.

Under the identification of $\tau Z_{n-1}(L)$ with $Z_{n-1}\big(\mathcal{D}(L)\big)$, let b' be the tuple from $\mathcal{D}(L)$ corresponding to b. We have that $P^e(b')=0$ for all $P(x) \in I(Z_{n-1}/k)$. On the other hand, as $Z_n \to Z_{n-1}$ is dominant (cf. Remark 6.7), $I(Z_{n-1}/k) = I(b_{n-1}/k)$. It follows that e on k extends to a ring homomorphism $\eta : k[b_{n-1}] \to \mathcal{D}(L)$, where b_{n-1} is the image of b under $Z_n \to Z_{n-1}$, by $\eta(b_{n-1}) = b'$. The assumption of σ-dominance implies that $\pi_i \circ \eta : k[b_{n-1}] \to L$ is injective, for each $i=1,\dots,t$, so that by Lemmas 4.8 and 4.10 we can extend η to a $\mathcal{D}$-field structure on some field L' extending L. In this $\mathcal{D}$-field, the fact that $\eta(b_{n-1}) = b'$ means that $\nabla(b_{n-1}) = b$. It then follows, that for any $r<n$,

$$\begin{aligned}\nabla\big(\hat{f}_{n,r}(b)\big) &= \nabla\big(\hat{f}_{n-1,r}(b_{n-1})\big)\\ &= \hat{f}_{n,r+1}\big(\nabla(b_{n-1})\big) \qquad \text{cf. Proposition 4.7(a) of [11]}\\ &= \hat{f}_{n,r+1}(b)\end{aligned}$$

Iterating, and recalling that $\hat{f}_{n,0}(b)=a$ and $\hat{f}_{n,n}(b)=b$, we get that $\nabla_n(a)=b$, as desired. □

6.3. **Jet spaces.** In [11] and [12], following the work of Pillay and Ziegler [14], we effected a linearisation of generalised Hasse-Schmidt varieties by introducing *jet spaces*. We now specialise this theory to our present context (Fact 6.9 below), and also prove a finiteness theorem (Proposition 6.15 below) that was not done in the earlier papers but is essential here. We continue to work in a fixed affine algebraic variety X over an inversive $\mathcal{D}$-subfield k.

To each point p of X, and for each $m > 0$, we can associate a linear algebraic variety called the *mth algebraic jet space of X at p*, denoted by $\operatorname{Jet}^m X_p$. It is a kind of higher-order tangent space; see §5 of [11] for a review of this notion. Now suppose that $\underline{Z}$ is a $\underline{\mathcal{D}}$-subvariety of X over k and $p \in \underline{Z}(\mathbb{U})$. In section §4 of [12] we constructed a certain linear $\underline{\mathcal{D}}$-subvariety of $\operatorname{Jet}^m X_p$ called the *mth $\underline{\mathcal{D}}$-jet space of $\underline{Z}$ at p* and denoted by $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p$. It follows from the construction that $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U})$ is a C-linear subspace of $\operatorname{Jet}^m X_p(\mathbb{U})$. Moreover, $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p$ is the fibre above p of a bundle $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z}) \to \underline{Z}$ which is a $\underline{\mathcal{D}}$-subvariety of the algebraic jet bundle $\operatorname{Jet}^m X \to X$. We refer the reader to [12] for the definitions and basic properties of these spaces, but let us record here the fact (Lemma 4.4 of [12]) that if p is k-generic in $\underline{Z}(\mathbb{U})$ then

$$\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U}) = \big\{\lambda \in \operatorname{Jet}^m X_p(\mathbb{U}) : \nabla_n(\lambda) \in \phi\big(\operatorname{Jet}^m(Z_n)_{\nabla_n(p)}(\mathbb{U})\big), \forall n \geq 0\big\}$$

where $\phi : \operatorname{Jet}^m(\tau_n X)_{\nabla_n(p)} \to \tau_n(\operatorname{Jet}^m X_p)$ is a certain surjective linear map constructed in [11] called the *interpolating map* (see §2.3 of [12] for a brief description of ϕ).

One thing that will be important for us is that if $\underline{Z}$ is σ-dominant then so is $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})$. Indeed, in Proposition 4.7 of [12] it is proved that the dominance of the $\hat{\pi}_0$ maps are preserved when one takes jets, and the very same proof works for the other $\hat{\pi}_i$ maps as well – the key lemma behind all these cases being the "compatibility of the interpolating map with comparing of prolongations" which is 6.4(c) of [11]. Similarly, the proof of 4.7 of [12] also shows that, if p is a k-generic rational point of $\underline{Z}$, and $\underline{Z}$ is σ-dominant, then so is $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p$.

We need one more piece of notation before stating the main result of [12] specialised to the present context. Given $\underline{Z} = (Z_n)_{n<\omega}$ and $r < \omega$, we let $\nabla_r \underline{Z} := (Z_{r+n})_{n<\omega}$. It is clear that $\nabla_r \underline{Z}$ is a $\underline{\mathcal{D}}$-subvariety of Z_r. If $\underline{Z}$ is σ-dominant, then this is also the case for $\nabla_r \underline{Z}$. Finally, assuming σ-dominance, one can show that the set of rational points of $\nabla_r \underline{Z}$ is exactly $\nabla_r\big(\underline{Z}(\mathbb{U})\big)$, see the proof[4] of 3.16 of [12].

Fact 6.9. *Suppose L and L' are inversive $\mathcal{D}$-subfields extending k, $\underline{Z}$ and $\underline{Z}'$ are absolutely irreducible $\underline{\mathcal{D}}$-subvarieties of X over L and L' respectively, and $p \in \underline{Z}(\mathbb{U}) \cap \underline{Z}'(\mathbb{U})$ is L-generic in $\underline{Z}$ and L'-generic in $\underline{Z}'$. If $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z})_{\nabla_r(p)}(\mathbb{U}) = \operatorname{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z}')_{\nabla_r(p)}(\mathbb{U})$ for all $m \geq 1$ and $r \geq 0$, then $\underline{Z} = \underline{Z}'$.*

Proof. This follows from the proof of Theorem 4.8 of [12] specialised to our context. The only reason we cannot apply that theorem directly is because the "richness" assumptions of that theorem are not necessarily satisfied here. However, richness is only used in the proof to ensure that all the relevant $\underline{\mathcal{D}}$-varieties have (densely) many rational points. Hence, because of Proposition 6.8 of this paper, all one needs for that proof to go through is that the relevant $\underline{\mathcal{D}}$-varieties here be σ-dominant. This is the case for $\underline{Z}$ and $\underline{Z}'$ because by assumption they have a generic rational

[4]Actually the proof in this case is much easier as the iterativity maps are trivial.

point, and as discussed in the preceding paragraphs taking jets and ∇_r preserves σ-dominance.

Let us also remark that Theorem 4.8 of [12] asks for p to be in the "good locus" of $\underline{Z}$ (and also of $\underline{Z}'$). That is, p should be smooth on X, $\hat{f}_n$ restricted to Z_n should be smooth at $\nabla_n(p)$, and also $\nabla_n(p)$ should land inside a certain L-definable nonempty Zariski open subset of Z_n mentioned in Lemma 4.4 of [12], for all $n > 0$. Our assumption here that p is L-generic (recalling also that we are in characteristic zero) ensures that p is in the good locus. $\square$

As one might expect, the linearisation that Fact 6.9 gives us is particularly useful when the $\underline{\mathcal{D}}$-jet spaces are finite dimensional as vector spaces over the constants. In the rest of this section we aim to prove that if $\underline{Z} = (Z_n)_{n<\omega}$ is "finite-dimensional" in the sense that $\dim Z_n$ is bounded independently of n, then $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U})$ is a finite dimensional C-vector space.

But this will require some work. First of all, recall from (2) that $1_{\mathcal{D}(A)} = \sum_{k=0}^{\ell-1} c_k\epsilon_k$ where $c_{\mathbf{0}} = 1$ and the other c_k's are constants in A. If we let $(\epsilon_\alpha : \alpha \in L_n)$ denote the basis for $\mathcal{D}_n(A)$ induced by ψ_n from the standard basis for A^{L_n}, and we iterate (2) n times, then we get

$$1_{\mathcal{D}_n(A)} = \sum_{\alpha\in L_n} c_\alpha\epsilon_\alpha \tag{7}$$

where if $\alpha = (i_1, \dots, i_m)$ then $c_\alpha = c_{i_1}c_{i_2}\cdots c_{i_m} \in A$ and $c_0 = 1$.

Suppose $Y \subseteq \mathbb{A}^\mu$ is an embedded affine algebraic variety over the constant field C. Then $s_n : \mathbb{S} \to \mathcal{D}_n$ induces a "zero section" $z_n : Y \to \tau_n Y$, which in co-ordinates is given by $z_n(x_1, \dots, x_\mu) = (c_\alpha x_i)_{1\le i\le\mu,\alpha\in L_n}$. We denote by z_nY the closed subvariety of τ_nY that is the image of z_n. Note that $(z_nY)_{n<\omega}$ is a $\underline{\mathcal{D}}$-subvariety whose rational points are precisely $Y(C)$, the constant points of Y.

Lemma 6.10. *Let $\mathbb{G}_{\mathrm{a}}^\mu$ denote the μth power of the additive group scheme. Suppose $\lambda_1, \dots, \lambda_\mu \in \mathbb{U}^\nu$ and $g : \mathbb{G}_{\mathrm{a}}^\mu \to \mathbb{G}_{\mathrm{a}}^\nu$ is given by*

$$g(x_1, \dots, x_\mu) = x_1\lambda_1 + \cdots + x_\mu\lambda_\mu \ .$$

If $\lambda_1, \dots, \lambda_\mu$ are linearly independent over the constant field then, for n sufficiently large, $\tau_n g : \tau_n(\mathbb{G}_{\mathrm{a}}^\mu) \to \tau_n(\mathbb{G}_{\mathrm{a}}^\nu)$ restricts to an embedding on $z_n\mathbb{G}_{\mathrm{a}}^\mu$.

Proof. Define a sequence of algebraic varieties by

$$V_n := \ker \tau_n(g) \restriction z_n\mathbb{G}_{\mathrm{a}}^\mu$$

and assume toward a contradiction that V_n is nontrivial for arbitrarily large $n < \omega$. The equations for V_n in terms of the co-ordinates $(x_{i,\alpha})_{1\le i\le\mu,\alpha\in L_n}$ for $\tau_n(\mathbb{G}_{\mathrm{a}}^\mu)$ are easily seen to be:

$$\begin{aligned} x_{i,\alpha} - c_\alpha x_{i,0} &= 0 \qquad 1 \le i \le \mu, \quad \alpha \in L_n \\ \sum_{i=1}^{\mu} \partial_\alpha(\lambda_i)x_{i,\alpha} &= 0 \qquad \alpha \in L_n \end{aligned}$$

Here, if $\alpha = (i_1, \dots, i_m)$ then $\partial_\alpha := \partial_{i_1}\cdots\partial_{i_m}$. We can simplify this by identifying $(x_{1,0}, \dots, x_{\mu,0})$ with the co-ordinates $(x_1, \dots, x_\mu)$ for $\mathbb{G}_{\mathrm{a}}^\mu$, and then substituting the first equations into the second, to get

$$x_{i,\alpha} - c_\alpha x_i = 0 \qquad 1 \leq i \leq \mu, \quad \alpha \in L_n \tag{8}$$

$$\sum_{i=1}^{\mu} c_\alpha \partial_\alpha(\lambda_i) x_i = 0 \qquad \alpha \in L_n \tag{9}$$

Note that the equations in (9) involve only $(x_1, \dots, x_\mu)$, and we can therefore define W_0 to be the linear algebraic subvariety of $\mathbb{G}_{\mathrm{a}}^{\mu}$ given by (9) but as α ranges through all of $L_{<\omega} := \bigcup_{n<\omega} L_n$.

Claim 6.11. *It suffices to prove that W_0 has a nonzero constant point.*

Proof of Claim 6.11. Indeed, if $0 \neq (d_1, \dots, d_\mu) \in W_0(C)$, then by (9) with $\alpha = 0$, and recalling that $c_0 = 1$, we get $\sum_{i=1}^{\mu} \lambda_i d_i = 0$, which contradicts the C-linear independence of $\lambda_1, \dots, \lambda_\mu$. □

To that end, set, for each $n > 0$, $W_n := z_n W_0 \subseteq \tau_n(\mathbb{G}_{\mathrm{a}}^{\mu})$. So W_n is defined by

$$x_{i,\alpha} - c_\alpha x_i = 0 \qquad 1 \leq i \leq \mu, \quad \alpha \in L_n \tag{10}$$

$$\sum_{i=1}^{\mu} c_\alpha \partial_\alpha(\lambda_i) x_i = 0 \qquad \alpha \in L_{<\omega} \tag{11}$$

Claim 6.12. *$W_n \subseteq \tau(W_{n-1})$, for all $n > 0$.*

Proof of Claim 6.12. Indeed, a typical new equation imposed by $\tau(W_{n-1})$ is of the form $\sum_{i=1}^{\mu} c_\alpha \partial_j \partial_\alpha(\lambda_i) x_{i,j} = 0$, where $j = 1, \dots, m$. But as $x_{i,j} = c_j x_i$ holds in W_n by (10), this becomes $\sum_{i=1}^{\mu} c_j c_\alpha \partial_j \partial_\alpha(\lambda_i) x_i = 0$. Setting $\alpha' = j^\frown\alpha$, this becomes $\sum_{i=1}^{\mu} c_{\alpha'} \partial_{\alpha'}(\lambda_i) x_i = 0$ which is satisfied by W_n as it is also of the form (11). □

Claim 6.13. *Fix $n > 0$ and z the zero section on $\tau_{n-1}(\mathbb{G}_{\mathrm{a}}^{\mu})$. Then $W_n = zW_{n-1}$.*

Proof of Claim 6.13. This is by simply inspecting the equations for zW_{n-1}. They include, first of all, all the equations in (11). They also include those of (10) but for $\alpha \in L_{n-1}$. Finally we have the new equations

$$x_{i,j^\wedge\alpha} - c_j x_{i,\alpha} = 0 \qquad 1 \leq i \leq \mu, 1 \leq j \leq \ell - 1, \quad \alpha \in L_{n-1}$$

But as $x_{i,\alpha} = c_\alpha x_i$ and $c_j c_\alpha = c_{j^\wedge\alpha}$, this agrees with (10) for L_n. That is, the equatons for zW_{n-1} agrees with those of W_n. □

Claim 6.14. *For each $i = 0, \dots, t$ and each $n > 0$, $\hat{\pi}_i : \tau W_{n-1} \to W_{n-1}^{\sigma_i}$ restricts to a dominant morphism from W_n to $W_{n-1}^{\sigma_i}$.*

Proof of Claim 6.14. Note first of all that $\hat{\pi}_i$ on τW_{n-1} is just the restriction of $\hat{\pi}_i$ on $\tau\tau_{n-1}(\mathbb{G}_{\mathrm{a}}^{\mu})$. But as $\tau_{n-1}(\mathbb{G}_{\mathrm{a}}^{\mu})$ is over the constants, $\tau_{n-1}(\mathbb{G}_{\mathrm{a}}^{\mu})^{\sigma_i} = \tau_{n-1}(\mathbb{G}_{\mathrm{a}}^{\mu})$, and hence we have $\hat{\pi}_i : \tau\tau_{n-1}(\mathbb{G}_{\mathrm{a}}^{\mu}) \to \tau_{n-1}(\mathbb{G}_{\mathrm{a}}^{\mu})$. Now, as $\pi_i : \mathcal{D}(A) \to A$ is an A-algebra morphisms, $\hat{\pi}_i \circ z$ is the identity on $\tau_{n-1}(\mathbb{G}_{\mathrm{a}}^{\mu})$. So

$$\begin{aligned} W_{n-1} &= \hat{\pi}_i\big(zW_{n-1}\big) \\ &= \hat{\pi}_i(W_n) \qquad \text{by Claim 6.13} \\ &\subseteq W_{n-1}^{\sigma_i} \end{aligned}$$

Since W_{n-1} and $W_{n-1}^{\sigma_i}$ are of the same dimension and irreducible, we have equality throughout. In particular, this proves the claim. □

Claims 6.12 and 6.14 say that $\underline{W} = (W_n)$ is a σ-dominant (absolutely irreducible) $\underline{\mathcal{D}}$-subvariety of $\mathbb{G}_{\mathrm{a}}^{\mu}$. Moreover, as the V_n are nontrivial for arbitrarily large n, one sees from the equations that all the W_n are nontrivial. From Proposition 6.8 it now follows that $\underline{W}$ has a rational generic (and hence nonzero) point. By Claim 6.11 then, the proof of Lemma 6.10 is complete. □

Proposition 6.15. *Suppose $\underline{Z}$ is a $\underline{\mathcal{D}}$-subvariety of X over k such that $\dim Z_n$ is bounded independently of n. Suppose $a \in \underline{Z}(\mathbb{U})$ is k-generic. Then for each $m \geq 1$, $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a(\mathbb{U})$ is a finite dimensional C-vector space.*

Proof. Fix $m \geq 1$. Note that as $\dim Z_n$ is bounded, so is $\dim(\mathrm{Jet}^m(Z_n)_{\nabla_n(a)})$, and hence also $\dim T_n$ where $T_n := \phi\big(\,\mathrm{Jet}^m(Z_n)_{\nabla_n(a)}(\mathbb{U})\big)$ is the sequence of algebraic varieties that defines $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a$. Let N be a bound on $\dim T_n$. We will show that the C-dimension of $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a(\mathbb{U})$ is bounded by N. Toward a contradiction, set $\mu := N+1$ and suppose $\lambda_1, \dots, \lambda_\mu \in \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a(\mathbb{U})$ are C-linearly independent. Consider the map $g : \mathbb{G}_{\mathrm{a}}^{\mu} \to \mathrm{Jet}^m(X)_a$ given by $(x_1, \dots, x_\mu) \mapsto \sum_{i=1}^{\mu} x_i\lambda_i$. Restricted to the C-points g is a morphism of $\underline{\mathcal{D}}$-subvarieties, $\mathbb{G}_{\mathrm{a}}^{\mu}(C) \to \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a(\mathbb{U})$. So at the nth level we have that $\tau_n g$ restricts to a morphism $z_n\mathbb{G}_{\mathrm{a}}^{\mu} \to T_n$. But by Lemma 6.10, for $n \gg 0$, this restriction is an embedding. Hence T_n has dimension at least μ, which is a contradiction. □

6.4. **A canonical base property.** Using jet spaces we obtain a description of canonical bases and deduce therefrom the Zilber dichotomy for finite-dimensional rank one types. Canonical bases for simple theories were introduced as hyperimaginary elements in [5], to which we refer the reader for further details. Here we show that the canonical bases in $\mathcal{D}$-CF_0 are interalgebraic with infinite sequences of real elements. As might be expected from our description of types in §5.3, the canonical base of a type $\mathrm{tp}(a/L)$ will need to take into account not just the L-loci of the $\nabla_n(a)$, but indeed of the $\Theta_r(a)$, where recall that

$$\Theta_r(a) := \big(\theta a : \theta \text{ a word of length} \leq r \text{ on } \{\partial_1, \dots, \partial_\ell, \sigma_1^{-1}, \dots, \sigma_t^{-1}\}\big).$$

Theorem 6.16. *Suppose L is an algebraically closed inversive $\mathcal{D}$-subfield and a is a finite tuple from $\mathbb{U}$. For each $r < \omega$ let $\underline{Z}^{(r)} := \underline{\mathcal{D}}\text{-locus}(\Theta_r a/L)$ in the sense that $Z_n^{(r)} = \mathrm{locus}\big(\nabla_n(\Theta_r a)/L\big)$ for all n. Then*

$$\mathrm{Cb}(a/L) \subseteq \mathrm{acl}\left(\{a\} \cup \bigcup_{m\geq 1, r\geq 0} \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z}^{(r)})_{\Theta_r a}(\mathbb{U})\right)$$

Regarding the quantifier-free canonical base, by which we mean the canonical parameter for $\underline{Z} := \underline{\mathcal{D}}$-locus$(a/L)$*, we have*

$$\mathrm{Cb}\left(\mathrm{qftp}(a/L)\right) \subseteq \mathrm{dcl}\left(\{a\} \cup \bigcup_{m\geq 1, r\geq 0} \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z})_{\nabla_r(a)}(\mathbb{U})\right)$$

Proof. First of all, let $K \subseteq L$ be the inversive $\mathcal{D}$-subfield generated by the minimal fields of definition of all the $Z_0^{(r)} = \mathrm{locus}(\Theta_r a/L)$, as $r < \omega$ varies.

Claim 6.17. $\mathrm{Cb}(a/L) \subseteq \mathrm{acl}(K)$ *and* $K \subseteq \mathrm{dcl}\left(\mathrm{Cb}(a/L)\right)$

Proof of Claim 6.17. Using Lemma 5.11 we can show that $a \mathop{\smile\!\!\!\!\!\!\mid}_K L$. We also know that $\mathrm{tp}(a/K^{\mathrm{alg}})$ is an amalgamation base because the independence theorem holds over algebraically closed sets. It follows that $\mathrm{Cb}(a/L)$ is in the definable closure of K^{alg} and hence in the algebraic closure of K. Conversely, suppose α is an automorphism of $(\mathbb{U}, \partial)$ that preserves the parallelism class of $\mathrm{tp}(a/L)$. That is, $\mathrm{tp}(a/L)$ and $\mathrm{tp}\left(\alpha(a)/\alpha(L)\right)$ have a common nonforking extension, say $\mathrm{tp}(b/F)$ where F is an inversive $\mathcal{D}$-field extending both L and $\alpha(L)$. Hence for each $r < \omega$

$$\begin{aligned} Z_0^{(r)} &= \mathrm{locus}(\Theta_r b/F) \quad \text{by Lemma 5.11} \\ &= \mathrm{locus}\left(\Theta_r \alpha(a)/\alpha(L)\right) \quad \text{by Lemma 5.11} \\ &= \alpha\big(\mathrm{locus}(\Theta_r a/L)\big) \\ &= \alpha(Z_0^{(r)}) \end{aligned}$$

So $\alpha \restriction_K = \mathrm{id}$, as desired. □

Given the above claim it suffices to show that K is in the definable closure of $\{a\} \cup \bigcup_{m\geq 1, r\geq 0} \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z}^{(r)})_{\Theta_r a}(\mathbb{U})$. Suppose that α is an automorphism that fixes a and all the $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z}^{(r)})_{\Theta_r a}(\mathbb{U})$ pointwise. Let $L' := \alpha(L)$ and $\underline{Y}^{(r)} := \left(\alpha(Z_n^{(r)})\right)_{n<\omega}$. Note that $\Theta_r a$ is generic in $\underline{Z}^{(r)}$ over L and in $\underline{Y}^{(r)}$ over L'. Note also that

$$\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_s \underline{Z}^{(r)})_{\nabla_s(\Theta_r a)}(\mathbb{U}) = \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_s \underline{Y}^{(r)})_{\nabla_s(\Theta_r a)}(\mathbb{U})$$

for all $m \geq 1$ and $s \geq 0$. Indeed, there is a co-ordinate projection taking $\Theta_{r+s}(a)$ to $\nabla_s(\Theta_r a)$ that will induce a definable surjection from $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z}^{(r+s)})_{\Theta_{r+s}(a)}(\mathbb{U})$ to $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_s \underline{Z}^{(r)})_{\nabla_s(\Theta_r a)}(\mathbb{U})$, and since the former is fixed pointwise by α, so is the latter. It follows by Fact 6.9 that $\underline{Z}^{(r)} = \underline{Y}^{(r)}$, and so in particular α fixes $Z_0^{(r)}$ for all $r \geq 0$. Hence $\alpha \restriction_K = \mathrm{id}$, as desired.

Carrying out the above argument with the inversive $\mathcal{D}$-subfield generated by the minimal fields of definition of all the $\mathrm{locus}(\nabla_n a/L)$, as $n < \omega$ varies, instead of with K, yields the desired description of the quantifier-free canonical bases. □

The following is a generalisation of Theorems 1.1 and 1.2 of [14].

Corollary 6.18 (The canonical base property for finite-dimensional types)**.** *Suppose* $\mathrm{tp}(a/k)$ *is finite-dimensional and* L *is an algebraically closed inversive* $\mathcal{D}$*-field extending* k *such that* $\mathrm{Cb}(a/L)$ *is interalgebraic with* L *over* k*. Then* $\mathrm{tp}(L/k\langle a\rangle)$ *is almost internal to the constants.*

Proof. Fix $r \geq 0$, and let $\underline{X}^{(r)} := \underline{\mathcal{D}}$-locus$(\Theta_r a/k)$ and $\underline{Z}^{(r)} := \underline{\mathcal{D}}$-locus$(\Theta_r a/L)$. The finite-dimensionality of $\mathrm{tp}(a/k)$ implies that $\dim X_n^{(r)}$ is bounded independently of n. By Proposition 6.15, for each $m \geq 1$, $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{X}^{(r)})_{\Theta_r(a)}(\mathbb{U})$ is a finite dimensional C-vector space that is type-definable over $k\langle a\rangle$. Let B be a countable set that contains a C-basis for $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{X}^{(r)})_{\Theta_r(a)}(\mathbb{U})$, for all m and r, and choose B so that $B \mathop{\smile\!\!\!\!\!\!|}_{k\langle a\rangle} L$. Then

$$\begin{aligned} L &\subseteq \mathrm{acl}\left(k\langle a\rangle \cup \bigcup_{m\geq 1, r\geq 0} \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z}^{(r)})_{\Theta_r a}(\mathbb{U})\right) \qquad \text{by Theorem 6.16} \\ &\subseteq \mathrm{acl}\left(k\langle a\rangle \cup B \cup C\right) \qquad \text{as the } \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z}^{(r)})_{\Theta_r(a)}(\mathbb{U}) \ \leq\ \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{X}^{(r)})_{\Theta_r(a)}(\mathbb{U}) \end{aligned}$$

So $\mathrm{tp}(L/k\langle a\rangle)$ is almost internal to the constant field C, as desired. □

Corollary 6.19 (The Zilber dichotomy for finite-dimensional types)**.** *Suppose p is a finite-dimensional* SU*-rank one type over a substructure of some model of* $\mathcal{D}$-CF_0*. Then p is either one-based or almost internal to the constants.*

Proof. Work in a saturated model $(\mathbb{U}, \partial)$ of $\mathcal{D}$-CF_0, and suppose $p = \mathrm{tp}(a/k)$ for some inversive $\mathcal{D}$-field k. If p is not one-based then there exists an algebraically closed inversive $\mathcal{D}$-field L extending k such that $\mathrm{Cb}(a/L)$ is interalgebraic with L over k, but $\mathrm{tp}(L/ka)$ is nonalgebraic. Since p is finite-dimensional the CBP (Corollary 6.18) applies and we have that $\mathrm{tp}(L/ka)$ is almost C-internal. So $L \subseteq \mathrm{acl}(kBac)$ where $L \mathop{\smile\!\!\!\!\!\!|}_{ka} B$ and c is a tuple from C. Hence $L \not\mathop{\smile\!\!\!\!\!\!|}_{kBa} c$. On the other hand, being (interalgebraic over k with) the canonical base, $L \subseteq \mathrm{acl}(ka_1 \dots a_n)$ for some independent realisations $a_1, \dots, a_n$ of p, which we may assume to be independent of Ba over k. So $(a_1, \dots, a_n) \not\mathop{\smile\!\!\!\!\!\!|}_{kBa} c$, and hence for some $i < n$, $a_{i+1} \in \mathrm{acl}(kBaa_1 \dots a_i c)$. But as $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_k Baa_1 \dots a_i$ by choice, this witnesses that p is almost internal to C. □

Remark 6.20. We do not know if the restriction of the above corollary to finite-dimensional types is necessary. In the case of partial differential fields, by which we mean differentially closed field of characteristic zero for finitely many commuting derivations, it follows from the analysis of regular non one based types in [10] that those of rank one are in fact finite-dimensional. We expect the same to hold here, though the methods of [10] do not seem to extend.

7. Appendix: On the assumptions

In proving the existence of the model companion we restricted ourselves to characteristic zero, and we also imposed on A the properties described in Assumptions 4.1. In this appendix, we discuss the extent to which these restrictions are necessary.

7.1. **No model companion in positive characteristic.** To begin with, in most cases the restriction to characteristic zero is necessary. While model companions are known to exist in positive characteristic for the differential and difference cases, at the level of generality considered in this paper model companions do not necessarily exist in characteristic $p > 0$. We will prove this by showing that the condition of having a pth root in some $\mathcal{D}$-field extension is not in general first-order.

For the time being Assumptions 4.1 remain in place.

Proposition 7.1. *Suppose (K, ∂) is a $\mathcal{D}$-field of characteristic $p > 0$ and $a \in K$. Then the following are equivalent:*

(i) *There is a $\mathcal{D}$-field extension of K in which a has a pth root.*
(ii) *For each $n < \omega$, $E_n(a) \in \mathcal{D}_n^p(K)$. Here $\mathcal{D}_n^p$ is the $\mathbb{S}$-algebra scheme that is the image of $\mathcal{D}_n$ under the $\mathbb{S}$-algebra morphism of raising to the power p.*

Proof. First of all, note that $E_n(a) \in \mathcal{D}_n^p(K)$ if and only if $E_n(a)$ is a pth power in $\mathcal{D}_n(K^{\mathrm{alg}})$, if and only if $E_n(a)$ is a pth power in $\mathcal{D}_n(L)$ for some field extension L of K.

(i) $\Longrightarrow$ (ii). Let (L, ∂) extend (K, ∂) with $b \in L$ such that $b^p = a$. Then

$$E_n(b)^p = E_n(b^p) = E_n(a)$$

in $\mathcal{D}(L)$ showing that $E_n(a)$ is a pth power for each $n < \omega$, as desired.

(ii) $\Longrightarrow$ (i). Passing to an elementary extension if necessary, we may assume that K is an $\aleph_0$-saturated as a field. Now for each n there exists $b_n \in \mathcal{D}_n(K^{\mathrm{alg}})$ such that $b_n^p = E_n(a)$. Since the minimal polynomial of b_0 over K is $x^p - a$, we can extend $E_n : K \to \mathcal{D}_n(K)$ to an A-algebra homomorphism $E_n' : K(b_0) \to \mathcal{D}_n(K^{\mathrm{alg}})$ by setting $E_n'(b_0) := b_n$.

Now, note then that $f_n(b_n)^p = E_{n-1}(a)$, and so by $\aleph_0$-saturation, we may assume that $f_n(b_n) = b_{n-1}$ for each $n > 0$. In other words, there exists $\{b^{(\alpha)} : \alpha \in L_{<\omega}\} \subseteq K^{\mathrm{alg}}$ such that

$$b_n = \sum_{\alpha \in L_n} b^{(\alpha)} \epsilon_\alpha$$

where $(\epsilon_\alpha : \alpha \in L_n)$ is the basis for $\mathcal{D}_n(A)$ over A induced by ψ_n from the standard basis for A^{L_n}. Let $L = K\big(b^{(\alpha)}\big)_{\alpha \in L_{<\omega}}$ and extend $e : K \to \mathcal{D}(K)$ to L by

$$e'(b^{(\alpha)}) := \sum_{j=0}^{\ell-1} b^{(j^\frown \alpha)} \epsilon_j$$

That this is an A-algebra homomorphism extending e follows from the fact that the E_n' defined above were A-algebra homomorphisms extending E_n. That $\pi \circ e = \mathrm{id}$ is clear from construction. We have thus given L a $\mathcal{D}$-field structure extending (K, ∂), and we have a pth root of a in L, namely b_0. $\square$

Using Proposition 7.1 we show that under a weak hypothesis on $\mathcal{D}(A)$, the theory of $\mathcal{D}$-fields does not have a model companion.

Proposition 7.2. *If p is a prime and there is some $\epsilon \in \mathcal{D}(A)$ with $\epsilon^p \neq 0$ but $\epsilon^N = 0$ for some N, then the theory of $\mathcal{D}$-fields of characteristic p does not have a model companion.*

In particular, consider Example 3.5(b): the class of fields of characteristic p equipped with a higher derivation of length $N > p$ does not have a model companion.

Proof. Since $\mathcal{D}$ is an $\mathbb{S}$-algebra, it is itself of characteristic p. Hence, the Frobenius defines a morphism of ring schemes $F : \mathcal{D} \to \mathcal{D}$ given on points as $F : \mathcal{D}(R) \to \mathcal{D}(R)$ via $a \mapsto a^p$ where the p^{th} power is taken with respect to the multiplication in $\mathcal{D}$. Let $\mathcal{N}$ be the nilradical of $\mathcal{D}$ considered as a subgroup scheme of $(\mathcal{D}, +)$. Visibly, $\mathcal{N}$ is mapped back to itself by F. Since the nilradical of $\mathcal{D}(A)$ is nontrivial, the kernel of $F : \mathcal{N} \to \mathcal{N}$ has positive dimension. Hence, its image has dimension strictly less than $\dim \mathcal{N}$. Let $\eta \in \mathcal{N}(A) \setminus F\mathcal{D}(A^{\mathrm{alg}})$, then 1, η and ϵ^p are linearly independent over A.

Fix a natural number $m > 0$, and let $F := A(x_1, \dots, x_m)$ be the field of rational functions in m variables over A. Define $e : A[x_1, \dots, x_n] \to \mathcal{D}(F)$ by $e(x_i) := x_i + x_{i+1}\epsilon^p$ for $i < m$ and $e(x_m) := x_m + \eta$. Since η and ϵ are nilpotent, we see that the composition of e with the reduction map $\mathcal{D}(F) \to \mathcal{D}(F)/\mathcal{N}(F)$ may be identified with the standard algebra structure map. Hence, each of the associated endomorphisms is simply the identity and we may therefore extend e to F to give F a $\mathcal{D}$-field structure.

Let for each natural number n, let $E_n : F \to \mathcal{D}_n(F)$ be the map obtained from e by iteration. One computes immediately for $n < m$ that $E_n(x_1) \in \mathcal{D}_n^p(F)$ but $E_m(x_1) \notin \mathcal{D}_m^p(F)$. As (F, ∂) may be embedded into an existentially closed $\mathcal{D}$-field, we see that relative to the theory of existentially closed $\mathcal{D}$-fields no finite subtype of the type $\Phi(x) := \{E_n(x) \in \mathcal{D}_n^p\}_{n=0}^{\infty}$ implies the full type. However, by Proposition 7.1, in an existentially closed $\mathcal{D}$-field the partial type $\Phi(x)$ is equivalent to the formula $(\exists y) y^p = x$. Hence, by compactness, the class of existentially closed $\mathcal{D}$-fields is not axiomatisable. That is, the theory of $\mathcal{D}$-fields does not have a model companion. □

7.2. **Removing Assumptions 4.1.** On the other hand, if we restrict to characteristic zero, then model companions exist even in the absence of Assumptions 4.1. As we do not yet see a pressing reason to develop the theory in full generality, we restrict ourselves here to a sketch of a proof.

We explain first why Assumption 4.1(ii) is unnecessary. That is, we will describe the model companion still assuming that A is a field of characteristic zero, but *without* assuming in the decomposition $\mathcal{D}(A) = \prod_{i=0}^{t} B_i$ that the residue field of each B_i is A.

For each i fix an irreducible polynomial $P_i(x)$ of degree d_i such that the residue field of B_i is the finite extension $A[x]/(P_i)$. Denote by $\mathcal{E}_i$ the $\mathbb{S}$-algebra scheme such that $\mathcal{E}_i(R) = R[x]/(P_i)$ for any A-algebra R. In particular this fixes a basis for $\mathcal{E}_i$. Note that we may assume $\mathcal{E}_0 = \mathbb{S}$; indeed, one of the B_i will still correspond to the kernel of π and hence will have residue field A.

We have as before $\mathcal{D} = \prod_{i=0}^{t} \mathcal{D}_i$ and $\theta_i : \mathcal{D} \to \mathcal{D}_i$, but now $\rho_i : \mathcal{D}_i \to \mathcal{E}_i$ are the $\mathbb{S}$-algebra homomorphisms which when evaluated at A are the residue maps on B_i. As before we set $\pi_i := \rho_i \circ \theta_i : \mathcal{D} \to \mathcal{E}_i$. Given a $\mathcal{D}$-ring (R, ∂). For each $i = 0, \dots, t$, instead of an associated endomorphism we now only have the *associated A-algebra homomorphisms* $\sigma_i := \pi_i^R \circ e : R \to \mathcal{E}_i(R)$, which with respect to the basis for $\mathcal{E}_i$ fixed above can be written as

$$\sigma_i(a) = \sum_{j=0}^{d_i - 1} \alpha_{ij}(a) x^j$$

The $\alpha_{ij} : R \to R$ will be linear maps that are A-linear combinations of the original operators ∂. Note that $d_0 = 1$, $\pi_0 = \pi$, and $\sigma_0 = \alpha_{0,0} = \mathrm{id}$.

Extending our earlier notation we now let $\mathcal{K}$ be the class of $\mathcal{D}$-rings (R, ∂) such that R is an integral A-algebra and $\sigma_i : R \to \mathcal{E}_i(R)$ is injective for $i = 1, \dots, t$. This is still an elementary class because the injectivity of σ_i is equivalent to $\bigcap_{j=0}^{d_i - 1} \ker(\alpha_{ij})$ being trivial, and the α_{ij} are 0-definable in (R, ∂).

More generally, the class $\mathcal{M}$ is now the class of triples (R, S, ∂) where $R \subseteq S$ are integral A-algebras and $\partial = (\partial_0, \dots, \partial_{\ell-1})$ is a sequence of maps from R to S such that $e : R \to \mathcal{D}(S)$ given by $e(r) := \partial_0(r)\epsilon_0 + \cdots + \partial_{\ell-1}(r)\epsilon_{\ell-1}$ has the following properties:

(i) e is an A-algebra homomorphism,
(ii) $\sigma_0 := \pi^S \circ e : R \to S$ is the identity on R, and
(iii) for each $i = 1, \dots, t$, $\sigma_i := \pi_i^S \circ e : R \to \mathcal{E}_i(S)$ is injective.

Lemma 7.3. (a) *If $(R, L, \partial) \in \mathcal{M}$ with L a field, then we can (uniquely) extend ∂ to the fraction field F of R so that $(F, L, \partial) \in \mathcal{M}$.*
(b) *Suppose $(F, L, \partial) \in \mathcal{M}$ where F and L are fields and L is algebraically closed. If $\sigma_i : F \to \mathcal{E}_i(L)$ are the embeddings associated to ∂, and σ_i' are extensions of σ_i to F^{alg}, then there is an extension ∂' of ∂ to F^{alg} with associated embeddings σ_i'.*

Proof. Part (a) is proved exactly as Lemma 4.8 was, and so we omit it here.

For part (b) it suffices to prove, for any given $a \in F^{\mathrm{alg}}$, that we can extend e to $F(a)$ in such a way that $\pi_i^L \circ e(a) = \sigma_i'(a)$ for each $i = 0, \dots, t$. Let $P(x) \in F[x]$ be the minimal polynomial of a over F and let $c_i := \sigma_i'(a) \in \mathcal{E}_i(L)$. Note that as $P(a) = 0$ but $\frac{d}{dx}P(a) \neq 0$ in F^{alg}, and since σ_i' is a ring homomorphism, we have that $P^{\sigma_i}(c_i) = 0$ while $\frac{d}{dx}P^{\sigma_i}(c_i)$ is a unit in $\mathcal{E}_i(L)$. We wish to lift this root to $\mathcal{D}_i(L)$. While it is not the case that $\rho_i^L : \mathcal{D}_i(L) \to \mathcal{E}_i(L)$ is the residue map of a local algebra, it is still surjective with nilpotent kernel. This is because the kernel of ρ_i^L is obtained from the kernel of ρ_i^A by tensoring with L over A, and the kernel of the latter is the maximal ideal of B_i which is nilpotent. Hence by a Hensel's Lemma type argument we can lift c_i to a root b_i of $P^{e_i}(x)$ in $\mathcal{D}_i(L)$. Then $b = (b_0, \dots, b_t) \in \mathcal{D}(L)$ is a root of $P^e(x) \in \mathcal{D}(L)[x]$, and we can extend e to $F(a)$ by sending a to b. By construction $\pi_i^L e(a) = \sigma_i'(a)$. □

Suppose now that (K, ∂) is an algebraically closed $\mathcal{D}$-field. Recall that we wrote each $B_i = A[x]/(P_i)$. Let $b_{i1}, \dots, b_{id_i}$ be the distinct roots of P_i in K. Then using these roots we can decompose $\mathcal{E}_i(K)$ into a power of K itself:

$$\mathcal{E}_i(K) = K[x]/(P_i) = \prod_{k=1}^{d_i} K[x]/(x - b_{ik}) = K^{d_i}$$

Composing the associated homomorphism σ_i with the co-ordinate projections we get a d_i-tuple of *associated endomorphisms* of K, $(\sigma_{i1}, \dots, \sigma_{id_i})$ where

$$\sigma_{ik} := \sum_{j=0}^{d_i-1} b_{ik}^j \alpha_{ij}$$

In fact, under the identification $\mathcal{E}_i(K) = K^{d_i}$ we have $\sigma_i = (\sigma_{i1}, \dots, \sigma_{id_i})$, and so by the *associated difference field* (K, σ) we mean the the field K equipped with all of these endomorphisms. It then also makes sense to say that (K, ∂) is *inversive* if each σ_{ik} is an automorphism. It is important to note, though, that the σ_{ik}, while still definable in (K, ∂), are now not 0-definable but b_{ik}-definable.

Lemma 7.4. *Suppose $(F, L, \partial) \in \mathcal{M}$ where F and L are fields. Then there exists an algebraically closed extension K of L and an extension of ∂ to an inversive $\mathcal{D}$-field structure on K.*

Proof. Replacing L with L^{alg} we may assume that L is algebraically closed. We can thus write $\sigma_i = (\sigma_{i1}, \dots, \sigma_{id_i})$ where the embeddings $\sigma_{ik} : F \to L$ are obtained by composing $\sigma_i : F \to \mathcal{E}_i(L)$ with the kth projection in the decomposition $\mathcal{E}_i(L) = L^{d_i}$. We can extend these σ_{ik} to automorphisms σ'_{ik} of some algebraically closed $K \supseteq L$. So $\sigma'_i := (\sigma'_{i1}, \dots, \sigma'_{id_i}) : K \to \mathcal{E}_i(K)$ extends σ_i. Now, as in Lemma 4.10, we fix a transcendence basis B for K over F and easily extend ∂ to $F[B]$ so that $(F[B], K, \partial) \in \mathcal{M}$ and the associate homomorphisms are $\sigma'_i \restriction F[B]$. By Lemma 7.3(a) we can extend ∂ to $F(B)$ preserving this property. By Lemma 7.3(b) we can extend ∂ further to a $\mathcal{D}$-structure on $K = F(B)^{\text{alg}}$ in such a way that the associated homomorphisms are $\sigma'_1, \dots, \sigma'_t$, and hence the associated endomorphisms are the automorphisms σ'_{ik}. □

Suppose now that X is an irreducible affine variety over an algebraically closed $\mathcal{D}$-field K. For each $i = 0, \dots, t$, we have the abstract prolongations with the induced morphisms as constructed in §4 of [11]:

$$\tau(X, \mathcal{D}, e) \xrightarrow{\widehat{\theta_i}} \tau(X, \mathcal{D}_i, e_i) \xrightarrow{\widehat{\rho_i}} \tau(X, \mathcal{E}_i, \sigma_i)$$

But we also have, for each fixed $k = 1, \dots, d_i$, the morphism

$$\tau(X, \mathcal{E}_i, \sigma_i) \longrightarrow X^{\sigma_{ik}}$$

Indeed, the kth factor projection $\mathcal{E}_i(K) = K^{d_i} \to K$ induces a map from $X\big(\mathcal{E}_i(K)\big)$, where X is viewed as a scheme over $\mathcal{E}_i(K)$ via base change coming from $\sigma_i : K \to \mathcal{E}_i(K)$, to $X^{\sigma_{ik}}(K)$.[5] Composing, we have for each i and k the morphism

$$\tau(X, \mathcal{D}, e) \xrightarrow{\widehat{\pi_{ik}}} X^{\sigma_{ik}}$$

Moreover, if we set $F := A(b_{ik})_{1 \le i \le t, 1 \le k \le d_i}$, and X moves uniformly within an F-definable family of varieties, then so do the $\widehat{\pi_{ik}} : \tau(X, \mathcal{D}, e) \to X^{\sigma_{ik}}$.

Now we can state the version of Theorem 4.5 without Assumption 4.1(ii).

Theorem 7.5. *Drop Assumptions 4.1, and assume only that A is a field of characteristic zero. Then $(K, \partial) \in \mathcal{K}$ is existentially closed if and only if*

I. *K is an algebraically closed field.*

II. *There exist distinct roots $b_{i,1}, \dots, b_{i,d_i}$ of P_i in K such that $\sigma_{ik} := \sum_{j=0}^{d_i-1} b_{ik}^j \alpha_{ij}$ is an automorphism of K, for all $i = 1, \dots, t$ and $k = 1, \dots, d_i$.*

III. *There exist distinct roots $b_{i,1}, \dots, b_{i,d_i}$ of P_i in K such that if X is an irreducible affine variety over K and $Y \subseteq \tau(X, \mathcal{D}, e)$ is an irreducible subvariety over K such that $\widehat{\pi_{ik}}(Y)$ is Zariski dense in $X^{\sigma_{ik}}$ for all $i = 0, \dots, t$ and $k = 1, \dots, d_i$, then there exists $a \in X(K)$ with $\nabla(a) \in Y(K)$.*

The theory of $\mathcal{D}$-fields of characteristic zero thus admits a model companion.

Theorem 7.5 is proved just as Theorem 4.5 was, using Lemmas 7.3 and 7.4 in place of 4.8, 4.9, and 4.10. That the given axioms are first-order also follows as before. We omit the details.

[5] This morphism does not follow formally from the comparing of prolongations done in [11] because the identification $\mathcal{E}_i(K) = K^{d_i}$ is over B, not A. To fit into the formalism of [11] we would thus require σ_i to be a B_i-algebra homomorphism, which it need not be as it may move the roots of P_i.

On the face of it, Assumption 4.1(i) is more serious than Assumption 4.1(ii), but we may reduce to the case where it holds. Indeed, if (K, ∂) is a $\mathcal{D}$-field for which A is not necessarily a field, then by regarding $\mathcal{D}$ as a ring scheme over the field of fractions of the image of A in K, we may see (K, ∂) as a $\mathcal{D}$-field in which Assumption 4.1(i) holds. That is, if we consider each possible way in which $\mathcal{D}(K)$ may split as a product of local rings via maps defined by linear equations defined over the algebraic closure of the field of fractions of the image of A, then we see that in every $\mathcal{D}$-field one of these splittings must hold. We obtain an axiomatisation in the absence of Assumptions 4.1 by taking each such possible form of the linear maps used for a splitting as an antecedent and then relativising the axiomatisation of Theorem 7.5.

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