Adding club subsets of ω_2 using conditions with finite working parts.

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Introduction.

When I returned to the CRM after a month's break in September 2003 I discovered that Sy Friedman had been squatting in 'my' office in my absence. The main trace he had left was a CRM preprint giving a forcing construction exploiting conditions with finite 'working parts.' This was uncanny as completely coincidentally I had been working on just such forcing constructions before I left.

After a couple of weeks I eventually got around to reading the preprint and started wondering about recasting the argument in my preferred formalism. I arrogantly assumed that this would allow one to smooth out parts of the proof and simplify the details of the definition of the forcing conditions (at the cost of taking the framework set out in §§1,2 below as given). However when I tried to write things down I found myself, to my chagrin, more or less cornered in making my definitions into most of the intricacies that Friedman had been. I suppose that this shows that the original proof is in some sense the natural one (or one of a family of 'natural' ones). Nevertheless I obstinately persisted and this note is the result.

There are a couple of excuses for making the note public. First of all, putting the constuction into the same framework as previous ones helps clarify its relationship to them. At the same time it opens up the possibility of mixing and matching, or using the construction as a component in more complicated $\mathbb{M}$-proper forcings. Thirdly, the argument that ω_2 is preservered is perhaps conceptually a little simpler than Freidman's. How strong these excuses are is for the reader to judge. A fourth reason is that the construction is, unlike Friedman's, repeatable. I discuss this further below.

Finally, the proof yields more than Friedman's does (at least explicitly).

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This is because hypothesis that Friedman used is that ω_2 is the L-sucessor of ω_1, whereas here all that is needed is the existence of a stationary $(\omega_1, 1)$-simplified morass. In order to establish the consistency of the *non*-existence of such a simplified morass one needs to assume the consistency of the existence of an inaccessible cardinal. The whys and wherefores of this are explained, for example, in the introduction to [SVM]. Thus ω_2 being L-accessible from ω_1 is ample for the proof given here.

This sort of weakening of the hypotheses tends to be regarded as being intrinsically good. I must confess I personally find the utility and aesthetic of the proof and technique more interesting than the formal weakening.

Actually one needs to be a little careful about what one claims here because Friedman's conclusion is that if there is a constructible very stationary set with a constructible witness to its 'very stationary-ness' then there is a constructible forcing to shoot a club set through it. The forcing in this paper will be easily seen to be constructible from the simplified morass (together with the very stationary set and its witness), while if ω_2 is the L-successor of ω_1 then the morass will be constructible. So strictly speaking the *result* here generalizes Friedman's rather than strengthening it.

Nevertheless, the 'side condition' parts of Friedman's forcing conditions are sets of countable constructible Σ_1-submodels of L_βs and one needs to know that if y is one of these models and $\gamma < \beta$ has cofinality ω_1 then $y \cap L_\gamma$ is another. Thus the properties of the L-hierachy itself seem to be firmly embedded in Friedman's construction. For example, if one has $\omega_2 = \omega_1^{L[A]}$ for some set A then one does not have the analogue of Friedman's result where 'constructible' is replaced by 'constructible in A' via an essentially unmodified version of his proof. This use of the L-hierachy is also why Friedman's construction is not repeatable. In contrast, since the stationary simplified morass remains a stationary simplified morass after the forcing in this paper the construction here can be repeated. *iteration?*

Most of the set theoretic notation used in the paper is standard, but it may be useful to remind the reader of a couple of items.

If $X \subseteq \mathrm{On}$ then $\mathrm{ssup}(X)$, the *strong supemum of* X, is the least ordinal α such that $X \subseteq \alpha$, and if τ, ν are ordinals with $\tau < \nu$ then $[\tau, \nu)$ is the interval $\{\xi \in \mathrm{On} \mid \tau \leq \xi < \nu\}$. For μ, κ cardinals, μ regular and $\mu < \kappa$ let $S^\kappa_\mu = \{\xi < \kappa \mid \mathrm{cf}(\xi) = \mu\}$. If the context is clear one simply writes S_μ for S^κ_μ.

Definitions, Theorems, Lemmas and so on are numbered separately in each section. Thus, for example, Fact (4) of §1 is referred to as Fact (4) within

§1 and as Fact (1.4) elsewhere. The symbol '▲' indicates the end of the statement of a result (a theorem, proposition, lemma or fact) or of a proof.

Definition. If $D \subseteq \omega_2$ then a stationary set $D_0 \subseteq D \cap S_{\omega_1}$ *witnesses* that D is *very stationary* if for each $\xi \in D_0$ the set $D \cap \xi$ contains a club subset of ξ.

§1. Background material on morasses and μ^+-$\mathbb{M}$-properness.

Definition 1. ([V]) Let κ be a regular cardinal.

$$\mathbb{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$$

is a $(\kappa, 1)$*-simplified morass* if $\langle \theta_\alpha \mid \alpha < \kappa \rangle$ is an increasing sequence of ordinals less than κ, $\theta_\kappa = \kappa^+$, and each $\mathcal{F}_{\alpha\beta}$ is a collection of maps from θ_α to θ_β such that the following properties hold:

(i) $\forall \alpha \leq \kappa \ \mathcal{F}_{\alpha\alpha} = \{ \mathrm{id} \}$

(ii) $\forall \alpha \leq \beta \leq \gamma \leq \kappa \ \mathcal{F}_{\alpha\gamma} = \{ g \cdot f \mid f \in \mathcal{F}_{\alpha\beta} \ \& \ g \in \mathcal{F}_{\beta\gamma} \}$

(iii) $\forall \alpha < \kappa \ \big((\mathcal{F}_{\alpha\alpha+1} = \{ \mathrm{id} \} \ \& \ \theta_{\alpha+1} = \theta_\alpha + 1 \big)$ or $\big(\mathcal{F}_{\alpha\alpha+1} = \{ \mathrm{id}, h \} \ \&$
$\exists \sigma < \theta_\alpha \ (h \restriction \sigma = \mathrm{id} \ \& \ \forall \tau \ (\sigma + \tau < \theta_\alpha \longrightarrow h(\sigma + \tau) = \theta_\alpha + \tau)))\big)$

(iv) $\forall \alpha \leq \kappa$ (α is a limit ordinal $\longrightarrow \forall \beta_0, \beta_1 < \alpha \ \forall f_0 \in \mathcal{F}_{\beta_0\alpha} \ \forall f_1 \in \mathcal{F}_{\beta_1\alpha}$
$\exists \gamma \in [\beta_0 \cup \beta_1, \alpha) \ \exists h \in \mathcal{F}_{\gamma\alpha} \ \exists g_0 \in \mathcal{F}_{\beta_0\gamma} \ \exists g_1 \in \mathcal{F}_{\beta_1\gamma}$
$(f_0 = h \cdot g_0 \ \& \ f_1 = h \cdot g_1))$

(v) $\bigcup \{ f``\theta_\alpha \mid \alpha < \kappa \ \& \ f \in \mathcal{F}_{\alpha\kappa} \} = \kappa^+$

Definition 2. Let $\mathbb{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$ be a $(\kappa, 1)$-simplified morass. Then $\mathcal{F} = \bigcup \{ (\alpha, f) \mid \alpha < \kappa \ \& \ f \in \mathcal{F}_{\alpha\kappa} \}$.

For $(\beta, f) \in \mathcal{F}$ and $\nu \leq \theta_\beta$ write $\varsigma_{f \restriction \nu}$ for $\mathrm{ssup}(\mathrm{rge}(f \restriction \nu))$.

Definition 3. A $(\kappa, 1)$-simplified morass, $\mathbb{M}$, with $\mathcal{F}$ as in Definition (2), is *stationary* if $\{ \mathrm{rge}(f) \mid \exists \alpha < \kappa \ (\alpha, f) \in \mathcal{F} \}$ is a stationary subset of $[\kappa^+]^{<\kappa}$.

Stationary $(\kappa, 1)$-simplified morasses exist in L, K_{DJ} and so on, for all regular cardinals κ, and the usual forcing ([V]) for adding $(\kappa, 1)$-simplified morasses adds stationary ones. (Notice that, as the forcing is κ-directed closed, it is consistent that κ is also supercompact.) The following well-known, fundamental fact due to Velleman always comes into play when dealing with simplified morasses.

Fact 4. (Velleman, [V, Lemma 3.2]) Let $\alpha \le \beta \le \kappa$, and f, $g \in \mathcal{F}_{\alpha\beta}$. If $\nu \in \mathrm{rge}(f) \cap \mathrm{rge}(g)$ there is some $\overline{\nu} < \theta_\alpha$ such that $f(\overline{\nu}) = \nu = g(\overline{\nu})$ and $f \restriction \overline{\nu} + 1 = g \restriction \overline{\nu} + 1$. ▲

Proof. By induction on β for each α. ▲

Another two useful facts from [V] are the following.

Fact 5. (Stanley, [V, Theorem 3.9]) If $\alpha \le \beta \le \kappa$, $f \in \mathcal{F}_{\alpha\beta}$ and $\nu < \theta_\alpha$ then there is some $g \in \mathcal{F}_{\alpha\beta}$ such that $g \restriction \nu = f \restriction \nu$ and $g(\nu + \xi) = \mathrm{ssup}(g``\nu) + \xi$ for $\nu + \xi < \theta_\alpha$. ▲

Proof. Again by induction on β for each α. ▲

Fact 6. (Velleman, [V, Corollary 3.5]) If $\langle f_i \mid i < \chi \rangle$ is a collection of maps with each $f_i \in \mathcal{F}_{\alpha_i\beta}$ for $i < \chi$ and $\chi < \mathrm{cf}(\beta)$ then there is some $\alpha \in [\sup(\{\alpha_i \mid i < \chi\}, \beta)$, some $f \in \mathcal{F}_{\alpha\beta}$ and maps $g_i \in \mathcal{F}_{\alpha_i\alpha}$ for all $i < \chi$ such that $f_i = f \cdot g_i$ for all $i < \chi$. In particular any collection of fewer than κ maps in $\mathcal{F}$ can be factored through some single common map $(\alpha, f) \in \mathcal{F}$. ▲

A small strengthening of Fact (4) is also well known.

Fact 7. Let $\alpha \le \beta \le \kappa$, and f, $g \in \mathcal{F}_{\alpha\beta}$. If ν, $\tau \le \theta_\alpha$ and $\mathrm{ssup}(f``\nu) = \mathrm{ssup}(g``\tau)$ then $\nu = \tau$ and $f \restriction \nu = g \restriction \nu$. ▲

Proof. By induction on β for each α. (For ν, $\tau < \theta_\alpha$ this is an immediate corollary of Facts (5) and (6), without the necessity of a separate inductive proof.) ▲

Next we need some notation that will be useful in specifically in this proof.

Definition 8. For $(\beta, f) \in \mathcal{F}$ and $\xi < \kappa^+$ define f *cut (down) at* ξ, written $f \downharpoonleft \xi$, as follows. Let $\nu \le \theta_\beta$ be maximal such that $f``\nu \subseteq \xi$. Then $f \downharpoonleft \xi$ is $f \restriction \nu$ and $(\beta, f) \downharpoonleft \xi$ is $(\beta, f) \restriction \nu$. Of course, $(\beta, f) \downharpoonleft \kappa^+$ is just (β, f); and using the notation of Definition (1.2) one has that $\varsigma_{f \downharpoonleft \xi}$ is $\mathrm{ssup}(f \cap \xi)$.

Definition 9. If $X \subseteq S^{\kappa^+}_\kappa$, $(\beta, f) \in \mathcal{F}$ and $\nu \le \theta_\beta$ then define $h_X(\beta, f \restriction \nu)$, the *X-height* of $(\beta, f) \restriction \nu$, to be the least $\xi \in X \cup \{\kappa^+\}$ such that $f``\nu \subseteq \xi$.

Definition 10. If $I \in [\kappa^+]^1 \cup [\kappa^+]^2$ write I^- for $\min(I)$ and I^+ for $\max(I)$.

Definition 11. If $a \subseteq [\kappa^+]^1 \cup [\kappa^+]^2$ let $X_a = \{I^- \mid I \in a\} \cap S_\kappa$.

Definition 12. If $a \subseteq [\kappa^+]^1 \cup [\kappa^+]^2$ and $A \subseteq \mathcal{F}$ let

$$\overline{A} = \left\{ (\beta, f) \downharpoonleft \xi \,\middle|\, (\beta, f) \in A \ \ \& \ \ \xi \in X_a \cup \{\kappa^+\} \right\}.$$

Fact 13. Note that if $(\beta, f) \in \mathcal{F}$, $\nu \leq \theta_\beta$ and $\beta \leq \gamma < \kappa$ then there are $g \in \mathcal{F}_{\beta\gamma}$ and $h \in \mathcal{F}_{\gamma\omega_1}$ such that $f \restriction \nu = h \cdot g \restriction \nu$ and $g \restriction \nu$ and $h \restriction \varsigma_{g\restriction\nu}$, $= h \restriction (\mathrm{ssup}(g``\nu))$, are uniquely defined. This is immediate from Fact (3).
▲

Now suppose that $\mathbb{M}$ is stationary and let $\mu = \kappa^-$ be the cardinal predecessor of κ. (Up until now everything mentioned was true for any regular κ, but the following definition as stated only makes sense for successor κ.)

Definition 14. Let λ be a regular cardinal greater than κ^+. Then $(\mathcal{N}, \epsilon) \prec (H_\lambda, \in)$ is a *good model* if $\overline{\overline{\mathcal{N}}} = \mu$, $\{\mathbb{M}, \mathcal{F}, c, \kappa, \kappa^+\} \cup \mu \subseteq \mathcal{N}$, $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$, $\delta = \mathcal{N} \cap \kappa \in \kappa$, there is some $F \in \mathcal{F}_{\delta\kappa}$ such that $\mathrm{rge}(F) = \mathcal{N} \cap \kappa^+$, and for each $(\alpha, f) \in \mathcal{F}$ with $\alpha < \delta$ if there is some $f' \in \mathcal{F}_{\alpha\delta}$ such that $f = F \cdot f'$ then (α, f), $\mathrm{rge}(f) \in \mathcal{N}$.

The following two observations about good models are useful in the next section.

Lemma 15. If $\mathcal{N}$ is good, $\alpha < \delta$, $g \in \mathcal{F}_{\alpha\kappa}$, $g' \in \mathcal{F}_{\alpha\delta}$, $g'' \in \mathcal{F}_{\delta\kappa}$ and $g = g'' \cdot g'$ then $h(g) = F \cdot g' \in \mathcal{F}_{\alpha\kappa} \cap \mathcal{N}$ (and so $(\alpha, h(g)) \in \mathcal{F} \cap \mathcal{N}$). ▲

Proof. As $\mathcal{N}$ is good, if $\alpha < \delta$ and $g' \in \mathcal{F}_{\alpha\delta}$ then $F \cdot g' \in \mathcal{N}$. ▲

Lemma 16. If $\mathcal{N}$ is good and $(\alpha, f) \in \mathcal{N}$ then $\exists f' \in \mathcal{F}_{\alpha\delta}$ $(f = F \cdot f')$. ▲

Proof. If $(\alpha, f) \in \mathcal{N}$ then, firstly, $f \in \mathcal{N}$, and, secondly, $\alpha < \delta$, since $\delta = \mathcal{N} \cap \kappa$. Since $\mathbb{M} \in \mathcal{N}$ and $\alpha \in \mathcal{N}$ one has that $\theta_\alpha < \delta$, and hence that $\theta_\alpha \subseteq \mathcal{N}$. So if $\xi < \theta_\alpha$ then $f(\xi) \in \mathcal{N}$. Hence $\mathrm{rge}(f) \subseteq \kappa^+ \cap \mathcal{N} = \mathrm{rge}(F)$.

Now factor f as $k \cdot h$, where $(\delta, k) \in \mathcal{F}$ and $h \in \mathcal{F}_{\alpha\delta}$. Let $\overline{\theta} = \mathrm{ssup}(h``\theta_\alpha)$. Then there is some $\zeta \leq \theta_\delta$ such that $\mathrm{ssup}(F``\zeta) = \mathrm{ssup}(k``\overline{\theta})$. Consequently, by Fact (7), one has that $\zeta = \overline{\theta}$ and $F \restriction \overline{\theta} = k \restriction \overline{\theta}$. Thus one also has that $f = F \cdot h$. ▲

However the main point of introducing the notion of a good model is in order to formulate the following definition and fact.

Definition 17. Let $\mathbb{P}$ be a forcing notion with $\mathbb{P} \in H_\lambda$ for some regular cardinal λ. $\mathbb{P}$ is κ-$\mathbb{M}$-*proper* if there is some $x \in [H_\lambda]^{\leq\mu}$ such that the following holds. Suppose $p \in \mathbb{P}$ and $\mathcal{N}$ is a good model with $\{\mathbb{P}, p\} \cup x \cup p \subseteq \mathcal{N}$. Then there is some $p^* \leq p$ which is $(\mathbb{P}, \mathcal{N})$-generic.

Fact 18. If $\mathbb{P}$ is κ-$\mathbb{M}$-proper then $\Vdash_{\mathbb{P}} \mu^+ = \kappa$. ▲

Fact 19. In order to prove that $\mathbb{P}$ preserves ω_2 it suffices to show that: If $p \in \mathbb{P}$, $\mathcal{D}$ is a dense and open (below p) subset of $\mathbb{P}$ and $\mathcal{N} \prec H_{\omega_3}$ is such that $\mathcal{D}$, p, $\mathbb{M}$, $\mathcal{F} \in \mathcal{N}$, $\overline{\overline{\mathcal{N}}} < \omega_2$, $\mathcal{N}^\omega \subseteq \mathcal{N}$, $\delta = \mathcal{N} \cap \omega_2 \in \omega_2$ and $\mathrm{cf}\,(\delta) = \omega_1$, there is some $p^* \leq p$ such that for any $q \leq p^*$ with $q \in \mathcal{D}$ there is some $s \in \mathcal{D} \cap \mathcal{N}$ such that q and s are compatible. ▲

Proof. This is well-known folklore, *cf.* the "general comment about ω_1-preservation" on the first page of [F]. ▲

From now onwards suppose that $\mathbb{M}$ is a stationary $(\omega_1, 1)$-simplified morass, so that $\mathcal{F} = \{\,(\beta, f) \mid \beta < \omega_1 \;\;\&\;\; f \in \mathcal{F}_{\beta\kappa}\,\}$.

§2. Adding a club subset of ω_2.

Theorem 1. Suppose there is a stationary $(\omega_1, 1)$-simplified morass. Let D be a very stationary subset of ω_2. Then there is an $\mathbb{M}$-proper, ω_2-preserving forcing of size ω_2 (and which is thus preserves all cardinals) such that if G is $\mathbb{P}$-generic over V then $V[G] \models$ "there is a club set $C \subseteq D$."

Definition 2. Define a notion of forcing $\mathbb{P}$ as follows. $p \in \mathbb{P}$ if $p = (a_p, A_p)$, where $a \in [[\omega_2]^1 \cup [\omega_2]^2]^{<\omega}$ and $A \in [\mathcal{F}]^{<\omega}$, and, writing X_p and h_p for X_{a_p} and $h_{X_{a_p}}$ respectively, the following properties hold.

(a) If I, $J \in a_p$ then either $I^+ < J^-$ or $J^+ < I^-$.

(b) If $I \in a_p$ and $(\beta, \phi) \in \overline{A_p}$ then

(i) $I^- \in \mathrm{rge}(\phi) \Longrightarrow I^+ \in \mathrm{rge}(\phi)$, and

(ii) $I^- \notin \mathrm{rge}(\phi)$ & $I^- < \varsigma_\phi \Longrightarrow \exists J \in a_p\; J^- = \min(\mathrm{rge}(\phi) \setminus I^-)$.

(So when $I^- \notin \mathrm{rge}(\phi)$ and $I^- < \varsigma_\phi$, one has, using (a), that $I^+ \notin \mathrm{rge}(\phi)$.)

(c) Suppose (β, f) and $(\gamma, g) \in A_p$ and $\xi \in X_p \cup \{\omega_2\}$ are such that $h_p(\beta, f \restriction \xi)$, $h_p(\gamma, g \restriction \xi) = \xi$. Let $\alpha < \xi$. Then

(i) $\mathrm{rge}(f \restriction \xi) \subseteq \mathrm{rge}(g \restriction \xi)$ or $\mathrm{rge}(g \restriction \xi) \subseteq \mathrm{rge}(f \restriction \xi)$, and

(ii) if $\varsigma_{f\restriction\alpha} = \alpha = \varsigma_{g\restriction\alpha}$ and $g(\alpha_\gamma) < f(\alpha_\beta) < \xi$ then $\forall \nu < \alpha\; \exists \tau \in (\nu, \alpha] \cap (\mathrm{rge}(g) \setminus \mathrm{rge}(f))$.

If q, $p \in \mathbb{P}$ then $q \leq p$ if $a_p \subseteq a_q$ and $A_p \subseteq A_q$.

Reformulations of (c).

First of all consider (c.i). Assume that $\beta \leq \gamma$ and set $\mathrm{dom}(\phi) = \mu$ and $\phi = \sigma \cdot \pi$, where there is some $g \in \mathcal{F}_{\beta\gamma}$ such that $\pi = g \restriction \mu$ and $\mathrm{dom}(\sigma) = \mathrm{ssup}(\pi``\mu) = \varsigma_{\pi\restriction\mu}$, and let $\mathrm{dom}(\psi) = \nu$. Then the conclusion to (c.i) is equivalent to the assertion that either $\varsigma_{\pi\restriction\mu} \leq \nu$ and $\phi = \psi \cdot \pi$ or $\nu \leq \mu$, $\psi = \sigma \restriction \nu$ and $\pi \restriction \nu = \mathrm{id}$.

Secondly, note that it is implicit in the hypotheses of (c.ii) that $\alpha < \varsigma_\phi, \varsigma_\psi$. By (c.i) they also give that $\mathrm{rge}(\phi) \subsetneq \mathrm{rge}(\psi)$ and $\beta < \gamma$ since one knows that $\min(\mathrm{rge}(\psi) \cap [\alpha, \xi)) = \psi(\alpha_\gamma) < \phi(\alpha_\beta) = \min(\mathrm{rge}(\phi) \cap [\alpha, \xi))$.

Thirdly, (c) is equivalent to the following apparent generalization. If (β, ϕ) and $(\gamma, \psi) \in \overline{A_p}$ and $h_p(\beta, \phi) = h_p(\gamma, \psi)$ then

(i) $\mathrm{rge}(\phi) \subseteq \mathrm{rge}(\psi)$ or $\mathrm{rge}(\psi) \subseteq \mathrm{rge}(\phi)$

(ii) if $\xi \in X_p \cup \{\omega_2\}$ and $\alpha < \xi$ are such that

$$\phi = f \downharpoonright \xi \;\&\; \psi = g \downharpoonright \xi \text{ for some } (\beta, f), (\gamma, g) \in A_p, \text{ and}$$

$$\varsigma_{\phi\downharpoonright\alpha} = \alpha = \varsigma_{\psi\downharpoonright\alpha} \text{ and } \psi(\alpha_\gamma) < \phi(\alpha_\beta) < \xi$$

then $\forall \nu < \alpha \; \exists \tau \in (\nu, \alpha] \cap (\mathrm{rge}(\psi) \setminus \mathrm{rge}(\phi))$.

This is because given such (β, ϕ), (γ, ψ) and ξ one has $\phi = \phi \downharpoonright h(\beta, \phi)$, $\psi = \psi \downharpoonright h(\gamma, \psi)$ and $\alpha < h(\beta, \phi) = h(\gamma, \psi) \leq \xi$

Finally, a stronger, yet still reasonable, conclusion for (c.ii) is to demand "$\psi(\alpha_\gamma) = \alpha$" in place of "$\exists \gamma \in (\nu, \alpha] \cap \mathrm{rge}(\psi) \setminus \mathrm{rge}(\phi)$." This condition will be satisfied automatically if the elements of A_p are all "good" maps in the sense of [M*2], for example. However the proof given below goes through without insisting on this more restrictive condition.

Overview of the relationship between the various clauses of the definition of $\mathbb{P}$. Let $\mathbb{P}_0$ consist of pairs $(a_p, A_p) \in [[\omega_2]^1 \cup [\omega_2]^2]^{<\omega} \times [\mathcal{F}]^{<\omega}$ satisfying (a), (b) and (c.i), with a similar ordering to that of $\mathbb{P}$. The proof below, in which (a), (b) and (c.i) are used in an inextricably intertwined way, shows that $\mathbb{P}_0$, as well as $\mathbb{P}$, is $\mathbb{M}$-proper and preserves ω_2.

However, the requirement that conditions in $\mathbb{P}$ satisfy (c.ii) is more or less orthogonal to this cardinal-preservation part of the proof of Theorem (6). One need only carry out the very simple check that the model $\mathcal{N}$ in the proof of $\mathbb{M}$-properness, Proposition (10) below, can be chosen so that the map (δ, F) satifies (c.ii) with respect to the maps $(\beta, f) \in A_p$ for the 'p' of that proof. This is essentially immediate. Apart from this, (c.ii) is self-propagating.

The reason one does need (c.ii) is to complete the proof that if G is $\mathbb{P}$-generic over V then $C_G = \{ I^- \mid \exists p \in G \; I \in a_p \}$ is indeed a club subset of ω_2.

I highlight the five key places in the proof of Theorem (6) where one is extending conditins and thes considerations are in play by tagging them (A)-(E).

Before embarking on the proofs that show that $\mathbb{P}$ preserves cardinals one may as well isolate a couple of lemmas concerning adding apices to A_p and large enough single ordinals to a_p for conditions $p \in \mathbb{P}$.

Definition 3. $(\gamma, g) \in$ '*calF* is a *candidate apex* for A_p if , $\bigcup a_p \subseteq \mathrm{rge}(g)$ and for all $(\beta, f) \in A_p$ one has $\beta < \gamma$ and there is some $k_f \in \mathcal{F}_{\beta\gamma}$ such that $f = g \cdot k_f$.

$p \in \mathbb{P}$ is *pointed* with *apex*

Lemma 4. Suppose that $p \in \mathbb{P}$ and $(\gamma, g) \in \mathcal{F}$ is a candidate apex for A_p. Then $p^* = (a_p, A_p \cup \{(\gamma, g)\}) \in \mathbb{P}_0$. ▲

Proof. (a) holds because it does for p and $a_{p^*} = a_p$.

(b). Let $I \in a_p$ and $\xi \in X_p \cup \{\omega_2\}$. By the assumption that $\bigcup a_p \subseteq \mathrm{rge}(g)$ one has that I^-, $I^+ \in \mathrm{rge}(g)$. If $I^- < \xi$ then $I^+ < \xi$ by (a) (since there is some $J \in a_p$ such that $\xi = J^-$) and so I^-, $I^+ \in \mathrm{rge}(g) \cap \xi = \mathrm{rge}(g \downharpoonleft \xi)$. While if $\xi \leq I^-$ then (b) is vacuously true. Thus (b) holds for the elements of $\overline{A_{p^*}} \setminus \overline{A_p}$. (b) holds for elements of $\overline{A_p}$ since it holds for p.

(c). Clearly if (β, ϕ), (ϵ, ψ) are either both elements of $\overline{A_p}$ or of $\overline{A_{p^*}} \setminus \overline{A_p}$ there is nothing (new) to prove. So suppose that $(\beta, f) \in A_p$ is such that $\phi = f \restriction \nu$ for some $\nu \leq \theta_\beta$ and that $(\epsilon, \psi) = (\gamma, g) \downharpoonleft \xi$ for $\xi = h_p(\beta, \psi)$ $= h_p(\gamma, g \downharpoonleft \xi)$. By the assumption on (γ, g) one has that $f = g \cdot k_f$. So $\phi = f \downharpoonleft \xi = (g \downharpoonleft \xi \cdot k_f) \restriction \nu$, as required. ▲

Remark. Thus $p^* \in \mathbb{P}$ if (γ, g) satisfies (c.ii) of the Definition of $\mathbb{P}$ with respect to the maps in A_p.

Definition 5. p is *pointed* with apex (γ, g) if $(\gamma, g) \in A_p$ and (γ, g) is a candidate apex for $A_p \setminus (\gamma, g)$.

Lemma 6. Suppose $p \in \mathbb{P}$ and $\mathrm{ssup}(\bigcup a_p \cup \bigcup \{\mathrm{rge}(f) \mid (\beta, f) \in A_p\}) \leq \zeta$. Then $p' = (a_p \cup \{\zeta\}, A_p) \in \mathbb{P}$. ▲

Proof. By the assumptions $I^+ < \zeta = \min\{\zeta\}$ for every $I \in a_p$, so (a) holds. Also by the assumptions, $\zeta \notin \mathrm{rge}(\phi)$ and $\mathrm{ssup}(\mathrm{rge}(\phi)) \leq \zeta$ for every $(\beta, \phi) \in \overline{A_p}$, so (b) holds vacuously. Finally $\overline{A_{p'}} = \overline{A_p}$ and $h_{p'}(\beta, \phi) = h_p(\beta, \phi)$ for $(\beta, \phi) \in \overline{A_p}$ unless $h_p(\beta, \phi) = \omega_2$ and $\zeta \in S_{\omega_1}$, in which case $h_{p'}(\beta, \phi) = \zeta$. Thus (c) also holds. ▲

Proposition 7. $\mathbb{P}$ is $\mathbb{M}$-proper. ▲

Proof. Let $p \in \mathbb{P}$ and let $\mathcal{N} \prec H_{\omega_3}$ be countable and good with p, $\mathbb{P}$ $\mathbb{M}$, $\mathcal{F}$, ω_1, $\omega_2 \in \mathcal{N}$, $\delta = \mathcal{N} \cap \omega_1 < \omega_1$. Let $(\delta, F) \in \mathcal{F}$ be such that and $\mathcal{N} \cap \omega_2 = \mathrm{rge}(F)$.

I also assume, by the stationarity of $\mathbb{M}$, that if $\nu < \theta_\delta$ and $\varsigma_{F\restriction\nu} < F(\nu)$ then $\mathrm{cf}\,(F(\nu)) = \omega_1$. (Recall that $\varsigma_{F\restriction\nu} = \mathrm{ssup}(F``\nu)$.)

(A). Let $p^* = (a_p, A_p \cup \{(\delta, F)\})$. By Lemmas (8) $p^* \in \mathbb{P}_0$. As each $(\beta, f) \in A_p$ and each $\nu < \theta_\beta$ are elements of $\mathcal{N}$ one has that $\varsigma_{f\restriction\nu} \in \mathcal{N} \cap \omega_2 = \mathrm{rge}(F)$. So condition (c.ii) of the definition of $\mathbb{P}$ holds for p^* and $p^* \in \mathbb{P}$. Clearly $p^* \leq p$. ▲(**A**)

I show that p^* is ($\mathbb{P}$, $\mathcal{N}$)-generic. Let $\mathcal{D} \in \mathcal{N}$ be a dense, open subset of $\mathbb{P}$ and let $q \leq p^*$ be such that $q \in \mathcal{D}$.

(B). Next set $q \restriction \mathcal{N} = (a_q \cap \mathcal{N}, A_q \cap \mathcal{N})$. Note that $\overline{A_q \cap \mathcal{N}} \subseteq \overline{A_q} \cap \mathcal{N}$, but that equality need not hold.

Claim 8. $q \restriction \mathcal{N} \in \mathbb{P}$ (and hence $q \leq q \restriction \mathcal{N}$). ▲

Proof. One has to show that $q \restriction \mathcal{N}$ satisfies (a)-(c) of the definition of $\mathbb{P}$. Clearly $q \restriction \mathcal{N}$ satisfies (a) since q does.

Now consider (b). Suppose $I \in a_q \cap \mathcal{N}$ and $(\beta, \phi) \in \overline{A_q \cap \mathcal{N}}$, and let $\xi \in (X_q \cap \mathcal{N}) \cup \{\omega_2\}$ and $(\beta, f) \in A_q \cap \mathcal{N}$, be such that $\phi = f \downharpoonright \xi$.

If $I^- \in \mathrm{rge}(\phi)$ then $I^+ \in \mathrm{rge}(\phi)$ by (a) and (b.i) for q. So suppose $I^- \notin \mathrm{rge}(\phi)$ and $I^- < \varsigma_\phi$. Thus there is some $J \in a_q$ with $J^- = \min(\mathrm{rge}(\phi))$ by (b.ii) for q. But then $J^+ \in \mathrm{rge}(\phi)$ by (b.i) for q. As $\mathcal{N}$ is good one has that $\mathrm{rge}(\phi) \subseteq \mathrm{rge}(f) \subseteq (\mathrm{rge}(F))$, so $J \in \mathcal{N}$. Hence $J \in a_q \cap \mathcal{N}$, showing (b.ii) in this instance as required.

Next look at (c.i). Suppose $(\beta, \phi) \in \overline{A_q \cap \mathcal{N}}$ and set $h_q(\beta, \phi) = \zeta$ and $h_{q\restriction\mathcal{N}}(\beta, \phi) = \xi$. Hence $\zeta \leq \xi$. Suppose $\zeta \notin \mathrm{rge}(F)$ and $\zeta < \varsigma_F$. So, by (b.ii) for q, there is some $J \in a_q$ such that $J^- = \min(\mathrm{rge}(F) \setminus \zeta)$, and so $J \in \mathcal{N} \cap a_q$.

Recall that if $\tau \in \mathrm{rge}(F)$ and $\mathrm{cf}\,(\tau) = \omega$ then $\tau \cap \mathrm{rge}(F)$ is unbounded in τ; so $\mathrm{cf}\,(J^-) = \omega_1$, and thus $J^- \in X_{q\restriction\mathcal{N}}$. Hence $J^- = h_{q\restriction\mathcal{N}}(\beta, \phi)$, that is, $\xi = \min(\mathrm{rge}(F) \setminus \zeta)$. Note that this implies that $[\zeta, \xi) \cap \mathrm{rge}(F) = \emptyset$, and that $\xi < \omega_2$.

Now suppose $h_{q\restriction\mathcal{N}}(\beta, \phi) = h_{q\restriction\mathcal{N}}(\gamma, \psi) < \omega_2$ and $h_q(\beta, \phi) < h_q(\gamma, \psi)$. Write ξ for $h_{q\restriction\mathcal{N}}(\beta, \phi)$. Then there is some $\tau \in \mathrm{rge}(\psi) \cap [h_q(\beta, \phi), h_q(\gamma, \psi))$. But

$\mathrm{rge}(\psi) \subseteq \mathrm{rge}(F)$ and so $\tau \in [h_q(\beta,\phi),\xi) \cap \mathrm{rge}(F)$, a contradiction.

Hence if $h_{q\restriction\mathcal{N}}(\beta,\phi) = h_{q\restriction\mathcal{N}}(\gamma,\psi) < \omega_2$ then $h_q(\beta,\phi) = h_q(\gamma,\psi)$, and so (c.i) holds by (c.i) for q.

If, alternatively, $\xi = \omega_2$ then ζ is the least element of $X_q \cup \{\omega_2\} \setminus \varsigma_F$, and again (c.i) holds by (c.i) for q.

Finally, (c.ii) for $q \restriction \mathcal{N}$ is immediate from (c.ii) for q since $\overline{A_q \cap \mathcal{N}} \subseteq \overline{A_q}$. ▲(**B**)

Notation 9.

Let	$\langle(\beta_i, f_i) \mid i < \chi\rangle$ enumerate $\langle(\alpha, f) \in A_q \mid \alpha < \delta\rangle$ for some $\chi < \mu$.
Let	$Y_i = \mathrm{rge}(f_i) \cap \mathrm{rge}(h(f_i))$ for $i < \chi$.
Note that	$Y_i = \mathrm{rge}(f_i) \cap \mathrm{rge}(F)$ for all $i < \chi$ as well.
Let	$\rho_i = \mathrm{ssup}(\{\rho < \theta_{\alpha_i} \mid f_i(\rho) = h(f_i)(\rho)\})$.
Then	$Y_i = \mathrm{rge}(f_i \restriction \rho_i) = \mathrm{rge}(\psi_{(\alpha_i,\rho_i),(\kappa,f_i(\rho_i))} \restriction \rho_i)$.
Let	$\beta^* = \mathrm{ssup}(\{\beta_i \mid i < \chi\})$.

Note that as Y_i is an initial segment of $\mathrm{rge}(h(f_i)) \subseteq \mathrm{rge}(F) = \mathcal{N}$ for $i < \chi$ (by Fact (1.4)), one has that $Y_i \subseteq \mathcal{N}$ for each $i < \chi$. Note also that $Y_i \in \mathcal{N}$ and as $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$ one has that $\langle Y_i \mid i < \chi\rangle \in \mathcal{N}$.

Let $\Phi(x)$ be the conjunction of the following:

(i) $x \in \mathcal{D}$, (ii) $x \leq q \restriction \mathcal{N}$, and

(iii) $\exists(\alpha^*, h^*) \in A_x$ with $\beta^\dagger < \alpha^*$ such that

(1) $\forall(\beta, f) \in A_x \ \big(\beta < \beta^* \longrightarrow \exists i < \chi \ (\mathrm{rge}(f) \cap \mathrm{rge}(h^*) = Y_i)\big)$,

(2) $\forall i < \chi \ \exists(\beta_i, f) \in A_x \ \big(\mathrm{rge}(f) \cap \mathrm{rge}(h^*) = Y_i \ \ \& \ \ \exists f'' \in \mathcal{F}_{\alpha^*\kappa}$
$\exists f' \in \mathcal{F}_{\beta\alpha^*} f = f'' \cdot f' \longrightarrow \mathrm{rge}(h^* \cdot f') \cap \mathrm{rge}(f) = Y_i\big)$,

(3) $a_{q\restriction\mathcal{N}} \subseteq \mathrm{rge}(h^*)$,

(4) $(a_x \setminus a_{q\restriction\mathcal{N}}) \cap \mathrm{rge}(h^*) = \emptyset$,

(**C**). Then $H_{\kappa^{++}} \models$ "$\Phi(q)$". So by the elementarity of $\mathcal{N}$ in $H_{\kappa^{++}}$ there is some $s \in \mathcal{N}$ with $\mathcal{N} \models \Phi(s)$. Set $a_r = a_s \cup a_q$ and $A_r = A_s \cup A_q$.

Claim 10. $r \in \mathbb{P}$ (and so $r \leq q$, s). ▲

Proof. (a). The only new cases to verify are when $\{I, J\} \notin [a_s]^2 \cup [a_q]^2$. So let $I \in a_s \setminus a_q$ and $J \in a_q \setminus a_s$. Note that I^-, $I^+ \in \mathrm{rge}(F)$.

If $J^- \geq \varsigma_F$ then clearly $I^+ < J^-$. Otherwise $J^- \notin \mathrm{rge}(F)$ and $J^- < \varsigma_F$. So, by (b.ii) for q, if $J^- \leq I^+$ then $J^+ < \min(\mathrm{rge}(F) \setminus J^-) \leq I^-$. ▲(a)

(b). Let $I \in a_r$ and let $(\beta, \phi) \in \overline{A_r}$, with $(\beta, \phi) = (\beta, f) \downharpoonleft \xi$ for some $(\beta, f) \in A_r$ and $\xi \in X_r \cup \{\omega_2\}$.

There are four cases to consider for each clause of (b):

(1)	$I \in a_s$ and $(\beta, f) \in A_s$
(2)	$I \in a_q$ and $(\beta, f) \in A_q$
(3)	$I \in a_q \setminus a_s$ and $(\beta, f) \in A_s \setminus A_q$
(4)	$I \in a_s \setminus a_q$ and $(\beta, f) \in A_q \setminus A_s$

(b.i) Suppose that $I^- \in \mathrm{rge}(\phi)$. Note, by (a), that $I^+ < \xi$.

Case (1). Since $\phi = f \downharpoonleft \xi$ one has $I^- \in \mathrm{rge}(f)$. Thus $I^+ \in \mathrm{rge}(f)$, by (b.i) for s and I, and hence $I^+ \in \mathrm{rge}(\phi) = \mathrm{rge}(f) \cap \xi$, as required.

Case (2). Exactly the same argument, with s replaced by q, also shows (b.i) holds here.

Case (3). One has $I^- \in \mathrm{rge}(\phi) \subseteq \mathrm{rge}(f) \subseteq \mathrm{rge}(F)$, since all maps in A_s factor through (δ, F). So, by (b.i) for q applied to I and (δ, F), one has that $I^+ \in \mathrm{rge}(F) \subseteq \mathcal{N}$. Hence $I \in a_q \cap \mathcal{N} \subseteq a_s$, contradicting the case hypothesis. Thus this case cannot occur.

Case (4). As $I \in a_s$ one has that I^-, $I^+ \in \mathrm{rge}(F)$, since $s \in \mathcal{N}$. Let ζ be least in $X_q \setminus (I^- + 1)$. By (a) one has that a_r is linearly ordered and so $I^+ < \zeta$. Thus $h_q(\delta, F \downharpoonleft \zeta) = h_q(\beta, f \downharpoonleft \zeta) = \zeta$. So $\mathrm{rge}(F \downharpoonleft \zeta) \subseteq \mathrm{rge}(f \downharpoonleft \zeta)$ or vice versa, by (c) for q.

In the former case $I^+ \in \mathrm{rge}(f)$. Hence $I^+ \in \mathrm{rge}(\phi), = \mathrm{rge}(f) \cap \xi$, as required.

Otherwise one has that $\beta < \delta$. Hence there is some $(\beta, g) \in A_s$ such that $(\beta, g) \downharpoonleft \zeta = (\beta, f) \downharpoonleft \zeta$ by clauses (iii.1) and (iii.2) of Φ, since there is some $i < \chi$ such that $\mathrm{rge}(f) \cap \zeta \subseteq Y_i$. As $I^- \in a_s \cap \mathrm{rge}(g)$ one has that $I^+ \in \mathrm{rge}(g)$ by (b.i) for s. So, again, $I^+ \in \mathrm{rge}(g) \cap \xi \cap \zeta \subseteq \mathrm{rge}(\phi)$. ▲(b.i)

(b.ii) Now suppose that $I^- \notin \mathrm{rge}(\phi)$ and $I^- < \varsigma_\phi (< \xi)$.

Case (1). One has $I^- < \varsigma_{f \downharpoonleft \xi}$ and $I^- \notin \mathrm{rge}(f)$, since $\phi = f \downharpoonleft \xi$. So, by (b.ii) for s, there is some $J \in a_s$ with $J^- = \min(\mathrm{rge}(f) \setminus I^-) = \min(\mathrm{rge}(\phi) \setminus I^-)$, as required.

Case (2). Exactly the same argument, with s replaced by q, again shows that (b.ii) holds.

Case (3). As $I \notin a_q \cap a_s$ it must be the case that I^-, $I^+ \notin \mathrm{rge}(F)$, by (b) for q, since $\varsigma_\phi \leq \varsigma_f < \varsigma_F$. So there is some $J \in a_q$ such that $\min(\mathrm{rge}(F) \setminus I^-) = J^-$.

As $J^- \in \mathrm{rge}(F)$ one has that $J^+ \in \mathrm{rge}(F)$ by (b.i) for q applied to J and (δ, F), and hence that $J \in a_q \cap \mathcal{N}, = a_q \cap a_s$. Clearly, as $\mathrm{rge}(\phi) \subseteq \mathcal{N}$, one has that $J^- \leq \min(\mathrm{rge}(\phi) \setminus I^-)$.

If equality holds there is nothing more to prove.

Otherwise, $J^- \notin \mathrm{rge}(\phi)$ and $J^- < \varsigma_\phi$. So there is some $K \in a_s$ such that $K^- = \min(\mathrm{rge}(\phi) \setminus J^-) = \min(\mathrm{rge}(\phi) \setminus I^-)$, by (b.ii) for s applied to (β, ϕ) and J, and this K witnesses (b.ii) for I and (β, ϕ).

Case (4). As $I \in a_s$ one has that I^-, $I^+ \in \mathrm{rge}(F)$. Let ζ be least in $X_q \setminus (I^+ + 1)$ and let $J \in a_q$ be such that $\zeta = J^-$.

If $\zeta = \min(\mathrm{rge}(\phi) \setminus I^-)$ there is nothing more to prove.

If $\zeta < \min(\mathrm{rge}(\phi) \setminus I^-)$ then there is some $K \in a_q$ such that $K^- = \min(\mathrm{rge}(f) \setminus J^-) = \min(\mathrm{rge}(\phi) \setminus J^-) = \min(\mathrm{rge}(\phi) \setminus I^-)$, by (b.ii) for q applied to J and (β, f).

Finally, if $\zeta > \min(\mathrm{rge}(\phi) \setminus I^-)$ then $h_q(F \downharpoonleft \zeta) = h_q(f \downharpoonleft \zeta)$. So one has that $\mathrm{rge}(F \downharpoonleft \zeta) \subseteq \mathrm{rge}(f \downharpoonleft \zeta)$ or vice versa, by (c) for q.

The former is impossible since $I^- \in \mathrm{rge}(F \downharpoonleft (\zeta \cap \xi)) \setminus \mathrm{rge}(\phi)$. Hence, as in the proof of the corresponding case of (b.i), there is some $(\beta, g) \in A_s$ such that $f \downharpoonleft \zeta = g \downharpoonleft \zeta$. By (b.i) for s applied to (β, g) and I there is some $J \in a_s$ such that $J^- = \min(\mathrm{rge}(g) \setminus I^-) = \min(\mathrm{rge}(\phi) \setminus I^-)$, as required. ▲(b.ii)

(c.i) Let (β, f), $(\gamma, g) \in \overline{A_r}$, and let $\xi \in X_r \cup \{\omega_2\}$ be such that $h(\beta, f \downharpoonleft \xi)$, $h(\gamma, g \downharpoonleft \xi) = \xi$. Let $\phi = f \downharpoonleft \xi$ and $\psi = g \downharpoonleft \xi$.

I start with a few useful elementary remarks.

Firstly, note that if $\zeta \in X_t$, where t is any of r, q and s, then $h_t(\beta, f \downharpoonleft \zeta) \leq \zeta$, and $(\beta, f) \downharpoonleft (h_t(\beta, f \downharpoonleft \zeta)) = (\beta, f) \downharpoonleft \zeta$.

Next, note that if $\xi \in X_s$ then $h_s(\beta, \phi) = h_s(\gamma, \psi) = \xi$. Similarly, if $\xi \in X_q$ then $h_q(\beta, \phi) = h_q(\gamma, \psi) = \xi$. Thus if all of the data comes from s or it all comes from q one has (c.i) by (c.i) for s or q, respectively.

Thirdly, note that if $\xi \in X_s$ then $h_q(\beta, \phi) = h_q(\gamma, \psi) = \min((X_q \cup \{\omega_2\}) \setminus \xi)$. Similarly, if $\xi \in X_q$ then one $h_s(\beta, \phi) = h_s(\gamma, \psi) = \min((X_s \cup \{\omega_2\}) \setminus \xi)$. Thus in each case one has both $h_q(\beta, \phi) = h_q(\gamma, \psi)$ and $h_s(\beta, \phi) = h_s(\gamma, \psi)$.

Eliminating duplicates and the two cases dealt with by (c) for s and for q respectively by using the second observation above, leaves (without loss of generality) the following four cases.

(1)	(β, f), $(\gamma, g) \in A_s$ and $\xi \in X_q$
(2)	(β, f), $(\gamma, g) \in A_q$ and $\xi \in X_s$
(3)	$(\beta, f) \in A_q$ & $(\gamma, g) \in A_s$ and $\xi \in X_s$
(4)	$(\beta, f) \in A_q$ & $(\gamma, g) \in A_s$ and $\xi \in X_q$

Case (1). Let $h_s(\beta, \phi) = h_s(\gamma, \psi) = \eta$. Clearly one has $\xi \leq \eta = h_s(\beta, \phi) \leq h_s(\beta, f \restriction \eta) \leq \eta$ and similary $\xi \leq \eta = h_s(\gamma, g \restriction \eta)$. So one can apply (c.i) for s to $(\beta, f \restriction \eta)$, $(\gamma, g \restriction \gamma)$ and η.

Case (2). Identical to *Case (1)* with the rôles of s and q exchanged.

Case (3). Let $\eta = h_q(\beta, \phi) = h_q(\gamma, \psi)$. Then $\eta = h_q(\delta, F \restriction \eta)$ since $\text{rge}(\psi) \subseteq \text{rge}(F) \cap \eta$. By (c.i) for q one has $\text{rge}(F) \cap \eta \subseteq \text{rge}(f) \cap \eta$ or $\text{rge}(f) \cap \eta \subsetneq \text{rge}(F) \cap \eta$.

In the former case $\text{rge}(\psi) = \text{rge}(\psi) \cap \xi \subseteq \text{rge}(F) \cap \eta \subseteq \text{rge}(f) \cap \eta \subseteq \text{rge}(\phi)$. In the latter case, by the properties of Φ, there is some $(\beta, f') \in A_s$ such that $f \restriction \eta = f' \restriction \eta$. Now one can finish the proof by applying *Case (2)*.

Case (4). Let $h_s(\beta, \phi) = h_s(\gamma, \psi) = \eta$. If $\eta = \xi$ then $\phi = f \restriction \eta$ and $\psi = g \restriction \eta$ and one may apply *Case (3)*. Otherwise $\xi < \eta$. Then $\xi = h_q(\delta, F \restriction \xi)$ since $\text{rge}(\psi) \subseteq \text{rge}(F)$. As in *Case (3)*, by (c.i) for q, this gives $\text{rge}(F) \cap \xi \subseteq \text{rge}(f) \cap \xi$ or $\text{rge}(f) \cap \xi \subsetneq \text{rge}(F) \cap \xi$. The remainder of the argument is now as in *Case (3)*. ▲(c.i)

(c.ii) In the cases where all of the data comes from s or from q one can simply apply (c.ii) for the relevant condition to obtain the required instance of (c.ii) for r. The remaining cases follow a similar pattern to those for (c.i).

Case (1). (β, f), $(\gamma, g) \in A_q$ and $\xi \in X_s \setminus X_q$.

Let $\zeta = \min(X_q \setminus \xi)$, so that $\zeta = h_q(\beta, \phi) = h_q(\gamma, \psi)$. One has that $\xi < h_q(\beta, f \restriction \zeta)$, $h_q(\gamma, g \restriction \zeta) \leq \zeta$. So $\zeta = h_q(\beta, f \restriction \zeta) = h_q(\gamma, g \restriction \zeta)$. Hence one can apply (c.ii) for q for $(\beta, f \restriction \zeta)$, $(\gamma, g \restriction \zeta)$ and ζ to obtain for each $\nu < \alpha$ some $\epsilon \in (\nu, \alpha) \cap (\text{rge}(g) \setminus \text{rge}(f))$.

Case (2). (β, f), $(\gamma, g) \in A_s$ and $\xi \in X_q \setminus X_s$.

Identical to *Case (1)* with the rôles of s and q reversed.

Case (3). $(\beta, f) \in A_q$, $(\gamma, g) \in A_s$.

Since $\beta < \gamma$ one has that $\beta < \delta$. As s satisfies properties (iii.1) and (iii.2) of the definition of Φ, and since $\text{rge}(f \downharpoonright \xi) \subseteq (\text{rge}(g \downharpoonright \xi) \subseteq \mathcal{N}$, there is some (β, f') with $(\beta, f) \downharpoonright \xi = (\beta, f') \downharpoonright \xi$. Now apply either (c) for s or *Case (2)* to (β, f'), (γ, g) and ξ to obtain the instance of (c.ii) required.

Case (4). $(\beta, f) \in A_s$, $(\gamma, g) \in A_q$.

Note that $f(\alpha_\beta) \in \text{rge}(g)$ and that $f(\alpha_\beta) \in \mathcal{N} \cap \omega_2 = \text{rge}(F)$.

If $\delta \leq \gamma$ this immediately gives that $\text{rge}(F) \cap f(\alpha_\beta) \subseteq \text{rge}(g) \cap f(\alpha_\beta)$.

As f, $\alpha_\beta \in \mathcal{N}$ one has $\varsigma_{f \restriction \alpha_\beta} \in \mathcal{N} \cap \omega_2 = \text{rge}(F)$, and thus the required instance of (c.ii) holds.

Finally, suppose that $\gamma < \delta$. Thus $\text{rge}(g) \cap f(\alpha_\beta) \subseteq \text{rge}(F) \cap f(\alpha_\beta)$. As in *Case (3)*, use the definition of Φ and hence s to obtain some $(\gamma, g') \in A_q$ such that $\text{rge}(g) \cap f(\alpha_\beta) = \text{rge}(g') \cap f(\alpha_\beta)$.

Then $\xi = h_r(\beta, \phi) \leq h_r(\gamma, g' \downharpoonright \xi) \leq \xi$. Thus one can apply (c.ii) for s or *Case (2)* as appropriate to (β, f), (γ, g') and ξ to obtain the instance of (c.ii) required. ▲((c.ii)). ▲(Claim (10), (**C**) / Proposition (7))

I next show that forcing with $\mathbb{P}$ presrves ω_2. The proof is similar to the proof of Proposition (7), but considerably simpler because the analogue of the crucial argument in Claim (10) now concerns a 'head-tail-tail' amalgamation.

(D). Proposition 11. $\mathbb{P}$ preserves ω_2. ▲

Proof. Let p, $\mathcal{D}$ and $\mathcal{N}$ be as in Fact (13). Let $p^* = (a_p \cup \{\delta\}, A_p)$. Lemma (6) shows that $p^* \in \mathbb{P}$ and so $p^* \leq p$. Let $q \leq p^*$ be such that $q \in \mathcal{D}$. Without loss of generality assume that q is pointed. Set $q \restriction \mathcal{N} = (a_q \cap \mathcal{N}, A_q \cap \mathcal{N})$.

Claim 12. $q \restriction \mathcal{N} \in \mathbb{P}$ and $q \leq q \restriction \mathcal{N}$.

Proof. The proof is very similar to the proof of Claim (8). In fact the proofs that $q \restriction \mathcal{N}$ satisfies (a) and (b.i) go through verbatim. For (b.ii) one can also argue exactly as in Claim (11) to get that if $I^- \notin \text{rge}(\phi)$ and $I^- < \text{ssup}(\text{rge}(\phi))$ there is some $J \in a_q$ such that $J^- = \min(\text{rge}(\phi))$ and $J^+ \in \text{rge}(\phi)$. But $\text{rge}(\phi) \subseteq \text{rge}(f) \subseteq \mathcal{N} \cap \omega_2$, so $J \in \mathcal{N}$.

Ad (c) for $q \restriction \mathcal{N}$ note that $X_{q \restriction \mathcal{N}} = X_q \cap \mathcal{N} = X_q \cap \delta$. So if (β, ϕ), $(\gamma, \psi) \in \overline{A_q \cap \mathcal{N}}$ and $h_{q \restriction \mathcal{N}}(\beta, \phi) = h_{q \restriction \mathcal{N}}(\gamma, \psi)$ then $h_q(\beta, \phi) = h_q(\gamma, \psi)$. Hence (c.i) holds by (c.i) for q. As in Claim (8), (c.ii) holds for $q \restriction \mathcal{N}$ since $\overline{A_q \cap \mathcal{N}} \subseteq \overline{A_q}$. ▲(Claim (12))

Notational definition 13. Noting that θ_β is countable, that $\mathrm{cf}(\delta) = \omega_1$, and that $\{\delta\} \in a_q$ so that there is no $I \in a_q$ with $I^- < \delta \leq I^+$, make the following definition in order to introduce notation in which to discuss some of the properties of q.

Let (β, F_q) be the apex of A_q and let $\nu < \theta_\beta$ be least such that $F_q(\nu) \geq \delta$. Let $F = f_\delta \restriction \nu$. Let $\varepsilon < \delta$, $n < k < \omega$, $B \subseteq k$, $\langle I_i \mid i < k \rangle$, $m < \omega$, $\langle f_j \mid j < m \rangle$ and $\langle \gamma_j \mid j < m \rangle$ be such that

- $(\bigcup a_q) \cap \mathcal{N}$, $\mathrm{rge}(F) \subseteq \varepsilon$,
- $\langle I_i \mid i < n \rangle$ enumerates $a_q \cap \mathcal{N}$ in increasing order,
- $\overline{\overline{a_q}} = k$ and if $\langle I_i \mid i < k \rangle$ enumerates a_q in increasing order, then $\langle \min(I_i) \mid i \in B \rangle$ enumerates X_q,
- $\langle (\gamma_j, f_j) \mid j < m \rangle$ enumerates $\{ (\gamma, f) \mid (\gamma, F_q \cdot f) \in A_q \}$, where $f_j \in \mathcal{F}_{\gamma_j \beta}$ for each $j < m$.

Let $\Psi(x)$ be the conjunction of the following four items:

$x \in \mathbb{P}$, $x \in \mathcal{D}$ and $x \leq q \restriction \mathcal{N}$;
x is pointed with apex $(\beta, F_x) \in \mathcal{F}$, such that
$(\beta, F_x) \restriction \nu = F$, and $A_x = \{ (\gamma_i, F_x \cdot f_i) \mid i < m \}$;
$\overline{\overline{a_x}} = k$, $\langle I_i \mid i < n \rangle$ is an initial segment of the increasing enumeration $\langle J_i \mid i < k \rangle$ of a_x (by least elements),
$X_x = \{ \min(J_i) \mid i \in B \}$ and $\bigcup \langle I_i \mid i < n \rangle = (\bigcup a_x) \cap \varepsilon$; and
$F_x(\nu) = \min(X_x) \setminus \varepsilon$.

Clearly $H_{\omega_3} \models \Psi(q)$. Note that F is a countable set of pairs of ordinals less than ε, and so, by the closure of $\mathcal{N}$ is an element of N. Let $s \in \mathbb{P} \cap \mathcal{N}$ be such that $\mathcal{N} \models \Psi(s)$. Note that $\mathcal{N}$ computes cofinalities of ordinals less than δ correctly, so, in particular, $\mathcal{N} \models \xi \in X_s$ if and only if $\xi \in X_s$.

Set $r = (a_s \cup a_q, A_s \cup A_q)$. Proposition (11) will be proved on showing that $r \in \mathbb{P}$.

That r satisfies (a) is clear since $a_s \cap a_q = a_q \cap \mathcal{N}$ and $\bigcup a_s \subseteq \delta$ while $\bigcup a_q \setminus a_{q \cap \mathcal{N}} \cap \delta = \emptyset$.

Now suppose that $(\beta, f) \in A_s$ and $\xi \in X_q$. Write (β, ϕ) for $\beta, f) \downharpoonleft \xi$. One has $\xi \in X_s \cap X_q$ or $\delta \leq \xi$. In either case $(\beta, f) \downharpoonleft \xi \in \overline{A_s}$. Moreover, if $\xi \in X_s \cap X_q$ then $(\beta, f) \downharpoonleft \xi \in \overline{A_q}$ as well, while clearly if $\delta \leq \xi$ then $(\beta, \phi) = (\beta, f)$. Also, $h_r(\beta, \phi) = h_s(\beta, \phi)$ if $h_s(\beta, \phi) < \omega_2$, while $h_r(\beta, \phi) = \delta$ if $h_s(\beta, \phi) = \omega_2$. Further, if $(\beta, f) \in A_s$ and $\xi \in X_s$ then, again, $h_r(\beta, \phi) = h_s(\beta, \phi)$.

On the other hand, if $(\beta, f) \in A_q$ and $\xi \in X_s \setminus X_q$ then $(\beta, f) \downharpoonleft \xi = (\beta, f) \downharpoonleft \delta$, so $(\beta, f) \downharpoonleft \xi \in \overline{A_q}$. Moreover, as $(\beta, f) \in A_q$ then there is some $i < m$ such that $f = F_q \cdot f_i$ and one has that $(\beta, f) \downharpoonleft \xi = (\beta, F_s \cdot f_i) \downharpoonleft \min(X_s \setminus X_q)$, and so $(\beta, f) \downharpoonleft \xi \in \overline{A_s}$.

Again writing (β, ϕ) for $(\beta, f) \downharpoonleft \xi$, one has that $h_r(\beta, \phi) = h_s(\beta, \phi)$.

Now suppose that $I \in a_r$ and $(\beta, \phi) \in \overline{A_r}$. By the observations of the previous paragraphs, (b) holds immediately for I and (β, ϕ) unless either $I \in a_s \setminus a_q$ and $(\beta, \phi) = (\beta, f) \downharpoonleft \xi$ for some $(\beta, f) \in A_q$ and $\xi \in X_q \setminus X_s$ or if $I \in a_q \setminus a_s$ and $(\beta, \phi) = \beta, f)$ for some $(\beta, f) \in A_s \setminus A_q$.

In the first of these two outstanding cases one has that $I^- \notin \mathrm{rge}(\phi)$ and $I^- < \mathrm{ssup}(\mathrm{rge}(\phi))$. Let $\zeta = \min(\mathrm{rge}(\phi) \setminus I^-)$. One has that $\delta \leq \zeta$. If equality holds then (b.ii) is establised. On the other hand if $\delta < \zeta$ one has that $\delta \notin \mathrm{rge}(\phi)$ and $\delta < \mathrm{ssup}(\mathrm{rge}(\phi))$. Recalling that $\{\delta\} \in a_q$ and applying (b.ii) for q one gets that there is some $J \in a_q$ such that $J^- = \min(\mathrm{ssup}(\mathrm{rge}(\phi)) \setminus \delta) = \min(\mathrm{ssup}(\mathrm{rge}(\phi)) \setminus I^-)$, as required

In the other outstanding case one has that $\mathrm{ssup}(\mathrm{rge}(f)) < \delta \leq I^-$, and so (b) holds vacuously.

Finally, (c) will be dealt with by reducing each possible case to an instance of (c) for either s or q.

Suppose $(\beta, \phi), (\gamma, \psi) \in \overline{A_r}$ and $h_r(\beta, \phi) = h_r(\gamma, \psi), = \tau$, say. If $\tau \in X_s \cap X_q$ then (β, ϕ), $(\gamma, \psi) \in \overline{A_s} \cap \overline{A_q}$ and (c) holds by (c) for s or for q. If $\delta < \tau$ then (β, ϕ), $(\gamma, \psi) \in \overline{A_q}$ and (c) holds by (c) for q.

Now, as $\mathcal{N} \models \Psi(s)$, one has $X_s \setminus X_q \neq \emptyset$. So if $(\beta, \phi) = (\beta, f) \downharpoonleft \delta$ for some $(\beta, f) \in A_q$ and $h_q(\beta, \phi) = \delta$ one has that $\mathrm{rge}(\phi) \subseteq \varepsilon$ and $h_r(\beta, \delta) \leq \min(X_s \setminus X_q)$. And, similarly to when one cuts down such at map $(\beta, f) \in A_q$ at some $\xi \in X_s \setminus X_q$, one has that $(\beta, \phi) \in \overline{A_s}$.

Hence, if $\tau = \delta$ then $h_s(\beta, \phi) = h_s(\gamma, \psi) = \omega_2$, (β, ϕ), $(\gamma, \psi) \in \overline{A_s}$ and (c) holds for (β, ϕ), (γ, ψ) by (c) for s.

Lastly, if $\varepsilon \leq \tau < \delta$ then (β, ϕ), $(\gamma, \psi) \in \overline{A_s}$ again, so (c) holds by (c) for s.

▲(**D**, Proposition (11))

Corollary 14. Forcing with $\mathbb{P}$ preserves cardinals. ▲

Proof. Propositions (7) and (11), respectively, show that ω_1 and ω_2 are preserved. Fact (1.7) shows that each map $(\beta, f) \in \mathcal{F}$ is uniquely determined by the ordinals β and $\mathrm{ssup}(\mathrm{rge}(f))$, so $\overline{\overline{\mathbb{P}}}$ is ω_2. ▲

It remains only to show that forcing with $\mathbb{P}$ does indeed add a club subset of ω_2.

(E). Let G be $\mathbb{P}$-generic and set $C_G = \{ I^- \mid \exists p \in G \; I \in a_p \}$. Clearly, by Lemma (6), C_G is unbounded in ω_2. Let $\mathrm{Lim}(C_G)$

Proposition 15. C_G is club in ω_2. ▲

Proof. Suppose not and, towards a contradiction, that $p \in \mathbb{P}$ is such that $p \Vdash$ "$\alpha \in \mathrm{Lim}(C_G)$ & $\alpha \notin C_g$." For $(\beta, \phi) \in \overline{A_p}$ write $\alpha(\phi)$ for $\min(\mathrm{rge}(\phi) \setminus \alpha)$ if $\mathrm{rge}(\phi) \setminus \alpha \neq \emptyset$. It is useful to start by giving a couple of simple auxiliary lemmas.

Lemma 16. For all $q \leq p$ and $I \in a_q$ one has $\alpha \notin [I^-, I^+]$. ▲

Proof. Otherwise $q \Vdash$ "$\alpha \in C_G$" or $q \Vdash$ "$\alpha \notin \mathrm{Lim}(C_G)$." ▲

The main part of the proof divides into three cases, (I)-(III) below, distinguished by using the following definitions. Set $\xi = \min(X_p \setminus \alpha)$ and define

$$S = \{ (\beta, f) \downharpoonright \xi \mid (\beta, f) \in A_p \;\&\; \mathrm{rge}(f) \cap [\alpha, \xi) \neq \emptyset \}.$$

Clearly if (β, ϕ), $(\gamma, \psi) \in S$ then $h_p(\beta, \phi) = h_p(\gamma, \psi) = \xi$ and so, by (c) for p, S is linearly ordered by inclusion of ranges of second co-ordinates.

Let $Y = \{ \alpha(\phi) \mid (\beta, \phi) \in S \;\&\; \varsigma_{\phi \downharpoonright \alpha} = \alpha \}$. Note that if (β, ϕ) and $(\gamma, \psi) \in \overline{A_p}$, $\alpha(\phi) < \alpha(\psi)$ and there is some $I \in a_p$ such that $\alpha(\phi) = I^-$ then there is some $J \in a_p$ such that $\alpha(\psi) = J^-$, by (b.ii) for p, I and ψ. So $Y_0 = \{ \alpha(\phi) \in Y \mid \neg \exists I \in \; a_p \; \alpha(\phi) = I^- \}$ is an initial segment of Y. Notice also that the same argument shows that $\max(Y_0)$ is less than the least I^- such that $\alpha < I^-$ and $I \in A_p$. Let $Y_1 = Y \setminus Y_0$. The three cases are: (I) $Y = \emptyset$, (II) $Y \neq \emptyset$ but $Y_0 = \emptyset$, and (III) $Y_0 \neq \emptyset$.

Lemma 17. One may as well assume that p is such that for all $(\beta, \phi) \in S$ either there is some $I \in a_p$ such that $\alpha(\phi) = I^-$ or $p \Vdash$ "$\alpha(\phi) \notin C_G$." ▲

Proof. As $\Vdash_{\mathbb{P}}$"$\alpha(\phi) \in C_G$ or $\alpha(\phi) \notin C_G$", if $p = p_0 \Vdash \alpha \in C_G$ there is some $p_1 \leq p_0$ such that there is some $I \in a_{p_0}$ with $I^- = \alpha(\phi)$. Write S_i and ξ_i for S, ξ as calculated in i. Note that $\xi_1 \leq \xi_0$.

Since S_0 is linearly ordered by inclusion of ranges of second co-ordinates one has that $\{ \alpha(\phi) \mid (\beta, \phi) \in S \}$. is a descending sequence under the linear order on S. So after finitely many steps the process of constructing p_{i+1} from p_i stabilises, *i.e.*, $\xi_{i+1} = \xi_i$ and $\{ \alpha(\phi) \mid (\beta, \phi) \in S_{i+1} \} = \{ \alpha(\phi) \mid (\beta, \phi) \in S_i \}$.

▲

Lemma 18. If $(\beta, f) \in S$, $\mathrm{ssup}(\mathrm{rge}(f) \cap \alpha) < \alpha < \mathrm{ssup}(\mathrm{rge}(f))$ then there is some $I \in a_p$ such that $\alpha(f) = I^-$ (and hence $\alpha < \alpha(f)$). ▲

Proof. Let $\mathrm{ssup}(\mathrm{rge}(f) \cap \alpha) = \tau$. One has $p \Vdash$ "$C_G \cap (\tau, \alpha) \neq \emptyset$." Let $q \leq p$ be such that there is some $I \in a_q$ with I^-, $I^+ \in (\tau, \alpha)$. By (b.ii) for q, I and (β, f) there is some $J \in a_q$ with $J^- = \min(\mathrm{rge}(f) \setminus I^-) = \min(\mathrm{rge}(f) \setminus \alpha) = \alpha(f)$.

So $p \not\Vdash$ "$\alpha(f) \notin C_G$." Thus, by Lemma (17), there is some $K \in a_p$ such that $\alpha(f) = K^-$. (And as $\alpha(f) \in C_G$ is witnessed by p one has $\alpha \neq \alpha(f)$.) ▲

(I). First of all, suppose $Y = \emptyset$, that is, that there is no $(\beta, \phi) \in S$ such that $\mathrm{ssup}(\mathrm{rge}(\phi) \cap \alpha) = \alpha$. Let $\nu = \mathrm{ssup}(\bigcup\{\mathrm{rge}(\phi) \cap \alpha \mid (\beta, \phi) \in \overline{A_p}\} \cup \bigcup\{I^+ \mid I \in a_p \ \&\ I^+ < \alpha\})$. Then I claim that for any $\gamma \in (\nu, \alpha)$ one has $(a_p \cup \{\gamma, \alpha\}, A_p) \in \mathbb{P}$.

(a) is clear by construction and Lemma (16).

(b.i) is vacuously true. For if $(\beta, f) \in A_p$ and $(\beta, f \downharpoonleft \xi) \notin S$ then either $h_p(\beta, f \downharpoonleft \xi) < \alpha$, or $h_p(\beta, f \downharpoonleft \xi) = \xi$, and $\mathrm{rge}(f) \cap \alpha$ is a (proper) subset of $\mathrm{rge}(\psi)$ for any $(\gamma, \psi) \in S$, by (c.i) for p. In either case one has $\varsigma_{f \downharpoonleft \xi} < \alpha$.

The proof for (b.i) shows that if $\gamma < \varsigma_\phi$ for $(\beta, \phi) \in \overline{A_p}$ then either $(\beta, \phi) \in S$ or else $\min(\mathrm{rge}(\phi) \setminus \gamma) \geq \xi$. In the former case (b.ii) holds by Lemma (18). In the latter case there are two possibilities. Either $\xi \in \mathrm{rge}(\phi)$, in which case ξ is a witness to (b.ii) as required. Or $\xi \notin \mathrm{rge}(\phi)$ and $\xi < \varsigma_\phi$, in which case one can obtain a J as required by applying (b.ii) to ξ and ϕ, since one has $\min(\mathrm{rge}(\phi) \setminus \xi) = \min(\mathrm{rge}(\phi) \setminus \gamma)$.

And (c) holds since it does for p.

Next suppose that there are $(\beta, \phi) \in S$ such that $\mathrm{ssup}(\mathrm{rge}(\phi) \cap \alpha) = \alpha$, so $Y \neq \emptyset$.

(II). If $Y \neq \emptyset$ but $Y_0 = \emptyset$ then I claim that $(a_p \cup \{\alpha\}, A_p) \in \mathbb{P}$. (a) holds by construction, (b.i) is trivially true, and (b.ii) is true by the case hypothesis if $(\beta, f) \in S$ and is true if $(\beta, f) \notin S$ and $\alpha < \mathrm{ssup}(\mathrm{rge}(f))$ since then either $\xi \in \mathrm{rge}(f)$ or one can apply (b.ii) for p, (β, f) and the $I \in a_p$ such that $\xi = I^-$. (c) is true because it is true for p.

(III). Lastly suppose $Y_0 \neq \emptyset$. Let ϕ_0 be such that $\alpha(\phi_0) = \max(Y_0)$ and ϕ_1 be such that $\alpha(\phi_1) = \min(Y_1)$. Add $\{\gamma, \alpha(\phi_0)\}$ to a_p for some $\gamma \in \mathrm{rge}(\phi_0) \setminus \mathrm{rge}(\phi_1)$ such that $\gamma \in (\nu, \alpha]$, where

$$\nu = \mathrm{ssup}(\bigcup\{\mathrm{rge}(f) \cap \alpha \mid (\beta, f) \in A_p \ \&\ \mathrm{ssup}(\mathrm{rge}(f) \cap \alpha) < \alpha\} \cup ((\bigcup a_p) \cap \alpha)).$$

There is such a γ by (c.ii) of the definition of what it is to be an element of $\mathbb{P}$. It is once more clear (similarly to the previous case, using the definitions of Y_0 and Y_1) that $(a_p \cup \{\gamma, \alpha(\phi_0)\}, A_p) \in \mathbb{P}$ and this contradicts $p \Vdash$ "$\alpha \in \mathrm{Lim}(C_G)$" if $\gamma < \alpha$, and contradicts $p \Vdash$ "$\alpha \notin C_G$" if $\gamma = \alpha$.

§3. Adding a club subset through a very stationary subset of ω_2.

Thus far a forcing has been given that adds a club subset of ω_2 with conditions with finite working parts. I now make cosmetic changes (which are essentially orthogonal to the bulk of the proof) to show that the club subset can be added through a *very stationary* subset D of ω_2.

What alterations are needed? The forcing to add a club subset of D, $\mathbb{Q}$, will be a refinement of $\mathbb{P}$. Before giving its formal definition and checking that it works as advertised, I try to give some motivation as to how the refinements are used.

The simplest change that one needs to make is to insist that if $p \in \mathbb{Q}$ and $I \in a_p$ then $I^- \in D$. This will ensure that if G is $\mathbb{Q}$-generic then $C_G \subseteq D$. (In this discussion names, such as C_G, refer to analogues for $\mathbb{Q}$ of objects used in the proof that $\mathbb{P}$ adds a club subset of ω_2 without collapsing cardinals.)

Focus, first of all, on the problem of ensuring that C_G is club. It is again clear that C_G will be unbounded by the analogue of Lemma (2.6), so one can concentrate on the changes necessary for the cases of the analogue of Proposition (2.15) that prove C_G is closed.

The case $Y = \emptyset$ does not, in fact, compell any changes. However, noting that the case $Y_0 = \emptyset$ could have been treated in a similar way to the case $Y_0 \neq \emptyset$ above, in order to deal with the case $Y \neq \emptyset$ one needs the following.

Suppose $\xi \in X_p$ and $\alpha < \xi$ is such that D is unbounded in α. Let $E = \{(\beta, \phi) \in \overline{A_p} \mid \varsigma_{\phi\downharpoonright\alpha} = \alpha \text{ and } \phi(\alpha_\beta) < \xi\}$, and suppose $E \neq \emptyset$. Set $E_0 = \{(\beta, \phi) \in E \mid \neg\exists I \in a_p \; \phi(\alpha_\beta) = I^-\}$ and $E_1 = E \setminus E_0$. Let $H_0 = \bigcap\{\mathrm{rge}(\phi) \mid (\beta, \phi) \in E_0\} \cap \alpha$ and $H_1 = \bigcup\{\mathrm{rge}(\phi) \mid (\beta, \phi) \in E_1\}$. Then for all $\nu < \alpha$ there is some $\tau \in (\nu, \alpha] \cap D \cap H_0 \setminus H_1$.

(Observe that the demand in the definition of E_1 that there is some $I \in a_p$ such that $I^- = \psi(\alpha_\gamma)$ conceals a demand that $\psi(\alpha_\gamma) \in D$.)

The natural way to modify the definition of $\mathbb{P}$ to ensure that this will be true is to refine (c.ii) appropriately. One then needs to check that the modification holds at each step of the proof in which one is checking that something is a condition, *i.e.*, p^*, $q \upharpoonright \mathcal{N}$, and r in the proof on $\mathbb{M}$-properness, the r in the proof of the preservation of ω_2, and the extensions on p in the proof of Proposition (2.15) itself. At first sight this looks like a huge number of places where something could go wrong and cause a cascade of further modifications.

Fortunately, things are not as horrid as they seem because the *only* place that checking (c.ii) holds is not simply a case of reducing to instances of (c.ii) known from the initial data is for p^*, where one has a reasonable amount of liberty in the choice of $(\delta, F)/\mathcal{N}$. Exploiting this one only needs to change (c.ii) a little, weakening it to insisting that it holds only when the map $(\beta, f) \in E_1$.

Finally, perhaps a moment should be devoted to defusing the worry that when in Proposition (2.15) one adds $\{\tau, \alpha(\phi_0)\}$ to a_p one then has to check (c.ii) for $\varsigma_{\phi_0 \downharpoonright \tau}$, which will require another elaboration, further checking and so on recursively. In order to check (c.ii) for $\varsigma_{\phi_0 \downharpoonright \tau}$ only one extra map needs to be avoided and since (the modification of) (c.ii) deals with all pairs of maps rather than just the analogues of ϕ_0, ϕ_1 it appears that there is actually nothing to worry about. However, sadly the additional demand that will be needed in the hypothesis of the analogue of (c.ii), mentioned in the previous paragraph, conflicts with this convenient ignore-everything-and-the-problem-will-go away "solution." So one does have to take some remedial action. The next most obvious/easiest attempt is to cut off the recursion bluntly at the first step by insisting that $\phi_0(\tau_\beta) = \varsigma_{\phi_0 \downharpoonright \tau}$, when what one would have to check would (at worst) be that $\tau \in D$, but already knowing that $\tau \in D$(!). And this does, it turns out, work. Well, so much for the discusion, now on with the proof.

I start with a (full) definition of $\mathbb{Q}$ and then discuss where the proofs for $\mathbb{P}$ need to be embroidered.

Definition 1. Let $D_1 = \{\alpha \in D \mid D \cap \alpha \text{ contains a club subset of } \alpha\}$. (Note that $D_0 \subseteq D_1$.)

Definition 2. $p \in \mathbb{Q}$ if $p = (a_p, A_p)$, where $a_p \in [[\omega_2]^1 \cup [\omega_2]^2]^{<\omega}$ and $A_p \in [\mathcal{F}]^{<\omega}$, and, writing X_p and h_p for X_{a_p} and $h_{X_{a_p}}$ respectively, the following properties hold.

(a) If I, $J \in a_p$ then either $I^+ < J^-$ or $J^+ < I^-$. Also if $I \in a_p$ then $I^- \in D$, and if, further, $\mathrm{cf}\,(I^-) = \omega_1$ then $I^- \in D_0$.

(b) If $I \in a_p$ and $(\beta, \phi) \in \overline{A_p}$ then

(i) $I^- \in \text{rge}(\phi) \Longrightarrow I^+ \in \text{rge}(\phi)$, and

(ii) $I^- \notin \text{rge}(\phi)$ & $I^- < \varsigma_\phi \Longrightarrow \exists J \in a_p\ J^- = \min(\text{rge}(\phi) \setminus I^-)$.

(So when $I^- \notin \text{rge}(\phi)$ and $I^- < \varsigma_\phi$, one has, using (a), that $I^+ \notin \text{rge}(\phi)$.)

(c) Suppose (β, f) and $(\gamma, g) \in A_p$ and $\xi \in X_p \cup \{\omega_2\}$ are such that $h_p(\beta, f \restriction \xi)$, $h_p(\gamma, g \restriction \xi) = \xi$. Let $\alpha < \xi$ be such that $\bigcup D \cap \alpha = \alpha$. Then

(i) $\text{rge}(f \restriction \xi) \subseteq \text{rge}(g \restriction \xi)$ or $\text{rge}(g \restriction \xi) \subseteq \text{rge}(f \restriction \xi)$,

(ii) if $\varsigma_{f\restriction\alpha} = \alpha$ and $\exists I \in a_p\,(I^- = f(\alpha_\beta)$ & $\alpha < I^- < \xi)$
then $\alpha \in D_1$,

(iii) if $\varsigma_{f\restriction\alpha} = \alpha$ and $f(\alpha_\beta) \in D_1$ then $\alpha \in D$ and

(iii) if f is as in (ii), $\alpha = \varsigma_{g\restriction\alpha}$, and $g(\alpha_\gamma) < f(\alpha_\beta)$
then $\forall \nu < \alpha\ \exists \tau \in (\nu, \alpha] \cap D \cap (\text{rge}(g) \setminus \text{rge}(f))\ \ g(\tau_\gamma) = \varsigma_{g\restriction\tau}$.

If q, $p \in \mathbb{Q}$ then $q \le p$ if $a_p \subseteq a_q$ and $A_p \subseteq A_q$.

The apparent asymetricality of (c.ii) provokes the query, "What happens when 'Y_1' $= \emptyset$?". The answer is that a special argument will take care of this situation the only time that it arises – in the proof of the analogue of Proposition (2.15) – and so fortunately one can avoid having to try to take care of it through the definition of $\mathbb{Q}$.

The conclusion of (c.iii), reading "then $\alpha \in D$," could be replaced by an unboundedness condition similar to the conclusion of (c.iv). This is briefly discussed at the end of the proof.

Lastly, even though (c.iv), the equivalent of the clause (c.ii) of the definition of $\mathbb{P}$, has been slightly mangled by the introduction of the requirement that $f(\alpha_\beta)$ is some I^-, the idea is essentially the same as before.

Theorem 3. $\mathbb{Q}$ adds a club through D.

Proof. Now I go through the proof of showing where alterations to the proof of Theorem (2.1) or further arguments are necessary. Start by considering the proof that $\mathbb{P}$ is $\mathbb{M}$-proper.

(A). The first step is to show $p^* \in \mathbb{Q}$. For $\alpha \in D_1$ let Γ_α be a club subset of $D \cap \alpha$. Make sure in choosing $\mathcal{N}$ that D, D_0, D_1 and $\langle \Gamma_\alpha \mid \alpha \in D_1 \rangle \in \mathcal{N}$.

The new points to check are that (δ, F) satisfies (c.ii) and (c.iii), and that if $(\beta, f) \in A_p$ then it and (δ, F) satisfy (c.iii).

Suppose that $I \in a_p$ and $\varepsilon < \theta_\delta$ are such that $F(\varepsilon) = I^-$ and $\varsigma_{F\restriction\varepsilon} < F(\varepsilon)$. Recall that by the choice of $\mathcal{N}$ this implies that $\mathrm{cf}(F(\varepsilon)) = \omega_1$. So, by (a) for p, one has $F(\varepsilon) \in D_0, \subseteq D_1$, and hence (c.ii) holds for (δ, F).

If $F(\varepsilon) \in D_1$ one has that $\mathcal{N} \models$ "$\Gamma_{F(\varepsilon)}$ is club in $F(\varepsilon)$". So $\Gamma_{F(\varepsilon)} \cap \mathcal{N}$ is club in $\varsigma_{F\restriction\varepsilon}$. But $\Gamma_{F(\varepsilon)} \cap \mathcal{N} \subseteq \Gamma_{F(\varepsilon)}$, so $\bigcup \Gamma_{F(\varepsilon)} \cap \mathcal{N} = \varsigma_{F\restriction\varepsilon} \in \Gamma_{F(\varepsilon)} \subseteq D$. Thus (c.iii) holds for (δ, F).

Now suppose $(\beta, f) \in a_p$, $\alpha = \varsigma_{f\restriction\varepsilon}$, and $\alpha < f(\varepsilon) = I^-$ for some $I \in a_p$. Then, by (c.ii) for p applied to (β, f), one $\alpha \in D$. As $\alpha \in \mathrm{rge}(F)$ (and so $\varsigma F \restriction \alpha_\delta = \alpha$) by the definition of F, α is an appropriate witness for (c.iv).

(B). Next consider the proof that $q \restriction \mathcal{N} \in \mathbb{Q}$. Again (c.ii) and (c.iii) are immediate from fact that $A_{q\restriction\mathcal{N}} \subseteq A_q$ and $a_{q\restriction\mathcal{N}} \subseteq a_q$. The rest of the proof is as before.

(C). Move on to the proof that r, the amalgamation of q and s, is a condition. The first thing to consider is whether (c.ii) can be reduced to (c.ii) for s and q.

Suppose that $(\beta, f) \in A_s$ and $I \in a_q \setminus A_s$. Then, as observed in *Case (3)* of the checking of (b.i), $I^- \notin \mathrm{rge}(f)$. So no new instances of (c.ii) arise for (β, f).

On the other hand, suppose $(\beta, f) \in A_q \setminus A_s$ and $I \in a_s \setminus a_q$ and $I^- \in \mathrm{rge}(f)$ then $I^- \in \mathrm{rge}(F)$. As observed in *Case (4)* of the checking of (b.i), if $\beta < \delta$ then there is $(\beta, g) \in A_s$ such that $f \downharpoonleft I^- + 1 = g \downharpoonleft I^- + 1$. Thus in this case no new instances of (c.ii) that are not covered by (c.ii) for s arise.

If $\delta \leq \beta$ and $\varsigma_{f\downharpoonleft I^-} < I^-$ then $\varsigma_{F\downharpoonleft I^-} < I^-$ as well. But then by the properties of F one has that $\mathrm{cf}(I^-) = \omega_1$ and, by (a) for s, hence $I^- \in D_0$, so (c.ii) holds.

The proof that (c.iv) holds is exactly as in the proof that the (c.ii) in the sense of $\mathbb{P}$ holds in Claim (2.10).

(D). Next comes the analogue of the proof of Proposition (2.11) which will show that ω_2 is preserved. Work as there, but, by the stationaryness of D_0, assume that the ordinal $\delta, = \mathcal{N} \cap \omega_2$, is an element of D_0. The entire proof then goes through with no essential changes (on noting that there is no problem about assuming that q is pointed – the proof is just as in the proof that the p^* of Proposition (2.7) is a condition.)

(E). Lastly consider the analogue of Proposition (2.15) Again assume towards a contradiction that $p \Vdash$ "$\alpha \in \mathrm{Lim}(C_G)$ & $\alpha \notin C_G$. Define S as in Proposition (2.15).

Observe that as $p \Vdash$ "$\alpha \in \mathrm{Lim}(C_G)$" one has that $p \Vdash$ "$\alpha \in \mathrm{Lim}(D)$". Hence D is unbounded in α. So in dealing with the case that there is no $(\beta, f) \in S$ such that $\varsigma_{\phi\downharpoonright\alpha} = \alpha$ simply choose $\tau \in (\nu, \alpha) \cap D$.

If $Y \neq \emptyset$ and $Y_0 = \emptyset$, then work exactly as in the proof of Proposition (2.15), exploiting the fact that $Y_1 \neq \emptyset$, so there is some map to which one can apply (c.ii) and then (c.iii) for p to ensure that $\alpha \in D$. Similarly, if Y_0, $Y_1 \neq \emptyset$ apply (c.iv) for p.

Finally consider the case $Y \neq \emptyset$ and $Y_1 = \emptyset$. Now the proof is completed by the following lemma.

Lemma 4. If $(\beta, \phi) \in S$ and $\varsigma_{\phi\downharpoonright\alpha} = \alpha$ then $\bigcup D \cap \mathrm{rge}(\phi \downharpoonright \alpha) = \alpha$.

Proof. Suppose $\tau \in D \cap \alpha$ but $\tau \notin \mathrm{rge}(\phi)$ and there is some $q \leq p$ such that $q \Vdash$ "$\exists I \in a_q\, \tau = I^-$". Then, by (b.ii), $q \Vdash$ "$\exists J \in a_q \min(\mathrm{rge}(\phi) \setminus \tau) = J^-$". But $a_q \subseteq D$ and $\mathrm{rge}(\phi) \cap (\tau, \alpha) \neq \emptyset$, so $\min(\mathrm{rge}(\phi) \setminus \tau) \in D \cap \alpha$. ▲

So one can choose $\tau \in (\nu, \alpha] \cap \mathrm{rge}(\phi_0) \cap \alpha$ where ν is as in the case Y_0, $Y_1 \neq \emptyset$. ▲

Alternative definition of $\mathbb{Q}$. The conclusion of (c.iii) in the definition of $\mathbb{Q}$, reading "then $\alpha \in D$," could be replaced by an unboundedness condition similar to the conclusion of (c.iv), specifically: "$\alpha \in D$ or $D \cap \alpha \setminus \mathrm{rge}(f)$ is unbounded in α." However then one would have to make the (harmless) assumption that successor elements of D have cofinality less than ω_1 *and* to add to the definition of $\mathbb{Q}$ the requirement that if $(\beta, f) \in a_p$ then $\mathrm{rge}(f)$ is closed under taking predecessors in D. One of these is necessary in order to be able to check that the (δ, F) in the proof of $\mathbb{M}$-properness satisfies the last clause of the conclusion of (c.iv) – that $\varsigma_{F\downharpoonright\tau} = F(\tau)$. I give the argument in case someone can spot a way of eliminating the additional closure requirement in this alternative definition.

There is some more work to be done here since the alternative version of (c.iii) also gives the possibility that $D \cap \alpha \setminus \mathrm{rge}(f)$ is unbounded in α. By elementarity and the facts that α, D and $(\beta, f) \in \mathcal{N}$ one has that $\mathcal{N} \models$ "$D \cap \alpha \setminus \mathrm{rge}(f)$" is unbounded in α. Hence $\mathrm{rge}(F) \cap D \cap \alpha \setminus \mathrm{rge}(f)$ is unbounded in α.

So far, so good. However one needs to be able to check that there are unboundedly many elements τ of this set such that $\varsigma_{F\upharpoonright\tau} = \tau$. If this fails then $\mathrm{cf}(\tau) = \omega_1$, by the assumption on F used above in the proof that (δ, F) satisfies (c.ii). By elementarity again, $\mathrm{rge}(F)$ is closed under both predecessors and successors in D. So one can wander forward through D (at least for up to ω many paces) looking to find a suitable replacement for τ in which $\mathrm{rge}(F)$ is unbounded.

With the auiliary closure requirement and the assumption that successor elements of D have countable cofinality in hand one can now finish the proof by merely taking one pace forward. For then this element of D is still in $\mathrm{rge}(F)$, is less than α, has countable cofinality as the successor of τ, and cannot be in $\mathrm{rge}(f)$ for otherwise τ would also be in $\mathrm{rge}(f)$ by the closure of $\mathrm{rge}(f)$ under predecessors.

Thus if $F(\alpha_\delta) < f(\alpha_\beta$ then (c.iv) is satisifed.

Finally, as observed, $\mathrm{rge}(F)$ itself is closed under predecessors in D by elementarity. ▲

§4. Forthcoming attractions in the same area.

For the next version of this paper:

Theorem. Suppose κ is a sucessor cardinal, $(\kappa^{++})^{<\kappa} = \kappa^{++}$ and there is a stationary $(\kappa, 1)$-simplified morass. Then there is an κ-$\mathbb{M}$-proper, κ^+-preserving, κ-closed forcing of size κ^+ (and which is thus preserves all cardinals) which adds a new club subset to any stationary subset of κ^+ which is closed under $< \kappa^-$ sequences and is very stationary.

(*"Proof" notes* Conditions consist of pairs (a, A) with $a \in [[\kappa^{++}]^1 \cup \kappa^{++}]^2]^{<\kappa}$ and $A \in [\mathcal{F}]^{<\kappa}$. One only needs the cardinal arithmetic assumption that $(\kappa^{++})^{<\kappa} = \kappa^{++}$ by the δ-system Lemma for morass maps.

One has a small difficulty to overcome in the analogue of Proposition (15) attacking the case that $\varsigma_{\phi\upharpoonright\alpha} < \alpha$ for all $(\beta, \phi) \in S$.

If $\nu = \mathrm{ssup}(\bigcup\{\mathrm{rge}(\phi) \cap \alpha \mid (\beta, \phi) \in \overline{A_p}\} \cup \bigcup\{I^+ \mid I \in a_p \ \& \ I^+ < \alpha\})$ then one cannot necessarily show $\nu < \alpha$ when $\mathrm{cf}(\alpha)$ is not greater than the size of the working parts of the conditions. That isn't too bad for simply adding club subsets (one simply adds the singleton $\{\alpha\}$ to a_p), but does cause problems when trying to shoot clubs through stationary (or even club) subsets of κ since one needs some way of proving that α is in the stationary set.

So I also insist that if the simplified morass, $\mathbb{M}$, is such that if $(\alpha, f) \in \mathcal{F}$ and

$\varsigma_{f\restriction\alpha} < f(\alpha)$ then $\mathrm{cf}(\alpha) = \kappa^-$ (and $\mathrm{cf}(f(\alpha)) \geq \kappa^-$). This avoids the above difficulty, is necessary for the $< \kappa$-closure argment and is unproblematic (the adjustment to ensure this is already presented in [V84]). ▲

Key open question. Is it possible to add a club subset of ω_3 with conditions with *finite* working parts?

In a separate paper:

Theorem. For any ordinal $\zeta < \omega_3$ there is a cardinal preserving forcing which adds a pcf structure on ζ.

Proof. I integrate the forcings from this paper, [M3] (which illustrates the $\mathbb{M}$-proper forcing version of adding a structure where the ordering matters and one is aiming for length less than ω_3 in place of a known forcing for length exactly ω_2) and [M2] (in which I force to add a pcf structure on ω_2).

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