Complex Individuals and Multigrade Relations

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Goodman and Leonard pointed out in 1940 that by using the calculus of individuals one could give formalizations within first-order logic of many idioms involving what they called "multigrade relations". These are relations such as 'are brothers', 'are compatriots', or 'built the bridge', which do not take any fixed number of arguments. One can say '*a* and *b* are compatriots' or '*a* and *b* and . . . and *z* are compatriots'. The purpose of this paper is to show that the reverse is also true; I give a formal account of multigrade relations and some related idioms, and show that there is a natural translation of the vocabulary of the calculus of individuals into the notation I provide which takes all the theorems of that calculus to valid sentences of the formalism.[1]

I. PLURAL SUBJECTS

The subject of a predicate such as 'live together' may be a string of names, such as 'Adam and Milly and Stephen', or a plural noun phrase such as 'the Mortons', or a string of plural noun phrases such as 'the Mortons and the MacDougals' or 'the Mortons and the MacDougals and some of the Hanrahans'. 'The Mortons' is not shorthand for 'Adam and Milly and Stephen' even if these are all and only the Mortons, for in using 'the Mortons' one leaves open who are Mortons and how many they are. Instead, the force of a sentence such as 'The Mortons live together' is to say something like 'there are some people, p_1, p_2, . . . , and p_1 is a Morton and so is p_2 and so are all the others, and p_1 and p_2 and . . . live together'. Notice that these idioms involve no presupposition that the subject covers only finitely many individuals; a multigrade relation can relate infinitely many relata. Notice also that multigrade relations can have a sort of finite adicity. For example, 'fought with' is in a clear sense dyadic, its arguments fall into two groups, although it can relate, say, four objects ('Adam and Bill fought with Yuri and Zero').

Two logical devices are involved here, each of which seems inevitable if multigrade relations are to be handled conveniently. One device is the use of what I shall call *M-quantifiers* to form closed sentences out of open sentences involving multigrade relations, just as universal and existential quantifiers form closed sentences out of open sentences involving n-ary relations. M-quantifiers are quite familiar in English; they are involved in many uses of 'some' and 'the' before plural nouns and in some uses of 'any' before a plural noun, e.g., 'any people who live together are likely to influence each other'. M-quantifiers are related to ordinary quantifiers in much the way that ordinary quantifiers are related to conjunctions and disjunctions. For just as the universal quantification '$(x)Px$' has roughly the force of an infinite conjunction '$P(a_1)$ & $P(a_2)$ & ...', where a_1, a_2, . . . are all the objects in the domain of discourse, so the universal M-quantifier has roughly the force of an infinite string of regular universal quantifiers. For example, 'any people who live together will influence each other' can be crudely analyzed as '$(x)(Tx \supset Ix)$ & $(x)(y)(Txy \supset Ixy)$ & $(x)(y)(z)(Txyz \supset Ixyz)$ & ...', where T and I are the multigrade relations of living together and of influencing one another, and there are as many conjuncts as the domain of discourse has members.[2] My notation for M-quantifiers is motivated by this analogy. One forms a universal quantification '$(x)Px$' by taking 'P _' (what all the conjoined instances have in common), filling the blank with a variable rather than a name, and then binding the variable with a quantifier that indicates that it is the conjunction of the instances that is intended. Analogously, from '$(x)Px$ & $(x)(y)Pxy$ & ...', one forms a universal M-quantification by taking '... P _' (what the conjoined instances have in common when the ...-place is understood to be filled with universal quantifiers binding variables in the _-place), filling the _ with a symbol for a generic string of variables and the ... with a symbol for a quantification of these variables, and then binding the result to indicate that all such instances are included, getting '$([x])P[x]$'. I write the existential M-quantifier as $E[x]P[x]$. (The square brackets roughly correspond to English pluralization.)

The other device that multigrade relations require is a way of forming multigrade relations out of ordinary predicates and relations. The necessity for a device with this effect may be seen as follows: If we formalize 'the Mortons live together' as '$([x])(M[x] \supset T[x])$', then M must represent the multigrade relation 'x_1 and x_2 and ... are Mortons' rather than the predicate 'x is a Morton'. But in the sentence 'all the Mortons have big noses', the phrase 'the Mortons' does correspond to the predicate 'x is a Morton'. Clearly, there is a lot to understand about the conditions under which an English expression performs one function rather than the other. I shall employ the notation P^* to indicate that the predicate P is being used as a multigrade relation which holds between some objects if they all satisfy P. Thus, 'the Mortons, who have large noses, live together' is formalized as '$(x)(Mx \supset Lx)$ & $([x])(M^*[x] \supset T[x])$'. If R is a dyadic relation, then R^* is a 'dyadic' multigrade relation (in the sense in which 'fought with' is) which holds between some objects and some other objects if each of the first objects bears R to some of the other objects. Note that this can turn a symmetrical relation into a nonsymmetrical one.

It is natural to wonder whether these devices may not be best understood in terms of set theory or higher-order logic. I think not. It will be easier to state my reasons after I have described the syntax and semantics of a formal language of multigrade relations. I then argue that this language is not plausibly interpreted as part of set theory or higher-order logic.

II. THE LANGUAGE L_M

L_M is an expansion of quantification theory to allow multigrade relations, M-quantifiers, and the *-operator. I use Mates's system $\mathcal{L}_I$ of quantificational logic with identity as a basis from which to construct L_M. The syntax of L_M is obtained by adding to $\mathcal{L}_I$ relational symbols $R_1, R_2, \ldots$ for multigrade relations, universal M-quantifiers $([v])$, and multigrade variables $[v]$ for each ordinary variable v. The formation rules are extended to permit atomic predicates to be followed by multigrade variables, M-quantifiers to precede matrices, and * to follow atomic predicates. Multigrade relations $R_1, R_2, \ldots$ and predicates followed by * may be followed by any number of ordinary or multigrade variables. '$E[v]$' is taken as an abbreviation for '$\sim([v])\sim$'. I write '$\overset{*}{=}$' for * applied to the identity relation. Note that '$([x])([y])P[x][y]$' is well-formed and that '$P[v]$' is well-formed even when 'P' is non-multigrade. Note also that I am taking all the multigrade relations R_i to be 'monadic'.

The definition of an interpretation and of truth under an interpretation are much what one might expect. It would not be hard to express them in completely set-theoretical terms; but my purposes are better served, and the definitions are prettier, if I make use of informal English multigrade relations and plural subjects, as follows:

An interpretation of L_{M} is provided by a non-empty domain D and an assignment F of a set of n-tuples of members of D to every n-place relation symbol, of a member of D to every ordinary variable v, of some members of D (in a certain order) to every multigrade variable $[v]$, and of a set of ordered sets of members of D to every multigrade relational symbol R_i. We assume an enumeration of all the ordinary variables; v_n is the nth variable in this enumeration. Then the standard definition of truth under an interpretation (see Mates [7], Ch. 4, Sect. 2) is augmented with

(a) An atomic sentence consisting of an atomic relation R (multigrade or not) followed by a string of ordinary and multigrade variables is true if and only if the sequence consisting of the object which F correlates with the variables, in the order in which the variables occur, is a member of $F(R)$, if R is multigrade; or if the tuple of the first n members of the sequence is a member of $F(R)$, if R is n-adic.

(b) A sentence consisting of an atomic relational symbol R followed by * followed by a string of ordinary or multigrade variables is true if and only if either the relational symbol is monadic and all of the objects which F assigns to the variables are members of $F(R)$, or it is polyadic and $F(R)$ relates each of the objects which F correlates with the first variable to some object which F correlates with the second variable, and to each object which F correlates with the third variable, and so on, for as many places as R has.

(c) $([v_n])A$ is true if and only if A is true under every interpretation of L_{M} in D which differs from F only in what it assigns to $[v_n]$.

(d) A sentence consisting of $\overset{*}{=}$ flanked by strings of variables is true if and only if each of the objects which F correlates with a variable to the left of $\overset{*}{=}$ is identical with some object which F correlates with a variable to the right of $\overset{*}{=}$.

I discuss the axiomatizability of the set of valid sentences of L_M in the next section.

L_M looks somewhat like a set theory or higher-order logic, with '$([x])$' taken as 'for any set x' or 'for any attribute x'. But there are three good reasons why the resemblance is only apparent. First, by interpreting $[x]$ as a variable over sets we do not turn multigrade relations into ordinary ones. For a single interpretation can make all of $\mathrm{E}[x]R[x]$, $\mathrm{E}[x]\mathrm{E}[y]R[x][y]$, etc., true. Thus, R would have to be a multigrade relation between sets, if it were any relation between sets, and this undercuts the motivation for construing it as a relation between sets at all. Second, if the multigrade variables are taken as ranging over sets, then the ordinary variables must be taken as ranging over sets, too. For schemata such as $(x)\mathrm{E}[y](Ax \equiv A[y])$ are valid. Third, the 'sets' that variables such as $[x]$ would range over would not distinguish between, e.g., $\{a, b, c\}$ and $\{a, \{b, c\}\}$. For $(x)(y)(z)\mathrm{E}[w](y \overset{*}{=} [w]\ \&\ z \overset{*}{=} [w]\ \&\ (v)(v \overset{*}{=} [w] \supset v = y \vee v = z)\ \&\ Axyz \equiv Ax[w])$ is a valid formula of L_M. If the variables are taken as ranging over sets, then $\overset{*}{=}$ must represent the subset relation, and thus $[w]$ in the formula just cited is $\{y, z\}$, and thus, since A is a predicate of sets, $\{x, y, z\}$ is indistinguishable from $\{x, \{y, z\}\}$.

The second and third of these reasons suggest a close connection with the calculus of individuals. I make this explicit in the next section.

III. L_M AND THE CALCULUS OF INDIVIDUALS

I take the calculus of individuals (with atoms) to be a first-order theory with the following axioms and axiom schema, involving the relation '$x < y$' to be interpreted as 'x is a part of y'. (See Goodman [3], Tarski [8], and Eberle [2].) To shorten the axioms, I use 'Atom x' as an abbreviation for '$(w)(w < x \supset x < w)$'.

(i) '$<$' is reflexive and transitive:
$(x)(y)(z)[x < x\ \&\ ((x < y\ \&\ y < z) \supset x < z)]$.

(ii) Distinct individuals contain distinct atoms:
$$(x)(y)(\mathrm{E}z)(\mathrm{E}w)[\mathrm{Atom}\, z \ \& \ \mathrm{Atom}\, w \ \& \ z < x \ \& \ w < y \ \& \ (\sim x = y \supset \sim z = w)].$$

(iii) For any satisfied predicate A, there is a sum consisting of all and only the As:
$$(\mathrm{E}x)Ax \supset (\mathrm{E}y)(z)(\mathrm{Atom}\, z \supset (z < y \equiv (\mathrm{E}w)(z < w \ \& \ Aw))).$$

The relations of mereological sum, $y = x + z$, and of overlap, $x \ \mathrm{Z} \ y$, may be defined in terms of $<$.

The basic idea in translating the calculus of individuals into L_{M} is to operate with multigrade relations such as '$x_1, x_2, \ldots$ make up a cat', where the relata are atoms, rather than with predicates such as 'x is a cat' which apply to nonatomic individuals. The relations must be multigrade, because there is no fixed number of parts that, e.g., a cat is composed of. Thus, to make assertions about cats we use multigrade variables ranging over their atomic parts rather than simple variables ranging over cats. A complex thing, that is, some atomic things, is a part of another complex thing, that is, some other atomic things, if the first things are among the other things. In other words, if each of the first things is identical with one of the second things. In other words, if the first things $\stackrel{*}{=}$ the second things. Thus, we interpret '$x < y$', where x and y range over possibly complex individuals, as '$[x] \stackrel{*}{=} [y]$', where $[x]$ and $[y]$ are multigrade variables ranging over atoms. 'Feet are parts of people' is taken as 'if some things make up a foot, then they are among some things that make up a person'. Identity between individuals x and y is interpreted as identity of the atoms of which they are composed, that is, as $[x] \stackrel{*}{=} [y]$ & $[y] \stackrel{*}{=} [x]$.

Under this interpretation (i), (ii), and (iii) are turned into valid formulas of L_{M}, obtained by replacing 'v' by '$[v]$' and '$x < y$' by '$[x] \stackrel{*}{=} [y]$'. The validity of (i) on this interpretation is evident. The validity of (ii) is clear when one sees that it says 'for any objects, there is an object among them'. (In this connection, note that 'Atom x' is translated as '$([w])([w] \stackrel{*}{=} [x] \supset [x] \stackrel{*}{=} [w])$', which is equivalent to '$(\mathrm{E}y)([x] \stackrel{*}{=} y)$', so that 'Atom' is translated by a predicate which is satisfied by a multigrade variable only when the interpretation correlates that variable with exactly one individual.) The validity of (iii) is clear when one sees that it says 'if any objects satisfy a (possibly multigrade) relation, then there are objects such that they are all the objects which satisfy the relation'.

The calculus of individuals is thus interpretable in L_{M}. The reverse is also true, if we restrict attention to symmetric multigrade relations.[3] We can translate L_{M}, with this restriction on its interpretation, into the calculus of individuals by replacing each multigrade quantifier and variable by a regular variable and quantifier and replacing each atomic formula $R_i v_1 \ldots v_n$ by $R_i(v_1+\ldots+v_n)$. $R^* v_1 \ldots v_n$ is replaced with $(y)((\mathrm{Atom}\ y\ \&\ y < (v_1+\ldots+v_n)) \supset Ry)$, if R is monadic, and with $(y)((\mathrm{Atom}\ y\ \&\ y < (v_1+\ldots+v_n)) \supset (\mathrm{E}z)(\mathrm{Atom}\ z\ \&\ z < (v_1+\ldots+v_n)\ \&\ Ryz))$, if R is dyadic (and similarly for other adicities, in accordance with clause (b) of the definition of truth for L_{M}). This translation takes sentences of L_{M} to sentences of the calculus of individuals (it is one-one into) and preserves logical consequence (relative to models with only symmetrical multigrade relations on the one side and models for which '$<$' satisfies the calculus of individuals on the other). Moreover, it takes a valid sentence of L_{M} to each axiom of the calculus of individuals. It follows that the valid sentences of L_{M} are axiomatizable by the sentences which the translation takes to the axioms of the calculus.

One must remember that this translation and axiomatization hold only on the restriction of L_{M} to symmetric multigrade relations. Without that restriction, L_{M} seems essentially stronger than the calculus of individuals. I do not know if it is then axiomatizable. One should also note that L_{M} cannot handle multigrade relations between complex individuals constructed out of the atoms over which its quantifiers range. If, for example, the domain consists of parts of the body, and we are treating 'is a person' as '$x_1, x_2, \ldots$ make up a person', then we cannot provide a good formalization of '$x_1, x_2, \ldots$ are compatriots'. The same is true of the calculus of individuals, and for a similar reason. If we construe '$Rx_1 \ldots x_n$' as '$R(x_1+\ldots+x_n)$', then we are in trouble with cases in which R holds between a, b, and c but not between a and the sum of b and c. To handle multigrade relations between complex individuals, we must extend L_{M} to include what one might call indefinitely multigrade relations, of the form '$R(x_1, x_2, \ldots; y_1, y_2, \ldots; \ldots)$'. Then '$x_1, x_2, \ldots$ are compatriots' would be represented as 'x_1 and x_2 and . . . make up a person, and so do y_1 and y_2 and . . . , and so do . . . , and the making-up-compatriots relation holds between x_1 ,x_2 ...; y_1 , y_2 ,...;...'. I don't doubt that a formal development of the idea would be very ugly.

IV. SOME METAPHYSICS

The translation of the calculus of individuals into L_M that I have described is a very natural one. Moreover, I believe that it gives a *better* exposition of the part-whole relation which is the object of Goodman's investigations than one can get with a formulation of the calculus of individuals as a theory in standard quantificational terms. For one of Goodman's reasons for preferring to operate with complex individuals rather than with sets is that if one admits the device of set abstraction one begins to climb

> up through an explosively expanding universe towards a prodigiously teeming Platonic Heaven. . . . [One] gets all these extra entities . . . by a magical process that enables him to make two or more distinct entities from exactly the same entities. ([4]: 159.)

The formation of sets is magic, because it gives one more than one had to start with. But why is the formation of complex individuals not magic too, since given a, b, and c it provides one, in addition, with $a + b$, $a + c$, $b + c$, and $a + b + c$, each of which is distinct from each of the three objects one began with? The query can be met by arguing that the *content* of $a + b + c$ is nothing more than a, b, and c. Now the notion of content is not particularly clear, but the reply is somewhat misleading besides, if it seems to allow that $a + b + c$ *is* something over and above a, b, and c, though its content is exhausted by theirs. For we need not allow this, as we can treat quantification over wholes in such a way that the assertion that $a + b + c$ exists is equivalent to the assertion that a, b, and c exist. L_M provides one way of doing this.

Moreover, on this formulation we do not have to tackle difficult issues about which of the relations of a system is its part-whole or its generating relation (see Goodman [4], Sect. 2 and Appendix). For the part-whole relation is not treated as a relation between elements of the domain of a theory expressed in L_M at all. We can, however, specify the conditions under which a relational term 'R' of a theory expressed in a standard quantificational language is construed as the part-whole relation by a translation of that theory into L_M. This is the case when 'Rxy' is uniformly translated as '$[x] \overset{*}{=} [y]$'. And, in general, one profitable way to study the claim that a relation generates a domain of entities out of certain objects taken as atoms is, I believe, to examine the intelligibility and plausibility of translations which take theories of the domain and the relation, expressed in quantificational languages, to theories in richer logical systems, in which only the atoms of the original generating relation are included in the range of the variables. One need not take such a translation as showing that there are no complex individuals; one may rather take it as part of an explanation of what it is for there to be complex individuals. It is an unmystifying way of saying that there are wholes and that all that is required for the wholes to exist is the existence of the parts.

REFERENCES

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[5] —— and H. Leonard, "The Calculus of Individuals and Its Uses," *Journal of Symbolic Logic* 5(1940).
[6] Kurt Gödel, "Russell's Mathematical Logic," in *The Philosophy of Bertrand Russell*, ed. by P. A. Schilpp (La Salle, Ill.: Open Court, 1944).
[7] Benson Mates, *Elementary Logic* (New York: Oxford University Press, 1965).
[8] Alfred Tarski, "Foundations of the Geometry of Solids," in *Logic, Semantics, Metamathematics* (Oxford: Clarendon Press, 1956): 24-29.

NOTES

[1] Richard Grandy made some useful suggestions about an earlier draft of this paper, and the referee for NOÛS pointed out a horrible mistake in it. Some interesting observations on multigrade relations and the part-whole relation are found in Chandler [1].

[2] The use of infinitely long formulas is just a heuristic, but my formal theory of multigrade relations and the use I put it to is related to an interpretation of the theory of types in infinitary logic suggested by Gödel (see [6]: 144). Note that the example could not be formalized as '$(x)(y)(Txy \supset Ixy)$' since three people who influence one another need not do so in pairs; x and y may together influence z though neither x nor y alone does.

[3] Symmetric in the extended sense that any two of their arguments may be

exchanged. Most English 'monadic' multigrade relations seem to be symmetric in this sense; if Adam and Bill and Charles live together, then Bill and Adam and Charles live together, etc. English 'dyadic' multigrade relations only occasionally are; if Adam fought with Bill and Charles, it does not follow that Bill fought with Adam and Charles.