James sequences and Dependent Choices

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We prove James's sequential characterization of (compact) reflexivity in set-theory $\mathbf{ZF}+\mathbf{DC}$, where $\mathbf{DC}$ is the axiom of Dependent Choices. In turn, James's criterion implies that every infinite set is Dedekind-infinite, whence it is not provable in $\mathbf{ZF}$. Our proof in $\mathbf{ZF}+\mathbf{DC}$ of James' criterion leads us to various notions of reflexivity which are equivalent in $\mathbf{ZFC}$ but are not equivalent in $\mathbf{ZF}$. We also show that the weak compactness of the closed unit ball of a (simply) reflexive space does not imply the Boolean Prime Ideal theorem : this solves a question raised in [6].

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1 Introduction

It has been proved by James (see [12]) that a real Banach space E is *reflexive* if and only if it satisfies the following Property :

> $(\mathbf{J})$: *There exists some real number* $\vartheta \in]0,1[$ *such that for every sequence* $(a_n)_{n\in\mathbb{N}}$ *of the unit sphere of* E*, there exists* $k \in \mathbb{N}$ *such that the distance between the respective convex hulls of* $\{a_i : i \leq k\}$ *and* $\{a_i : i > k\}$ *is* $\leq \vartheta$.

Notice that Property $(\mathbf{J})$ has a *countable character* : the Banach space E satisfies $(\mathbf{J})$ if and only if every closed separable subspace of E satisfies $(\mathbf{J})$. Say that a real Banach space E is *J-reflexive* if E satisfies Property $(\mathbf{J})$. Various proofs of James's above characterization of reflexivity occur in the literature (see for example two different proofs by James in [12] and [13], or the short new proof by Oja in [16]), however, all these proofs rely on some form of the *Axiom of Choice* ($\mathbf{AC}$), and they involve various notions of reflexivity for Banach spaces E (weak compactness of the closed unit ball of E, Šmulian property on E, surjectivity of the canonical mapping from E to its second dual, ...). Of course, equivalences between these notions of reflexivity are provable in set-theory with choice $\mathbf{ZFC}$, but generally, they do not hold in set-theory without choice $\mathbf{ZF}$; most of them rely on the axiom of *Dependent Choices* ($\mathbf{DC}$), the axiom of *Hahn-Banach* ($\mathbf{HB}$) or the *Boolean Prime Ideal* ($\mathbf{BPI}$) -see Section 2.1-.

In different ways, James ([13] or [2] p. 51-56) and Oja ([16]) both proved in $\mathbf{ZF}+\mathbf{HB}+\mathbf{DC}$ that every J-reflexive Banach space E is *onto-reflexive*, *i.e.* : "The *canonical mapping* $j_E : E \to E''$ is onto." (see Notation 2.6). In [12]-Theorem **1**- which relies on [11]-Lemma **1**-, James proved in $\mathbf{ZF}+\mathbf{DC}$ that given a J-reflexive Banach space E, every closed separable subspace V of E is ω*-reflexive*, *i.e.* : "Every descending sequence $C_0 \supseteq C_1 \supseteq C_2 \cdots \supseteq C_n \ldots$ of nonempty closed bounded convex subsets of V has a nonempty intersection." In this paper, given a normed space E, we define the *convex topology* on E as the topology whose closed sets are intersections of finite unions of closed convex subsets of E, and we introduce the following strong notion of reflexivity for E, which we call *convex-reflexivity* : "The closed unit ball of E is compact in the convex topology." We then prove in $\mathbf{ZF}+\mathbf{DC}$ the following statement (see Theorem 3.9 of Section 3) :

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$\mathbf{J2C}$, **(James Criterion)** *Every J-reflexive Banach space is convex-reflexive.*

This result improves the classical proofs because, on the one hand, convex-reflexivity implies both ω-reflexivity and onto-reflexivity in $\mathbf{ZF}$ (see Proposition 4.7) but none of the converse implications holds in $\mathbf{ZF}$ (see Proposition 4.11) and on the other hand, our proof links in $\mathbf{ZF}+\mathbf{DC}$ convex-reflexivity, J-reflexivity and well-founded trees ; moreover, it yields in $\mathbf{ZF}$ a *rank of J-reflexivity* for J-reflexive Banach spaces with a dense well-orderable subset. Also notice that our proof can be shortened if one uses $\mathbf{AC}$ (see Remark 3.10), yielding a short new proof of James's criterion in $\mathbf{ZFC}$.

Our paper is organized as follows : in Section 2 we recall various weak forms of the Axiom of Choice, we set out some definitions and notation, and we lay out various "weak" topologies on a normed space, in Section 3 we show in $\mathbf{ZF}+\mathbf{DC}$ how to saturate a filter according to some numerical constraint -without using any maximal filter- (Section 3.2), and we prove the statement $\mathbf{J2C}$ in $\mathbf{ZF}+\mathbf{DC}$ (see Theorem 3.9 in Section 3). On the way we get in $\mathbf{ZF}$ similar results for spaces with a dense well-orderable subset. In Section 4 we compare in $\mathbf{ZF}$ various notions of reflexivity, and we obtain several $\mathbf{ZF}$-equivalent characterizations of J-reflexivity (see Section 4.2). In Section 5, we prove that $\mathbf{J2C}$ implies the following weak form of the axiom $\mathbf{DC}$: "*Every infinite set is Dedekind-infinite.*" It follows that $\mathbf{J2C}$ is not provable in $\mathbf{ZF}$. Finally, in Section 6, we solve Question **2.11** of [6].

2 Preliminaries

2.1 Some consequences of $\mathbf{AC}$, some models of $\mathbf{ZF}$

2.1.1 Consequences of $\mathbf{AC}$

In this section, we recall various weak forms of the Axiom of Choice which will be used in this paper : see [14], and also [10] for a recent account on the relative strength of numerous consequences of the Axiom of Choice. Given a non-empty family $(A_i)_{i\in I}$ of non-empty sets, any element $f \in \prod_{i\in I} A_i$ is called a *choice function* for the family $(A_i)_{i\in I}$.

($\mathbf{AC}$, **Axiom of Choice***). For every non-empty family* $(A_i)_{i\in I}$ *of non-empty sets,* $\prod_{i\in I} A_i$ *is non-empty.*

The following well-known consequence of $\mathbf{AC}$, is not provable in $\mathbf{ZF}$, and it does not imply $\mathbf{AC}$:

($\mathbf{BPI}$, **Boolean Prime Ideal axiom***). Every non-trivial Boolean algebra has a prime ideal.*

In $\mathbf{ZF}$, the axiom $\mathbf{BPI}$ is known (see [10]) to be equivalent to the Tychonov axiom, *For every family* $(X_i)_{i\in I}$ *of compact Hausdorff spaces, the topological product* $\prod_{i\in I} X_i$ *is compact.*

The Hahn-Banach axiom is not provable in $\mathbf{ZF}$, it is a consequence of $\mathbf{BPI}$ but does not imply it :

($\mathbf{HB}$, **axiom of Hahn-Banach, analytic form***). If* E *is a normed space over the field of real numbers* $\mathbb{R}$*, if* $p : E \to \mathbb{R}$ *is a sublinear mapping and if* $f : F \to \mathbb{R}$ *is a linear mapping defined on a subspace* F *of* E *satisfying* $\forall x \in F, f(x) \leq p(x)$*, then, there exists a linear mapping* $g : E \to \mathbb{R}$ *extending* f *and dominated by* p *(i.e.* $\forall x \in E\ g(x) \leq p(x)$*).*

Here, a mapping $p : E \to \mathbb{R}$ is said to be *sublinear* if for every $x, y \in E$ and every $\lambda \in \mathbb{R}_+$, $p(x+y) \leq p(x)+p(y)$ and $p(\lambda x) = \lambda p(x)$.

Here are some "countable" axioms of choice.

($\mathbf{DC}$, **axiom of Dependent Choices***). For every non-empty set* X *and every binary relation* R *on* X *satisfying* $\forall x \in X\ \exists y \in X\ xRy$*, there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements of* X *such that* $\forall n \in \mathbb{N}\ x_n R x_{n+1}$*.*

($\mathbf{AC}(\mathbb{N})$, **Countable axiom of Choice***). For every sequence* $(A_n)_{n\in\mathbb{N}}$ *of nonempty sets, the product* $\prod_{n\in\mathbb{N}} A_n$ *is nonempty.*

A set I is *Dedekind-finite* (D-finite for short) if there is no one-to-one mapping from $\mathbb{N}$ into I. In the opposite case, the set I is *Dedekind-infinite* (D-infinite).

($\mathbf{D}$, **Dedekind-infinite***). "Every infinite set is Dedekind-infinite".*

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($\mathbf{AC}(\mathbb{N},\mathbf{fin})$*,* **Countable Axiom of Choice for Finite sets***). For every sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty finite sets, the set $\prod_{n\in\mathbb{N}} A_n$ is nonempty.*

Notice that $\mathbf{AC}(\mathbb{N},\mathbf{fin})$ is equivalent to the fact that any countable union of finite sets is countable.

It is known that, (see [14]), $\mathbf{DC} \Rightarrow \mathbf{AC}(\mathbb{N}) \Rightarrow \mathbf{D} \Rightarrow \mathbf{AC}(\mathbb{N},\mathbf{fin})$ and that none of the converse implications holds. Links between axioms $\mathbf{DC}$, $\mathbf{D}$, $\mathbf{HB}$, and $\mathbf{BPI}$ are well known : see [14], and also [10]. In particular, it is known that $\mathbf{BPI}$ does not imply $\mathbf{D}$ and that $\mathbf{DC}$ does not imply $\mathbf{HB}$.

The following models will be referred to later on to compare in $\mathbf{ZF}$ various notions of reflexivity.

2.1.2 The "basic Cohen model"

This model satisfies $\mathbf{ZF}$+$\neg\mathbf{D}$. More precisely, there is a dense subset I of $\mathbb{R}$ which is Dedekind-finite (see [14] p. 66). Notice that since $I = \sqcup_{n\in\mathbb{Z}}(I \cap [n, n+1))$, it admits an onto mapping $f : I \to \mathbb{N}$. However, since I is totally orderable, $\mathbf{AC}(\mathbb{N},\mathbf{fin})$ restricted to non-empty, finite subsets of I holds. (In fact, the model at stake satisfies $\mathbf{BPI}$.)

2.1.3 The "Second Cohen model"

This model (see [14]) satisfies $\mathbf{ZF} + \neg\mathbf{AC}(\mathbb{N},\mathbf{fin})$: indeed, in this model, there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of two-element subsets of $\mathcal{P}(\mathbb{R})$, the set of all subsets of $\mathbb{R}$, satisfying $\prod_{n\in\mathbb{N}} A_n = \varnothing$.

2.1.4 A model due to Pincus

In [17] is given a model of $\mathbf{ZF}$+$\neg\mathbf{HB}$.

2.2 Well-founded relations

Given a binary relation $\prec$ on a set E, we denote by $\succ$ the *reverse* relation (thus $x \succ y$ iff $y \prec x$). A binary relation $\prec$ on a set E is *well-founded* if every nonempty subset $A \subseteq E$ has a $\prec$-minimal element, *i.e.* an element $a \in A$ such that $\forall x \in A\ (x \prec a \Rightarrow x = a)$. Let $\overline{\mathbb{N}} := \mathbb{N} \cup \{\mathbb{N}\}$. Given some $N \in \overline{\mathbb{N}}$, an *ascending sequence* of $(E, \prec)$ (or a $\prec$-ascending sequence of E) is a sequence $(x_n)_{n\in N}$ of E such that for every n, if $n+1 \in N$ then $x_n \prec x_{n+1}$. An *increasing sequence* of $(E, \prec)$ is a sequence $(x_n)_{n\in N}$ of E such that for every n, if $n+1 \in N$ then ($x_n \prec x_{n+1}$ and $x_n \neq x_{n+1}$). A *descending sequence* of $(E, \prec)$ is an ascending sequence of $(E, \succ)$. A *decreasing sequence* of $(E, \prec)$ is an increasing sequence of $(E, \succ)$. Clearly, a well-founded binary relation has no infinite decreasing sequence, and the converse statement is equivalent to $\mathbf{DC}$ (see Proposition 2.1).

Notation ($X^{<\omega}$ **)** Given some set X, we denote by $X^{<\omega}$ the set $\cup_{n\in\mathbb{N}} X^n$ of finite sequences of X. For every $\sigma \in X^{<\omega}$, the natural number n such that $\sigma \in X^n$ is the *length* of σ and is denoted by $|\sigma|$.

A *tree* of finite sequences on a set X is a subset T of $X^{<\omega}$ which is closed by restriction *i.e.* if for every $\sigma, \sigma' \in X^{<\omega}$, $(\sigma \subseteq \sigma' \in T) \Rightarrow \sigma \in T$; the tree T is endowed with the binary relation $\subseteq$. Given some trees $S, T \subseteq X^{<\omega}$, say that S is a *subtree* of T if $S \subseteq T$.

The following Proposition is straightforward :

Proposition 2.1 ($\mathbf{DC}$ and well-founded relations) *The following statements are equivalent :*

1. *Axiom* **DC** *;*
2. *Every binary relation which is not well-founded has an infinite decreasing sequence ;*
3. *For every tree $(T, \subseteq)$, if the reverse relation $(T, \supsetneq)$ is not well-founded, then T has an infinite increasing sequence $(\sigma_n)_{n\in\mathbb{N}}$.*

Thus, in $\mathbf{ZF}$+$\mathbf{DC}$, a binary relation $\prec$ on a set X is well-founded if and only if $(X, \prec)$ has no infinite decreasing sequence. However, if X is well-orderable, this equivalence holds in $\mathbf{ZF}$.

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2.3 Lattices and filters

Given a set X, we denote by $\mathcal{P}(X)$ the set of all subsets of X. A *nonempty* subset $L \subseteq \mathcal{P}(X)$ is a *lattice of subsets of* X if L is closed under finite unions and finite intersections ; in particular, $\varnothing \in L$ and $X \in L$. A *nonempty* subset $\mathcal{F}$ of the lattice L is called a *filter* of L when the following three conditions are satisfied :

$$\varnothing \notin \mathcal{F} \tag{1}$$

$$\forall A \in \mathcal{F}\ \forall B \in \mathcal{F}\ \ A \cap B \in \mathcal{F} \tag{2}$$

$$\forall A \in \mathcal{F}\ \forall B \in L\ \ (A \subseteq B \Rightarrow B \in \mathcal{F}) \tag{3}$$

A *filter on the set* X is a filter of the lattice $\mathcal{P}(X)$.

2.4 Topological spaces

A family $(A_i)_{i \in I}$ of subsets of a set X satisfies the *finite intersection property* if for every nonempty finite subset F of I, the set $\cap_{i \in F} A_i$ is non-empty. A topological space X is *compact* if every family of closed subsets of X satisfying the finite intersection property has a nonempty intersection (the space X is not required to be Hausdorff). Given a subset A of a topological space X, we denote by $\overline{A}$ its topological closure.

2.5 Metric spaces

Notation (ball, diameter, radius) Given a metric space (X, d), for every $a \in X$, and every $r \in \mathbb{R}_+$, we denote by $\Gamma(a, r)$ the closed ball $\{x \in X : d(a, x) \leq r\}$, and by $B(a, r)$ the open ball $\{x \in X : d(a, x) < r\}$. For every subsets A, B of X, we denote by $d(A, B)$ the *distance between* A *and* B :

$$d(A, B) := \inf_{x \in A, y \in B} d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$$

For every subset $A \subseteq X$, the *diameter* of A is

$$diam(A) := \sup_{x, y \in A} d(x, y) \ \ \in\ \mathbb{R}_+ \cup \{+\infty\} \tag{4}$$

the *radius* of A is :

$$rad(A) := \inf\{r \in \mathbb{R}_+ : \exists a \in X\ A \subseteq \Gamma(a, r)\} \ \ \in\ \mathbb{R}_+ \cup \{+\infty\} \tag{5}$$

Notice that for every non-empty set A, $rad(A) \leq diam(A) \leq 2\ rad(A)$. In particular, A is not bounded if and only if $diam(A) = rad(A) = +\infty$. For every $\mathcal{A} \subseteq \mathcal{P}(X)$, let

$$\delta(\mathcal{A}) := \inf\{diam(A) : A \in \mathcal{A}\} \tag{6}$$

$$\rho(\mathcal{A}) := \inf\{rad(A) : A \in \mathcal{A}\} \tag{7}$$

Then, $\rho(\mathcal{A}) \leq \delta(\mathcal{A}) \leq 2\rho(\mathcal{A})$.

A filter $\mathcal{F}$ of subsets of the metric space X is a *Cauchy filter* if its diameter $\delta(\mathcal{F})$ is 0, or, equivalently, if $\rho(\mathcal{F})$ is 0. The metric space X is *complete* if for every Cauchy filter $\mathcal{F}$ on X, the set $\cap\{\overline{F} : F \in \mathcal{F}\}$ is nonempty (and in this case it is a singleton). A *Cauchy sequence* of the metric space X is a sequence $(x_n)_{n \in \mathbb{N}}$ of X such that the filter generated by $\{\{x_k : k \geq n\} : n \in \mathbb{N}\}$ is Cauchy.

Notation (ε-neighborhood) Given some real number $\varepsilon > 0$, and some subset A of the metric space (X, d), denote by A_ε the *ε-neighborhood* of A : $A_\varepsilon := \{x \in X : d(x, A) \leq \varepsilon\}$. Notice that A_ε is a closed subset of X.

A subset K of a metric space is *precompact* if for every real number $\varepsilon > 0$, there exists a finite subset $F \subseteq K$ such that K is contained in F_ε. Every compact metric space is precompact. Every subset of a precompact metric space is precompact. The convex hull of every finite subset of a normed space is compact, hence precompact.

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2.6 Continuous dual of a normed space

In this paper *all vector spaces are spaces over the field* $\mathbb{R}$ of real numbers.

Given a normed vector space $(E, \|.\|)$, we denote by Γ_E (*resp.* S_E) the closed unit ball $\{x \in E : \|x\| \leq 1\}$ (*resp. the unit sphere* $\{x \in E : \|x\| = 1\}$) of E. The normed space E is a *Banach space* if it is complete (*w.r.t.* Cauchy filters of the metric space E).

We denote by E' the *continuous dual* of E endowed with the dual norm : thus, E' is the vector space of continuous linear mappings $f : E \to \mathbb{R}$, endowed with the norm given by $\|f\| := \sup_{x\in\Gamma_E} |f(x)|$ for every $f \in E'$. The *second dual* of E is the continuous dual of E', and it is denoted by E''.

Notation (canonical mapping) For every $x \in E$, we denote by $\tilde{x} : E' \to \mathbb{R}$ the "evaluation mapping at point x", associating the number $f(x)$ to each $f \in E'$. We denote by j_E the *canonical mapping* from E to the second dual E'', associating to every $x \in E$ the evaluation mapping $\tilde{x}$.

Clearly, j_E is linear, continuous and $\|j_E\| \leq 1$. In $\mathbf{ZF}{+}\mathbf{HB}$, one proves that j_E is *isometric* : $\forall x \in E,\ \|\tilde{x}\| = \|x\|$.

Remark 2.2 In $\mathbf{ZF}{+}\neg\mathbf{HB}$, one can prove the existence of an infinite dimensional Banach space E such that $E' = \{0\}$ (see Lemma **5** p. 12 in [9] or Theorem **2** in [15]). For such a space E, $E'' = \{0\}$ so j_E is onto but j_E is not one-to-one. In [1], Section **4**, several other examples of pathologies occurring in functional analysis without the axiom of choice are provided.

Question 1 Given a normed space E, if the canonical mapping $j_E : E \to E''$ is isometrical, then j_E is one-to-one. Is the converse statement true ?

2.7 Weak topologies on a normed space

Given some normed vector space E, and some vector subspace $V \subseteq E'$, denote by $\sigma(E, V)$ the coarsest topology $\mathcal{T}$ on E such that every $f \in V$ is continuous from $(E, \mathcal{T})$ to $\mathbb{R}$. It is easy to prove that E endowed with this topology $\mathcal{T}$ is a linear topological vector space (*tvs*) : this means that the addition $+ : E \times E \to E$ and the scalar multiplication $. : \mathbb{R} \times E \to E$ are continuous. Moreover the *tvs* E is locally convex.

The *weak topology* on the normed space E is the topology $\sigma(E, E')$. Thus, the weak topology on E is generated by the (strict) hemi-spaces :

$$H^s_{f,\lambda} := \{x \in E : f(x) < \lambda\}, \quad f \in E',\ \lambda \in \mathbb{R}$$

Moreover, closed sets of $\sigma(E, E')$ are intersections of finite unions of (large) hemi-spaces of the following form :

$$H^l_{f,\lambda} := \{x \in E : f(x) \leq \lambda\}, \quad f \in E',\ \lambda \in \mathbb{R}$$

The *weak* topology* on the normed space E' is the topology $\sigma(E', j_E[E])$. The weak* topology on E' is also denoted by $\sigma(E', E)$. Thus, hemi-spaces of E' of the following type form a sub-basis of open subsets of $\sigma(E', E)$:

$$H^s_{\tilde{x},\lambda} = \{f \in E' : f(x) < \lambda\}, \quad x \in E,\ \lambda \in \mathbb{R}$$

Question 2 Let E be a normed space such that there exists a bounded convex subset with non-empty interior which is weakly closed. Then, clearly, singletons of E are weakly closed. Are closed balls of E weakly closed ?

2.8 Convex topology on a normed space

Definition 2.3 (convex lattice of a normed space) Given a normed space E, we denote by L_E the lattice generated by closed convex subsets of E. Notice that L_E is the set of finite unions of closed convex subsets of E : we call it the *convex lattice of* E.

The following Fact will be used in Lemma 3.4 :

Fact 1 *Let C be a convex subset of a normed space E, and $\varepsilon \in \mathbb{R}_+^*$. Then the ε-neighborhood C_ε of C is convex (whence $C_\varepsilon \in L_E$).*

P r o o f. Given two convex subsets A, B of E, then $A + B$ is convex. Now, $C_\varepsilon = \bigcap_{\varepsilon < s} \big(C + \Gamma(0, s)\big)$ whence C_ε is convex. □

Definition 2.4 (convex topology) The *convex topology* on a normed space E is the topology for which a subset of E is closed if and only if it is the intersection of a subset of the convex lattice of E.

Remark 2.5 (convex-topology *versus* weak topology) The convex topology on a normed space E is intermediate between the weak topology $\sigma(E, E')$ and the *strong topology* (associated to the norm of E). Also notice that closed balls of E, closed subspaces of E (for example finite dimensional subspaces of E) are closed for the convex topology. In $\mathbf{ZF}+\mathbf{HB}$, every closed convex subset of a normed space E is weakly closed, in which case the convex topology on E and the weak topology on E are equal. However, the weak topology on some infinite dimensional normed spaces E may be trivial in $\mathbf{ZF}+\neg\mathbf{HB}$: indeed, in such a theory, there exists an infinite dimensional Banach space E satisfying $E' = \{0\}$ (see Remark 2.2) ; such a space has only two weakly open subsets ($\varnothing$ and E), hence the weak topology is strictly coarser than the convex topology on E !

Notice that given a Banach space E, the convex topology and the weak topology may differ even when j_E is isometrical (see Remark 4.9, Section 4.4).

Remark 2.6 In a Hilbert space, the weak topology and the convex topology are equal in **ZF** (because in a Hilbert space, every closed convex subset is weakly closed, see [9]).

2.9 ϑ-sequences

James' criterion of "reflexivity" introduced in [12] and [13] is formulated in terms of ϑ-sequences.

2.9.1 ϑ-sequences and triangular sequences

Recall that (see Section 2.2) we denote by $\overline{\mathbb{N}}$ the set $\mathbb{N} \cup \{\mathbb{N}\}$.

Notation For every subset A of a vector space, we denote by $conv(A)$ the convex hull of A, and we denote by $span(A)$ the vector subspace linearly spanned by A.

Definition 2.7 (ϑ-sequence) Given some $\vartheta \in \mathbb{R}_+^*$, and some $N \in \overline{\mathbb{N}}$, a sequence $(a_k)_{k \in N}$ of the normed space E is a *ϑ-sequence* if for every integer $i < N$, the distance between $span\{a_k : k < i\}$ and $conv\{a_k : i \leq k < N\}$ is $> \vartheta$. Notice that the empty sequence $\varnothing$ is a *ϑ-sequence*.

Notice that if $\vartheta > 0$, every ϑ-sequence is linearly independent. This notion of ϑ-sequence can be reformulated using *triangular sequences* :

Definition 2.8 (triangular sequence) Let E be a normed space, $\vartheta \in]0, 1[$, and $N \in \overline{\mathbb{N}}$. Given a sequence $(a_k)_{k<N}$ of E, and a sequence $(f_k)_{k<N}$ of the closed unit ball of the continuous dual of $span\{a_k : k < N\}$, say that the sequence $(a_k, f_k)_{k<N}$ is *ϑ-triangular* when for every $k, l < N$, $f_k(a_l) = 0$ if $l < k$, and $f_k(a_l) > \vartheta$ if $k \leq l$.

Thus the sequence $(f_k)_{k<N}$ witnesses $(a_k)_{k<N}$ being a ϑ-sequence. Conversely indeed :

Proposition 2.9 *Let E be a normed space, $\vartheta \in]0, 1[$, $N \in \overline{\mathbb{N}}$, and σ be a sequence $(a_i)_{i<N}$ of E. Let $V := span\{a_i : i < N\}$. The following properties are equivalent :*

1. *The sequence σ is a ϑ-sequence of V ;*

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2. *There exists a sequence $(f_k)_{k<N}$ of $S_{V'}$ such that $(a_i, f_i)_{i<N}$ is ϑ-triangular.*

3. *There exists a sequence $(f_k)_{k<N}$ of $\Gamma_{V'}$ such that $(a_i, f_i)_{i<N}$ is ϑ-triangular.*

Proof. 1. $\Rightarrow$ 2. Assume that $(a_i)_{i<N}$ is a ϑ-sequence of V. Since V is separable, one can build in **ZF** some sequence $(f_k)_{k<N}$ of $S_{V'}$ satisfying $f_k = 0$ on $span\big(\{a_i : 0 \leq i < k\}\big)$ and $f_k > \vartheta$ on $conv(\{a_i : k \leq i < N\})$.

2. $\Rightarrow$ 3. Straightforward.

3. $\Rightarrow$ 1. Assume that there exists a sequence $(f_k)_{k<N}$ of $\Gamma_{V'}$ such that $(a_i, f_i)_{i<N}$ is a ϑ-triangular sequence of V. Given some integers k, n such that $k < n < N$, if $(\lambda_i)_{i<k} \in \mathbb{R}^k$, if $(\mu_i)_{k \leq i<n} \in {\mathbb{R}_+}^{n-k}$ with $\sum_{k \leq i<N} \mu_i = 1$, then $f_k(-\sum_{i<k} \lambda_i.a_i + \sum_{k\leq i<n} \mu_i.a_i) = f_k(\sum_{k\leq i<n} \mu_i.a_i) > \vartheta$ hence

$$\left\| \sum_{i<k} \lambda_i.a_i - \sum_{k\leq i<n} \mu_i.a_i \right\| > \vartheta$$

(because $\|f_k\| \leq 1$). □

Fact 2 (norming a ϑ-sequence) *Let $\varepsilon, \vartheta, \vartheta_\varepsilon > 0$ satisfying $\vartheta(1+\varepsilon) < \vartheta_\varepsilon < 1$. Let $n \in \mathbb{N}$, and $(a_i)_{i<n}$ be a ϑ_ε-sequence of the closed ball $\Gamma(0, 1+\varepsilon)$ of the normed space E. Then $(\frac{a_i}{\|a_i\|})_{i<n}$ is a ϑ-sequence of S_E.*

Proof. Indeed, let $V := span\{a_i : i < n\}$ and let $(f_i)_{i<n}$ be some sequence of $S_{V'}$ such that $(a_i, f_i)_{i<n}$ is ϑ_ε-triangular. Then, $(\frac{a_i}{\|a_i\|}, f_i)_{i<n}$ is ϑ-triangular because for every $i < n$, for every $(\lambda_t)_{t<i} \in \mathbb{R}^i$, for every real numbers $\lambda_i, \dots, \lambda_{n-1} \in \mathbb{R}_+$ satisfying $\sum_{i\leq t<n} \lambda_t = 1$, $f_i(\sum_{i\leq t<n} \lambda_t \frac{a_t}{\|a_t\|} - \sum_{t<i} \lambda_t \frac{a_t}{\|a_t\|}) = f_i(\sum_{i\leq t<n} \lambda_t \frac{a_t}{\|a_t\|}) \geq \sum_{i\leq t<n} \lambda_t \frac{\vartheta_\varepsilon}{\|a_t\|} \geq \sum_{i\leq t<n} \lambda_t \frac{\vartheta_\varepsilon}{1+\varepsilon} = \frac{\vartheta_\varepsilon}{1+\varepsilon} \geq \vartheta$. □

2.9.2 Associated sequences

"Associated sequences" of convex sets will be used to build ϑ-sequences, see Definition 3.5 and Lemma 3.7 :

Definition 2.10 (associated sequence) Given some real number $\vartheta > 0$, some $N \in \overline{\mathbb{N}}$, and some sequence $(a_i)_{i<N}$ of the normed space E, say that a $\subseteq$-descending sequence $(C_i)_{i<N}$ of closed convex sets is *ϑ-associated to* $(a_i)_{i<N}$ if for each $i < N$ the following two conditions are satisfied :

$$span_\vartheta(\{a_t : t < i\}) \cap C_i = \varnothing \tag{8}$$

$$a_i \in C_i \qquad (\text{whence } \overline{conv}(\{a_t : i \leq t\}) \subseteq C_i\ ) \tag{9}$$

Notice that given a (finite or infinite) sequence $(a_i)_{i<N}$ of a normed space E, the following three conditions are equivalent : "$(a_i)_{i<N}$ is a ϑ-sequence" ; "$\big(\overline{conv}(\{a_t : i \leq t\})\big)_{i<N}$ is ϑ-associated to the sequence $(a_i)_{i<N}$" ; "There exists a $\subseteq$-descending sequence of closed convex subsets of E which is ϑ-associated to $(a_i)_{i<N}$".

3 DC implies J2C

3.1 Stationary sets

Given a lattice L of subsets of some set X, a filter $\mathcal{F}$ of the lattice L, we denote by $\mathcal{S}(\mathcal{F})$ the set of *$\mathcal{F}$-stationary sets*, *i.e.* elements of the lattice L meeting every element of $\mathcal{F}$. Then, for every $A, B \in L$, the following easy conditions are fulfilled :

$$\mathcal{F} \subseteq \mathcal{S}(\mathcal{F}) \tag{10}$$

$$(A \in \mathcal{S}(\mathcal{F}) \text{ and } A \subseteq B) \Rightarrow B \in \mathcal{S}(\mathcal{F}) \tag{11}$$

Moreover,

$$A \cup B \in \mathcal{S}(\mathcal{F}) \Rightarrow (A \in \mathcal{S}(\mathcal{F}) \text{ or } B \in \mathcal{S}(\mathcal{F})) \tag{12}$$

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In particular, if $A \cup B = X$ then $A \in \mathcal{S}(\mathcal{F})$ or $B \in \mathcal{S}(\mathcal{F})$.

$$(A \in \mathcal{S}(\mathcal{F}) \text{ and } B \in \mathcal{F}) \Rightarrow A \cap B \in \mathcal{S}(\mathcal{F}) \tag{13}$$

Notice that the filter $\mathcal{F}$ of L is a *maximal* filter of the lattice L if and only if $\mathcal{S}(\mathcal{F}) = \mathcal{F}$.

3.2 Saturating filters *w.r.t.* numerical constraints

Given a lattice L of subsets of a set X, and a mapping $\phi : \mathcal{P}(L) \to [0, +\infty]$, define the mapping $\Phi : \mathcal{P}(\mathcal{P}(L)) \to [0, +\infty]$ by $\Phi(\mathcal{A}) = \inf\{\phi(A) : A \in \mathcal{A}\}$ and say that a filter $\mathcal{F}$ of L is *ϕ-saturated* if $\Phi(\mathcal{S}(\mathcal{F})) = \Phi(\mathcal{F})$. For example, if ϕ is the "radius" function rad (*resp.* "diameter" function $diam$) -see Section 2.5- then, Φ is the function ρ (*resp.* δ). Notice that for a given function ϕ, the function Φ is order reversing from $(\mathcal{P}(\mathcal{P}(L)), \subseteq)$ to $([0, +\infty], \leq)$. Besides, every maximal filter of L is trivially saturated *w.r.t.* any such ϕ.

Remark 3.1 Given some ϕ-saturated filter $\mathcal{G}$, every filter $\mathcal{H}$ containing $\mathcal{G}$ is also ϕ-saturated : indeed,

$$\Phi(\mathcal{S}(\mathcal{H})) \leq \Phi(\mathcal{H}) \leq \Phi(\mathcal{G}) = \Phi(\mathcal{S}(\mathcal{G})) \leq \Phi(\mathcal{S}(\mathcal{H}))$$

Lemma 3.2 *Let (X, d) be a metric space, let L be a lattice on X and let $\phi : \mathcal{P}(L) \to [0, +\infty[$.*

1. *In $\mathbf{ZF}+\mathbf{DC}$, every filter $\mathcal{F}$ of L is contained in a ϕ-saturated filter of L.*
2. *The same conclusion holds in $\mathbf{ZF}$ in the particular case of the "radius" function ρ, whenever L contains all closed balls of X and X has a dense well orderable subset.*

Proof. (1). For every $\varepsilon > 0$, for every filter $\mathcal{G}$ of L, there exists some $F \in \mathcal{S}(\mathcal{G})$ such that $\phi(F) \leq \Phi(\mathcal{S}(\mathcal{G})) + \varepsilon$, thus, the filter $\tilde{\mathcal{G}}$ generated by $\mathcal{G}$ and F satisfies $\Phi(\tilde{\mathcal{G}}) \leq \Phi(\mathcal{S}(\mathcal{G})) + \varepsilon$. Using $\mathbf{DC}$, define an infinite $\subseteq$-ascending sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of filters such that $\mathcal{F}_0 = \mathcal{F}$ and such that for each $n \in \mathbb{N}$,

$$\Phi(\mathcal{F}_{n+1}) \leq \Phi(\mathcal{S}(\mathcal{F}_n)) + \frac{1}{n+1} \tag{14}$$

Then, for every $n \in \mathbb{N}$, the filter $\mathcal{G} := \cup_{i \in \mathbb{N}} \mathcal{F}_i$ satisfies the following inequality :

$$\Phi(\mathcal{S}(\mathcal{F}_n)) \leq \Phi(\mathcal{S}(\mathcal{F}_{n+1})) \leq \Phi(\mathcal{S}(\mathcal{G})) \leq \Phi(\mathcal{G}) \leq \Phi(\mathcal{F}_{n+1}) \tag{15}$$

Using inequalities (14) and (15) for every $n \in \mathbb{N}$, it follows that $\Phi(\mathcal{S}(\mathcal{G})) = \Phi(\mathcal{G})$.

(2). By definition of ρ, for every $\varepsilon > 0$, for every filter $\mathcal{G}$ of L, there exists some $a \in X$ and some $r \in \mathbb{R}_+$ satisfying $\Gamma(a, r) \in \mathcal{S}(\mathcal{G})$ and $r \leq \rho(\mathcal{S}(\mathcal{G})) + \varepsilon$; thus $\Phi(\tilde{\mathcal{G}}) \leq \Phi(\mathcal{S}(\mathcal{G})) + \varepsilon$ where $\tilde{\mathcal{G}}$ is the filter generated by $\mathcal{G}$ and $\Gamma(a, r)$. Given some well-ordered dense subset D of X, the proof of (1) goes through in $\mathbf{ZF}$: define $\mathcal{F}_{n+1}$ as the filter generated by $\mathcal{F}_n$ and $\Gamma(a, r)$, where (a, r) is the first element of $D \times \mathbb{Q}$ satisfying $\Gamma(a, r) \in \mathcal{S}(\mathcal{F}_n)$ and $r \leq \rho(\mathcal{S}(\mathcal{F}_n)) + \frac{1}{n+1}$. □

Lemma 3.3 (Filters with positive radius) *Let (X, d) be a metric space, L be a lattice of closed subsets of X containing the closed balls of X, and $\mathcal{F}$ be a filter of L. For every real number r satisfying $0 \leq r < \rho(\mathcal{S}(\mathcal{F}))$, and every precompact subset K of X, $K_r \notin \mathcal{S}(\mathcal{F})$.*

Proof. If K is finite, then $K_r \notin \mathcal{S}(\mathcal{F})$ because of Property (12) for $\mathcal{F}$-stationary sets (see Section 3.1). In the general case, let r' be a real number such that $r < r' < \rho(\mathcal{S}(\mathcal{F}))$; using precompactness of K, let F be a finite subset of K such that $K \subseteq F_{r'-r}$; then $K_r \subseteq F_{r'}$. Using the first case, $F_{r'} \notin \mathcal{S}(\mathcal{F})$, so $K_r \notin \mathcal{S}(\mathcal{F})$. □

Lemma 3.4 (Filters with positive radius in normed spaces) *Let E be a normed space, and $\mathcal{F}$ be a filter of the convex lattice L_E containing some bounded set. Then, for every finite-dimensional vector subspace $V \subseteq E$, and every real number r satisfying $0 \leq r < \rho(\mathcal{S}(\mathcal{F}))$, $V_r \notin \mathcal{S}(\mathcal{F})$.*

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P r o o f. Use precompactness of every bounded subset of the finite dimensional space V. More precisely, let R be some real number such that $\Gamma(0,R) \in \mathcal{F}$. Observe that

$$V_r \cap \Gamma(0,R) \subseteq [V \cap \Gamma(0,R+r)]_r$$

Since $V \cap \Gamma(0,R+r)$ is precompact, Lemma 3.3 implies that $[V \cap \Gamma(0,R+r)]_r \notin \mathcal{S}(\mathcal{F})$. It follows that $V_r \cap \Gamma(0,R) \notin \mathcal{S}(\mathcal{F})$ whence $V_r \notin \mathcal{S}(\mathcal{F})$. □

Definition 3.5 (Tree of finite ϑ-sequences) Given a normed space E, some $\vartheta \in \mathbb{R}_+^*$ and some subset A of E, we denote by $T_{A,\vartheta}$ the set of finite ϑ-sequences of A ; thus $(T_{A,\vartheta}, \subseteq)$ is a subtree of $(A^{<\omega}, \subseteq)$.

Definition 3.6 (Subtree $(Z_{\mathcal{G},A,\vartheta}, \subseteq)$) Given some normed space E, some $\vartheta \in \mathbb{R}_+^*$, some subset A of E and some collection $\mathcal{G}$ of subsets of E, we denote by $Z_{\mathcal{G},A,\vartheta}$ the set of finite ϑ-sequences $(a_i)_{i<n}$ of A admitting an associated sequence $(C_i)_{i<n}$ such that for all $i < n$, $C_i \in \mathcal{G}$: notice that $(Z_{\mathcal{G},A,\vartheta}, \subseteq)$ is a subtree of $(T_{A,\vartheta}, \subseteq)$.

Lemma 3.7 *Let E be a Banach space. Assume that there exists a filter $\mathcal{G}$ of L_E, containing some closed bounded convex subset C, and satisfying $\rho(\mathcal{G}) = \rho(\mathcal{S}(\mathcal{G})) > 0$. Let $\vartheta \in]0, \rho(\mathcal{G})[$.*

1. *Every element of the tree $Z_{\mathcal{S}(\mathcal{G}),C,\vartheta}$ has a successor in this tree.*
2. *If D is a dense well-orderable subset of C, then every element of the tree $Z_{\mathcal{S}(\mathcal{G}),D,\vartheta}$ has a successor in this tree.*

P r o o f. 1. Let $\sigma = (a_i)_{i<n} \in Z_{\mathcal{S}(\mathcal{G}),C,\vartheta}$. Let $V := span\{a_i : i < n\}$ and let $(C_i)_{i<n}$ be some $\subseteq$-descending sequence of convex sets $\in \mathcal{S}(\mathcal{G})$ which is ϑ-associated to σ ; let $\mathcal{H}$ the filter generated by $\mathcal{G} \cup \{C_i : i < n\}$: notice that, using Remark 3.1, $\rho(S(\mathcal{H})) = \rho(\mathcal{H}) = \rho(\mathcal{G})$. Using Lemma 3.4, the closed convex set V_ϑ does not belong to $\mathcal{S}(\mathcal{H})$, whence there exists some $F \in \mathcal{H}$ such that $V_\vartheta \cap F = \varnothing$; since F is a finite union of closed convex sets, there exists also some convex set $K \in \mathcal{S}(\mathcal{H})$ satisfying $V_\vartheta \cap K = \varnothing$ (see Property (12) of $\mathcal{H}$-stationary sets). Now, using Property (13) of $\mathcal{H}$-stationary sets, $C_n := K \cap \bigcap_{i<n} C_i \cap C \in \mathcal{S}(\mathcal{H}) \subseteq \mathcal{S}(\mathcal{G})$: let $a_n \in C_n$. Then $(a_i)_{i \le n}$ is a successor of σ in $Z_{\mathcal{S}(\mathcal{G}),C,\vartheta}$ and $(C_i)_{i \le n}$ is ϑ-associated to $(a_i)_{i \le n}$.

2. The proof is similar : just use some real number $\eta > 0$ such that $V_{\vartheta+\eta}$ does not belong to $\mathcal{H}$; this yields some convex set $K \in \mathcal{S}(\mathcal{H})$ satisfying $V_{\vartheta+\eta} \cap K = \varnothing$ whence $V_\vartheta \cap K_\eta = \varnothing$; then $C_n := (K \cap \bigcap_{i<n} C_i)_\eta \cap C$ meets D and $V_\vartheta \cap C_n = \varnothing$. □

Remark 3.8 Given some $\vartheta \in]0,1[$, a sequence $\sigma \in T_{S_E,\vartheta}$ may fail to admit any successor in $T_{S_E,\vartheta}$: for example if the norm $\|.\|$ is *locally uniformly rotund* at point $a \in S_E$ (*cf.* [8] p. 42), namely if $\forall \varepsilon > 0\ \exists \eta_\varepsilon > 0\ \forall x \in E\ \left((|\left\|\frac{x+a}{2}\right\| - 1| < \eta_\varepsilon \text{ and } |\,\|x\| - 1| < \eta_\varepsilon) \Rightarrow (\|x-a\| < \varepsilon)\right)$, then, given some $\varepsilon \in]0,1[$, for every $\vartheta \in]\varepsilon \vee (1-\eta_\varepsilon), 1[$, the 1-sequence (a) does not have any successor in $T_{S_E,\vartheta}$; indeed, for $x \in S_E$, $(\|x-a\| \ge \vartheta > \varepsilon) \Rightarrow 1 - \left\|\frac{x+a}{2}\right\| \ge \eta \Rightarrow \left\|\frac{x+a}{2} - 0\right\| \le 1 - \eta = \vartheta$, thus $d(conv\{x,a\}, span\{\varnothing\}) \le \vartheta$ so that (a,x) cannot be a ϑ-sequence.

Theorem 3.9 (DC implies J2C) *Let E be a Banach space. Assume that C is a closed bounded convex subset of E which is not compact in the convex topology.*

1. *In $\mathbf{ZF}+\mathbf{DC}$, there exists some $\vartheta_0 > 0$ and some ϑ_0-sequence in C. Moreover, if $\Gamma(0,1)$ is not compact in the convex topology, then, for every $\vartheta \in]0,1[$, the unit sphere S_E contains a ϑ-sequence.*
2. *If C has a dense well-orderable subset D, then there exists some $\vartheta_0 > 0$, and some ϑ_0-sequence in D, which is definable from E, ϑ_0, D and any well-ordering on D. Moreover, if $\Gamma(0,1)$ is not compact in the convex topology, then for every $\vartheta \in]0,1[$, every dense well-orderable subset D of S_E contains a ϑ-sequence, which is definable from E, ϑ, D and any well-ordering on D.*

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P r o o f. (1) Let $\mathcal{F}$ be a filter of the lattice L_E containing C, such that $\cap\mathcal{F} = \varnothing$. By Lemma 3.2.1, let $\mathcal{G}$ be some filter of L_E containing $\mathcal{F}$ and satisfying $\rho(\mathcal{G}) = \rho(\mathcal{S}(\mathcal{G}))$: then $\cap\mathcal{G} \subseteq \cap\mathcal{F} = \varnothing$. Since $\mathcal{G}$ contains the bounded set C, $\rho(\mathcal{G}) < +\infty$. Since E is a Banach space, $\mathcal{G}$ is not a Cauchy filter, whence $0 < \rho(\mathcal{G})$. Let $\vartheta_0 \in]0, \rho(\mathcal{G})[$. Using Lemma 3.7.1, every element of the tree $Z_{\mathcal{S}(\mathcal{G}),C,\vartheta_0}$ has a successor in this tree ; using $\mathbf{DC}$, there exists an infinite increasing sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $(Z_{\mathcal{S}(\mathcal{G}),C,\vartheta_0}, \subsetneq)$; then $(a_n)_{n\in\mathbb{N}} := \cup_{n\in\mathbb{N}}\sigma_n$ is an infinite ϑ_0-sequence of C.

Now, assume that $\Gamma(0,1)$, and then, closed balls of E with radius > 0 fail to be compact in the convex topology. Let $\vartheta \in]0,1[$. Let $\varepsilon > 0$ satisfying $\vartheta(1+\varepsilon) < 1$ and let $\vartheta_\varepsilon \in]\vartheta(1+\varepsilon), 1[$. Applying $\mathbf{DC}$, build some filter $\mathcal{G}$ of L_E containing some closed ball and satisfying both relations $\rho(\mathcal{G}) = \rho(\mathcal{S}(\mathcal{G}))$ and $\cap\mathcal{G} = \varnothing$. Up to an homothetie, one may assume that $\rho(\mathcal{G}) = 1$. Up to a translation, one may assume that $\Gamma(0, 1+\varepsilon) \in \mathcal{G}$. Using $\mathbf{DC}$ and Lemma 3.7.1, this yields an infinite ϑ_ε-sequence $(a_n)_{n\in\mathbb{N}}$ of $\Gamma(0, 1+\varepsilon)$. Using Fact 2, $(\frac{a_n}{\|a_n\|})_{n\in\mathbb{N}}$ is a ϑ-sequence of S_E.

(2) The proof is similar to (1), using Lemma 3.2-(2) instead of Lemma 3.2-(1), and Lemma 3.7.2 instead of Lemma 3.7.1. □

Remark 3.10 A shorter proof of $\mathbf{J2C}$ in $\mathbf{ZFC}$ can be obtained by using a *maximal* filter $\mathcal{F}$ of L_E satisfying $\cap\mathcal{F} = \varnothing$. Notice that the existence of a maximal filter containing a proper filter in an arbitrary lattice of sets is equivalent to $\mathbf{AC}$ (see [3]).

4 Various notions of reflexivity

4.1 Various notions of compactness

Given a closed convex subset C of a Banach space, say that C is ω*-compact* if every $\subseteq$-descending sequence $(C_n)_{n\in\mathbb{N}}$ of non-empty closed *convex* subsets of C has a non-empty intersection.

Proposition 4.1 *Let C be some closed bounded convex subset C of a Banach space E. Consider the following notions of compactness for C :*

a) *C is compact in the convex topology.*

b) *C is ω-compact.*

c) *Every closed convex subset of C with a dense well-orderable subset is ω-compact.*

d) *Every separable closed convex subset of C is ω-compact.*

e) *For every sequence $(x_n)_{n\in\mathbb{N}}$ of C, $C \cap \bigcap_{n\in\mathbb{N}} \overline{conv}\{x_k : k \geq n\}$ is non-empty.*

f) *For every sequence $(x_n)_{n\in\mathbb{N}}$ of C,*

$$\inf_{n\in\mathbb{N}} d\big(conv\{x_k : k < n\}, conv\{x_k : k \geq n\}\big) = 0$$

g) *For every sequence $(x_n)_{n\in\mathbb{N}}$ of C,*

$$\inf_{n\in\mathbb{N}} d\big(span\{x_k : k < n\}, conv\{x_k : k \geq n\}\big) = 0$$

1. *In $\mathbf{ZF}+\mathbf{DC}$, $g) \Rightarrow a)$.*
2. *In $\mathbf{ZF}$, if C has a dense well-orderable subset, then $g) \Rightarrow a)$.*
3. *In $\mathbf{ZF}$, $a) \Rightarrow b) \Rightarrow c) \Leftrightarrow d) \Leftrightarrow e) \Leftrightarrow f) \Leftrightarrow g)$.*

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P r o o f. 1. and 2. : Use Theorem 3.9 in Section 3.

3. We first prove e) $\Rightarrow$ f). The five other direct implications are straightforward. Now assume that for every sequence $(x_n)_{n\in\mathbb{N}}$ of C, the set $C\cap\bigcap_{n\in\mathbb{N}}\overline{conv}\{x_k : k\geq n\}$ is non-empty. Given a sequence $(x_n)_{n\in\mathbb{N}}$ of C, there exists some $l\in C\cap\bigcap_{n\in\mathbb{N}}\overline{conv}\{x_k : k\geq n\}$. Then, $l\in\overline{\cup_{n\in\mathbb{N}^*}conv\{x_k : k<n\}}\cap\bigcap_{n\in\mathbb{N}^*}\overline{conv}\{x_k : k\geq n\}$. It follows that $\inf_{n\in IN} d\big(conv\{x_k : k<n\}, conv\{x_k : k\geq n\}\big)=0$.

The proof of $g)\Rightarrow c)$ follows from 2. □

Definition 4.2 (J-compactness) Say that a closed convex subset C of a normed space is *J-compact* (James-compact) if for every $\vartheta>0$, C does not contain any ϑ-sequence.

Thus, if C is some closed convex subset of a Banach space E, each Property c), d), e), f) and g) of Proposition 4.1 is equivalent in **ZF** to J-compactness of C.

Remark 4.3 (index of J-compactness) In **ZF**, every well-founded binary relation R on a set E has a *rank* : there exists a smallest ordinal $\alpha=rk(R)$ for which there is a (unique) mapping $f:E\to\alpha$ satisfying for all $x,y\in E$, (xRy and $x\neq y$) $\Rightarrow f(x)<f(y)$. Given a Banach space E and some J-compact closed bounded convex subset C of E, define the *index of J-compactness* of C as

$$ind_J(C) := \sup\{rk(T_{C,\vartheta},\supsetneq) : \vartheta>0\} = \{rk(T_{C,\frac{1}{n}},\supsetneq) : n\in\mathbb{N}^*\}$$

Given some *analytic* well founded binary relation R on a complete metric space X having a dense well orderable subset D of cardinal α, the rank of the well-founded relation R is some ordinal $<\alpha^+$, and there is a one-to-one mapping from $rk(R)$ to α, which is definable from E, D and any bijection between D and α (see [7] where no choice is needed in the proof). In particular, given a Banach space E and some closed bounded convex subset C of E which is J-compact, if C has a dense well-orderable subset D with cardinal α, then for every $\vartheta>0$, the rank r_ϑ of the (analytic) well-founded binary relation $(T_{C,\vartheta},\supsetneq)$ is some ordinal $<\alpha^+$, and there is a one-to-one mapping from r_ϑ to α, which is definable from E, D, ϑ and any bijection between D and α. Let δ be the *density of* the topological space C, *i.e.* the first ordinal equipotent with some dense subset of C. Now given a sequence $(r_n)_{n\in\mathbb{N}}$ of ordinals $<\delta^+$, *if there exists a sequence* $(j_n)_{n\in\mathbb{N}}$ *such that each* j_n *is a one-to-one mapping from* r_n *into* δ, then $\sup_{n\in\mathbb{N}} r_n<\delta^+$. It follows that $ind_J(C)<\delta^+$. In particular, given a *separable* closed bounded convex subset C of a Banach space which is J-compact, then $ind_J(C)$ is a countable ordinal. [There exist models of **ZF** containing a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of countable ordinals with least upper bound the first uncountable ordinal ω_1.]

4.2 J-reflexivity

Lemma 4.4 *Given a Banach space E, the following properties are equivalent.*

1. *For every closed subspace V of E with a dense well-orderable subset, the closed unit ball of V is compact in the convex topology.*
2. *For every separable closed subspace V of E, the closed unit ball of V is ω-compact.*
3. *For every sequence $(x_n)_{n\in\mathbb{N}}$ of Γ_E, $\bigcap_{n\in\mathbb{N}}\overline{conv}(\{x_k : k\geq n\})$ is non-empty.*
4. *For every sequence $(x_n)_{n\in\mathbb{N}}$ of Γ_E,*
$$\inf_{n\in\mathbb{N}} d\big(conv(\{x_k : k<n\}), (conv(\{x_k : k\geq n\})\big)=0$$
5. *There exists $\vartheta\in]0,1[$ such that for every sequence $(x_n)_{n\in\mathbb{N}}$ of S_E,*
$$\inf_{n\in\mathbb{N}} d\big(span\{x_k : k<n\}, conv\{x_k : k\geq n\}\big)<\vartheta$$

P r o o f. Each property 1., 2., 3., 4., is equivalent to J-compactness of Γ_E (see Proposition 4.1). Finally, 5. $\Rightarrow$ 1 follows from Theorem 3.9.2. □

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According to our definition of J-reflexivity at the beginning of our paper, say that a Banach space E is *J-reflexive* if E satisfies one of the above equivalent properties. Recall that J-reflexivity has a countable character : a Banach space E is J-reflexive if and only if every closed separable subspace of E is J-reflexive. The *index of J-reflexivity* of a J-reflexive Banach space is the index of J-compactness of its closed unit ball.

Remark 4.5 Given some sequence $(x_n)_{n\in\mathbb{N}}$ of a vector space E, say that some sequence $(b_n)_{n\in\mathbb{N}}$ of E is a *block-sequence* of $(x_n)_{n\in\mathbb{N}}$ if there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ and a sequence $(\lambda_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that

$$\forall k \in \mathbb{N} \quad b_k = \sum_{n_k \leq i < n_{k+1}} \lambda_i x_i$$

Moreover, if for every $k \in \mathbb{N}$, $0 \leq \lambda_k$ and $\sum_{n_k \leq i < n_{k+1}} \lambda_i = 1$, then say that $(b_n)_{n\in\mathbb{N}}$ is a *convex block-sequence* of $(x_n)_{n\in\mathbb{N}}$. Now, given a Banach space E, J-reflexivity of E (see Property 3. in Lemma 4.4) can be formulated as follows : "Every bounded sequence of E admits a convex block-sequence which (strongly) converges."

4.3 Various distinct notions of reflexivity

Definition 4.6 (various notions of reflexivity) Say that a Banach space E is :

- *convex-reflexive* if the closed unit ball Γ_E of E is compact in the convex-topology ;
- *ω-reflexive* if Γ_E is ω-compact ;
- *simply-reflexive* if the canonical mapping $j_E : E \to E''$ is isometric and onto ;
- *onto-reflexive* if the canonical mapping j_E is onto.

We will now compare in **ZF** these various notions of "reflexivity", see Figure 1. Notice that non-implications in Figure 1 follow from Proposition 4.11.

Proposition 4.7 *Let E be a Banach space.*

1. *In* $\mathbf{ZF}$*, "E is convex-reflexive'" $\Rightarrow$ "E is ω-reflexive" $\Rightarrow$ "E is J-reflexive".*
2. *In* $\mathbf{ZF}+\mathbf{DC}$*, if E is J-reflexive then E is convex-reflexive.*
3. *In* $\mathbf{ZF}$*, if E has a dense well-orderable subset and if E is J-reflexive, then E is convex-reflexive.*
4. *Convex-reflexivity $\Rightarrow$ onto-reflexivity.*
5. *Simple-reflexivity $\Rightarrow$ onto-reflexivity.*

P r o o f. 1., 2. and 3. : use Proposition 4.1.

4. Assume that the Banach space E is convex-reflexive *i.e.* that Γ_E is compact in the convex topology ; *a fortiori*, the ball Γ_E is $\sigma(E, E')$-compact. Endow the second dual E'' with the weak* topology $\sigma(E'', E')$. Since the canonical mapping $j_E : (E, \sigma(E, E')) \to (E'', \sigma(E'', E'))$ is continuous, $j_E[\Gamma_E]$ is compact in E'' ; it follows that $j_E[\Gamma_E]$ is closed in the Hausdorff space E''. Besides, according to Goldstine's lemma (which is provable in **ZF**, *cf.* [4]), the subset $j_E[\Gamma_E]$ is dense in $\Gamma_{E''}$. It follows that $j_E[\Gamma_E] = \Gamma_{E''}$ whence E is onto-reflexive.

5. is trivial. □

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4.4 Some classical normed spaces

Given a set I, we denote by $\mathcal{P}_f(I)$ the set of finite subsets of I. Also recall that $\ell^0(I)$ is the following normed vector space endowed by the "sup" norm :

$$\ell^0(I) := \{(x_i)_{i\in I} : \forall\varepsilon > 0\ \exists F \in \mathcal{P}_f(I)\ \forall i \in I\backslash F\ |x_i| \leq \varepsilon\}$$

The continuous dual of $\ell^0(I)$ is (isometrically isomorphic with) the following normed vector space endowed with the "sum" norm :

$$\ell^1(I) := \{(x_i)_{i\in I} : \sum_{i\in I} |x_i| < +\infty\}$$

Moreover, the continuous dual of $\ell^1(I)$ is the following space endowed with the "sup" norm :

$$\ell^\infty(I) := \{(x_i)_{i\in I} : \sup_{i\in I} |x_i| < +\infty\}$$

For every $p \in]1, +\infty[$, the vector space

$$\ell^p(I) := \{(x_i)_{i\in I} : \sum_{i\in I} |x_i|^p < +\infty\}$$

is endowed with the N_p norm : $N_p((x_i)_{i\in I}) = \left(\sum_{i\in I} |x_i|^p\right)^{1/p}$.

Denote by $\mathbb{R}^{(I)}$ the vector space of all mappings $f \in \mathbb{R}^I$ such that the set $\{i \in I : f(i) \neq 0)\}$ is finite. Clearly $\mathbb{R}^{(I)}$ is dense in each space $\ell^p(I)$ for $p \in \{0\} \cup [1, +\infty[$. It follows that the Banach space $\ell^0(I)$ is the completion of the normed space $\mathbb{R}^{(I)}$ endowed with the "sup" norm and for every $p \in [1, +\infty[$, the Banach space $\ell^p(I)$ is the completion of the normed space $\mathbb{R}^{(I)}$ endowed with the N_p norm.

If I is finite, then, the continuous dual of $\ell^\infty(I)$ is (isometrically) isomorphic with $\ell^1(I)$. Using $\mathbf{HB}$, for every infinite set I the continuous dual of $\ell^\infty(I)$ strictly contains $\ell^1(I)$ whence $\ell^1(I)$ is not onto-reflexive in $\mathbf{ZF}+\mathbf{HB}$. However :

Proposition 4.8 *There are models of* $\mathbf{ZF}+\mathbf{DC}$ *where the continuous dual of* $\ell^\infty(\mathbb{N})$ *is* $\ell^1(\mathbb{N})$.

P r o o f. According to Pincus and Solovay (see [18], "Discussion" p. 187), there exists a model of $\mathbf{ZF}+\mathbf{DC}$ where every measure on $\mathbb{N}$ is trivial. Here, a *measure* on a set I is a mapping m associating to every subset A of I a real number $m(A) \in \mathbb{R}_+$ and satisfying for every disjoint subsets A, B of I the equality $m(A \cup B) = m(A) + m(B)$. In particular $m(\varnothing) = 0$. Clearly, for every $a \in I$, the "Dirac mapping" δ_a associating to every subset A of I the real number 1 if $a \in A$ and 0 if $a \notin A$ is a measure on I. More generally, if J is a subset of I and if $(\lambda_i)_{i\in J}$ is a family of $[0, 1]$ such that $\sum_{i\in J} \lambda_i = 1$ then $\sum_{i\in J} \lambda_i\delta_{a_i}$ is a measure on I which is said to be *discrete* or "trivial". Notice that every measure on a finite set is trivial. We now show that in a model of $\mathbf{ZF}$ where every measure on $\mathbb{N}$ is trivial, the continuous dual of $\ell^\infty(\mathbb{N})$ is $\ell^1(\mathbb{N})$. Denote by $(e_n)_{n\in\mathbb{N}}$ the canonical basis of $\mathbb{R}^{(\mathbb{N})}$: for each $n \in \mathbb{N}$, the mapping e_n is defined by $e_n(t) = 1$ if $t = n$ and $e_n(t) = 0$ if $t \neq n$. Say that a linear mapping $\Phi : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ is *positive* if for every $x \in \ell^\infty(\mathbb{N})$, $x \geq 0 \Rightarrow \Phi(x) \geq 0$. Notice that for every continuous linear mapping $\Phi : \ell^\infty(\mathbb{N}) \to \mathbb{R}$, there exist positive linear mappings $\Phi^+, \Phi^- : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ such that $\Phi = \Phi^+ - \Phi^-$ (see [5]). We are to show that every *positive* linear mapping $\Phi : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ is in $\ell^1(\mathbb{N})$. For every $n \in \mathbb{N}$, let $\lambda_n := \Phi(e_n)$. Let us show that $\Lambda := (\lambda_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and that for every $a \in \ell^\infty(\mathbb{N})$, $\Phi(a) = j_E(\Lambda)(a)$. First, $\sum_{n\in\mathbb{N}} |\lambda_n| = \sum_{n\in\mathbb{N}} \lambda_n = \sup_{A\in\mathcal{P}_f(\mathbb{N})} \Phi(1_A) \leq \Phi(1_\mathbb{N}) \leq \|\Phi\| < +\infty$. Moreover, for every positive element $a = (a_i)_{i\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$, $j_E(\Lambda)(a) = \sum_{i\in\mathbb{N}} \lambda_i a_i = \sup_{A\in\mathcal{P}_f(\mathbb{N})} \sum_{i\in A} \lambda_i a_i = \sup_{A\in\mathcal{P}_f(\mathbb{N})} \Phi(\sum_{i\in A} a_i) \leq \Phi(a)$ (because $\sum_{i\in A} a_i \leq a$) whence $j_E(\Lambda)(a) \leq \Phi(a)$. It follows that the linear mapping $d := \Phi - j_E(\Lambda)$ is *positive*. Let m be the measure on $\mathbb{N}$ which is associated to d *via* the formula $m(A) = d(1_A)$ for every subset A of $\mathbb{N}$. For every $n \in \mathbb{N}$, $m(\{n\}) = u(e_n) - \lambda_n = 0$. Since every measure on $\mathbb{N}$ is trivial, it follows that the measure m is null. In particular $m(\mathbb{N}) = 0$ whence $d(1_\mathbb{N}) = 0$. It follows that the positive linear mapping d is null. □

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Remark 4.9 Given a Banach space E such that the canonical mapping $j_E : E \to E''$ is isometric (whence the topology $\sigma(E, E')$ is Hausdorff), then, it is not provable in $\mathbf{ZF}$ that the convex topology and the weak topology $\sigma(E, E')$ are equal. For example, consider a model of $\mathbf{ZF}$ where the continuous dual of the space $E = \ell^\infty(\mathbb{N})$ is $\ell^1(\mathbb{N})$ (see Proposition 4.8). Then the subspace $\ell^0(\mathbb{N})$ is a closed convex subset of $\ell^\infty(\mathbb{N})$, but $\ell^0(\mathbb{N})$ is not closed in the weak topology $\sigma(E, E')$: in fact, using Goldstine's Lemma (*cf.* [4]), $\ell^0(\mathbb{N})$ is dense in $\ell^\infty(\mathbb{N})$ for the *-weak topology $\sigma(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N})) = \sigma(E, E')$.

4.5 Counter-examples about "reflexivity" of spaces

Lemma 4.10 *Let I be an infinite set.*

1. (a) *The space $\ell^0(I)$ is not onto-reflexive.*
   (b) *The closed unit ball of $\ell^1(I)$ is not compact in the weak topology (in particular, $\ell^1(I)$ is not convex-reflexive).*
   (c) *If there exists an onto-mapping $f : I \to \mathbb{N}$, then $\ell^1(I)$ is not ω-reflexive. Moreover, if there is a finite-to-one such f (*i.e.* if for every $n \in \mathbb{N}$ the set $f^{-1}(\{n\})$ is finite) then $\ell^0(I)$ is not ω-reflexive. In particular, $\ell^0(\mathbb{N})$ is not ω-reflexive.*

2. *If I is Dedekind-finite, and if $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$ restricted to finite non-empty subsets of I holds, then, the unit sphere of $\ell^0(I)$ does not contain any infinite linearly independent sequence $(a_n)_{n\in\mathbb{N}}$; in particular, for every real number $p \in \{0\} \cup [1, +\infty[$, the space $\ell^p(I)$ is J-reflexive.*

P r o o f. 1a. Since I is infinite, $1_I \in \ell^\infty(I) \backslash \ell^0(I)$, whence $\ell^0(I)$ is not onto-reflexive.

1b. For every $i \in I$, let $K_i := \{f \in \Gamma_{\ell^1(I)} : \sum_{k\in I} f(k) = 1 \text{ and } f(i) = 0\}$. Then the family of (weakly) closed convex sets $(K_i)_{i\in I}$ of $\Gamma_{\ell^1(I)}$ satisfies the finite intersection property, but the set $\cap_{i\in I} K_i$ is empty.

1c. Likewise, given an onto mapping $f : I \to \mathbb{N}$, for each $n \in \mathbb{N}$, denote by I_n the set $f^{-1}(\{n\}) := \{i \in I : f(i) = n\}$, and let $K_n := \{f \in \Gamma_{\ell^1(I)} : \sum_{k\in I} f(k) = 1 \text{ and } (\forall i \in I_n, f(i) = 0)\}$; then the $\subseteq$-descending sequence $(K_n)_{n\in\mathbb{N}}$ of nonempty closed convex subsets of $\Gamma_{\ell^1(I)}$ has an empty intersection. Now, if f is finite-to-one, each set $L_n := \{f \in \Gamma_{\ell^0(I)} : \forall i \in \bigcup_{k\le n} I_k \ f(i) = 1\}$ is a closed convex subset of $\Gamma_{\ell^0(I)}$, and the sequence $(L_n)_{n\in\mathbb{N}}$ is a $\subseteq$-descending. Since $\bigcap_{n\in\mathbb{N}} L_n = \varnothing$, the space $\ell^0(I)$ is not ω-compact.

2. Recall that in the "basic Cohen model", (see Section 2.1.2), there is an infinite Dedekind-finite set such that $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$ restricted to finite non-empty subsets of I holds. Given such a set I and some $a = (a_i)_{i\in I} \in \ell^0(I)$, the *support* $s(a) := \{i \in I : a_i \neq 0\}$ of a is a countable union of finite sets : in fact, $s(a) = \cup_{k\in\mathbb{N}} F_k(a)$ where $F_k(a) := \{i \in I : |a(i)| > \frac{1}{k+1}\}$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of the unit sphere of $\ell^0(I)$. Then $S := \cup_{n\in\mathbb{N}} \ s(a_n) = \cup_{n\in\mathbb{N}} \cup_{k\in\mathbb{N}} F_k(a_n)$ whence S is finite. Now $span\{a_n : n \in \mathbb{N}\} \subseteq \ell^0(S)$ whence $span\{a_n : n \in \mathbb{N}\}$ is finite-dimensional. □

Proposition 4.11 1. *In $\mathbf{ZF}$, J-reflexivity does not imply ω-reflexivity. J-reflexivity does not imply onto-reflexivity.*

2. *In $\mathbf{ZF}$, ω-reflexivity does not imply convex-reflexivity.*

3. *In $\mathbf{ZF}+\mathbf{DC}$, simple-reflexivity does not imply J-reflexivity. In particular, in $\mathbf{ZF}+\mathbf{DC}$, onto-reflexivity does not imply convex-reflexivity.*

P r o o f. 1. Consider a model of $\mathbf{ZF}$ where there exists some infinite, Dedekind-finite set I satisfying $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$ and admitting an onto mapping $f : I \to \mathbb{N}$, for example the basic Cohen model (see Section 2.1.2). Using Lemma 4.10.2, the space $\ell^1(I)$ is J-reflexive but using Lemma 4.10.1c, $\ell^1(I)$ is not ω-reflexive. Moreover, the space $\ell^0(I)$ is J-reflexive (Lemma 4.10.2) but $\ell^0(I)$ is not onto-reflexive (Lemma 4.10.1a).

2. Consider some model of $\mathbf{ZF}+\neg\mathbf{AC}(\mathbb{N}, \mathbf{fin})$, for example the second Cohen model (see Section 2.1.3). Then, in such a model, there exists, (see [9]-Theorem **9**-), a Hilbert space H with a $\subseteq$-descending sequence $(F_n)_{n\in\mathbb{N}}$ of nonempty weakly closed subsets of the closed unit ball Γ_H of H satisfying $\cap_{n\in\mathbb{N}} F_n = \varnothing$. However, the space H is ω-reflexive and even more : every family of closed *convex* subsets of Γ_H satisfying the finite intersection

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property has a non-empty intersection (see [9]-Corollary **5** -).
3. The space $\ell^1(\mathbb{N})$ is not J-reflexive since the canonical basis of $\ell^1(\mathbb{N})$ is a 1-sequence. However, there is a model of $\mathbf{ZF}+\mathbf{DC}$ where the continuous dual of $\ell^\infty(\mathbb{N})$ is $\ell^1(\mathbb{N})$ (for example in Pincus and Solovay's model, see Proposition 4.8). In such a model, the space $\ell^1(\mathbb{N})$ is simply reflexive. □

5 Weaker axioms than $\mathbf{J2C}$

Recall that given a normed space $(E, \|.\|)$, the mapping

$$\delta_E : \varepsilon \mapsto \inf \{1 - \left\| \frac{x+y}{2} \right\| : x \in \Gamma_E,\ y \in \Gamma_E,\ \|x-y\| \geq \varepsilon\}$$

is called the *modulus of convexity* of E. Notice that $\delta_E :]0,2] \to \mathbb{R}_+$ is continuous and order-preserving. The space E is *uniformly convex* if and only if $\forall \varepsilon > 0,\ \delta_E(\varepsilon) > 0$. In other words, E is uniformly convex if there exists a mapping $\delta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that, for every real number $\varepsilon > 0$, for every $x, y \in \Gamma_E$,

$$\|x-y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon) \tag{16}$$

Any mapping $\delta :]0,2] \to \mathbb{R}_+^*$ satisfying (16) is called a *witness of uniform convexity for* E, and then, δ_E is the best witness of uniform convexity. The following Lemmas are well-known :

Lemma 5.1 *Every Hilbert space is uniformly convex, with modulus of convexity* $\varepsilon \mapsto 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$.

P r o o f. See [2] p. 189-190. □

Lemma 5.2 *Every uniformly convex Banach space is J-reflexive. More precisely, let $(E, \|.\|)$ be a uniformly convex normed space, with a continuous witness of uniform convexity δ. Then, there exists a real number $\vartheta \in]0,1[$ such that $1 \leq \vartheta + \delta(\vartheta)$. For such a ϑ, the space E does not have any ϑ-triangular sequence of length 2.*

P r o o f. The function $g : \varepsilon \mapsto \varepsilon + \delta(\varepsilon)$ is continuous and $g(1) > 1$, hence there exists $\vartheta \in]0,1[$ such that $1 \leq g(\vartheta)$. If $((a_1, a_2), (f_1, f_2))$ is a ϑ-triangular sequence, then $\left\| \frac{a_1+a_2}{2} \right\| \geq \vartheta \geq 1 - \delta(\vartheta)$ and $\|a_1 - a_2\| \geq \vartheta$: this is contradictory since by definition of δ, $\|a_1 - a_2\| \geq \vartheta \Rightarrow \left\| \frac{a_1+a_2}{2} \right\| < 1 - \delta(\vartheta)$. □

Now, consider the two following statements :

$\mathbf{A1}$: *Hilbert spaces are convex-reflexive.*

$\mathbf{A2}$: *Uniformly convex Banach spaces are convex-reflexive.*

$\mathbf{RCuc}$: *"The closed unit ball of a uniformly convex Banach space is weakly compact".*

The implications $\mathbf{DC} \Rightarrow \mathbf{RCuc}$ and $\mathbf{DC} \Rightarrow \mathbf{A1}$ were obtained in [6].

Theorem 5.3 1. $\mathbf{J2C} \Rightarrow \mathbf{A2} \Rightarrow \mathbf{RCuc} \Rightarrow \mathbf{A1} \Rightarrow \mathbf{AC}(\mathbb{N}, \mathbf{fin})$.

2. $\mathbf{J2C} \Rightarrow \mathbf{D}$.

P r o o f. 1. The statement $\mathbf{J2C} \Rightarrow \mathbf{A2}$ follows from Lemma 5.2. The implication $\mathbf{A2} \Rightarrow \mathbf{RCuc}$ is straightforward since the weak topology on a normed space is contained in the convex topology. The implication $\mathbf{RCuc} \Rightarrow \mathbf{A1}$ follows from Lemma 5.1 and the fact that in a Hilbert space, the weak topology and the convex topology are equal. Finally $\mathbf{A1} \Rightarrow \mathbf{AC}(\mathbb{N}, \mathbf{fin})$ has been proved in [9].
2. Let I be an infinite set. Then, using Lemma 4.10.1b, $\ell^1(I)$ is not convex-reflexive. Let $\vartheta \in]0,1[$. Using $\mathbf{J2C}$, the closed unit sphere of $\ell^1(I)$ has an infinite ϑ-sequence $(a_n)_{n\in\mathbb{N}}$. Such a sequence is linearly independent. Using Lemma 4.10.2 and the consequence $\mathbf{AC}(\mathbb{N}, \mathbf{fin})$ of $\mathbf{J2C}$, it follows that I is Dedekind-infinite. □

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Remark 5.4 In particular, $\mathbf{J2C}$ is not provable in $\mathbf{ZF}+\mathbf{BPI}$ since in the basic Cohen model, $\mathbf{BPI}$ holds whereas $\mathbf{D}$ fails.

Question 3 Does the statement $\mathbf{J2C}$ imply $\mathbf{DC}$?

Question 4 Does $\mathbf{J2C}$ imply $\mathbf{AC}(\mathbb{N})$?

Question 5 Does $\mathbf{AC}(\mathbb{N})$ imply $\mathbf{A2}$ or $\mathbf{A1}$?

Question 6 Are statements $\mathbf{A2}$ and $\mathbf{A1}$ equivalent ?

6 A statement strictly weaker than BPI

It is well known (see [14]) that the axiom $\mathbf{BPI}$, being equivalent to the compactness of $\{0,1\}^I$ for every set I, implies numerous compactness results. In particular, the classical proof of Alaoglu's theorem shows that $\mathbf{BPI}$ implies the following result of functional analysis :

($\mathbf{RC}$, **Reflexive compactness***). The closed unit ball of every simply-reflexive Banach space is weakly compact.*

We will now prove that $\mathbf{RC}$ does not imply $\mathbf{BPI}$, and this solves Question **2.11** of [6].

Proposition 6.1 *The axiom* $\mathbf{HB}$ *implies that every closed subspace of an onto-reflexive Banach space is onto-reflexive.*

P r o o f. Given a closed subspace V of E, let $r : E' \to V'$ be the "restriction" mapping which associates to each $f \in E'$ the mapping $f_{\restriction V}$. Given some $\Phi \in V''$, $\Phi \circ r \in E''$; since E is onto-reflexive, let $a \in E$ such that $\Phi \circ r = \tilde{a}$, whence $\Phi = \tilde{a}_{\restriction V}$. We now show that $a \in V$; seeking a contradiction, assume that $a \in E \backslash V$: then, using $\mathbf{HB}$ Property for E, let $f \in E'$ such that $f[V] = 0$ and $f(a) > 0$. We now get a contradiction, since $\Phi(f_{\restriction V}) = f(a) > 0$, and on the other hand, $\Phi(f_{\restriction V}) = \Phi(0) = 0$. □

Theorem 6.2 1. *The axiom* $\mathbf{HB}$ *implies that every onto-reflexive Banach space is J-reflexive.*

2. $(\mathbf{HB}+\mathbf{DC})$ *implies that every onto-reflexive Banach space is convex-reflexive (whence* $(\mathbf{HB}+\mathbf{DC})$ *implies* $\mathbf{RC}$*).*

3. $\mathbf{RC}$ *does not imply* $\mathbf{BPI}$.

P r o o f. 1. Let E be an onto-reflexive Banach space. We show that E is J-reflexive. Seeking a contradiction, assume that for some $\vartheta \in]0,1[$, E has a ϑ-triangular sequence $(a_n, f_n)_{n \in \mathbb{N}}$ (see Proposition 2.9). Let V be the closed subspace generated by $\{a_n : n \in \mathbb{N}\}$, and let $C := conv\big(\{f_n : n \in \mathbb{N}\}\big)$. Then $d(C,0) \geq \theta$, hence, using $\mathbf{HB}$, there exists some $\Phi \in \Gamma_{V''}$ such that $\Phi[C] \geq \vartheta$. Using $\mathbf{HB}$, Proposition 6.1 implies that the subspace V is onto-reflexive, whence there exists some $x \in V$ such that $\Phi = \tilde{x}$; let $n \in \mathbb{N}$ and $(\lambda_i)_{0 \leq i \leq n} \in \mathbb{R}^{n+1}$ such that $\left\| x - \sum_{0 \leq i \leq n} \lambda_i a_i \right\| < \vartheta$. Then $|\Phi(f_{n+1})| = |f_{n+1}(x)| = |f_{n+1}(x - \sum_{0 \leq i \leq n} \lambda_i a_i)| < \vartheta$: this is contradictory with $\Phi[C] \geq \vartheta$.
2. Use Theorem 3.9 and 1.
3. Use 2. and the fact that, ([10]), $(\mathbf{HB}+\mathbf{DC})$ does not imply $\mathbf{BPI}$. □

Question 7 It follows from our study that, in $\mathbf{ZF}+\mathbf{DC}+\mathbf{HB}$, convex-reflexivity, ω-reflexivity, simple-reflexivity, J-reflexivity and onto-reflexivity are equivalent. Is there some "classical" notion of reflexivity which is not equivalent to these notions in $\mathbf{ZF}+\mathbf{DC}+\mathbf{HB}$?

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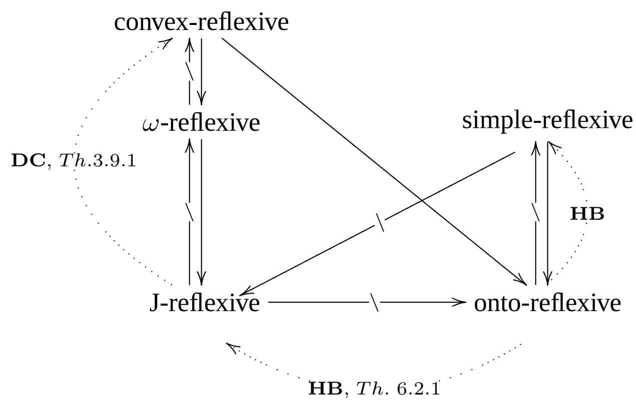

Fig. 1 Reflexivity in **ZF**.

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