Natorp's Neo-kantian Mathematical Philosophy of Science

Thomas Mormann

1. The Mathematical Philosophy of Science of the Marburg Neo-kantianism. According to classical wisdom, the most exact science of all sciences is mathematics. Thus it is hardly surprising also in the Marburg account of philosophy of science mathematics played a pre-eminent role. It should be noted from the beginning, however, that the Marburg account of philosophy of science, and of philosophy of mathematics in particular, had some rather peculiar features that distinguishes it from contemporary mainstream ideas of what is philosophy of mathematics. From the Marburg neo-Kantian perspective any philosophically acceptable "perspective on mathematics" must take into account not only mathematics but mathematics and the exact sciences, in particular physics. This was succinctly expressed by Cassirer in his early programmatic paper *Kant und die moderne Mathematik* (1907):

> Der Blick der Philosophie darf – wenn man dieses Verhältnis einmal schroff und paradox ausdrücken will – weder auf die Mathematik noch auf die Physik gerichtet sein; er richtet sich einzig auf den Zusammenhang beider Gebiete. (Cassirer 1907, 48)

Dealing with the issue of what is to be understood by the "mathematical philosophy of science of the Marburg Neo-kantianism", first of all it is expedient to point out that the Marburg perspective is characterized by some peculiarities that distinguish it from the contemporary understanding of the philosophy of science, and philosophy of mathematics in particular.

For the philosophers of the Marburg School mathematics was the constitutive method of empirical science *überhaupt.* Mathematics was the guarantor and expression for the scientific character of these disciplines. At the same time, for them, the insight into this fact, secured the scientificity of philosophy (of science) itself. From a neo-Kantian perspective any philosophically acceptable "perspective on mathematics" had to take into account not only mathematics but mathematics and the exact sciences, i.e., the mathematized sciences, in particular physics. This "paradoxical" stance was not Cassirer's idiosyncracy, all members of the Marburg school subscribed to a similar conception, in particular, Cohen and Natorp. As I want to show in the following it gave the Marburg philosophy of science a peculiar flavor that justifies its characterization as a "mathematical philosophy of science".

In this paper I'd like to deal with Natorp's version of the Marburg mathematical philosophy of science which up to now has been a rather neglected topic of the (history of) philosophy of science.[1] First, a trivial clarification: Mathematical philosophy of science is not just philosophy of mathematics. Indeed, a mathematical philosophy mathematics is to be understood in a more complex way:

(1) Philosophy of mathematics proper is only one part of a comprehensive mathematical philosophy of science in general.

(2) The application of mathematics in the empirical sciences is a central issue for mathematical philosophy of science.

(3) The methods of mathematical philosophy of science are inspired by mathematics.

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[1] Arguably, Cassirer's version of the mathematical philosophy of science of the Marburg school is the best known, Cohen's the most obscure, and Natorp's is the most ignored version of the Marburg school's mathe-matical philosophy of science. As I want to show in the following, this does not entail that Natorp's version is the most uninteresting version.

Although all members of the Marburg school subscribed to a mathematical philosophy of science in this sense, this does not entail, of course, that the mathematical philosophy of science of the Marburg school was a monolithic doctrine remaining constant during the whole existence of the school and interpreted in the same way by all of its members.

True, the Marburg philosophers attempted to make this impression to non-members of the school. But a closer look reveals that from early on there existed more or less subtle differences and inconsistencies between the various versions of the Marburg mathematical philosophy as they were put forward by Cohen, Cassirer, and Natorp. In this paper, I'll like to concentrate on Natorp, but this clearly requires to deal with Cohen and Cassirer to some extent as well.

Somewhat paradoxically, from the very beginning the Marburg mathematical philosophy of science had a rather tensed relationship with then contemporary mathematics. It may be compared with an unhappy love-affair – from the side of the Marburg school. While the Marburg philosophers never get tired of emphasizing the fundamental importance of mathematics for the sciences and for reason in general, mathematicians and logicians used to keep a negative attitude toward the advances of the Marburg philosophers - criticising them often as mathematically incompetent (cf. Frege (1885), Cantor (1884), Fraenkel (1967)). This stretched from the harsh criticisms that Cohen's *Das Prinzip der Infinitesimalmethode und seine Geschichte* (1883) received from most mathematicians to the rejection of Natorp's intended "synthese" of Cantor's and Veronese's accounts (cf. Natorp (1910)) put forward by mathematicians such as Abraham Fraenkel and Abraham Robinson in the 1960s (cf. Fraenkel (1968), Robinson (1966)).

To some extent, these criticisms were justified, in other respects they were unfair and overstated ignoring the different intentions of mathematicians and philosophers. One reason for these sharp disagreements was that at the turn of the previous century the concepts of the infinitely large and the infinitely small (infinitesimal) were hotly discussed among mathematics itself, and the Marburg school in some sense clashed with what was to become the mainstream of 20th century mathematics. In an oversimplified way this mainstream conception can be characterized by the thesis that the concept of the infinitely small was, as Russell put it, an inconsistent pseudoconcept that had to be eliminated from the discourse of mathematics. In contrast, Hermann Cohen in *Das Prinzip* (Cohen(1883)) had launched forward the bold claim that the concept of the infinitesimal was the central notion of philosophy of science *überhaupt*.

For a long time the generally accepted conviction of mathematicians and philosophers was that the anti-infinitesimalists were the winners of this dispute. Meanwhile, the situation has become more complicated: The concept of the infinitesimal has been rehabilitated in mathematics. It is a recognized topic in the disciplines of Non-Archimedean mathematics, in particular in non-Standard Analysis, and Smooth Analysis, where systems containing infinitesimals of different kind are studied. Today, infinitesimals are recognized as mathematically respectable objects of the same dignity as finite or infinitely large magnitudes. REF, REF, REF.

By hindsight, then, Natorp's *Logische Grundlagen* (1910) may be considered as having been ahead of its time, insofar as it is one of the very few treatises of philosophy of science that at least partially understood the relevance of non-Archimedean mathematics. This is not to deny that Natorp's account is deeply flawed insofar, as it got the relation between infinitely small and infinitely large (Cantorian) numbers quite wrong.

Natorp's *Grundlagen* was published in the same year as Cassirer's *Substanzbegriff und Funktionsbegriff*, namely in 1910. Since then, it stood always in the shadow of Cassirer's more brillant opus. It would be unjustified, however, to consider it straightforwardly as obsolete: In certain aspects Natorp's *Grundlagen* was more modern, mathematically more profound, and, certainly, more faithful to the original spirit of the Marburg Neo-kantianism than Cassirer's *Substance and Function*, or so I want to argue. More precisely, Natorp intended to elucidate and precisify Cohen's often obscure approach and, at the same time, he attempted to preserve the conceptual essence of Cohen's approach.[2]

The outline of the paper is as follows: In the next section 2 *The Transcendental Method and Natorp's Knowledge* we deal with Natorp's "*knowledge equation"*, a mathematical model of the "transcendental method" that Natorp considered as the core of the Marburg. The following section 3 *The Object of Knowledge as an Infinite Task: Two Opposing Elucidations* deals with two opposing elucidations of how science copes with this "infinite task": As Carnap contended in the *Aufbau* (Carnap 1928), this endeavour can be divided into quite distinct parts, namely, a finite "constitution" of the object and an infinite incompleatable empirical description. In contrast, and more in the original spirit of Cohen and Natorp the physicist and philosopher Margenau in *The Nature of Physical Reality* (Margenau 1950) conceived the infinity of this "Aufgabe" as an infinite dialectical process in which relative "data" and "conceptual constructs" determine each other. Section 4 *Eliminating the Dichotomy between Anschauung and Begriff* deals with one of the most important features that distinguishes the Marburg Neo-kantianism for Kantian orthodoxy, namely the abandonment of the crucial difference of intuition

[2] Natorp was well aware of the fact that he did not fully succeed in this respect. In a letter to Görland (November 21, 1902) (cf. Holzhey (1986, 302)) he characterized Cohen's way of philosophizing as follows: "Er ist u[nd] bleibt Poet in der *Art* seines Philosophierens, obwohl in sehr vielen Fällen die Ergebnisse sich nachher auch auf logischen Wegen darstellen lassen; in einigen aber vielleicht nicht, wenigstens reichen die *mir* zugänglichen Pfade der Logik nicht zu *allen* seinen Resultaten, obwohl zu recht vielen." Natorp's admitted inability to fully reformulate Cohen's "poetic way of philosophizing" in a more sober logical way had the consequence that he did not publish a fuller account of Cohen's *Logik* in his *Die logischen Grundlagen der exakten Wissenschaften* (Natorp 1910)) as he had planned originally (cf. Holzhey (1986, Section II).

and concept. Section 5 is concerned with non-Archimedian geometrical systems that played a central role in Natorp's defense of Cohen's "infinitesimal" metaphysics. Section 6 *Infinitesimals, the Revolution of Rigor, and Natorp's Failed Synthesis* discusses Natorp's attempted synthesis of Carnap's and Veronese's account of infinitely large and infinitely small (infinitesimals) numbers.

2. The Transcendental Method and Natorp's Knowledge Metaphor. The Neokantian approach to epistemology and philosophy aimed to be faithful to the spirit but not necessarily to the letter of Kant's philosophy. For Natorp this meant to restitute the "transcendental method" as the true core of the Kantian approach, and to give up all of ingredients of Kant's system that did not sit well with that method. The transcendental method deals with the problem of the possibility of experience. The NeoKantians interpreted Kant as contending that the object of experience is determined by the laws and methods of the knowing subject. Thereby the object no longer is something given ("gegeben") but something "posed" ("aufgegeben") (cf. Kinkel 1923, p. 405). Conceiving Neokantian philosophy as based on the transcendental method has two implications:

(i) Philosophy recognizes the historical, societal and scientific context in which it exists. It is aware that it is rooted in the specific theoretical and practical experiences of its time and refuses to build up "high towers of metaphysical speculations" (cf. Natorp 1912, p.195, Kinkel 1923, p. 402/403).

(ii) Philosophy accepts the facts of science, morality, art and religion. The task of philosophy is to carry out a *deductio iuris* of these facts, i.e., it has to provide a kind of "logical analysis" which shows the reasons why these facts are possible thereby revealing what is the “quid iuris“ of them. In still other words, and going beyond the epistemological sphere, philosophy has to show the lawfulness and reasonableness of the cultural achievements of mankind.

Thereby the philosophy of critical idealism (as the self-proclaimed true heir of Kant’s philosophy) is lead to a “genetic“ epistemology and theory of science that regards the ongoing process of scientific and cultural creation as essential, not its temporary results. These are to be considered as being of secondary importance. As Natorp famously put it with respect to scientific knowledge: the fact of scientific knowledge is always “becoming fact“ (“Werdefaktum”) and is never “closed“ or “finished“. There never is something “given“ that is not transformed in the ongoing and strictly speaking infinite process of cognition.

The rejection of a non-conceptual given in any form brings the Marburg brand of Neokantianism in open conflict with some of the corner-stones of Kant’s epistemology, to wit, the dualism of “scheme“ and “intuition“, and related dualisms such as that of “spontaneity“ and “receptivity“ of thinking. Natorp was well aware of this fact: “Maintaining this dualism of epistemic factors (receptivity and spontaneity, T.M.) is virtually impossible if one takes serious the core idea of the transcendental method.” (Natorp 1912, 9).

Subscribing to a “genetic“ account of knowledge that emphasizes the process character of knowledge gives the relation of knowledge priority over its relata, to wit, the knowing subject and the object of knowledge. Both are constituted in the ongoing process of knowledge. Taken for themselves they are just abstractions from the more basic relation of knowlede. Although it may sometimes be expedient to treat the subject of knowledge and the object of knowledge separately, this separation is to be considered only as a methodological device by which one may distinguish between two complementary accounts: one in which the object occupies centre stage, and one which emphasizes the role of the cognizing subject. Speaking in a Kantian framework, object-oriented accounts emphasize the role of receptivity of cognition, in particular perception, while subject-oriented, epistemic account are inclined to lay stress upon the constructive aspects of cognition. According to the Neokantian doctrine both accounts are incomplete and therefore mistaken. For the Neokantianism, ontology and epistemology are two sides of the same coin. Ontology without epistemology would be some kind of magic, which leaves unexplained how knowledge gets access to its object, while epistemology without ontology would be without content, since it denies the objec- tual character of cognition. Expressed in Kantian language, object-oriented approaches tend to emphasize the receptivity of cognition. According to them, cognition is essentially a passive and receptive behaviour. The thinking mind is confronted with something outside and independent of the sphere of reason. Ignoring more subtle differences this amounts to some kind of "copy theory" or "mirror theory" of knowledge. Subject-oriented approaches, on the other hand, emphasize the spontaneity of cognition. According to them, cognizing is essentially to be considered as a creative activity. Such a conception does not admit a "given" as a mind-independent presupposition of the cognizing process. Rather, the given ("das Gegebene") is to be conceived of as the product ("Ergebnis") of the immanent determination of thought. Thereby, subject-oriented approaches are in danger of underestimating the resisting power of the real world in favor of the unrestricted creative power of the knowing mind. According to Natorp, critical idealism, employing the "transcendental method" as its fundamental guide-line, avoids the shortcomings and deficits of both the subject-oriented and the object-oriented accounts .

Natorp famously (or notoriously) used to elucidate the transcendental method with the aid of a metaphor, namely, the metaphor of the "knowledge equation" (cf. also Kinkel

(1923, 405)). It compared the evolution of science with the solution of numerical equations. More precisely, the "x" of a polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0 = 0$. According to it, coming to know an object – the "Erkenntnisobjekt" - was analogous to the process of solving such a numerical equation, i.e., determining the "unknown x". To be specific, consider the equation

$$x^3 + x^2 + x + 1 = 0$$

It has the solutions x = -1, +i, -i). In line with Natorp's didactic intentions this equation sought to convey several ideas concerning knowledge and its objects:

1. The fact that it has several different solutions indicates that the process of research may not lead to unique results.

2. The fact that two of the knowledge solutions, namely +i and –i, are imaginary, reflects the fact that the research process may lead to an expansion of the original fundamental concepts one started with: The admission of complex numbers as solutions of an equation with only real parameters transcends the conceptual space in which the equation was originally formulated insofar, as all these parameters are real numbers, even integers.

According to Natorp, the structure of the knowledge equation, namely, its peculiar inter-weaving of determined and yet to be determined ingredients, has important consequences for the concept of knowledge:

> If the object is to be the x of the equation of inquiry, then it must be possible <u>to determine the meaning of this x by the nature of this equation (i.e., by the inquiry itself) in relation to its known factors (our fundamental concepts)</u>. From this it must follow whether and in what sense the solution of this problem is possible for us. This <u>is the very idea of the transcendental or critical method</u> (Natorp 1903, 10).

The transcendental method does not aim to extend our knowledge beyond the limits of the scientific method. Rather, it seeks to clarify the limits of scientific knowledge. It is called "transcendental" since it goes beyond the cognition that is immediately directed onto the objects, but aims to obtain information about the general direction of the path to be taken. It does not provide us with any specific knowledge about an object beyond experience. Hence, following the established Kantian terminology it is transcendental, but not transcendent.

Natorp and his fellow philosophers of the Marburg school viewed the object of knowledge, not as an unproblematic starting point of the ongoing process of scientific investigation, but rather as its ideal limit. For the Marburg Neo-kantianism the object was not given, it was a problem to be solved. In various versions, this equational account of knowledge can be found in virtually all of Natorp's epistemological writings (cf. Natorp (1902, 1910, 1912, 1927)).

One might object that Natorp's equational model of scientific cognition is too simple in the sense that empirical objects hardly ever show up as solutions of a finite equation such as the one considered above. Hence, what is still missing in this version of the metaphor is the "infinite character" of the knowledge equation. Natorp was aware of this shortcoming and tried to remedy it by emphasizing the knowledge equation as an infinite *Aufgabe*. Elaborating the equational model, he pointed out that the knowledge equation was not to be understand simply as a finite *Aufgabe* (problem) but as an infinite *Aufgabe* (problem) that could be solved only approximately by finite creatures like us. Thereby he sought to be escape from the trap of an overstated, "Hegelian" rationalism for which all problems were merely conceptual problems.[3] Against Hegel's intellectualism he wanted to characterized the Marburg approach as a more modest approach way as follows:

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[3] For a then contemporary critical discussion of the affinities between Hegelianism and the Marburg school from the point of view of Southwest Neo-kantianism see Marck (1913).

> Although we conceive of the object of knowledge (= x), similarly as Hegel does, only in relation to the functions of knowledge itself, and consider it . . . as the x of the equation of knowledge, . . . we have understood that this equation is of such a kind that it leads to an infinite calculation. This means that the x is never fully determined by the parameters a, b, c, ... of the equation. Moreover, the sequence of the parameters is not to be thought as "closed" but rather extendable further and further. (Natorp 1912, 211-212)

According to Natorp, the transcendental method was the point of departure (from Kant) and the guide-line of the scientific philosophy of the Marburg school. It distinguished the transcendental approach of the Marburg philosophers from other rival ones that Natorp characterized by the usage of psychological, metaphysical, and logical methods, respectively (cf. Natorp, 1912, 194ff).

3. The object of knowledge as an infinite Task: Two opposing elucidations. Not only the members of the Marburg school considered the "knowledge equation" as a characteristic and convenient metaphor for the epistemology of the Marburg school, also philosophers who did not belong to the school used it as an expededient means to separate their own approach from that of the Marburg philosophers. A good example is provided by Carnap's *Logical Construction of the World* (*Aufbau*) (Carnap 1928). One of Carnap's aim in this work was to separate the empiricist account of the *Aufbau* from Marburg Neo-kantianism by dismissing of the knowledge equation as follows:

> According to the conception of the Marburg School (cf. Natorp's *Grundlagen*, 18ff) the object is the eternal X, its determination is an incompleatable task. In opposition to this it is to be noted that finitely many determinations suffice for the constitution of the object – and thus for its univocal description among the objects in general. Once such a description is set up the object is no longer an X, but rather something

> univocally determined – whose complete description then certainly still remains an incompleteable task. (Aufbau, § 179)

In *A Parting of the Ways* (2000) Michael Friedman argues that this attempt of Carnap to dissociate himself from the Marburg school fails, since, due to certain technical difficulties of the *Aufbau* (already noted by Quine), it can be shown that Carnap did not succeed in constructing stable physical objects at a definite rank in the type-theoretical hierarchy of the quasi-analytical reconstruction of the world (cf. Friedman (2000, 83). Thereby, Friedman concludes

> Carnap's construction of the physical world therefore appears never to close off at a definite rank in the hierarchy of types ... . And this means, of course, that the Marburg doctrine of the never completed "X" turns out to be correct, at least so far as physical (and hence all higher level) objects are concerned. (Friedman 2000, 84).

Whether this argument against Carnap's finite constructivism is thus definite as Friedman claims need not be discussed here. The more modest point I want to make is that in any case Carnap himself admitted that he did not escape the "infinite" character of the task of determing the "knowledge object" claimed by the Marburg school. He simply divided the process of determining the object into two essentially different parts, namely, a finite formal one, namely, that of constitution, and an infinite one, that of empirical determination. How these two parts are related does not say. Rather, he is content to invoke the metaphor of determining the geographical coordinates of the location of a physical object and the open-ended, infinite process of determining its empirical properties. Metaphorically, it may be obvious that the former can be determined without the latter, but a second thought reveals that this is far from obvious.

In some sense, then, the *Aufbau* falls back onto a Kantian position conceiving cognition, i.e., the determination of the object of knowledge, as a bipartite process consisting of two separate ingredients, namely empirical determination and conventional constitution. Arguably, the Marburg Neo-kantian account of scientific knowledge, which denied any kind of Kantian dichotomy, was much farther away from traditional Kantianism and closer to what is really going in in science than Carnap's early logical empiricism. The *Aufbau's* unintended (and by its author vigorously denied) accordance with the original Marburg account concerning the infinite character of the determination of the knowledge object evidences that this feature is well entrenched in our understanding of the knowledge process. This does not mean, of course, that the Marburg description of this process is fully satisfying.

One of the very few attempts to improve and elaborate the original Marburg from outside is the one that undertook the physicist and philosopher Henry Margenau in many of his works from the midst of the 1930s onwards (cf. Margenau (1935, 1950)).

A quite different attempt to come to elucidate the infinite character of empirical knowledge assumed by the Marburg philosophy of science was proposed by the physicist and philosopher Henry Margenau (1901 – 1997). Margenau's work is particularly interesting since he has been one of the very few "working physicists" sympathetic and knowledgeable of Neokantian philosophy of science. As far as I know, he was the only philosopher-scientist who ever took notice of Cohen's *Die Logik der reinen Erkenntnis* (Cohen 1902). He attempted to elucidate Natorp's "infinite *Aufgabe*" of determining the knowledge object in an unending process in terms of a dialectics between the perceptual and a domain of conceptual constructs, respectively. In contrast to Kantian orthodoxy, according to Margenau, percepts (*Anschauungen*) and concepts (*Begriffe*) were to be distinguished but nevertheless belonged to the same connected realm:

> Sensation as part of the process of knowledge is not wholly *sui generis* and a passage from the qualities that signify an act of clear perception to those

characterizing pure thought may well be gradual. ....

On the one hand, many *concepts* have sensory-empirical aspects because of their reference to the immediately given ... and ... on the other hand *sensory data* require concepts for their interpretation. Torn out of its context in experience, the immediately given becomes as grotesque as its counterpart, the rational, has often been when nourished in seclusion. Unless one is careful not to disturb the natural setting of data and thought, one's philosophy is artificial and certainly unrepresentative of science. (Margenau 1950, 55)

Margenau explicitly and approvingly mentioned Cohen's basic "thesis of the origin" (*Die These des Ursprungs*): "Only thinking can generate what can be considered as being." Even more remarkable, he claimed that this thesis, usually considered as evidence of Cohen's extravagant and overstated idealism, was fully compatible with Locke's empiricist *dictum* "Nihil est in intellectu, quod prius non fuerit erat in sensu". Going beyond Nartorp, Margenau was not content to explain the infinite determination of the knowledge object by the metaphor of a mathematical equation, rather, he relied on a detailed "dialectical" description of the historical evolution of theories. In its most elementary form it went like this:

We observe a falling body, or many different falling bodies, we then take the typical body into mental custody and endow it with the abstract properties expressed in the law of gravitation. It is no longer the body we originally perceived, for we have added properties which are neither immediately evident nor empirically necessary. If it be doubted that these properties are in a sense arbitrary we need merely recall the fact that there is an alternate, equally or even more sucessful physical theory – that of general relativity – which ascribes to the typical bodies the power of influencing the metric of space, i.e., entirely different properties from those expressed in Newton's law of gravitation (Margenau 1935, 57)

Margenau continued by explaining the general process of the dialectical generation of knowledge in the empirical sciences as follows:

The full course of physical explanation ... begins in the range of perceptible

> awareness, swings over into what we shall now term the field of symbolic construction, and returns to perceptibe awareness, or as we have said nature ... The essential feature of physical explanation is evidently the transition from nature to the realm of construct, and the reverse. (ibid., 59).

This kind of dialectic determination of the object of knowledge, in Margenau's terms the ongoing process of the explanation of the physical object, can already be found in Natorp, although in more abstract, mathematical terms. For Natorp's Neo-kantianism this dialectic between "percept" and "concept" also affected the Kantian dichotomy between intuition (*Anschauung*) and understanding (*Begriff*) leading for the Marburg account to the collapse of the architectonic of the original Kantian system.

<u>4. Eliminating the dichotomy between *Anschauung* and *Begriff*.</u> For Natorp, one of the all-important consequences of the transcendental method was that the Kantian dualism between intuition and thought had to be given up. Natorp, as all members of the Marburg school considered Kant's philosophy a promising starting point for modern epistemology and philosophy of science, but not as a doctrine that had to be followed literally. Like all neo-Kantians he emphatically subscribed to the slogan of 'going with Kant beyond Kant'. The most important deviation from Kantian orthodoxy was to give up Kant's sharp separation between understanding and sensibility as two faculties of the mind. Beginning with Cohen, the Marburg philosophers replaced Kant's two faculties of the mind by a single comprehensive activity of 'pure thought' (*reines Denken*). [4]Pure thought primarily expressed itself in the progressive evolution of the mathematized empirical sciences. According to Cohen's well-known slogan, philosophy had to take

[4] As has been observed by many scholars giving up the Kantian dualism between a logical or conceptual faculty of pure understanding and an intuitive or non-conceptual faculty of pure sensibility amounts to an important difference between Kantian and Neo-kantian philosophy (cf. Edel (1991. 60ff), Friedman (2000, 27f)).

‘the fact of science’ as the starting point for its considerations. This attitude was but a consequence of the ‘transcendental method’ considered by the neo-Kantians to be the core of Kantian philosophy. According to it, philosophy of science, like philosophy in general, did not operate in empty space, but had to rely on the historically established facts of science, ethics, art, and religion that provided it with its proper content (*cf.* (Natorp, 1912, 196–197). The task of philosophy was to ‘justify’ these facts by elucidating their reasonableness, thereby and real sense of them. In other words, philosophy had to explicate the meaning of human culture, and in particular, the meaning of science.

In a nutshell then, for Cohen, Natorp, and Cassirer the task of philosophy of science was to make explicit the method of science as ‘the method of an infinite and unending creative evolution of reason. Fulfilling this task was the indestructible core of Kant’s philosophy’ (Natorp, 1912, 200). In line with this dynamical concept of science the Marburg school did not conceive the ‘fact of science’ as something static to be found “out there”. Rather, science was to be conceived as a ‘fact in becoming’ (*Werdefaktum*). This led to a genetic epistemology that regarded the process of scientific evolution as essential, not so much its temporary results.[5]

Thus, on the one hand, Natorp fully endorsed Cohen’s elimination of the Kantian dualism between understanding and sensibility:

> So bleibt “Anschauung” nicht länger als denkfremder Faktor in der Erkenntnis dem Denken gegenüber – und entgegenstehend, sondern ist Denken, nur nicht blosses Gesetzesdenken, sondern volles Gegenstandsdenken; Anschauung verhält sich zum Denken des Begriffs, wie

[5] Cf. Mormann and Katz (2013).

> zum Gesetze der Funktion die Funktion selbst in ihrer Ausübung, im Vollzug. (Natorp 1912, 204)

On the other hand, however, the original Kantian distinction between understanding and sensibility reappears, or is maintained in attenuated form, as the distinction between two aspects of "Denken", namely, between "*Gesetzesdenken"* and *"Gegenstandsdenken*". Interestingly, Natorp attempted to elucidate this contrast by relying on the notion of function. According to him, the relation between intuition and understanding can be thought as being analogous to the relation between two aspects of a function, namely, the aspect of a function as as an object, and the aspect of a function as a tool (for calculation). Whether this attempt of saving some version of the traditional Kantian dichotomy is successful, seems dubious.

5. Archimedean and Non-Archimedean Systems. Today, virtually every philosopher agrees with Cassirer's thesis:

> In all the history of mathematics there are few events of such immediate and decisive importance for the shaping and development of the problem of knowledge as the discovery of the various forms of non-Euclidean geometries. (Cassirer 1950, 21)

This does not entail that all philosophers are seriously interested in the existing multiplicity of non-Euclidean geometries. Many just acknowledged this multiplicity just as a fact, without paying much attention to its consequences. Actually, this is the attitude of most philosophers of today who take into account only Riemannian, i.e. locally Euclidean geometry - due to its relevance for Einstein's general theory of relativity.

Most other types of non-Euclidean geometries are ignored, or, at least, are considered as philosophically uninteresting, they are taken as a topic of technical and formal interest for mathematicians only. One of the very few exceptions to this general philosophical attitude was Natorp. More precisely, in *Die logischen Grundlagen* he pointed out that the Marburg account of philosophy of science, which ascribed a central role to the concept of the infinitesimal, had to make philosophical sense of a "highly non-Euclidean" even non-Riemannian geometries, namely, non-Archimedean geometries. Today, for philosophers, "non-Euclidean geometry" usually means "non-Euclidean Riemannian geometry" as it is used in Einstein's theory of general relativity. This amounts to a considerable restriction of what may be understood as a, possibly empirically, meaningful general concept of space. Not only it excludes many examples of non-Euclidean geometries, *a fortiori* it dismisses modern general mathematical theories of spatial concepts such as topology as philosophically irrelevant.

In mathematics systems of magnitudes that contain infinitely small magnitudes are called Non-archimedean systems, systems of magnitudes with only finite elements are called Archimedean systems. A paradigmatic example of an Archimedean system is the system of natural number ($\mathbf{N}$, $<$) with the relation $<$ defined as $a < c := \exists b \in \mathbf{N} \ (a + b = c)$. A non-Archimean system would be a system that contains magnitudes that are – absurdly - infinitely close to 0 but nevertheless are distinct from each other and 0, to borrow a definition from Quine.[6]

The "official" characterization of geometrical Archimedean magnitudes can be found in Hilbert's *Foundation of Geometry* (1899):

> The Axiom of Archimedes (Hilbert 1899, V.1, p. 26). If AB and CD are any segments then there exists a number n such that n segments CD

[6] Quine (1976, §51, 428).

constructed contiguously from A, along the ray from A through B, will pass beyond the point B.

To us, who live in a Euclidean world, this axiom seems to be very natural. Infinitesimals, i.e., magnitudes that do not satisfy this axioms, appear to be absurd. If CD were a magnitude that does not satisfy the axiom of Archimedes with respect to AB, then CD would be infinitesimally small with respect to AB, or, from the opposite perspective, AB would be infinitely large with respect to CD, since no multiple of CD could ever be larger than AB.

The non-satisfaction of the Archimedean axiom asserts that there are infinitely small and infinitely large lines, i.e., exactly the kind of magnitudes that Russell, Carnap, Quine and many others considered as “absurd”, probably due to the fact that we have difficulties to imagine them and they allegedly contradict ordinary imagination.

Since the 1880s it was well-known that Non-archimedean systems existed. In §12 of *Foundations of Geometry* Hilbert showed that there existed models of geometry that satisfied all axioms of Euclidean geometry except the Archimedean one. As a mathematician Hilbert was very open-minded with respect to Non-archimedean systems. He explicitly pointed out that these systems may be interesting not only for mathematical reasons but also from a philosophical point of view. The independence of the Archimedean axiom from the other geometrical axioms could be of principal interest also for physics:

[The logical independence of the Archimedean axiom] leads to the following result: the fact that we reach by concatenating terrestrial distances the dimensions and the distances of celestial bodies, i.e., that we can determine lengths in space by terrestial measures, as well as the fact that the distances in the interior of atoms can be expressed by the yardstick, is not at all a logical consequence of the theorems on triangle congruences and geometrical configurations, but only a result of empirical investigation. The validity of the Archimedean axiom in nature requires confirmation by experiment in the same familiar sense as the theorem on the sum of angles in the triangle (Hilbert (1917, 408-409)).

Also Poincaré, with a somewhat different focus, had emphasized several time that Non-archimedean systems might be interesting for the empirical sciences (cf. Poincaré (1906)). Already in *Science and Hypothesis* he had pointed out that psycho-physical continua determined by sensations had a quite different structure than punctiform mathematical continua.[7] Giovanelli has shown that Cohen himself obtained important impulses for his Non-archimedean "metaphysics of infinitesimals" from Fechner's experiments concerning very small differences in psycho-physical continua.[8]

on psycho-physical continua. as they showed up in Fechner's experiments according to which (cf. Fechner (1860)). This means that Cohen's approach did not emerge solely from mathematical and philosophical speculations but from its very beginnings was not unrelated to considerations concerning the empirical sciences (cf. Giovanelli (2016)).

In order to ensure the mathematical meaningfulness of Non-Archimedean structures it is sufficient to consider general systems of magnitudes. Already the later logical empiricist Hans Hahn had done this in his trail-blazing paper *Über die nichtarchimedischen Grössensysteme* (Hahn (1907)). Hahn had defined "systems of magnitudes" (*Grössensysteme*) as linearly ordered commutative groups G = (G, +, <, 0) endowed with an associative and commutative addition "+" and a linear order "<" that satisfies the familiar axioms as they hold, for example, for the system of integer numbers Z. If the neutral element of G is denoted by 0, the elements a of G that satisfy 0 < a are called positive, and elements b with b < 0 are called negative. A system of magnitudes satisfies the Archimedean axiom if the following holds:

The Archimedean axiom for algebraic systems of magnitudes). Let a and b

[7] Ibid., 22f.
[8] Cf. Giovanelli (2016, 10).

be positive elements of a system of magnitudes G and a < b. The system G is an Archimedean system iff there is a natural number n such na > b (na := a + a + ... + a (n-times)).

Obviously, the system of integers (**Z**, + , < , 0) is an Archimedean system. It is quite elementary, however, to construct from this Archimedean system non-Archimedean systems: Let (**Z x Z**, +., 0) be the Cartesian product of pairs of integers {(a, b); a, b ∈ **Z**} may be endowed with a Non-archimedean order by the following recipe:

(a, b) < (a', b') := a < a' or a = a' and b < b'

Then for all n, the pairs one has (0, n) < (1, 1), i.e., relative to (1,1) the elements (0, n) are infinitesimally small, and, viceversa, (1,1) is infinitely large relative to (0, n). One may object that the system of magnitudes (**Z x Z**, <, 0) is rather "artificial". But this is hardly a strong argument against the mathematical meaningfulness of Non-Archimedean systems. It must be admitted, however, that this kind of systems does not provide any better understanding of the infinitesimal magnitudes dx, dy, ... on which the traditional infinitesimal calculus is based. At least, systems such as (**Z x Z**, <, 0) and its ilk show that the concept of an infinitesimal magnitude is not openly "absurd" as many of its foes contended. Rather, their "absurdity" is the result of an uncritical adherence to an obsolete criterion of "imaginability". Hahn admitted that non-archimedean systems can be interpreted as generalizations of Archimedean systems. Moreover, he was completely clear about the possible empirical relevance of non-archimedean systems.[9] In a popular lecture he explicitly pointed out:

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[9] Hahn's recognition of the possible empirical (and philosophical) relevance of non-archimedean systems remained a mavericks' position in logical empiricist philosophy of science. The logical empiricist mainstream, represented by Carnap, stuck to the verdict that the concept of infinitely small magnitudes simply was a meaningless pseudo-concept (cf. Carnap (1928), Quine (1976)).

> Spaces can be devised in which the Archimedean postulate is replaced by its opposite, that is, in which there are lengths that are greater than any multiple of a given length. Hence in these spaces infinitely large and infinitely small lengths can exist… In a "non-Archimedean" space, lengths can be measured, and a system of analytical geometry can be developed. Of course, the real numbers of ordinary arithmetic are of no help in this geometry but one uses "non-Archimedean" number systems, which can be interpreted and applied in calculations as well as the real numbers of ordinary arithmetic. (Hahn 1934, 99)

Hahn pointed out, however, that up to now there was no evidence that non-Archimedean systems could have applications outside mathematics. This reminds of the situation at the end of the 19th century when the analogous question of the applicability of non-Euclidean but nevertheless Archimedean geometries was on the agenda. Similarly as Hahn around 1930 didn't see any possibility of a real, i.e., empirical application of non-archimedean systems, Poincaré around 1900 had seen no option for applying non-euclidean, Riemannian geometries, but did not exclude this possibility once and for all. As is well known, a few years after his death, Non-Euclidean (Archimedean) geometry turned out to be an essential conceptual tool for general relativity theory.

Although the mathematical meaningfulness of infinitesimal magnitudes could hardly be denied after Hilbert's *Foundations of Geometry* and Hahn's *Non-Archimedean Systems of Magnitudes* nevertheless there seemed to remain an unsurmountable weakness of Non-Archimedean systems which had been formulated by Felix Klein, the inaugurator of the *Erlangen Program* already in 1908. Klein asked, whether the Non-archimedean systems of magnitudes could be used to reformulate the traditional infinitesimal calculus in such a way that it satisfied the modern standards of rigor put forward by Dedekind and Weierstraß. Surveying the results obtained so far he resignedly concluded that this was not the case (cf. Klein 1908 (2016, 236)). Twenty years later, Hermann Weyl in *Philosophy of Mathematics and Natural Sciences* arrived at the same negative conclusion (cf. Weyl (1928)).

This shortcoming of Non-Archimedean theories was overcome only in the 1960s by Abraham Robinson. He achieved to construct Non-Archimedean systems of magnitudes for which the basic results of classical infinitesimal calculus could be proved in a rigorous manner.

In a sense, then, one may say that Natorp was ahead of his time. He was one of the very few philosophers who realized that Non-Archimedean mathematics might be philosophically and scientifically relevant beyond the limited realm of pure mathematics. In *Grundlagen* he even (unsucessfully) attempted to bring about something like a synthesis of the various accounts of the infinitely large and the infinitely small being discussed at the turn of the 20th century.

6. The Infinitesimal, the Revolution of Rigor, and Natorp's Failed Synthesis. The origin of the mathematical philosophy of science of the Marburg school was Cohen's *Das Prinzip der Infinitesimalmethode und seine Geschichte* (1883). The central claim of *Das Prinzip der Infinitesimalmethode* was that the infinitesimal was the central concept of philosophy of science *überhaupt*:

> The foundation of the concept of the infinitesimal is an issue of philosophy in a double sense. First, the conscience of traditional logic is not soothed until this basic notion of mathematized natural science has not been characterized and explained philosophically as far as possible. Further, there remains an irretrievable gap in the foundations of knowledge as long as this fundamental instrument as a presupposition of the mathematical and thus empirical knowledge has not been recognized and demarcated. (Cohen (1883, §1))

For Cohen the philosophical elucidation of the concept of the infinitesimal was of outmost importance for any philosophy of science worth its salt. From *Das Prinzip der Infinitesimalmethode* 1883 till the end of his philosophical career he unflinchingly abided by this thesis. Few followed him on this road. Logicians, mathematicians and philosophers, who did not belong to the Marburg school, used to harshly criticize *Das* Prinzip.

Recently, Giovanelli characterized *Das Prinzip* as a remarkably “unsuccessful book” (cf. Giovanelli 2016). In a sense, this verdict is correct, particularly, if one takes into account its negative reception from the side of mathematicians, but in another sense it needs to be qualified. “Unsuccessful” books are usually forgotten and don’t leave a trace. This was not the fate of Cohen’s book. On the contrary, all members of the Marburg school defended it – admittedly, more or less energetically.Cassirer dissociated his account from Cohen’s *Das Prinzip* rather early without ever criticizing it explicitly[10]. In contrast, Natorp kept faith with Cohen and defended Cohen’s infinitesimal-centered philosophy of science against all attacks from outside, for instance, against the fierce criticism that Russell launched forward in *Principles of Mathematics* (Russell 1903).

Russell’s violent attack on Cohen was not original, actually it relied on arguments put forward by the mathematicians Cantor, Weierstraß and others. Quite generally the concept of the infinitesimal was going through hard times in the last decades of the 19th century, when the “great revolution of rigor” took place.

The received historical narrative concerning infinitesimals in that period runs as follows. The idea of infinitesimals has been with us since antiquity. Mathematicians have used one or another variety of infinitesimals or indivisibles without really understanding what

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[10] In *Substanzbegriff und Funktionsbegriff* (Cassirer 1910) the infinitesimal calculus is only mentioned in passing as one calculus among many others. This implicit betrayal of one of the Marbung School’s central dogmas did not go unnoticed by Cohen. In a letter to Cassirer he praised Cassirer’s *Substanzbegriff und Funktionsbegriff* in general but criticized that the concept of function instead the infinitesimal occupied center stage in this work (cf. Mormann and Katz (2013, XXX).

they were doing. Eventually, infinitesimals fell into disrepute for logical and philosophical reasons, as enunciated by Berkeley and others.

Despite Berkeley's allegedly devastating criticism, mathematicians continued to use infinitesimals in the 19th century with more or less good intellectual conscience. Finally, according to the traditional narrative, Cauchy, followed by Cantor, Dedekind, and Weierstrass, succeeded in formulating a rigorous foundation for the calculus in terms of the epsilon-delta approach. Thereupon infinitesimals were "officially" expelled from the realm of legitimate mathematics once and for all.

Natorp neither clung to the obsolete intuitive, logically dubious approach of infinitesimals that Cohen had proposed in *Das Prinzip,* (Cohen 1883), nor considered infinitesimals as "inconsistent fictions", as Vaihinger proposed, nor did whiggishly subscribe to the new orthodoxy of the "great triumvirate" (Cantor, Dedekind, Weierstrass) that insisted on the elimination of infinitesimals from any respectable mathematical discourse in favor of an approach based on Weierstrass's epsilontics. Instead, in *Grundlagen* Natorp attempted to do justice to infinitely large numbers, infinitesimals and limit concepts. More precisely, he attempted to reconcile the various doctrines of the infinitely large and the infinitely small put forward by Cantor, Dedekind, Weierstraß, Stolz, du Bois-Reymond, Veronese and others, in one comprehensive synthesizing Neo-kantian framework.

In an orthodox Neo-kantian vein Natorp argued as follows: If one takes the trans–cendental method seriously the concept of the infinitesimal could not be founded on any kind of intuition, as Cohen had claimed in *Das* Prinzip. Rather, the infinitesimal had to have its origin in "pure thought". Thus, Cohen had been right when he had changed his mind in *Die Logik der reinen Erkenntnis* (Cohen 1902) and claimed that the infinitesimal is to be grounded in the principle of the origin. One of the aims of Natorp's

Grundlagen was to elucidate Cohen's revised account based on a rather obscure "principle of the origin".

More precisely, Natorp undertook the bold attempt to reconcile Cantor's and Veronese's accounts (Veronese 1894) in the sense that he ascribed to Veronese the merit of having completed Cantor's work by extending his work on infinitely large numbers to the infinitely small. Thereby he presented Veronese as perfector of the Cantorian revolution (cf. (Natorp 1910 (184, 200)).[11]

Natorp's argument for this thesis was based on a vague and intuitive "Gedanken–experiment" according to which the infinitely large numbers and the infinitesimal are somehow "dual" to each other: Clearly, for any $n \in \mathbf{N}$ the unit interval [0,1] can be divided into 2^n subintervals $[k/2^n, k+1/2^n]$ of length $1/2^k$. Without any further argument, Natorp assumed that this process could be extended to infinity ("∞") somehow such that [0,1] could be divided into infinitely many subintervals of infinitesimal length $1/2^\infty$. Thus, according to Natorp, $1/2^\infty$ might be considered as infinitesimally small with respect to 1. This process could be continued starting with an infinitesimal interval $[0, 1/2^\infty]$ to yield a magnitude even infinitesimally small with respect to $1/2^\infty$ and so on. Thereby one obtains a series of infinitesimals directly corresponding to Cantors ordinals (cf. Natorp (1910, 195)). This construction of Natorp is, of course, a gross mathematical blunder. There is no direct correpondence between Cantor's infinite ordinals and the various non-Archimedean systems of infinitesimals of Veronese and others.[12]

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[11] This is, in every respect, a quite untenable interpretation. Cantor himself vigorously denied that Veronese's various kinds of infinitesimals had anything to do with his infinitely large (ordinal or cardinal) numbers. For a discussion of Veronese's philosophical background see Cantú (2010).

[12] For a modern account of the rise of non-Archimedean mathematics in general and systems of non-Archimedean magnitudes in particular, see Ehrlich (1994, 2006). There is no reason to discuss these issues in any detail here. Natorp's mathematically simplistic philosophical discussion of non-Archimedean systems in *Grundlagen* does not require this.

Natorp's immature attempt of a synthesis was severely critisized by mathematicians. Abraham Fraenkel is an example. In his *Lebenskreise. Aus den Erinnerungen eines jüdischen Mathematikers (1967)* Fraenkel harshly criticised the mathematical philosophy of science of the Marburg School:

> In Cohen's reference to the "fact" of mathematics and the mathematical natural sciences I missed any discussion of consistency ... In particular, I was deeply concerned how the Marburg school treated the infinitely small, beginning with Cohen's "Prinzip der Infinitesimalmethode (1883) up to Natorp's "Die Logischen Grundlagen der exakten Wissenschaften" (1910), where the infinitesimal was directly correlated with Cantor's transfinite numbers.(Fraenkel 1965,107)

To be sure, Fraenkel did not militate (as Cantor did) against infinitesimals in general. Rather, he rightly rejected Natorp's mistaken "synthesis". Moreover, he enthusiastically welcomed the quite unexpected achievement of his former student Abraham Robinson:

> A quite different, legitimate and surprising rehabilitation of the actual infinitely small recently has been achieved – from 1960 onwards – by my former student Abraham Robinson, now professor at the University of California in Los Angeles. (ibid.)

7. Concluding Remarks. Natorp's mathematical philosophy of science may be considered as the most radical version of the Marburg mathematical philosophy of science. He understood mathematical philosophy of science as a philosophy, for which mathematics occupied centre stage in philosophy of science and epistemology, and even in wider sense, namely, as a representative of reason in general.

Grundlagen was the last serious defense of Cohen's "infinitesimal metaphysics" which centered around the concept of the infinitesimal. For this purpose in *Grundlagen* Natorp used all logical and mathematical means available at that time, in particular, the various theories of the then contemporary theories of infinity. In this sense, Natorp's philosophy of science attempted to be a truly timely philosophy of science.[13] To be sure, mathematically, it was deeply flawed, but, as it seems to me, it was a failure that does not deserve to be completely forgotten.

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[13] For instance, Natorp's *Grundlagen* was the first Neo-kantian work of philosophy of science that dealt with Einstein's (special) theory of relativity - Cassirer's *Substanz-begriff und Funktionsbegriff* that was published in the same year 1910, only dealt with classical physics.

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