On computable automorphisms of the rational numbers

A. S. Morozov, Sobolev Institute of Mathematics, Novosibirsk,* and

J. K. Truss, University of Leeds

Abstract

The relationship between ideals **I** of Turing degrees and groups of **I**–recursive automorphisms of the ordering on rationals is studied. We discuss the differences between such groups and the group of all automorphisms, prove that the isomorphism type of such a group completely defines the ideal **I**, and outline a general correspondence between principal ideals of Turing degrees and the first–order properties of such groups.

1 Introduction

In this paper we study certain subgroups of the group $\mathrm{Aut}\,\langle\mathbb{Q},\leq\rangle$ of all automorphisms of the set of rational numbers as an ordered set, namely those defined by ideals **I** of the Turing degrees. This group, written $\mathrm{Aut}_{\mathbf{I}}\,\langle\mathbb{Q},\leq\rangle$, consists of all members of $\mathrm{Aut}\,\langle\mathbb{Q},\leq\rangle$ which (under a suitable coding) have Turing degree lying in **I**. Our main results are that the ideal can be 'recovered' from $\mathrm{Aut}_{\mathbf{I}}\,\langle\mathbb{Q},\leq\rangle$:

Theorems 2.22 and **2.23**: For ideals **I** and **J** of Turing degrees,

$$\mathrm{Aut}_{\mathbf{I}}\,\langle\mathbb{Q},\leq\rangle \cong \mathrm{Aut}_{\mathbf{J}}\,\langle\mathbb{Q},\leq\rangle \Leftrightarrow \mathbf{I}=\mathbf{J}$$

and

$$\mathrm{Aut}_{\mathbf{I}}\,\langle\mathbb{Q},\leq\rangle \text{ is embeddable in } \mathrm{Aut}_{\mathbf{J}}\,\langle\mathbb{Q},\leq\rangle \Leftrightarrow \mathbf{I}\subseteq\mathbf{J}.$$

In addition we prove the following results on arithmetical classes of sets (of natural numbers):

Theorem 2.19: Let $\mathcal{A}$ be an arithmetical class of sets. Then there is a first order sentence φ of the language of group theory such that for all Turing degrees **d**

$$\mathrm{Aut}_{\mathbf{d}}\,\langle\mathbb{Q},\leq\rangle \models \varphi \Leftrightarrow \mathbf{d}\cap\mathcal{A}\neq\emptyset.$$

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Theorem 2.20: If φ is a first order sentence of the language of group theory, then the union of the class of Turing degrees $\mathbf{d}$ such that $\mathrm{Aut}_{\mathbf{d}}\langle\mathbb{Q},\leq\rangle \models \varphi$ is an arithmetical family of sets.

The key idea involved in all these proofs is to interpret various concepts inside $\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle$, specifically the natural numbers, and first order arithmetic, as vehicles for reconstructing the ideal. The methods for doing this are rather standard in the theory of ordered permutation groups ([1] and [5]), though we give the necessary details, and our treatment is self-contained.

We use bold Roman letters to stand for Turing degrees, and $\leq$ for the partial ordering induced by the relation of Turing reducibility. The smallest Turing degree, which consists of just the recursive sets, is denoted by $\mathbf{0}$. We refer the reader to [4] for background on recursion theory. We let $c : \omega\times\omega \overset{1-1, onto}{\longrightarrow} \omega$ be a computable pairing function, with computable inverses $(\cdot)_0, (\cdot)_1 : \omega \to \omega$ such that $(c(x,y))_0 = x$, $(c(x,y))_1 = y$, and $c((x)_0,(x)_1) = x$ hold for all $x, y \in \omega$ (see [2]).

Let us fix some computable 1–1 onto map $\nu : \omega \to \mathbb{Q}$, that is a map ν such that, given $k \in \omega$ one can algorithmically find integers m and n such that $n > 0$ and $\nu(k) = \frac{m}{n}$. Our results will not depend on the precise choice of the numbering ν, since one can easily verify that for any other numbering μ with this property there is a recursive 1–1 function f such that $\nu f = \mu$, that is, the numbers of elements with respect to one such numbering can be translated to the numbers with respect to the other by a computable procedure. Note that f is a computable permutation.

If for some $A \subseteq \mathbb{Q}$ the set $\nu^{-1}(A)$ is recursive or r.e., we say that the set A is *recursive* (*r.e.* respectively). If a sequence of rational numbers $(a_i)_{i<\omega}$ has the property that the mapping $i \mapsto \nu^{-1}(a_i)$ is recursive, we call this sequence *computable*. Note that these definitions do not depend on the precise choice of ν. Similar remarks apply to all subsequent definitions.

Let $\mathbf{I}$ be an *ideal* of Turing degrees, i.e., $\mathbf{0} \in \mathbf{I}$, $\mathbf{a} \leq \mathbf{b} \in \mathbf{I} \to \mathbf{a} \in \mathbf{I}$, and $\mathbf{a}, \mathbf{b} \in \mathbf{I} \to \sup\{\mathbf{a},\mathbf{b}\} \in \mathbf{I}$. We say that a function is *computable with respect to* $\mathbf{I}$ provided it is computable relative to some element of $\mathbf{I}$. We use the same terminology when speaking of recursivity, recursive enumerability, etc. In all cases, 'with respect to $\mathbf{I}$' or 'in $\mathbf{I}$' means 'in some element of $\mathbf{I}$'.

The set of all $\mathbf{I}$–recursive order-preserving permutations of the rationals is denoted by $\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle$. Clearly this is a group under composition. If $\mathbf{I} = \{\mathbf{s} \mid \mathbf{s} \leq \mathbf{d}\}$, for some Turing degree $\mathbf{d}$, we write $\mathrm{Aut}_{\mathbf{d}}\langle\mathbb{Q},\leq\rangle$ for $\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle$; in the case $\mathbf{d} = \mathbf{0}$, i.e., if $\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle$ is the group of all recursive automorphisms, we may also denote it by $\mathrm{Aut}_{\mathbf{r}}\langle\mathbb{Q},\leq\rangle$. Note that if $\mathbf{I}$ contains all Turing degrees then $\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle = \mathrm{Aut}\,\langle\mathbb{Q},\leq\rangle$.

A real r is called $\mathbf{I}$*–recursive* provided that it defines a recursive cut, i.e., the sets $\{x \in \mathbb{Q} \mid x \leq r\}$ and $\{x \in \mathbb{Q} \mid x > r\}$ are $\mathbf{I}$–recursive.

Since we are studying subgroups of the group Aut $\langle \mathbb{Q}, \leq \rangle$ of all order-preserving permutations of $\mathbb{Q}$, we shall need various pieces of terminology concerning this group, taken from [1]. If $g \in$ Aut $\langle \mathbb{Q}, \leq \rangle$ then for any orbit X of $\mathbb{Q}$ under the action of g, the least convex set containing X is called an *orbital.* Thus the orbital containing x is equal to $\{y \in \mathbb{Q} : \exists m, n \in \mathbb{Z} : g^m x \leq y \leq g^n x\}$. There are three kinds of possible orbitals, those on which g is increasing, decreasing, or constant (in the last of which the orbital just consists of a single point fixed by g), and we say that the orbital has *parity* $+1$, -1, or 0 in the three cases respectively. A standard result of Holland's (which in fact applies more generally) says that two elements of Aut $\langle \mathbb{Q}, \leq \rangle$ are conjugate if and only if there is an order-isomorphism between their sets of orbitals which preserves parity. Special sorts of elements of Aut $\langle \mathbb{Q}, \leq \rangle$ (which are needed extensively in the remainder of the paper) are called *bumps.* These are elements having just one non-trivial (parity $\neq 0$) orbital, so called in view of the shape of their graphical representation. If $f \in$ Aut $\langle \mathbb{Q}, \leq \rangle$ then we denote by $\bar{f}$ its continuous extension to $\mathbb{R}$, and we define the *support of* f to be $\mathtt{sp}\,(f) = \left\{ x \in \mathbb{R} \,\middle|\, \bar{f}(x) \neq x \right\}$.

2 Some definable properties

Let P be some property of rational numbers. We shall call a rational number r *ν–minimal with the property P* if P is satisfied by r and not by any rational with smaller ν–number.

Lemma 2.1 *Assume a and b are $\mathbf{I}$–recursive reals and $a < b$. Then there are bumps f_+ and f_- in* Aut $\langle \mathbb{Q}, \leq \rangle$ *such that* $\mathtt{sp}\,(f_+) = \mathtt{sp}\,(f_-) = (a, b)$ *and* $\forall x \in (a,b)\,(f_+(x) > x)$, $\forall x \in (a,b)\,(f_-(x) < x)$.

Proof: The proof is more-or-less standard and we give it here in order to verify that in fact we are performing an algorithmic procedure with respect to $\mathbf{I}$, and because similar arguments will be used several times in the sequel. We construct $f = f_+$ using an effective back-and-forth argument, and we may let $f_- = f_+^{-1}$.

At the end of each step we shall have a finite mapping f_s, whose domain and range are subsets of (a, b), such that the following are true:

1. $\forall x \in \mathtt{dom}\,(f_s)\,(f_s(x) > x)$;
2. $\forall x, y \in \mathtt{dom}\,(f_s)\,(x < y \rightarrow f_s(x) < f_s(y))$,
3. if we let $a_s = \min \mathtt{dom}\,(f_s)$ and $b_s = \max \mathtt{range}\,(f_s)$ then $b_s = f_s^n(a_s)$ holds for some $n < \omega$.

Description of the construction.

Step 0.
Take some $a_0, b_0 \in (a, b)$ so that $a_0 < b_0$ and let $f_0 \rightleftharpoons \{\langle a_0, b_0 \rangle\}$.

Step s+1.
If $a_s = \min \mathtt{dom}(f_s)$ and $b_s = \max \mathtt{range}(f_s)$ we let $f'_s \rightleftharpoons f_s \cup \{\langle p, a_s \rangle, \langle b_s, q \rangle\}$, where p and q are the ν–minimal rationals in (a, a_s) and (b_s, b) respectively.

Take the ν–minimal rational number $r \in (a_s, f_s^{-1}(b_s)) \setminus \mathtt{dom}(f_s)$ (for $s > 0$; if $s = 0$ this set is empty). Our goal is to add it to the domain of f_s and preserve the properties above. Consider the $a', b' \in \mathtt{dom}\, f_s$ such that $a' < b'$, $(a', b') \cap \mathtt{dom}\, f_s = \emptyset$, and $r \in (a', b')$. Take the ν–minimal rational number t lying in $(r, \infty) \cap (f_s(a'), f_s(b'))$, and add the pair $\langle r, q \rangle$ to f'_s: $f''_s \rightleftharpoons f'_s \cup \{\langle r, q \rangle\}$.

Similarly we can add the ν–minimal rational in $(f_s(a_s), b_s)$ to the range of f''_s so that the result by f''_s will satisfy the requirements above. Denote the result by f_{s+1}.

This completes the construction.

Let f be identity outside (a, b) and coincide with $\bigcup_{s<\omega} f_s$ on (a, b).

One can check by induction that if f_s satisfies requirements 1–3 above then so does f_{s+1}. Since at each stage of the construction we took rationals with minimal ν–numbers, we have that $\lim_{s\to\infty} b_s = b$ and $\lim_{s\to\infty} a_s = a$; for the same reason we see that $\mathtt{dom}(f) = \mathtt{range}(f) = \mathbb{Q}$.

We check that

$$(a, b) = \left\{ x \mid \exists n \, (f^{-n}(c) < x < f^n(c)) \right\} \tag{1}$$

for any $c \in (a, b)$, i.e., (a, b) is an orbital. By the construction, $f(a_{s+1}) = a_s$, $b_{s+1} = f(b_s)$. Take an arbitrary $c \in (a, b)$. Then there are s and n such that $f^n(a_s) < c < f^{n+1}(a_s)$. By applying f^k to all members of this inequality, we obtain $f^{k+n}(a_s) < f^k(c) < f^{n+1+k}(a_s)$. Taking limits as $k \to \infty$ and $k \to -\infty$, we have $\lim_{k\to-\infty} f^k(c) = a$ and $\lim_{k\to\infty} f^k(c) = b$, which implies (1).

Obviously, f is an element of $\mathrm{Aut}_{\mathbf{I}} \langle \mathbb{Q}, \leq \rangle$, since by construction it belongs to $\mathrm{Aut} \langle \mathbb{Q}, \leq \rangle$ and is computable with respect to $\mathbf{I}$. □

The following lemma, that asserts homogeneity of $\mathbb{Q}$ with respect to $\mathrm{Aut}_{\mathbf{I}} \langle \mathbb{Q}, \leq \rangle$, may be proved by similar methods.

Lemma 2.2 *Suppose that $a_0 < a_1 < \ldots < a_n$ and $b_0 < b_1 < \ldots < b_n$ are $\mathbf{I}$–recursive reals such that $a_i \in \mathbb{Q} \Leftrightarrow b_i \in \mathbb{Q}$, for $i = 0, \ldots, n$. Then there is $f \in \mathrm{Aut}_{\mathbf{I}} \langle \mathbb{Q}, \leq \rangle$ such that $\bar{f}(a_i) = b_i$, for all i.*

Moreover, a stronger statement for computable sequences of rationals is also true for $\mathrm{Aut}_{\mathbf{I}} \langle \mathbb{Q}, \leq \rangle$.

Lemma 2.3 *Suppose that* $(a_i)_{i\in\mathbb{Z}}$ *and* $(b_i)_{i\in\mathbb{Z}}$ *are* $\mathbf{I}$*–recursive sequences of rationals, both unbounded above and below, such that*

$$\forall i \in \mathbb{Z}\,(a_i \leq a_{i+1} \;\wedge\; b_i \leq b_{i+1} \;\wedge\; (a_i < a_{i+1}) \Leftrightarrow (b_i < b_{i+1}))\,.$$

Then there is $f \in \mathrm{Aut}_{\mathbf{I}}\,\langle\mathbb{Q},\leq\rangle$ *such that* $f(a_i) = b_i$*, for all* i*.*

As usual, $[f,g]$ stands for the group commutator $f^{-1}g^{-1}fg$ and f^g denotes the conjugate $g^{-1}fg$ of f by g.

Lemma 2.4 *The property 'p either has no orbitals of parity* $+1$ *or has no orbitals of parity* -1*' (or, equivalently,* $\forall u \in \mathbb{Q}(x(u) \geq u) \vee \forall u \in \mathbb{Q}(x(u) \leq u)$*) is first–order definable in* $\mathrm{Aut}_{\mathbf{I}}\,\langle\mathbb{Q},\leq\rangle$ *by the formula*

$$\mathtt{Comp}\,(x) \quad \rightleftharpoons \quad \exists y \exists z \left(y \neq 1 \;\wedge\; z \neq 1 \;\wedge\; \forall t\, \left[y, z^{x^t}\right] = 1\right).$$

Proof: The proof is the same as that of Lemma 3.1 in [5], except that in some places the computable versions of classical facts, here formulated as Lemmas 2.1 and 2.2, are needed. '`Comp`' stands for 'comparability with the identity'. □

Lemma 2.5 *The following property of* $x, y \in \mathrm{Aut}_{\mathbf{I}}\,\langle\mathbb{Q},\leq\rangle$*:*

$$\forall u \in \mathtt{sp}\,(x) \forall v \in \mathtt{sp}\,(y)(u < v) \vee \forall u \in \mathtt{sp}\,(x) \forall v \in \mathtt{sp}\,(y)(u > v)$$

(or, roughly speaking, $\mathtt{sp}\,(x) < \mathtt{sp}\,(y) \vee \mathtt{sp}\,(x) > \mathtt{sp}\,(y)$*) is first order definable in* $\mathrm{Aut}_{\mathbf{I}}\,\langle\mathbb{Q},\leq\rangle$ *by the formula*

$$\mathtt{Apart}\,(x,y) : \exists z \left(\mathtt{Comp}\,(z) \;\wedge\; z \neq 1 \;\wedge\; \forall t\, \left(\left[x, y^{z^t}\right] = 1\right)\right).$$

Proof: This is the same as that of [5, Lemma 3.2]. All additional permutations needed for this proof exist by Lemma 2.2. □

Lemma 2.6 *The property 'x has bounded support' is first order definable in* $\mathrm{Aut}_{\mathbf{I}}\,\langle\mathbb{Q},\leq\rangle$ *by the formula*

$$\exists z\,(\mathtt{Comp}\,(z) \;\wedge\; \mathtt{Apart}\,(f, f^z) \;\wedge\; \mathtt{Apart}\,(f, f^{z^{-1}})).$$

Proof: This is immediate from the previous lemma. □

We can now illustrate some properties of computable automorphisms that differ from those of the whole of $\mathrm{Aut}\,\langle\mathbb{Q},<\rangle$.

Consider the formula

$$\begin{aligned}\mathtt{Atom}\,(f) : \quad & \mathtt{Comp}\,(f) \;\wedge\; \neg(\exists f_0 \exists f_1\,(f_0 \neq 1 \;\wedge\; f_1 \neq 1 \;\wedge \\ & \mathtt{Apart}\,(f_0, f_1) \;\wedge\; f = f_0 f_1)).\end{aligned}$$

In the whole group Aut $\langle \mathbb{Q}, < \rangle$, this formula clearly defines bumps, but this is false in $\mathrm{Aut}_{\mathbf{r}} \langle \mathbb{Q}, \leq \rangle$, as we now show.

Proposition 2.7 *There is $f \in \mathrm{Aut}_{\mathbf{r}} \langle \mathbb{Q}, \leq \rangle$ such that*

1. *the set $\left\{ x \in \mathbb{R} \mid \bar{f}(x) = x \right\}$ is unbounded above and below;*
2. $\forall x \in \mathbb{Q} \, (f(x) > x)$;
3. $\mathrm{Aut}_{\mathbf{r}} \langle \mathbb{Q}, \leq \rangle \models \mathtt{Atom}(f)$.

Remark. All reals x such that $\bar{f}(x) = x$ are non–recursive, otherwise we could decompose f into two disjoint parts f_0 and f_1 so that it would satisfy the formula $\neg\mathtt{Atom}(f)$. Moreover, the fixed point set of $\bar{f}$ has no isolated point. For if on the contrary a real x_0 with $\bar{f}(x_0) = x_0$ is an isolated point in the fixed point set, consider $a, b \in \mathbb{Q}$ such that $a < x_0 < b$ and x_0 is the only point of $[a, b]$ not in $\mathrm{sp}\,(f)$. In this case x_0 will be a recursive real, since the sets

$$\{x \in \mathbb{Q} \mid x < x_0\} = \{x \mid \exists n \in \omega \, (x < f^n(a))\}$$

and

$$\{x \in \mathbb{Q} \mid x > x_0\} = \left\{ x \mid \exists n \in \omega \, (x > f^{-n}(b)) \right\}$$

are recursively enumerable, their union is the whole $\mathbb{Q}$, and consequently they are recursive, contrary to what we said above.

Lemma 2.8 *There is a recursively enumerable equivalence relation θ on $\mathbb{Q}$ such that*

1. *all classes of θ are convex;*
2. *if A is a recursive set of rationals which is a union of classes of θ then $A = \emptyset$ or $A = \mathbb{Q}$;*
3. *all classes of θ are open and bounded.*

Proof: First we define a uniform enumeration of all disjoint pairs of r.e. subsets of $\mathbb{Q}$. Let W_n be nth r.e. set (see [4] or [2]). Given $n \in \omega$, enumerate the sets $\widehat{U}_n$ and $\widehat{V}_n$ as follows: simultaneously enumerate sets $W_{(n)_0}$ and $W_{(n)_1}$ so that at each step at most one new element appears either in the enumeration of $W_{(n)_0}$ or in the enumeration of $W_{(n)_1}$. If at some step this newly enumerated element is still not enumerated either in $\widehat{U}_n$ or in $\widehat{V}_n$, and appears first in $W_{(n)_0}$ then we add it to $\widehat{U}_n$ and add it to $\widehat{V}_n$ otherwise. Note that for any two disjoint r.e. sets A and B there is $n \in \omega$ such that $\widehat{U}_n = A$ and $\widehat{V}_n = B$.

The finite parts of the sets $\widehat{U}_n$ and $\widehat{V}_n$ enumerated up to step t are denoted by $\widehat{U}_n^t$ and $\widehat{V}_n^t$, respectively.

Let $r_k \rightleftharpoons \nu(k)$ be the kth rational number, and let

$$\begin{aligned} U_n^t &\rightleftharpoons \left\{ r_k \,\middle|\, k \in \widehat{U}_n^t \right\}, \\ V_n^t &\rightleftharpoons \left\{ r_k \,\middle|\, k \in \widehat{V}_n^t \right\}, \\ U_n &\rightleftharpoons \bigcup_{t\in\omega} U_n^t = \left\{ r_k \,\middle|\, k \in \widehat{U}_n \right\}, \\ V_n &\rightleftharpoons \bigcup_{t\in\omega} V_n^t = \left\{ r_k \,\middle|\, k \in \widehat{V}_n \right\}. \end{aligned}$$

For us it is important that if $\mathbb{Q}$ is the union of disjoint recursive sets A_0 and A_1 then $A_0 = U_n$ and $A_1 = V_n$ for some $n \in \omega$.

Our task is to avoid the situation when U_n and V_n form a nontrivial partition of $\mathbb{Q}$ such that U_n and V_n are unions of classes of θ. In the course of the forthcoming construction, we shall search for pairs of rationals a, b such that a is suspected of belonging to one class of the partition, while b belongs to the other. In this situation we shall add all pairs between a and b to θ, i.e., the whole interval $[a, b]$ will be a subset of its equivalence class. In order to obtain a nontrivial equivalence, we only add rather small such intervals, in such a way that their whole measure will be finite. This will ensure that there are pairs of nonequivalent elements.

Define the set of pairs P, that will generate the required equivalence, in steps, and simultaneously enumerate the set S of natural numbers, for which there is no reason to execute our construction at further steps. The finite parts of P and S that have been enumerated up to step t are denoted by P_t and S_t respectively.

Step 0.
$P_0 \rightleftharpoons \emptyset$, $S_0 \rightleftharpoons \emptyset$.

Step $t > 0$.
For all $n < t$ such that there are $a \in U_n^t$ and $b \in V_n^t$ with $|a - b| < \frac{1}{2^n}$ and $n \notin S_{t-1}$, we consider such a pair $\langle a, b\rangle$ having minimal code. Let $a' = \min\{a, b\}$, $b' = \max\{a, b\}$, and add the pair $\langle a', b'\rangle$ to P, and n to S. Thus, at the end of this step the sets P_t and S_t are defined.

If we define P to be $\bigcup_{t\in\omega} P_t$, then the construction ensures that P is r.e.

Let θ be the equivalence relation containing P which is the transitive closure of θ^*, where

$$\langle x, y\rangle \in \theta^* \quad \rightleftharpoons \quad \exists\, \langle a, b\rangle \in P\, (a \le x, y \le b).$$

Clearly, θ is r.e. and all its classes of are convex. Note that each equivalence class of θ is the union of intervals of the form $[a, b]$, $\langle a, b\rangle \in P$, and has measure at most $\sum_{k\in\omega} \frac{1}{2^k} = 2$.

Each $r \in \mathbb{Q}$ is in some nontrivial class of θ because, if n_0 is such that $U_{n_0} = \left\{ r - \frac{1}{m+1} \,\middle|\, m < \omega \right\}$ and $V_{n_0} = \left\{ r + \frac{1}{m+1} \,\middle|\, m < \omega \right\}$, then some pair $\left\langle r - \frac{1}{m_0}, r + \frac{1}{m_1} \right\rangle$ will be added to P and, thus, r will be θ–equivalent to any $q \in \left[r - \frac{1}{m_0}, r + \frac{1}{m_1} \right]$. This also shows that $\theta \neq \{\langle x, x\rangle \mid x \in \mathbb{Q}\}$.

If $\mathbb{Q} = A_0 \cup A_1$ is a partition of $\mathbb{Q}$ into non-empty recursive sets A_0 and A_1, let n be such that $U_n = A_0$ and $V_n = A_1$. At some step of the construction there will be $a \in A_0$ and $b \in A_1$ such that $|a - b| < \frac{1}{2^n}$, thus at some step an element from A_0 and an element from A_1 become equivalent and therefore neither A_0 nor A_1 can be union of equivalence classes of θ. □

Lemma 2.9 *Suppose that* $\mathbf{I}$ *is an ideal of Turing degrees, and* A *is an* $\mathbf{I}$*–recursive subset of* $\mathbb{Q}$ *which is a union of some open classes of an* $\mathbf{I}$*–r.e. equivalence relation* η*. Then there is* $f \in \mathrm{Aut}_{\mathbf{I}}\langle \mathbb{Q}, \leq\rangle$ *such that*

1. *f is the identity outside A;*
2. *$f(x) > x$ for all $x \in A$;*
3. *a subset of A is a nontrivial orbital of f if and only if it is a class of η.*

Proof: This lemma is proved by a stepwise construction similar to that of Lemma 2.1. □

Proposition 2.7 follows from this lemma, since no automorphism f corresponding to the θ of Lemma 2.8 can be expressed as a product of two automorphisms f_0 and f_1 in the group $\mathrm{Aut}_{\mathbf{r}}\langle \mathbb{Q}, \leq\rangle$ so that $\mathrm{Aut}_{\mathbf{r}}\langle \mathbb{Q}, \leq\rangle \models \mathtt{Apart}\,(f_0, f_1)$. For if such f_0 and f_1 exist, the set $\mathtt{sp}\,(f_0) = \{x \in \mathbb{Q} \mid f_0(x) > x\}$ is recursive, and thus θ fails to satisfy property 2 of Lemma 2.8. The remaining parts of the proposition are trivial. Nevertheless, it can be easily seen that f can be so decomposed within $\mathrm{Aut}_{\mathbf{I}}\langle \mathbb{Q}, \leq\rangle$. □

The proposition can be relativized to $\mathrm{Aut}_{\mathbf{d}}\langle \mathbb{Q}, \leq\rangle$, for any Turing degree $\mathbf{d}$. The only change necessary is to consider the $\mathbf{d}$–r.e. family of sets $\{W_n^{\mathbf{d}}\}_{n<\omega}$ instead of $\{W_n\}_{n<\omega}$.

Corollary 2.10 *For any Turing degree* $\mathbf{d}$*,* $\mathrm{Aut}_{\mathbf{d}}\langle \mathbb{Q}, \leq\rangle$ *is not an elementary substructure of* $\mathrm{Aut}\,\langle \mathbb{Q}, \leq\rangle$*.*

The property we used to show that $\mathrm{Aut}_{\mathbf{d}}\langle \mathbb{Q}, \leq\rangle$ is not an elementary substructure of $\mathrm{Aut}\,\langle \mathbb{Q}, \leq\rangle$ is rather complicated. Now we give an example of a very easy property to establish this that uses only one existential quantifier.

Proposition 2.11 *For each Turing degree* $\mathbf{d}$ *there are automorphisms* $f, g \in \mathrm{Aut}_{\mathbf{d}} \langle \mathbb{Q}, \leq \rangle$ *which are conjugate in* $\mathrm{Aut} \langle \mathbb{Q}, \leq \rangle$ *but not in* $\mathrm{Aut}_{\mathbf{d}} \langle \mathbb{Q}, \leq \rangle$.

Proof: We give a sketch proof. Fix a $\mathbf{d}$–recursive enumerable non–$\mathbf{d}$–recursive set A. We start the step–by–step process of construction of a $\mathbf{d}$–recursive permutation f such that for all x holds $f(x) > x$ and whose support is $(0,1) \cup (1,2) \cup (2,3) \cup \ldots$, i.e., we start a countable family of processes as in Lemma 2.1, for intervals $(0,1)$, $(1,2)$, $(2,3)\ldots$ In the course of the construction, we slightly change this process, introducing new points in addition to $0, 1, 2, \ldots$ If at some step of the construction n appears in the enumeration of A, we take a ν–minimal rational point $r \in (n, n+1)$ which is less than all elements in the domain of the finite part of f enumerated so far in $(n, n+1)$, and then continue the construction of the automorphism f as two processes: construction of bumps on the intervals (n, r) and $(r, n+1)$. The resulting $\mathbf{d}$–recursive element f of $\mathrm{Aut} \langle \mathbb{Q}, \leq \rangle$ will therefore have the property that its support is the union of open intervals with rational endpoints which are ordered in type ω. Moreover, the set $S(f) = \{ \langle r_0, r_1 \rangle \mid \exists r \in \mathbb{Q} \, (r_0 < x < r_1 \ \wedge \ f(x) = x)\}$ is not $\mathbf{d}$–recursive, otherwise we would have $n \in A \Leftrightarrow \langle n, n+1 \rangle \in S$, which is impossible, since A is not $\mathbf{d}$–recursive. An immediate check proves that this property of non–recursivity of $S(f)$ is preserved under conjugation within $\mathrm{Aut}_{\mathbf{d}} \langle \mathbb{Q}, \leq \rangle$.

We construct the other automorphism g as an element of $\mathrm{Aut}_{\mathbf{d}} \langle \mathbb{Q}, \leq \rangle$ so that $\forall x (f(x) \geq x)$ and $\mathtt{sp}\,(f) = (0,1) \cup (1,2) \cup (2,3) \cup \ldots$

Thus $S(g)$ is recursive while $S(f)$ is not, so f and g are not conjugate in $\mathrm{Aut}_{\mathbf{d}} \langle \mathbb{Q}, \leq \rangle$. On the other hand, f is conjugated to g in $\mathrm{Aut} \langle \mathbb{Q}, \leq \rangle$ by Holland's criterion [5]. $\Box$

We abbreviate $\exists t (x^t = y)$ by $x \sim y$, i.e., $x \sim y$ means that x is a conjugate of y. We have already shown that some of the usual methods fail to work if we restrict the complexity of automorphisms. In spite of this, it is possible to define some important kinds of automorphisms, such as bumps, for instance:

Lemma 2.12 *The property 'x is a bump' (bounded or unbounded) is definable in* $\mathrm{Aut}_{\mathbf{I}} \langle \mathbb{Q}, \leq \rangle$ *by the formula*

$$\begin{aligned} \mathtt{bump}\,(x): \quad & \mathtt{Atom}\,(x) \ \wedge \ x \neq 1 \ \wedge \\ & \forall y \, (y \sim x \rightarrow (\mathtt{Atom}\,(xy) \rightarrow xy \sim x)). \end{aligned}$$

Proof: Assume that x satisfies this formula but fails to be a bump. Since $\mathtt{Atom}\,(x)$, there are no $a, b \in \mathtt{sp}\,(x)$, $a < b$, and non-empty open interval $(c, d) \subset (a, b)$ such that $\mathtt{sp}\,(x) \cap (c, d) = \emptyset$. Moreover, recall that by definition the property $\mathtt{Atom}\,(x)$ implies $\mathtt{Comp}\,(x)$. Without loss of generality, we may assume that $x(r) \geq r$, for all rationals r. ¿From this we deduce that the reals $r_- =$

$\inf \mathtt{sp}(x)$ and $r_+ = \sup \mathtt{sp}(x)$ (if finite) are $\mathbf{I}$–recursive. To see this, fix some rational $c_0 \in \mathtt{sp}(x)$. Now the $\mathbf{I}$–recursivity of r_- and r_+ follows from

$$c < r_- \quad \Leftrightarrow \quad c < c_0 \ \wedge \ x(c) = c,$$

$$c > r_+ \quad \Leftrightarrow \quad c > c_0 \ \wedge \ x(c) = c,$$

for all $c \in \mathbb{Q}$ (since as previously remarked, all rationals from (r_-, r_+) are in $\mathtt{sp}(x)$). Using the enumeration of rationals by natural numbers, we can construct an $\mathbf{I}$–computable sequence of rationals $\ldots < a_{-1} < b_{-1} < a_0 < b_0 < a_1 < b_1 < \ldots$, indexed by $\mathbb{Z}$, so that $\lim_{n\to\infty} a_{-n} = r_-$ and $\lim_{n\to\infty} a_n = r_+$, and $b_i = x(a_i)$, for all $i < \omega$. By Lemma 2.3, there is $h \in \mathrm{Aut}_{\mathbf{I}}\langle \mathbb{Q}, \leq\rangle$ such that h is identity on the complement of (r_-, r_+) and $h(a_i) = b_i$, $h(b_i) = a_{i+1}$, for all $i < \omega$.

To check that $g = h^{-1}xh \cdot x$ is a bump, it suffices to show that $\bar{g}(t) > t$, for all $t \in \mathbb{R} \cap (r_-, r_+)$. First suppose $\bar{x}(t) = t$ for some such t. Then there is an $i < \omega$ such that $b_i < t < a_{i+1}$. Hence

$$t < a_{i+1} \leq \bar{h}^{-1}\bar{x}\bar{h}(b_i) < \bar{h}^{-1}\bar{x}\bar{h}(t) \leq \bar{h}^{-1}\bar{x}\bar{h} \cdot \bar{x}(t) = \bar{g}(t).$$

Now suppose $\bar{x}(t) > t$. Then $\bar{g}(t) = \bar{h}^{-1}\bar{x}\bar{h}\bar{x}(t) \geq \bar{x}(t) > t$.

In each case $\bar{g}(t) > t$. For the reals t outside the interval (r_-, r_+), $\bar{g}(t) = t$ holds. Thus a conjugate of x multiplied by x is a bump and is conjugate to x. Therefore, x is itself a bump.

It can be easily seen that any bump $x \in \mathrm{Aut}_{\mathbf{I}}\langle \mathbb{Q}, \leq\rangle$ satisfies the formula $\mathtt{bump}(x)$ on $\mathrm{Aut}_{\mathbf{I}}\langle \mathbb{Q}, \leq\rangle$, so this completes the proof. □

Lemma 2.13 *Assume* $\mathbf{I}$ *is an ideal of Turing degrees. Then*

1. *bounded bumps are distinguished in* $\mathrm{Aut}_{\mathbf{I}}\langle \mathbb{Q}, \leq\rangle$ *by the formula*

$$\mathtt{BBump}(x): \quad \mathtt{bump}(x) \ \wedge \ \exists y\, (\mathtt{Comp}(y) \ \wedge \ \mathtt{Apart}(x, x^y) \ \wedge \ \mathtt{Apart}(x, x^{y^{-1}})),$$

2. *the relation*

   'b is a bounded bump such that b and xb^{-1} have disjoint supports'

   is first–order definable in $\mathrm{Aut}_{\mathbf{I}}\langle \mathbb{Q}, \leq\rangle$ *by the formula*

$$\mathtt{BumpIn}(b, x): \quad \mathtt{BBump}(b) \ \wedge \ \exists x_0 \exists x_1 (\mathtt{Apart}(x_0, x_1) \ \wedge \ \mathtt{Apart}(x_0, b) \ \wedge \ \mathtt{Apart}(x_1, b) \ \wedge \ x = x_0 b x_1),$$

3. *the relation 'b is the leftmost or rightmost bump of x' is first–order definable in* $\mathrm{Aut}_{\mathrm{I}}\langle\mathbb{Q},\leq\rangle$ *by the formula*

$$\mathtt{RLMostBump}(b,x): \quad \mathtt{BumpIn}(b,x) \ \wedge\ \mathtt{Apart}(b,xb^{-1}),$$

4. *the relation 'For all* $t\in\mathbb{R}$ $\bar{b}(t)>t$ *or for all* $t\in\mathbb{R}$ $\bar{b}(t)<t$*' is first–order definable in* $\mathrm{Aut}_{\mathrm{I}}\langle\mathbb{Q},\leq\rangle$ *by the formula*

$$\mathtt{FullBump}(x): \quad \mathtt{bump}(x) \ \wedge\ \neg\exists y\,(y\neq 1 \ \wedge\ \mathtt{Apart}(y,x)),$$

5. *the relation 'x is a bump and it is neither a bounded bump nor a full bump' is first–order definable in* $\mathrm{Aut}_{\mathrm{I}}\langle\mathbb{Q},\leq\rangle$ *by the formula*

$$\mathtt{SBump}(x): \quad \mathtt{bump}(x) \ \wedge\ \neg\mathtt{BBump}(x) \ \wedge\ \neg\mathtt{FullBump}(x).$$

Proof: These are all immediate from the characterizations of `Comp`, `Apart`, and `bump` given above. □.

Now we are able to interpret the natural numbers $\mathbb{N}$ in our group. This is needed to speak about definability on $\mathbb{N}$.

Let

$$\mathtt{Norm}_1(x,y,z) \ \rightleftharpoons\ \mathtt{Comp}(x)\ \wedge\ \mathtt{Comp}(y)\ \wedge$$
$$\mathtt{RLMostBump}(y,x)\ \wedge\ \mathtt{FullBump}(z)\ \wedge\ x^z = xy^{-1},$$

and

$$\mathtt{Norm}(x,y,z) \rightleftharpoons (\exists u)(\exists v)(\mathtt{Norm}_1(x,y,z)\wedge \mathtt{Norm}_1(u,y,v)\wedge z=v^2).$$

Lemma 2.14 *Suppose that* $\mathrm{Aut}_{\mathrm{I}}\langle\mathbb{Q},\leq\rangle \models \mathtt{Norm}_1(a,b,z)$*. Then all bumps* b^{z^n} *have disjoint supports, these supports are ordered either like* ω *(in the case* $z>1$*) or like* ω^* *(in the case* $z<1$*), and* $a=\prod_{n<\omega} b^{z^n}$*. If* $\mathrm{Aut}_{\mathrm{I}}\langle\mathbb{Q},\leq\rangle \models \mathtt{Norm}(a,b,z)$ *then in addition for any two consecutive bumps* b^{z^n} *and* $b^{z^{n+1}}$ *of* a *there is a non-trivial interval between their supports.*

Proof: Without loss of generality assume $z>1$. Then b is the leftmost bump in a, and the next bump is b^z, the bump after it is b^{z^2}, the next bump is b^{z^3}, etc. It remains to show that a can be exhausted in this way. But since z has a unique orbital, $\lim_{n\to\infty}\bar{z}^n(t)=\infty$, for each $t\in\mathbb{R}$, and thus, for each $q\in\mathbb{Q}$ there is $n\in\omega$ such that $\mathtt{sp}(b^{z^n})>q$ which yields what is required.

The statement concerning `Norm` follows since if $z=v^2$ then the bumps of $\prod_{n<\omega} b^{z^n}$ are the alternate bumps of $\prod_{n<\omega} b^{v^n}$. □

If $\mathrm{Aut}_{\mathtt{I}}\langle \mathbb{Q}, \leq \rangle \models \mathtt{Norm}(a,b,z)$ then we call a,b,z *normal* parameters. The automorphism a in this case is a product of bumps b^{z^n} for $n \in \omega$ ordered either in type ω or ω^*. We deal with the first case only, since the two cases are isomorphic to each other (under an automorphism induced by the mapping $q \mapsto \widehat{q}$, $\widehat{q}(x) = q(-x)$).

Fix some normal parameters $\bar{p} = a, b, z$. Now we define the ordering on these bumps:

$$x \prec_{\bar{p}} y \quad \rightleftharpoons \quad \mathtt{BumpIn}(x,a) \ \wedge \ \mathtt{BumpIn}(y,a) \wedge \exists t\,(t \sim z \ \wedge \ x^t = y).$$

Using this ordering we can define the successor relation on bumps of a by:

$$\mathtt{sc}(x,y) \quad \rightleftharpoons \quad x \prec_{\bar{p}} y \ \wedge \ \neg\exists u(x \prec_{\bar{p}} u \prec_{\bar{p}} y).$$

The corresponding function will be referred to as s, i.e., $\mathtt{sc}(x,y) \Leftrightarrow s(x) = y$.

We write $0_{\bar{p}}$ for the minimal element b, which may also be characterized by the formula $\forall t \neg(t \prec_{\bar{p}} x)$. More generally we write $n_{\bar{p}}$ for the nth bump.

Now we can define operations of addition and multiplication on bumps of a (denoted by $+_{\bar{p}}$ and $\cdot_{\bar{p}}$ respectively):

$$\begin{aligned} x +_{\bar{p}} y = w \quad \rightleftharpoons \quad & \exists u\,(u 0_{\bar{p}} u^{-1} = x \ \wedge \\ & \forall t \prec_{\bar{p}} y(\mathtt{BumpIn}(utu^{-1},a) \ \wedge \ u(s(t))u^{-1} = s(utu^{-1})) \ \wedge \\ & uyu^{-1} = w), \end{aligned}$$

$$\begin{aligned} x \cdot_{\bar{p}} y = w \quad \rightleftharpoons \quad & \exists u\,(u 0_{\bar{p}} u^{-1} = 0_{\bar{p}} \ \wedge \\ & \forall t \prec_{\bar{p}} y(\mathtt{BumpIn}(utu^{-1},a) \ \wedge \ u(s(t))u^{-1} = utu^{-1} +_{\bar{p}} x) \\ & \wedge \ uyu^{-1} = w). \end{aligned}$$

Lemma 2.15 *The predicate*

$$\begin{aligned} \mathtt{same}(x,y) \quad \rightleftharpoons \quad & \mathtt{FullBump}(y) \ \wedge \\ & (\forall t\,(y(t) > t \ \wedge \ x(t) \geq t) \vee \forall t\,(y(t) < t \ \wedge \ x(t) \leq t)) \end{aligned}$$

is first–order definable in $\mathrm{Aut}_{\mathtt{I}}\langle \mathbb{Q}, \leq \rangle$ *by the formula*

$$\mathtt{FullBump}(y) \ \wedge \ \forall y' \sim y(y'x \sim y).$$

Proof: If $\mathtt{same}(x,y)$ then this formula is obviously satisfied.

On the other hand, assume that $\mathtt{FullBump}(y) \ \wedge \forall y' \sim y(y'x \sim y)$, $\forall t\,(y(t) > t)$, but nevertheless $\neg(x \geq 1)$. In this case $x(t) < t$ for some $t \in \mathbb{Q}$. Take a conjugate y' of y so that $x(t) < y'x(t) < t$. Then $\neg(y'x \geq 1)$ and $y'x$ cannot be conjugate to y.

The other case $\forall t\,(y(t) < t)$ and $\neg(x \leq 1)$ is similar. $\Box$

Assume $\bar{p} = a, b, z$ are normal parameters as above.

Now we define representations in $\mathrm{Aut}_{\mathbf{I}}\langle \mathbb{Q}, \leq \rangle$ of a class of functions $\omega \to \omega$ which will be enough to recover the ideal $\mathbf{I}$.

Let

$$C(\bar{p}) \quad \rightleftharpoons \quad \left\{ f \;\middle|\; \mathtt{same}\,(f, z) \;\wedge\; \forall t\,[\mathtt{BumpIn}\,(t, a) \rightarrow \mathtt{BumpIn}\,(ftf^{-1}, a)] \right\}.$$

Suppose that $z > 1$. Then $C(\bar{p})$ is the set of all nonnegative $f \in \mathrm{Aut}_{\mathbf{I}}\langle \mathbb{Q}, \leq \rangle$ that take bumps to bumps by conjugation.

Now assume $f \in C(\bar{p})$. Define the value of $F(f, m, \bar{p})$ to be n if and only if f takes the mth bump $m_{\bar{p}}$ of a to its nth bump $n_{\bar{p}}$ by conjugation, i.e., $fm_{\bar{p}}f^{-1} = n_{\bar{p}}$.

Lemma 2.16 *The set of functions* $\{\lambda m.F(f, m, \bar{p}) \mid f \in C(\bar{p})\}$, *where* $\bar{p} = a, b, z$ *are normal parameters, consists of all monotonic* I*–recursive embeddings* $\omega \to \omega$.

Proof: We remark that to prove this lemma we need to know that there is some 'space' between consecutive bumps of the normal parameters, since we may have to fit in many bumps of one automorphism in between consecutive bumps of another. This is why we modified the definition of $\mathtt{Norm}_1$ to give $\mathtt{Norm}$.

Following this idea, we have to verify the computability of various notions. For the sake of definiteness, assume $a, b \geq 1$ and $\forall t\,(z(t) > t)$.

If $q \in \mathbb{Q}$ belongs to the support of $m_{\bar{p}}$ then we say that q *is in the* m*th bump of* a. Note that if q is rational, then the following relations are $\mathbf{I}$–recursive:

1. q is in the mth bump of a;
2. q lies between the mth and $(m+1)$th bumps of a;
3. q is less than all bumps of a;
4. $q_0, q_1 \in \mathbb{Q}$ are in the same bump of a.

Indeed, (1) is equivalent to $z^m b z^{-m}(q) > q$; (2) is equivalent to $a(q) = q$ and $z^m(q_0) < q < z^{m+1}(q_0)$, where q_0 is an arbitrary fixed rational in the (unique) orbital of b (= the first orbital of a); (3) is equivalent to $a(q) = q \;\wedge\; q < q_0$, where q_0 is as above; to prove that (4) is recursive, note that this statement is equivalent to

$$a(q_0) \neq q_0 \;\wedge\; a(q_1) \neq q_1 \rightarrow$$

$$\neg\exists m \exists n (m \neq n \;\wedge\; q_0 \text{ belongs to } m\text{th bump of } a \;\wedge$$

$$q_1 \text{ belongs to } n\text{th bump of } a)$$

as well as to

$$\exists m\, (z^m b z^{-m}(q_0) > q_0 \;\wedge\; z^m b z^{-m}(q_1) > q_1),$$

i.e., it and its complement are **I**–r.e.

Making use of these facts, one can for each monotonic **I**–recursive embedding $\omega \rightarrow \omega$ construct $\widehat{f} \in C(\bar{p})$ so that $f = \lambda m.F(\widehat{f}, m, \bar{p})$.

On the other hand, if $f \in C(\bar{p})$ then clearly the function $\lambda m.F(f, m, \bar{p})$ is a monotonic embedding $\omega \rightarrow \omega$ and

$$F(f, m, \bar{p}) = n \Leftrightarrow \exists q_0 \exists q_1 \in \mathbb{Q}(q_0 \text{ is in the } m\text{th bump } \;\wedge\; q_1 \text{ is in the } n\text{th bump } \;\wedge\; f(q_0) \text{ and } q_1 \text{ are in the same bump}).$$

This shows that the set $\{\langle m, n\rangle \mid F(f, m, \bar{p}) = n\}$ is **I**–r.e. and thus the function $f = \lambda m.F(f, m, \bar{p})$ is **I**–recursive. □

We say that $f \in C(\bar{p})$ *codes a set* $A \subseteq \omega$ if

$$\forall m\, (m \in A \Leftrightarrow F(f, m, \bar{p}) + 1 < F(f, m+1, \bar{p})).$$

Lemma 2.17 *A subset A of ω is coded by some $f \in C(\bar{p})$ if and only if A is recursive in* **I**. *Each $f \in C(\bar{p})$ codes some* **I***–recursive set.*

Proof: Immediate. □

Lemma 2.18 *Let $R_{\mathbf{I}}$ be the set of all I–recursive subsets of ω. Then the many–sorted structure*

$$\langle \mathrm{Aut}_{\mathbf{I}} \langle \mathbb{Q}, \leq \rangle, \omega, R_{\mathbf{I}}; +, \cdot, F, \in \rangle$$

is definable in $\mathrm{Aut}_{\mathbf{I}} \langle \mathbb{Q}, \leq \rangle$ *from any normal parameters $\bar{p} = a, b, z$. The formulae that provide this interpretation do not depend either on I or on the choice of normal parameters $\bar{p}$.*

Proof: We mean of course that ω is represented as the set

$$\{t \mid \mathrm{Aut}_{\mathbf{I}} \langle \mathbb{Q}, \leq \rangle \models \mathtt{BumpIn}\,(t, a)\} = \{n_{\bar{p}} : n \in \omega\}.$$

The operations $+$ and $\cdot$ were defined above. The natural ordering $<$ can also be defined from $+$. For $f \in C(\bar{p})$ we let $F(f, m, \bar{p}) = n \Leftrightarrow f m_{\bar{p}} f^{-1} = n_{\bar{p}}$ (recall that $m_{\bar{p}}$ and $n_{\bar{p}}$ are bumps). Elements of $R_{\mathbf{I}}$ correspond to classes of functions $f \in C(\bar{p})$ that define the same set, i.e., f_0 and f_1 in $C(\bar{p})$ denote the same set if and only if

$$\forall m\,(F(f_0, m, \bar{p}) + 1 < F(f_0, m+1, \bar{p}) \Leftrightarrow F(f_1, m, \bar{p}) + 1 < F(f_1, m+1, \bar{p})).$$

In view of the interpretations above, this statement can be expressed by a formula. It remains to express the relation 'm belongs to the set represented by f', for which we use the formula $F(f, m, \bar{p}) + 1 < F(f, m+1, \bar{p})$.

Theorem 2.19 *Let $\mathcal{A}$ be an arithmetical class of sets. Then there is a first order sentence φ of the language of group theory such that for all Turing degrees* $\mathbf{d}$

$$\mathrm{Aut}_{\mathbf{d}}\,\langle \mathbb{Q}, \leq \rangle \models \varphi \Leftrightarrow \mathbf{d} \cap \mathcal{A} \neq \emptyset.$$

Proof: Let $I = \{\mathbf{b} \mid \mathbf{b} \leq \mathbf{d}\}$. We need the fact that Turing reducibility $X \leq_T Y$ can be expressed in the language of arithmetic with additional unary predicates for X and Y (see [4]). By Lemma 2.18, there is a formula $\psi(\bar{p})$ that expresses $\exists X \in R_{\mathbf{I}}((\forall Y \in R_{\mathbf{I}}\; Y \leq_T X) \;\wedge\; X \in \mathcal{A})$ with respect to any triple of normal parameters. The required group-theoretical formula is then $\exists \bar{p}(\mathtt{Norm}\,(\bar{p}) \wedge \psi(\bar{p}))$. □

Theorem 2.20 *If φ is a first order sentence of the language of group theory, then the union of the class of Turing degrees* $\mathbf{d}$ *such that* $\mathrm{Aut}_{\mathbf{d}}\,\langle \mathbb{Q}, \leq \rangle \models \varphi$ *is an arithmetical family of sets.*

Proof: The proof is rather standard, see for instance [3]. The idea of the proof is that first we prove that the predicate 'm is a number of an X–recursive function that defines an X–recursive automorphism of $\langle \mathbb{Q}, \leq \rangle$ on ν–numbers' is expressible in arithmetic with an auxiliary unary predicate X; then we express the statement that m and n define the same X–recursive automorphisms of $\langle \mathbb{Q}, \leq \rangle$; next the statement that k is the number of an X-recursive composition of X-recursive automorphisms with numbers m and n. Finally, we express the statement that the group of all X–recursive permutations satisfies the sentence φ. □

Corollary 2.21 *Suppose that* $\mathbf{d}$ *contains an arithmetical set, or is a Turing degree of the form $\mathbf{0}^{(\alpha)}$, where α is a recursive ordinal. Then there is a first order sentence φ such that for all ideals* $\mathbf{I}$ *of Turing degrees*

$$\mathrm{Aut}_{\mathbf{I}}\,\langle \mathbb{Q}, \leq \rangle \models \varphi \Leftrightarrow I = \{\mathbf{s} \mid \mathbf{s} \leq \mathbf{d}\}.$$

This follows from fact that such degrees $\mathbf{d}$ are arithmetical classes of sets (see [4]).

Theorem 2.22 *For all ideals of Turing degrees* $\mathbf{I}$ *and* $\mathbf{J}$

$$\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle \cong \mathrm{Aut}_{\mathbf{J}}\langle\mathbb{Q},\leq\rangle \Leftrightarrow \mathbf{I}=\mathbf{J}.$$

Proof: This follows from the interpretation of $R_{\mathbf{I}}$ in $\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle$, since $R_{\mathbf{I}}$ is thus uniquely defined by the isomorphism type of $\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle$. □

Theorem 2.23 *For all ideals of Turing degrees* $\mathbf{I}$ *and* $\mathbf{J}$

$$\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle \textit{ is embeddable in } \mathrm{Aut}_{\mathbf{J}}\langle\mathbb{Q},\leq\rangle \Leftrightarrow I\subseteq J.$$

holds.

Proof: We give a sketch proof of $\Rightarrow$. The idea is to code degrees into word problems.

We show that if an arbitrary set $A\subseteq\omega$ is r.e. in $\mathbf{I}$ then it is r.e. in $\mathbf{J}$. From this it follows that any set recursive in $\mathbf{I}$ is also recursive in $\mathbf{J}$, which implies $\mathbf{I}\subseteq\mathbf{J}$.

First, consider $c_0, c_1 \in \mathrm{Aut}_{\mathbf{r}}\langle\mathbb{Q},\leq\rangle$ such that c_0 and c_1 are the identity outside $(0,1)$ and for all $x\in\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle$, $[x,c_0]=[x,c_1]=1$ holds if and only if x is the identity on $(0,1)$. Such c_0, c_1 exist; for instance, if we let $c_0^*(x)=x+1$ and $c_1^*(x)=2x$, one can easily check that $[y,c_0^*]=[y,c_1^*]=1\Rightarrow y=1$. Take an arbitrary computable 1–1 order–preserving mapping $\mathbb{Q}\to(0,1)$. Now transfer c_0^* and c_1^* to $(0,1)$ by means of this mapping, and then expand the resulting mappings in a trivial way to the whole $\mathbb{Q}$ to give the required c_0 and c_1.

The next step is to construct $b\in\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle$ that codes the set A. We construct b so that all its nontrivial orbitals are subsets of intervals of the type $(n,n+1)$ for $n\in\omega$, and b has a nontrivial orbital on $(n,n+1)$ if and only if $n\in A$. The idea is to construct step by step identity mappings on each open interval $(n,n+1)$, $n\in\omega$, and if, in the process of simultaneous enumeration of A relative to $\mathbf{I}$, n is enumerated into A, then construct a nontrivial bump on the rest of $(n,n+1)$.

Let $z(x)=x+1$. The elements $z^m c_0 z^{-m}$ and $z^m c_1 z^{-m}$ have the property that an element x commutes with both of them if and only if x is the identity on $(m,m+1)$. Thus, $[z^m c_0 z^{-m},b]=[z^m c_1 z^{-m},b]=1$ holds if and only if $m\notin A$. Considering the images $\mathbf{c_0},\mathbf{c_1},\mathbf{b},\mathbf{z}\in\mathrm{Aut}_{\mathbf{J}}\langle\mathbb{Q},\leq\rangle$ of c_0,c_1,b,z, respectively, under an arbitrary embedding of $\mathrm{Aut}_{\mathbf{I}}\langle\mathbb{Q},\leq\rangle$ into $\mathrm{Aut}_{\mathbf{J}}\langle\mathbb{Q},\leq\rangle$, we have the same equivalence:

$$[\mathbf{z}^m\mathbf{c_0}\mathbf{z}^{-m},\mathbf{b}][\mathbf{z}^m\mathbf{c_1}\mathbf{z}^{-m},\mathbf{b}]=1\Leftrightarrow m\notin A.$$

It remains to observe that the left hand condition is co-r.e. in $\mathbf{J}$, which follows from the recursion–theoretic fact that there are $\mathbf{J}$–computable total functions $F_0(m,x)$ and $F_1(m,x)$ such that

$$\nu^{-1}[\mathbf{z}^m\mathbf{c_0}\mathbf{z}^{-m},\mathbf{b}](q)=F_0(m,\nu^{-1}(q))$$

and

$$\nu^{-1}[\mathbf{z}^m \mathbf{c_1} \mathbf{z}^{-m}, \mathbf{b}](q) = F_1(m, \nu^{-1}(q)),$$

for all $q \in \mathbb{Q}$.

By the above we have that $m \notin A$ is equivalent to

$$[\mathbf{z}^m \mathbf{c_0} \mathbf{z}^{-m}, \mathbf{b}] = [\mathbf{z}^m \mathbf{c_1} \mathbf{z}^{-m}, \mathbf{b}] = 1 \Leftrightarrow (\forall x\, F_0(m, x) = x) \ \wedge \ (\forall x\, F_1(m, x) = x),$$

from which it is easy to see that this condition is co-r.e., and hence A is r.e. relative to $\mathbf{J}$.

The proof of $\Leftarrow$ is trivial. □

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