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On Σ-definability without equality over the real numbers

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In [5] (1982) it has been shown that for first-order definability over the reals there exists an effective procedure which by a finite formula with equality defining an open set produces a finite formula without equality that defines the same set. In this paper we prove that there exists no such procedure for Σ-definability over the reals. We also show that there exists even no uniform effective transformation of the definitions of Σ-definable sets (i. e., Σ-formulas) into new definitions of Σ-definable sets in such a way that the results will define open sets, and if a definition defines an open set, then the result of this transformation will define the same set. These results highlight the important differences between Σ-definability with equality and Σ-definability without equality.

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1 Introduction

Some of the significant results in first-order definability over the reals are quantifier elimination [14], which can be considered as a semantic characterisation of definable sets, and existence of an effective procedure which by a finite formula with equality defining an open set produces a finite formula without equality defining the same set [5].

One of the most interesting and practically important types of definability is Σ-definability. The concept of Σ-definability is closely related to the generalised computability on abstract structures [1, 7, 13], in particular on the real numbers [9, 10, 12]. Notions of Σ-definable sets or relations generalise those of computably enumerable sets of natural numbers and play a leading role in the specification theory that is used in the higher order computation theory on abstract structures [8].

There are known semantic characterisations of sets over the reals which are Σ-definable with equality [7] and Σ-definable without equality [10]. It is natural to ask whether there is an effective procedure which by a Σ-formula with equality defining an open subset of $\mathbb{R}^n$ produces a Σ-formula without equality defining the same set. In the present paper we give a negative answer to this question. We also show that there is no uniform effective transformation of the definitions of Σ-definable sets (i. e., Σ-formulas) into new definitions of Σ-definable sets in such a way that the results will define open sets, and if a definition defines an open set, then the result of this transformation will define the same set. For related results we refer to [3, 4, 15, 16, 17], where computability of the interior of a given set was studied in different frameworks.

2 Basic definitions and notions

In this paper we consider the ordered structure of the real numbers,

$$\langle \mathbb{R}, 0, 1, +, \cdot, <, = \rangle \rightleftharpoons \langle \mathbb{R}, \sigma_0 \rangle.$$

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We extend the real numbers by the set of hereditarily finite sets $\mathrm{HF}(\mathbb{R})$ which is rich enough for information to be coded and stored. We construct the set $\mathrm{HF}(\mathbb{R})$ of hereditarily finite sets over the reals as follows:

1. $\mathrm{HF}_0(\mathbb{R}) \rightleftharpoons \mathbb{R}$.
2. $\mathrm{HF}_{\mathrm{n}+1}(\mathbb{R}) \rightleftharpoons \mathcal{P}_\omega(\mathrm{HF}_\mathrm{n}(\mathbb{R})) \cup \mathrm{HF}_\mathrm{n}(\mathbb{R})$, where $n \in \omega$ and for every set B, $\mathcal{P}_\omega(B)$ is the set of finite subsets of B.
3. $\mathrm{HF}(\mathbb{R}) \rightleftharpoons \bigcup_{m\in\omega} \mathrm{HF}_m(\mathbb{R})$.

We define $\mathbf{HF}(\mathbb{R})$ as the following model:

$$\mathbf{HF}(\mathbb{R}) \rightleftharpoons \langle \mathrm{HF}(\mathbb{R}), U, \sigma_0, \emptyset, \in \rangle \rightleftharpoons \langle \mathrm{HF}(\mathbb{R}), \sigma \rangle,$$

where the constant $\emptyset$ stands for the empty set and the binary predicate symbol $\in$ has the set-theoretic interpretation. We also add a 1-ary predicate symbol U naming the set of urelements (the real numbers).

The natural numbers $0, 1, \ldots$ are identified with the (finite) ordinals in $\mathrm{HF}(\mathbb{R})$, i. e., $\emptyset, \{\emptyset, \{\emptyset\}\}, \ldots$, so in particular, $n + 1 = n \cup \{n\}$ and the set ω is a subset of $\mathrm{HF}(\mathbb{R})$.

The atomic formulas are $U(x)$, $p(\bar{x}) = q(\bar{y})$, $p(\bar{x}) < q(\bar{y})$, $z \in s$, where s ranges over sets and p, q are polynomials with integer coefficients.

The set of Δ_0-*formulas* is the closure of the set of atomic formulas under operators $\wedge, \vee, \neg$, bounded quantifiers $(\exists x \in y)$ and $(\forall x \in y)$, with $(\exists x \in y)\Psi$ meaning the same as $\exists x(x \in y \wedge \Psi)$ and $(\forall x \in y)\Psi$ meaning the same as $\forall x(x \in y \rightarrow \Psi)$, where y ranges over sets.

The set of Σ-*formulas* is the closure of the set of Δ_0-formulas under $\wedge$, $\vee$, $(\exists x \in y)$, $(\forall x \in y)$, where y ranges over sets, and $\exists$.

The set of $\Sigma_<$-*formulas* is the subset of Σ-formulas that have positive occurrences of the predicate "$<$" and do not have occurrences of the predicate "$=$".

Definition 2.1

1. A relation $B \subseteq \mathrm{HF}(\mathbb{R})^n$ is Δ_0-*definable* (Σ-*definable*) if there exists a Δ_0-formula (Σ-formula) $\Phi(\bar{a})$ with

$$\bar{b} \in B \Leftrightarrow \mathbf{HF}(\mathbb{R}) \vDash \Phi(\bar{b}).$$

2. A function $f : \mathrm{HF}(\mathbb{R})^n \longrightarrow \mathrm{HF}(\mathbb{R})^m$ is Δ_0-*definable* (Σ-*definable*) if there exists a Δ_0-formula (Σ-formula) $\Phi(\bar{c}, \bar{d})$ such that

$$f(\bar{a}) = \bar{b} \Leftrightarrow \mathbf{HF}(\mathbb{R}) \vDash \Phi(\bar{a}, \bar{b}).$$

In sequel we say that a relation is Σ-*definable without equality* if it is definable by a $\Sigma_<$-formula. The following theorems reveal algorithmic properties of Σ-formulas over $\mathbf{HF}(\mathbb{R})$.

Theorem 2.2 (Semantic characterisation of Σ-definability [7]) *A set $A \subseteq \mathbb{R}^n$ is Σ-definable if and only if there exists an effective sequence of quantifier-free formulas in the language σ_0, $\{\Phi_s(x)\}_{s\in\omega}$, such that*

$$x \in A \Leftrightarrow \mathbf{HF}(\mathbb{R}) \vDash \bigvee_{s\in\omega} \Phi_s(x).$$

Remark 2.3 The semantic characterisation of Σ-definability reveals correspondence between Σ-definability and BSS-semidecidability [2, 3]. It is easy to see that for the real numbers with equality these concepts coincide.

Theorem 2.2 implies the following.

Corollary 2.4 *The set of pairs $\langle \ulcorner\varphi(x)\urcorner, n\rangle$, where $\ulcorner\varphi(x)\urcorner$ denotes the codes of a formula φ with at most one free variable x, n is a natural number, and $\mathbf{HF}(\mathbb{R}) \vDash \varphi(n)$, is computably enumerable.*

Theorem 2.5 (Semantic characterisation of Σ-definability without equality [10]) *A set $B \subseteq \mathbb{R}^n$ is Σ-definable without equality if and only if there is an effective sequence of quantifier-free formulas in the language σ_0 with positive occurrences of "$<$" and without occurrences of "$=$", $\{\Psi_s(x)\}_{s\in\omega}$, such that*

$$x \in B \Leftrightarrow \mathbf{HF}(\mathbb{R}) \vDash \bigvee_{s\in\omega} \Psi_s(x).$$

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Remark 2.6 It is worth noting that Theorem 2.2 and Theorem 2.5 give us effective procedures which generate formulas approximating Σ-relations and provide tools for descriptions of the results of effective infinite approximating processes by finite formulas.

For $\bar{a} \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}$, let $B(\bar{a}, \varepsilon) = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x} - \bar{a}\| < \varepsilon\}$.

Definition 2.7 A set $S \subseteq \mathbb{R}^n$ is said to be *computably enumerable* (*c. e.*) *open* if there are computable families $(\bar{a}_i)_{i\in\omega} \in (\mathbb{Q}^n)^\omega$ and $(\varepsilon_i)_{i\in\omega} \in \mathbb{Q}^\omega$ such that $S = \bigcup_{i\in\omega} B(\bar{a}_i, \varepsilon_i)$.

Now we give a different view to the sets which are Σ-definable without equality. We consider

$$\mathbf{HF}(\mathbb{Q}) \rightleftharpoons (\mathrm{HF}(\mathbb{Q}), U, \in, \sigma_{\mathbb{Q}}) \rightleftharpoons (\mathrm{HF}(\mathbb{Q}), U, \in, 0, 1, +, \cdot, <, =).$$

Define the predicate $P^r(p, q) \subseteq \mathbb{R} \times \mathbb{Q}^2$ as follows:

$$P^r(p, q) \rightleftharpoons U(p) \wedge U(q) \wedge (p < r < q).$$

As variables for upper indices in the predicates $P^X(x, y)$, we here use capital letters $X, Y, \dots$, maybe with indices, and small letters, maybe with indices, for the rest cases. We assume the set of capital variables and the set of small variables to be disjoint.

Define the class of $\Delta_0^{\mathbb{R}}$-formulas as the smallest class of formulas that contains all atomic formulas of the signature $\sigma_{\mathbb{Q}}$, all formulas $P^X(y, z)$, and is closed under conjunctions, disjunctions, negations, and bounded quantifications with small variables $(\forall x \in y)$ and $(\exists x \in y)$.

The class of $\Sigma^{\mathbb{R}}$-formulas is defined as the smallest class of formulas which is closed under conjunctions, disjunctions, bounded quantifications with small variables $(\forall x \in y)$, $(\exists x \in y)$, and existential quantifications.

A $\Sigma^{\mathbb{R}}$-formula φ is called a *positive* $\Sigma^{\mathbb{R}}$*-formula* if all the occurrences of predicates $P^X(x, y)$ in this formula are positive. Such formulas are referred to as $\Sigma_+^{\mathbb{R}}$*-formulas*.

For each $\Sigma_+^{\mathbb{R}}$-formula $\varphi(X_1, \dots, X_m, y_1, \dots, y_n)$ and for each $r_1, \dots, r_m \in \mathbb{R}$, $q_1, \dots, q_n \in \mathbf{HF}(\mathbb{Q})$, the relation $\mathbf{HF}(\mathbb{Q}) \vDash \varphi(r_1, \dots, r_m, q_1, \dots, q_n)$ is defined in a natural way by induction.

A set $S \subseteq \mathbb{R}^n$ is $\Sigma_+^{\mathbb{R}}$*-definable* if there are a $\Sigma_+^{\mathbb{R}}$-formula $\varphi(\bar{X}, \bar{y})$ and a tuple of parameters $\bar{q} \in \mathbf{HF}(\mathbb{Q})$ such that

$$S = \{\bar{r} \in \mathbb{R} \mid \mathbf{HF}(\mathbb{Q}) \vDash \varphi(\bar{r}, \bar{q})\}.$$

Taking into account that all elements of $\mathbf{HF}(\mathbb{Q})$ are Σ-definable over $\mathbf{HF}(\mathbb{Q})$, we may assume that the tuple $\bar{q}$ is empty.

Let $\bar{q} = \langle q_0', q_0'', q_1', q_1'', \dots, q_m', q_m'' \rangle \in \mathbb{Q}^{2m}$. We define $B(\bar{q})$ to be the set

$$B(\bar{q}) \rightleftharpoons \{\bar{r} = \langle r_1, \dots, r_m \rangle \in \mathbb{R}^m \mid \textstyle\bigwedge_{i=1}^m (q_i' < r_i < q_i'')\}.$$

Lemma 2.8 *If $B \subseteq \mathbf{HF}(\mathbb{Q})$ is Σ-definable, then B is $\Sigma_+^{\mathbb{R}}$-definable.*

Proof. The claim is straightforward from definitions. □

Theorem 2.9 *A set $S \subseteq \mathbb{R}^m$ is $\Sigma_+^{\mathbb{R}}$-definable if and only if there exists a Σ-definable function $f : \omega \longrightarrow \mathbb{Q}^{2m}$ such that $S = \bigcup_{i\in\omega} B(f(i))$. Moreover, given a $\Sigma_+^{\mathbb{R}}$-formula, we can effectively construct an algorithm to compute this function f.*

Proof.

$(\Rightarrow)$ First let us note that if $S \subseteq \mathbb{R}^m$ is $\Sigma_+^{\mathbb{R}}$-definable, then there exists an effective sequence of existential formulas which defines the set S (see [7]). We fix a Σ-definable numbering $\nu : \omega \longrightarrow \mathbb{Q}$. In order to illustrate how to construct f we consider the following example. We suppose $S \subseteq \mathbb{R}$ is definable by the formula

$$\varphi(X) \rightleftharpoons \bigvee_{n\in\omega} \exists q_1^1 \exists q_2^1 \exists q_1^2 \exists q_2^2 (P^X(q_1^1, q_2^1) \wedge P^X(q_1^2, q_2^2) \wedge p_1^1(\nu n) = q_1^1 \wedge p_2^1(\nu n) = q_2^1$$
$$\wedge\, p_1^2(\nu n) = q_1^2 \wedge p_2^2(\nu n) = q_2^2),$$

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where $p^i_j(x)$ is a polynomial with integer coefficients. Define

$$f(n) = \begin{cases} (\nu n, \nu n) & \text{if } p^1_2(\nu n) < p^2_1(\nu n) \vee p^2_2(\nu n) < p^1_1(\nu n), \\ (p^2_1(\nu n), p^1_2(\nu n)) & \text{if } p^2_1(\nu n) < p^1_2(\nu n), \\ (p^2_1(\nu n), p^2_2(\nu n)) & \text{if } p^1_1(\nu n) < p^2_1(\nu n) \wedge p^2_2(\nu n) < p^1_2(\nu n), \\ (p^1_1(\nu n), p^2_2(\nu n)) & \text{if } p^1_1(\nu n) < p^2_2(\nu n), \\ (p^1_1(\nu n), p^1_2(\nu n)) & \text{if } p^2_1(\nu n) < p^1_1(\nu n) \wedge p^1_2(\nu n) < p^2_2(\nu n). \end{cases}$$

It is easy to see that f is Σ-definable. In the general case, the proof is routine based on the same ideas.

(⇐) Suppose that there exists a Σ-definable function $f : \omega \longrightarrow \mathbb{Q}^{2m}$ such that $S = \bigcup_{i\in\omega} B(f(i))$. By Lemma 2.8, the $\Sigma^{\mathbb{R}}_+$-formula

$$\varphi(X) \rightleftharpoons \exists n \exists q^1_1 \exists q^1_2 \ldots \exists q^m_1 \exists q^m_2 (n \in \omega \wedge f(n) = (q^1_1, q^1_2, \ldots, q^m_1, q^m_2) \wedge \textstyle\bigwedge_{i\le m} P^X(q^i_1, q^i_2))$$

defines the set S. □

The following proposition shows that the concept of $\Sigma^{\mathbb{R}}_+$-definability corresponds to Σ-definability without equality.

Theorem 2.10 *A set $S \subseteq \mathbb{R}^n$ is $\Sigma^{\mathbb{R}}_+$-definable if and only if it is Σ-definable without equality.*

Proof.

(⇒) By Theorem 2.9 and Theorem 2.2, it follows that each $\Sigma^{\mathbb{R}}_+$-definable set is Σ-definable without equality.

(⇐) If a set is Σ-definable without equality, it is definable by a computable infinite disjunction $\bigvee_{i\in\omega} \psi_i(\bar{x})$ of finite conjunctions of formulas of the kind $f(\bar{x}) < g(\bar{x})$. By using decidability of the elementary theory of $\mathbb{R}$, we can, for each such formula $\psi_i(\bar{x})$, effectively enumerate the set S_i of all $\bar{q} \in \mathbb{Q}^{2m}$ such that

$$(\forall \bar{x} \in B)(\bar{q})\psi_i(\bar{x}),$$

moreover, it could be easily verified that $\psi_i(\bar{x})$ is equivalent to $\bigvee_{\bar{q}\in S_i} (\bar{x} \in B(\bar{q}))$. □

Corollary 2.11 *A set $S \subseteq \mathbb{R}^n$ is Σ-definable without equality if and only if S is c. e. open.*

The following result shows that openness of a Σ-definable set is necessary but not sufficient to be Σ-definable without equality. It is worth noting that this claim can be shown using Remark 2.3 and [3, Theorem 3]. In order to stay in the framework of Σ-definability, we prove the following proposition.

Proposition 2.12 *There exists an open set $S \subseteq \mathbb{R}$ such that*

1. *S is Σ-definable;*
2. *S is not c. e. open.*

Proof. Fix some computable one-one onto mapping $q : \omega \longrightarrow \mathbb{Q}$ (we denote $q(m) = q_m$). For $n \in \omega$, let

$$S_n = \textstyle\bigcup_{i\in W_n} B(q_{\ell(i)}, q_{r(i)}),$$

where W_n is the nth c. e. set. Denote by W^t_n a finite part of W_n enumerated at the first t steps. Let

$$S^t_n = \textstyle\bigcup_{i\in W^t_n} B(q_{\ell(i)}, q_{r(i)}).$$

Note that each S_n is c. e. open and for each c. e. open set S there exists n such that $S = S_n$. Moreover, the relation $a \in S^t_n$, $a \in \mathbb{Q}$, $n, t \in \omega$, is computable.

Now we simultaneously run ω processes. A process with number n is assigned to its own interval $(n, n+1)$. At each step, it may generate subintervals of $(n, n+1)$. Namely, at step t, it first generates open intervals

$$I^-_{n,t} = (n, n + \frac{1}{2} - \frac{1}{t+4}) \quad \text{and} \quad I^+_{n,t} = (n + \frac{1}{2} + \frac{1}{t+4}, n+1).$$

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Next, if $n + \dfrac{1}{2} \in S_n^t$, then we take the minimal $i \in W_n^t$ such that $n + \dfrac{1}{2} \in B(q_{\ell(i)}, q_{r(i)}) \subseteq S_n^t$ and generate a new interval $B(n + \dfrac{1}{2}, \varepsilon)$ so that there exists $c_n \in \mathbb{Q}$ such that

$$c_n \in B(q_{\ell(i)}, q_{r(i)}) \subseteq S_n \setminus (B(n + \frac{1}{2}, \varepsilon) \cup \bigcup_{t' \leqslant t} (I_{n,t'}^- \cup I_{n,t'}^+)).$$

We can effectively select such an $\varepsilon = q_k$ and $c_n = q_l$ with minimal possible numbers k and l. If c_n was defined at this step, then we stop the nth process forever. Otherwise we pass to the next step.

Now define the set S as union of all intervals generated by all these processes and single-point sets $\{n + \dfrac{1}{2}\}$, $n \in \omega$. Clearly, S is Σ-definable over $\mathbf{HF}(\mathbb{R})$.

We claim that S is open and does not coincide with each S_n, $n \in \omega$, thus S is not effectively open. The only points which are not evidently internal points of S are points of the kind $n + \dfrac{1}{2}$, $n \in \omega$. If $n + \dfrac{1}{2} \notin S_n$, then the nth process generates infinitely many intervals $I_{n,t}^-$ and $I_{n,t}^+$. One can easily see that in this case,

$$\{n + \frac{1}{2}\} \cup \bigcup_{t \in \omega} (I_{n,t}^- \cup I_{n,t}^+) = (n, n+1)$$

and thus $n + \dfrac{1}{2}$ is an internal point of S. If $n + \dfrac{1}{2} \in S_n$, then at some step an open interval containing this point is generated and thus it is an internal point of S again.

Assume S to coincide with some S_n. If $n + \dfrac{1}{2} \in S_n$, then by construction $c_n \in S_n \setminus S$. If $n + \dfrac{1}{2} \notin S_n$, it remains to note that by definition of S, we have $n + \dfrac{1}{2} \in S$. It follows that $S \neq S_n$. □

3 The main results

The following theorem shows that there is no reasonable effective transformation of Σ-formulas such that the result of this transformation extracts an open subset of the set defined by the initial formula and does not change this subset in case the initial formula already defines an open subset of $\mathbb{R}$. First we need a lemma.

Lemma 3.1 *For every $a < b$, there exist Σ-definable functions $\alpha_i^+(a,b), \alpha_i^-(a,b), \beta_i^+(a,b), \beta_i^-(a,b)$ defined on $\omega \times \mathbb{R}^2$ such that*

$$\begin{aligned} a = \alpha_0^-(a,b) < \beta_0^-(a,b) < \alpha_1^-(a,b) < \beta_1^-(a,b) < \cdots \\ < \frac{a+b}{2} \\ < \cdots < \beta_1^+(a,b) < \alpha_1^+(a,b) < \beta_0^+(a,b) < \alpha_0^+(a,b) = b \end{aligned}$$

and $\lim_{i \to \infty} \alpha_i^+(a,b) = \lim_{i \to \infty} \alpha_i^-(a,b) = \lim_{i \to \infty} \beta_i^+(a,b) = \lim_{i \to \infty} \beta_i^-(a,b) = \dfrac{a+b}{2}$.

We leave the proof to the reader.

Let us denote $\varphi(x)^{\mathbf{HF}(\mathbb{R})}[x] \rightleftharpoons \{x \mid \mathbf{HF}(\mathbb{R}) \vDash \varphi(x)\}$.

Theorem 3.2 *There is no effective transformation $\varphi \longmapsto \varphi^\circ$ of Σ-formulas with at most one free variable such that the following hold.*

1. *For every Σ-formula $\varphi(x)$, the set $\varphi^\circ(x)^{\mathbf{HF}(\mathbb{R})}[x]$ is open and $\varphi^\circ(x)^{\mathbf{HF}(\mathbb{R})}[x] \subseteq \varphi(x)^{\mathbf{HF}(\mathbb{R})}[x]$.*
2. *For every Σ-formula $\varphi(x)$, if the set $\varphi(x)^{\mathbf{HF}(\mathbb{R})}[x]$ is open, then $\varphi^\circ(x)^{\mathbf{HF}(\mathbb{R})}[x] = \varphi(x)^{\mathbf{HF}(\mathbb{R})}[x]$.*

Proof. Let $f : \omega \longrightarrow \omega$ be a computable function whose range is not computable. Let

$$A_n = \{1\} \cup \bigcup_{t \in \omega, n \notin \{f(0), \dots, f(t)\}} ((\alpha_t^-(0,2), \alpha_{t+1}^-(0,2)] \cup [\alpha_{t+1}^+(0,2), \alpha_t^+(0,2))),$$

where $\alpha_t^{\pm}(0,2)$ are taken from Lemma 3.1. Then clearly A_n is open if and only if $n \notin \operatorname{range}(f)$. One easily ascertains that there is a computable family $\varphi_n(x)$ of Σ-formulas such that $\varphi_n(x)^{\mathbf{HF}(\mathbb{R})}[x] = A_n$ for all $n \in \omega$.

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The following condition could be easily verified:

$$1 \in \mathrm{Int}(A_n) \Leftrightarrow n \notin \mathrm{range}(f) \Leftrightarrow A_n \text{ is open.}$$

Suppose now that there exists an effective transformation $^\circ$ satisfying the condition of the theorem. Then

$$n \notin \mathrm{range}(f) \Leftrightarrow \mathbf{HF}(\mathbb{R}) \vDash \varphi_n^\circ(1),$$

which by Corollary 2.4 implies that the set $\mathrm{range}(f)$ is computable, which is a contradiction. □

Consider an example. Let $\varphi(x)$ be a Σ-formula saying that $(x \in (0,2) \wedge x \neq 1) \vee (x \in (0,2) \wedge x = 1)$.

If we try to satisfy this formula in a direct way by some $x \in (0,1)$, then we should first examine the first part, namely $x \in (0,2) \wedge x \neq 1$. This check will be successful for $x \in (0,1) \cup (1,2)$. Next, we should satisfy the second part, $x \in (0,2) \wedge x = 1$, which either will be unsuccessful for $x \neq 1$ or gets stuck when $x = 1$. However, we could satisfy this formula with elements of the set $(0,1) \cup (1,2)$ only. But it is evident that $\varphi(x)$ is logically equivalent to the formula $x \in (0,2)$, which also defines an open set.

Thus, we can propose the following uniform way to extract open parts of the formulas, which, as we believe, should work more or less reasonably. First we present a Σ-formula $\varphi(x)$ as infinite disjunction $\bigvee_{i\in\omega} \psi_i(x)$ of a computably enumerable family $(\psi_i(x))_{i\in\omega}$ of a quantifier-free formulas; the algorithm enumerating members of this disjunction could be found uniformly in $\varphi(x)$. Then we enumerate all pairs $\langle a, b\rangle$ of rationals such that

$$(\forall x \in (a,b)) \bigvee_{i<t} \psi_i(x)$$

for some t. Show it to be possible. The last condition could be reduced to an equivalent quantifier-free formula with no free variables uniformly in a, b, t, that is, this formula could be effectively checked uniformly in t, a, b. This yields an algorithm to enumerate all such pairs $\langle a, b\rangle$. Let $(\langle a_i, b_i\rangle)_{i\in\omega}$ be such an enumeration. The result of the transformation of φ is the infinite c. e. disjunction $\bigvee_{i\in\omega}(a_i < x < b_i)$, whose algorithm enumerating its members could be found uniformly in $\varphi(x)$. By the remarks on the uniformity, this infinite disjunction could be presented as an equivalent Σ-formula if needed.

Of course, if we consider the following definition of the set $(0,2)$,

$$x = 1 \vee \bigvee_{n\in\omega}((0 < x < 1 - \frac{1}{n+1}) \vee (1 + \frac{1}{n+1} < x < 2)),$$

then the above algorithm will produce a formula that defines the set $(0,1) \cup (1,2)$, but intuitively, the above definition does not give us an opportunity to ascertain that $1 \in (0,2)$ as well. As a corollary, we get the following result.

Theorem 3.3 *There is no effective transformation $\varphi \longmapsto \varphi^\circ$ of Σ-formulas with at most one free variable such that, for every Σ-formula $\varphi(x)$, if the set $\varphi(x)^{\mathbf{HF}(\mathbb{R})}[x]$ is open, then*

1. *φ° is a $\Sigma_<$-formula;*
2. *$\varphi^\circ(x)^{\mathbf{HF}(\mathbb{R})}[x] = \varphi(x)^{\mathbf{HF}(\mathbb{R})}[x]$.*

To formulate the further results, we need to recall some definitions from the theory of numberings [6]. Informally, in the theory of numberings, we simultaneously consider objects and their codes, which are natural numbers. Let S be an arbitrary set. Any onto mapping $\nu : \omega \longrightarrow S$ is called its *numbering*. The pair (S, ν) is called a *numbered set*. Assume that (S_0, ν_0) and (S_1, ν_1) are numbered sets and $p : S_0 \longrightarrow S_1$ is a mapping. Then p is called a *morphism from* (S_0, ν_0) *to* (S_1, ν_1) if it is is computable on the codes of objects, i. e., if there is a computable function f such that $p\nu_0 = \nu_1 f$. A morphism p from (S, ν) to (S, ν) is called a *retraction* if $p^2 = p$. Informally, a retraction transforms all objects into objects of the set $p(S) \subseteq S$ in such a way that each object in the set $p(S)$ remains unchanged.

Define a numbering of the class $\Sigma(\mathbb{R})$ of all Σ-definable sets over $\mathbf{HF}(\mathbb{R})$ as follows. First we consider some Gödel numbering γ of Σ-formulas with at most one free variable x, i. e., an arbitrary mapping γ from ω onto the set of all such formulas such that, given a formula, one can effectively write down its Gödel number, and given an arbitrary $n \in \omega$, one can effectively write down γ_n (the formula with the number n). All the numberings with this property are equivalent.

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Define the numbering ν in a natural way as $\nu(m) = (\gamma_m)^{\mathbf{HF}(\mathbb{R})}[x]$.

Now we investigate if it is possible to uniformly algorithmically transform the definitions of Σ-definable subsets (i. e., Σ-formulas) into new definitions of Σ-definable subsets such that the results will define open sets, and if a definition defines an open set, the result of this transformation will define the same set. Clearly, this question could be reformulated as follows: "does there exist a retraction $p : (\Sigma(\mathbb{R}), \nu) \longrightarrow (\Sigma(\mathbb{R}), \nu)$ whose image is the set of all open Σ-definable sets?"

The answer turns out to be negative.

Theorem 3.4 *For all retractions $p : (\Sigma(\mathbb{R}), \nu) \longrightarrow (\Sigma(\mathbb{R}), \nu)$, the set of all open Σ-definable sets does not coincide with the set $p(\Sigma(\mathbb{R}))$.*

Proof. Assume that such a retraction exists. One can easily see that in this situation there exists an effective transformation $\varphi \longmapsto \varphi^\lambda$ of Σ-formulas with at most one free variable x and a mapping $X \longmapsto X^\lambda$ on Σ-definable subsets of $\mathbb{R}$ such that

1. for all Σ-formulas $\varphi(x)$, $(\varphi^\lambda)^{\mathbf{HF}(\mathbb{R})}[x]$ is an open subset of $\mathbb{R}$;
2. for all Σ-formulas $\varphi(x)$, $(\varphi^{\mathbf{HF}(\mathbb{R})}[x])^\lambda = (\varphi^\lambda)^{\mathbf{HF}(\mathbb{R})}[x]$;
3. for all Σ-definable open sets X, $X^\lambda = X$.

Lemma 3.5 *If $X \subseteq Y$ are Σ-definable subsets of $\mathbb{R}$, then the set $X^\lambda \setminus Y^\lambda$ contains no algebraic reals.*

Proof. Assume the contrary, i. e., that there are sets X and Y defined by Σ-formulas $\varphi_X(x)$ and $\varphi_Y(x)$, respectively, such that the set $X^\lambda \setminus Y^\lambda$ contains an algebraic point α.

Fix a c. e. non-computable set S of natural numbers and consider a Σ-formula $\theta_n(x)$ equivalent to

$$\varphi_X(x) \vee (\varphi_Y(x) \wedge n \in S).$$

Since the membership in S could be expressed by a Σ-formula, the formula θ_n could be written down effectively uniformly in n. Obviously,

$$\theta_n^{\mathbf{HF}(\mathbb{R})}[x] = \begin{cases} X & \text{if } n \notin S, \\ Y & \text{if } n \in S. \end{cases}$$

It follows from the properties of λ that

$$(\theta_n^\lambda)^{\mathbf{HF}(\mathbb{R})}[x] = \begin{cases} X^\lambda & \text{if } n \notin S, \\ Y^\lambda & \text{if } n \in S. \end{cases}$$

Thus, we have

$$\mathbf{HF}(\mathbb{R}) \models (\theta_n)^\lambda(\alpha) \Leftrightarrow n \notin S.$$

Inasmuch as the left-hand condition is a c. e. condition on n, we obtain that $\omega \setminus S$ is c. e., which is a contradiction. The proof of the lemma is complete. □

Lemma 3.6 *Assume that $a_0, b_0, c, a_1, b_1 \in \mathbb{Q}$ and that $a_0 < b_0 < c < a_1 < b_1$. Then*

$$[(a_0, b_0) \cup \{c\} \cup (a_1, b_1)]^\lambda \subseteq (a_0, b_0) \cup (a_1, b_1).$$

Proof. Take an arbitrary $\varepsilon \in \mathbb{Q}$ so that

$$b_0 < c - \varepsilon < c + \varepsilon < a_1.$$

Since λ preserves open sets, we have

$$[(a_0, b_0) \cup (c - \varepsilon, c + \varepsilon) \cup (a_1, b_1)]^\lambda = (a_0, b_0) \cup (c - \varepsilon, c + \varepsilon) \cup (a_1, b_1).$$

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Inasmuch as

$$(a_0, b_0) \cup \{c\} \cup (a_1, b_1) \subseteq (a_0, b_0) \cup (c - \varepsilon, c + \varepsilon) \cup (a_1, b_1),$$

by Lemma 3.5 the set

$$\begin{aligned} &[(a_0, b_0) \cup \{c\} \cup (a_1, b_1)]^\lambda \setminus [(a_0, b_0) \cup (c - \varepsilon, c + \varepsilon) \cup (a_1, b_1)]^\lambda \\ &\quad = [(a_0, b_0) \cup \{c\} \cup (a_1, b_1)]^\lambda \setminus ((a_0, b_0) \cup (c - \varepsilon, c + \varepsilon) \cup (a_1, b_1)) \end{aligned} \tag{1}$$

does not contain algebraic points. Show that this set (1) is empty. Indeed, if it is nonempty, then since the set

$$[(a_0, b_0) \cup \{c\} \cup (a_1, b_1)]^\lambda$$

is open, the set (1) contains an open interval, which implies that it contains an algebraic real, and this contradicts Lemma 3.5.

Thus, for any small enough rational ε,

$$[(a_0, b_0) \cup \{c\} \cup (a_1, b_1)]^\lambda \subseteq (a_0, b_0) \cup (c - \varepsilon, c + \varepsilon) \cup (a_1, b_1).$$

From this we derive that

$$[(a_0, b_0) \cup \{c\} \cup (a_1, b_1)]^\lambda \subseteq (a_0, b_0) \cup \{c\} \cup (a_1, b_1).$$

Since the left-hand part of this inclusion is open, it follows that the right-hand part of it cannot contain c, i. e.,

$$[(a_0, b_0) \cup \{c\} \cup (a_1, b_1)]^\lambda \subseteq (a_0, b_0) \cup (a_1, b_1).$$

The proof of the lemma is complete. □

Let f be a computable function with non-computable range. Denote by $\varphi_n(u)$ a Σ-formula expressing that

$$(u = 1) \vee (\exists t \in \omega)[n \notin \{f(0), \dots, f(t-1)\} \wedge (0 < u < 1 - \frac{1}{t}) \wedge (1 + \frac{1}{t} < u < 2)].$$

Note that one can choose these formulas φ_n in such a way that the mapping $n \longmapsto \varphi_n$ will be computable.

One can easily see that if $n \notin \mathrm{range}(f)$, then $(\varphi_n)^{\mathbf{HF}(\mathbb{R})}[u] = (0, 2)$, and that if $n \in \mathrm{range}(f)$, then

$$(\varphi_n)^{\mathbf{HF}(\mathbb{R})}[u] = (0, r_0) \cup \{1\} \cup (r_1, 2),$$

for appropriate $r_0, r_1 \in \mathbb{Q}$ such that $0 < r_0 < 1 < r_1 < 2$.

Now with Lemma 3.6 we obtain that if $n \notin \mathrm{range}(f)$, then $(\varphi_n^\lambda)^{\mathbf{HF}(\mathbb{R})}[u] = (0, 2)$, and that if $n \in \mathrm{range}(f)$, then $(\varphi_n^\lambda)^{\mathbf{HF}(\mathbb{R})}[u] \subseteq (0, 1) \cup (1, 2)$.

Thus, for all $n \in \omega$,

$$1 \in (\varphi_n^\lambda)^{\mathbf{HF}(\mathbb{R})}[u] \Leftrightarrow n \notin \mathrm{range}(f),$$

which means that for all $n \in \omega$,

$$\mathbf{HF}(\mathbb{R}) \models \varphi_n^\lambda(1) \Leftrightarrow n \notin \mathrm{range}(f).$$

Since the left-hand part of this equivalence is c. e., the set $\omega \setminus \mathrm{range}(f)$ is c. e. It follows that $\mathrm{range}(f)$ is computable, which is a contradiction. The proof of Theorem 3.4 is complete. □

Actually, we have simultaneously proven the following.

Theorem 3.7 *For all retractions $p : (\Sigma(\mathbb{R}), \nu) \longrightarrow (\Sigma(\mathbb{R}), \nu)$, the set of all Σ-definable without equality sets does not coincide with the set $p(\Sigma(\mathbb{R}))$.*

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The following result shows that, in the general case, we cannot even hope for the existence of an internal part of a Σ-definable set which is maximal by inclusion among its Σ-subsets.

Theorem 3.8 *There exists a set $S \subseteq \mathbb{R}$ such that the following hold.*

1. *S is Δ-definable.*
2. *Neither the closures nor the inner parts of the sets $S, \mathbb{R} \setminus S$ are Σ-definable.*
3. *If V is either S or $\mathbb{R} \setminus S$, then the class*

$$\{X \subseteq V \mid X \text{ is } \Sigma\text{-definable without equality}\}$$

has no maximal element by inclusion.

4. *If V is either S or $\mathbb{R} \setminus S$, then the class*

$$\{X \supseteq V \mid (X \text{ is closed}) \wedge (X \text{ is } \Sigma\text{-definable})\}$$

has no minimal element by inclusion.

Proof. Fix a computable function $f : \omega \longrightarrow \omega$ whose range is not computable.

First we show how for any two real numbers a, b, $a < b$, and $n \in \omega$, we could separate the interval $[a, b)$ into two Σ-definable sets $A_n(a, b)$ and $B_n(a, b)$ so that

$$A_n(a,b) \cap B_n(a,b) = \emptyset, \quad A_n(a,b) \cup B_n(a,b) = [a,b), \quad \text{and}$$
$$\frac{a+b}{2} \in \mathrm{Int}(A_n(a,b)) \Leftrightarrow n \notin \mathrm{range}(f).$$

Denote $c = \dfrac{a+b}{2}$. Let

$$A_n(a,b) = \{c\} \cup \bigcup_{t\in\omega, n\notin\{f(0),\dots,f(t)\}} ([\alpha_t^-(a,b), \alpha_{t+1}^-(a,b)) \cup [\alpha_{t+1}^+(a,b), \alpha_t^+(a,b))),$$
$$B_n(a,b) = \bigcup_{t\in\omega, n\in\{f(0),\dots,f(t)\}} [\alpha_t^-(a,b), \alpha_t^+(a,b)) \setminus \{c\}.$$

Clearly, both sets are Σ-definable, they are disjoint, and their union equals to $[a, b)$. Moreover, there exist Σ-formulas $\varphi_A(n, a, b, x)$ and $\varphi_B(n, a, b, x)$ such that

$$A_n(a,b) = \varphi_A(n,a,b,x)^{\mathbf{HF}(\mathbb{R})}[x] \quad \text{and} \quad B_n(a,b) = \varphi_B(n,a,b,x)^{\mathbf{HF}(\mathbb{R})}[x].$$

Obviously,

$$\frac{a+b}{2} \in \mathrm{Int}(A_n(a,b)) \Leftrightarrow n \notin \mathrm{range}(f).$$

Next we show how given any two real numbers a, b, $a < b$, and $n \in \omega$, we could uniformly construct Σ-subsets $C_n(a, b)$ and $D_n(a, b)$ so that

$$C_n(a,b) \cup D_n(a,b) = [a,b), \quad C_n(a,b) \cap D_n(a,b) = \emptyset, \quad \text{and}$$
$$\frac{a+b}{2} \in \mathrm{cl}(C_n(a,b)) \Leftrightarrow n \notin \mathrm{range}(f).$$

Let

$$C_n(a,b) = \bigcup_{t<\omega, n\notin\{f(0),\dots,f(t)\}} ([\alpha_t^-(a,b), \beta_t^-(a,b)) \cup [\beta_t^+(a,b), \alpha_t^+(a,b)))$$
$$D_n(a,b) = \bigcup_{t<\omega, n\in\{f(0),\dots,f(t)\}} [\beta_t^-(a,b), \beta_t^+(a,b)).$$

Clearly, both sets are Σ-definable, they are disjoint, and their union equals to $[a, b)$. Moreover, there exist Σ-formulas $\varphi_C(n, a, b, x)$ and $\varphi_D(n, a, b, x)$ such that

$$C_n(a,b) = \varphi_C(n,a,b,x)^{\mathbf{HF}(\mathbb{R})}[x] \quad \text{and} \quad D_n(a,b) = \varphi_D(n,a,b,x)^{\mathbf{HF}(\mathbb{R})}[x].$$

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Obviously,

$$\frac{a+b}{2} \in \mathrm{cl}(C_n(a,b)) \Leftrightarrow n \notin \mathrm{range}(f).$$

Now define the set S as follows:

$$S = \bigcup_{i\in\omega} A_i(8i, 8i+2) \cup \bigcup_{i\in\omega} B_i(8i+2, 8i+4) \cup \bigcup_{i\in\omega} C_i(8i+4, 8i+6) \cup \bigcup_{i\in\omega} D_i(8i+6, 8i+8).$$

Suppose that the interior of S is Σ-definable. Then we have

$$8i+1 \in S \Leftrightarrow i \notin \mathrm{range}(f),$$

which contradicts Corollary 2.4. Similarly, the assumption that the closure of S is Σ-definable leads to the condition

$$8i+5 \in S \Leftrightarrow i \notin \mathrm{range}(f),$$

which contradicts Corollary 2.4. The proofs that the sets $\mathrm{Int}(\mathbb{R} \setminus S)$ and $\mathrm{cl}(\mathbb{R} \setminus S)$ are not Σ-definable could be done in a similar way. The rest statements of the theorem are easily verified. □

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