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Set Theory, Topology, and the Possibility of Junky Worlds

Abstract. A possible world is a junky world if and only if each thing in it is a proper part. The possibility of junky worlds contradicts the principle of general fusion. Bohn (2009) argues for the possibility of junky worlds, Watson (2010) suggests that Bohn‘s arguments are flawed. This paper shows that the arguments of both authors leave much to be desired. First, relying on the classical results of Cantor, Zermelo, Fraenkel, and von Neumann, this paper proves the possibility of junky worlds for certain weak set theories. Second, the paradox of Burali-Forti shows that according to the Zermelo-Fraenkel set theory **ZF**, junky worlds are possible. Finally, it is shown that set theories are not the only sources for designing plausible models of junky worlds: Topology (and possibly other "algebraic" mathematical theories) may be used to construct models of junky worlds. In sum, junkyness is a relatively widespread feature among possible worlds.

Key words: junky worlds, principle of universal fusion, Zermelo natural numbers, von Neumann ordinal numbers, Burali-Forti paradox, topological junky worlds.

1. Introduction

By definition, the possible world J is a junky world if and only if every part of J is a proper part of some other part of J. The possibility of junky worlds contradicts the principle of general fusion because the fusion of all parts of a world, if it exists, cannot be a proper part of a part of this world. In other words, the possibility of a junky world is not compatible with Lewis’s thesis that unrestricted fusion holds in mereology or that every collection of objects of a possible world composes a further object of this world (cf. Lewis 1991).

Recently, some authors have doubted the general validity of the principle of universal fusion.

Bohn (2009) argued in favor of the possibility of junky worlds, whereas Watson argued that Bohn's arguments are flawed (cf. Watson 2010). In this paper, it is shown that both Bohn and Watson's arguments leave much to be desired, especially because they do not clearly demonstrate the set theory on which their arguments are based. This paper uses some classical results of Cantor, Zermelo, Fraenkel, and von Neumann for certain set theories to prove the possibility of junky worlds. In particular, the paradox of Burali-Forti shows that, if one relies on Zermelo-Fraenkel set theory ZF, junky worlds are possible.

Set theory is not, however, the only source of models of junky worlds and may not even be the best. In this paper, we argue that elementary topology, and possibly other "algebraic" theories (cf. Shapiro (1997))[1], may give us interesting models of junky worlds.

To set the stage, let us recall some mereological terminology. Every possible world W defines a mereological system (PTW, $\leq$) of parts of W, where PTW is the class of parts of W and $\leq$ is a binary parthood relation defined for parts a and b of W. Then, "$a \leq b$" is to be read as "a is a part of b or equal to b", and "$a < b$" is to be read as "a is a proper part of b" (i.e., "a is a part of b and $a \neq b$"). A mereologically meaningful parthood relation $\leq$ must be at least reflexive and transitive, and a "classical" mereological system (M, $\leq$) is characterized by the requirement that the parthood relation $\leq$ defines the structure of a Boolean algebra (with the bottom element 0 deleted) on the class M of mereological individuals. This Boolean algebra is complete if and only if every family $F \subseteq M$ of mereological individuals has a fusion in M.

After these terminological preparations, a junky world J can be precisely defined as a world whose mereological system (PTJ, $\leq$) satisfies the condition that every part of J is a proper part of some other part of J: $(x)\ (x \in PTJ \Rightarrow \exists y\ (y \in PTJ \text{ AND } x < y))$.

Before exploring the technical details, it is necessary to clarify the notion of "possibility" that is used in the paper. Roughly, our notion of possibility can be characterized as a notion of logico-mathematical possibility. Accordingly, a necessary requirement for a world W to be

[1] Algebraic theories may be characterized as mathematical theories with many intended models, in contrast to non-algebraic theories that aim, at least prima facie, to be about a unique model. Examples of algebraic theories are group theory, lattice theory, topology, and graph theory.

considered a possible world is that its mereological structure (PTW, ≤) be mathematically coherent and mereologically plausible. In other words, what is important for such a structure is not the nature of its objects but the pattern in which the objects are arranged. In brief, the present paper uses an account of possible worlds that subscribes to an "ante-rem structuralism" that has been put forward by authors such as Resnik (1981, 1982), Hellman (1989), and Shapiro (1983).[2] Of course, there may be other requirements that "really possible" worlds must satisfy, but they will be addressed only briefly in this paper.

2. Bohn's argument for junky worlds and Watson's counterargument revisited

Bohn's argument for the possibility of junky worlds proceeds in several steps. First, he invites us to consider the following "scenario":

> ... Our universe is a miniature replica universe housed in a particle of a bigger replica universe, which is again a miniature replica universe housed in a particle of an even bigger replica universe, and so on ad infinitum (Bohn 2009, 28).

According to Bohn, first, this scenario allows us to classify worlds as possible when everything is a proper part (i.e., junky worlds; ibidem, 28). Second, Bohn notes that the concept of a junky world does not lead to contradictions, and finally, he argues that famous philosophers, such as Leibniz, endorsed the thesis that junky worlds are possible or even contended that our world is a junky world. According to Bohn, therefore, we are entitled to assert that junky worlds are possible. This assertion entails that the principle of unrestricted mereological fusion does not hold universally because the fusion of all parts of a junky world does not exist.

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[2] A structuralist philosophy of mathematics avant la lettre may be ascribed already to Cassirer, see, in particular, his opus magnum *Substanzbegriff und Funktionsbegriff* (Cassirer 1910 (1953)). Following Dedekind, Cassirer emphasized that only the ordinal structure of natural numbers is mathematically relevant, not their intrinsic nature as "objects" (cf. Cassirer (1953, chapter II, section 2).

Watson (2010) disagrees with the above arguments and contends that Bohn's scenario argument is flawed because it uses the following, "obviously invalid" argument:

> Take a set, say {}, and suppose that it's a member of some bigger set, {{}}, and suppose that that set is a member of some bigger set still, {{{}}}, and so on ad infinitum. Call the sets in this series 'the constructed sets'. It follows that there's no set that contains all the constructed sets, because there's no constructed set that isn't a member of some other constructed set. (Watson 2010, 79)

This argument, which allegedly captures the essence of Bohn's argument for the possibility of junky worlds, is said to be invalid because

> the set that contains all the constructed sets need not itself be a constructed set – despite there being no constructed set that has as members all of the constructed sets, there is nevertheless some non-constructed set that does have as members all the constructed sets. (Watson 2010, 80)

First, Watson's series of "constructed sets" (i.e., {}, {{}}, {{{}}}, ...) does not reflect the mereological structure that Bohn's scenario intends to convey. Bohn's world consists of a series of nested universes in which each universe is a part of the following ones. Watson's collection of "constructed sets", however, lacks any structure that can be interpreted as a reasonable mereological parthood relation $\leq$. The following argument proves this point. It is expected that for a series of "constructed sets", the relationship of set-theoretical inclusion $\subseteq$ will play the role of the mereological parthood relation $\leq$, assuming that the parts of a set are its (non-empty) subsets (cf. Lewis 1991). There are, however, no non-trivial inclusion relationships between the "constructed sets". When we denote the "constructed sets" with a_i, beginning with $a_0 = \{\}$, $i = 0, 1, 2, \ldots$, direct inspection reveals that $i \neq j$ yields $a_i \cap a_j = \emptyset$.

One may think that Watson opted for the elementhood relation $\in$ as a mereological parthood relation of his collection of "constructed sets". This assumption, however, leads to trouble because, although there are subsequent members of the series $a_i \in a_{i+1}$ and $a_{i+1} \in a_{i+2}$, we do not obtain $a_i \in a_{i+2}$ because $\in$ is not transitive for the series of "constructed sets". Hence, the relation $\in$ does not satisfy the minimal adequacy conditions for a reasonable parthood relationship. Furthermore, one might object that in Bohn's scenario, all nested universes are assumed to have non-trivial mereological structures (i.e., the mereological systems $(PTa_i, \leq)$ should be non-trivial). This assumption is a plausible requirement, but Watson's "constructed sets" are all singletons; in other words, they do not have any non-trivial proper parts.

In sum, Watson's series of "constructed sets" is not a reasonable mereological system that reflects the intended mereological structure of Bohn's scenario of a world of nested universes.

3. Watson's argument, amended with von Neumann ordinals

At first view, it seems that the shortcomings of Watson's model can easily be remedied. Instead of the series of "constructed sets" Ø, {Ø}, {{Ø}}, ..., Watson should have chosen the following series:

(VN) $\quad\quad \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots,$

This series is defined by the following: starting with Ø, define the successor of a defined member x of (VN) as $x \cup \{x\}$. It is easy to verify that the elements of this series are linearly ordered with respect to proper set-theoretical inclusion $\subset$ as the proper parthood relation $<$ of this mereological system. Indeed, the series (VN) is well-known as the well-ordered series of finite von Neumann ordinal numbers, as introduced by von Neumann in the 1920s and usually denoted by 0, 1, 2, ... Inductively, they are defined by $0 := \emptyset$ and $n+1 := n \cup \{n\}$, and they are ordered by set-theoretical inclusion $0 \subset 1 \subset 2 \subset 3 \subset \ldots$.[3]

Every von Neumann ordinal n is a set with n elements, and the mereological system (PTn, ≤) of its parts is the power set of n (i.e., (PTn, ≤) is a Boolean algebra with 2^n elements). As n ≤ n+1, the von Neumann ordinal n is an element of the mereological system (PT(n+1), ≤). Hence, we obtain a series of nested universes, each with a non-trivial mereological structure and a proper part of all those that follow. This series may be reasonably seen as capturing the mereological essentials of Bohn's scenario.

In sum, by replacing the Zermelo natural numbers with finite von Neumann ordinals, Watson's argument against Bohn's claim that his scenario suggests the possibility of a junky world can be reformulated as follows (cf. Watson 2010, 80):

> Although there is no finite von Neumann ordinal n that contains all finite von Neumann ordinals, there is, nevertheless, a set that contains all finite von Neumann ordinals.

Furthermore, the series (VN) of finite von Neumann ordinals not only has a reasonable mereological structure, but the set of all finite von Neumann ordinals is a von Neumann ordinal (i.e., a well-ordered transitive set). It is called the first infinite von Neumann ordinal and is usually denoted by ω. Evidently, it can serve as a top of Bohn's world of nested universes. Thus, Bohn's scenario - at least in this interpretation - fails to support the possibility of junky worlds.

In other words, with the help of von Neumann, Watson seems to eventually accomplish his objective, which is to provide a good model of nested universes that does not lead to a junky

[3] In addition, Watson's "constructed sets" may claim some prominence in the history of set theory, as Zermelo used them in his formulation of the axiom of infinity. (cf. Zermelo 1908). Hence, from now on, let us refer to Watson's "constructed sets" as "Zermelo (natural) numbers". Structurally, Zermelo's and von Neumann's natural numbers are, of course, equivalent. In the famous paper *What Numbers Could Not Be* (Benacerraf 1965), Benacerraf compellingly argued that the co-existence of Zermelo's and von Neumann's natural numbers suggests a structural interpretation of mathematics, according to which mathematical structures are important mathematical objects.

world. It seems that there is no good reason why the worlds suggested by Bohn's scenarios should be junky worlds - or so it seems. However, this point is not the end of the story.

4. Bohn's scenario amended by using von Neumann ordinals

Turning to von Neumann for help turns out to be double-edged sword. In arguing for the possibility of junky worlds, Bohn could have drawn from von Neumann's arguments for his thought experiment. First, why is "obvious" that the set ω of all finite von Neumann ordinals exists? Apparently, Watson sees no problem in this assumption because he feels that set theory justifies it. That is, given any countable collection of sets, all reasonable set theories recognize that the collection of these sets is also a set. While this claim is essentially correct, it is helpful to recall in some detail how the (arguably standard) set theory **ZF** of Zermelo-Fraenkel ensures the existence of the first infinite von Neumann ω (i.e., the set of the finite ordinals 0, 1, 2, ...).

Based on the original version of **ZF**, two axioms play a crucial role in proving this assertion. First, we invoke the infinity axiom to ensure the sethood of the infinite collection of Zermelo natural numbers (i.e., Watson's "constructed sets"). Second, we use Fraenkel's replacement schema to ensure that not only is the collection of Zermelo natural numbers a set but that the collection ω of the finite von Neumann ordinals is a set as well. In Zermelo's original system **Z** (cf. Zermelo 1908), Fraenkel's replacement schema was missing, which means that in **Z**, the set of finite von Neumann ordinals was <u>not</u> a set at all.

This shortcoming may be remedied by slightly altering Zermelo's infinity axiom and reformulating it to ensure the existence of the set ω of finite von Neumann ordinals. This alteration is achieved by replacing the finite Zermelo natural numbers Ø, {Ø}, {{Ø}}, ... by the finite von Neumann ordinals Ø, {Ø}, {Ø, {Ø}}, ... to formulate the axiom of infinity. In an obvious sense, the resulting theory is equivalent to Zermelo's original theory. Hence, in the following, both theories are denoted by **Z**. We may then say that Watson's amended argument against Bohn is valid, not only for **ZF** but also for the much weaker (modified) theory **Z**.

Although Zermelo's axiom of infinity is, from the mathematician's point of view, quite weak, it is not metaphysically sacrosanct. It is logically possible that the axiom of infinity would fail so that, for instance, the hereditarily finite sets form a logically possible junky world among whose parts there are no infinite sets. Moreover, there may be powerful reasons to assume that only finite worlds are physically possible. Then, Bohn's nested series of universes would provide an example of a junky world. Arguing in this fashion, however, amounts to giving up a strictly structural account of possible worlds. Therefore, this line of thought will not be pursued further in this paper.

In the rest of this paper, we show that accepting **Z** or **ZF** does not signal the end of the possibility of junky worlds. Bohn's scenario can be improved so that it withstands Watson's amended argument not only for **Z** but also for **ZF**.

Based on work by Cantor and von Neumann, we know that the series of finite ordinals 0, 1, 2, ... is only the beginning of an infinite series of von Neumann ordinal numbers such as

$$0, 1, 2, \ldots \omega, \omega +1, \omega + 2, \ldots \omega + \omega, \omega + \omega + 1, \ldots \omega^2, \ldots \omega^\omega, \ldots .$$

Here, the infinite von Neumann ordinals $\omega + (n + 1)$ are defined by $\omega+(n+1) := \omega+n \cup\{\omega+n\}$, for n = 0, 1, 2, ... This series of ordinals can be used to strengthen Bohn's argument for the possibility of junky worlds as follows. Assume that Bohn's world of nested universes is nested according to the series $0, 1, 2, \ldots \omega, \omega +1, \omega + 2, \ldots$. Then, according to Zermelo set theory Z, such a world may be a junky world because there are models of **Z** for which the collection of the ordinals $0, 1, 2, \ldots \omega, \omega +1, \omega + 2, \ldots$ does not exist. In other words, in **Z,** there may be no <u>set</u> that contains all of them. In these models, the required fusion $\omega + \omega$ does not exist. Hence, under the assumption of the validity of **Z,** this world violates the principle of universal fusion.

On the other hand, if we accept Fraenkel's replacement schema and rely on **ZF** instead of **Z**, the resulting world is <u>not</u> junky because the union $\omega + \omega$ can easily be proven to exist as a set in **ZF**. Thus, the union of all nested universes does serve as a maximal part of the world suggested by Bohn's amended scenario. In other words, whether a Bohn scenario based on the sequence 0, 1, 2, ... ω, ω +1, ω + 2, ... leads to a junky world depends on the set theory that is applied.

A partisan of the universal fusion principle may not be overly impressed by this result and may argue that **Z** is not a serious candidate for a general background theory for mathematics or for the metaphysics of modality. Hence, it may be considered irrelevant that a weak set theory such as **Z** could entail the possibility of junky worlds as long as the standard set theory **ZF** (= **Z** + Fraenkel's replacement schema) guarantees a top element.

To refute this argument, the next section shows that for **ZF**, we can design a Bohnian scenario that lacks a top element, thereby definitely establishing the possibility of junky worlds.

5. Burali-Forti and some arguments for the possibility of junky worlds

Admittedly, relying on a set theory that accepts ω as a set but that denies this status to ω + ω is unusual. Quite generally, the theory **Z** is not considered as an appealing theory compared to its more powerful successor, **ZF**.[4] Hence, for the rest of this paper let us subscribe to Zermelo-Fraenkel set theory **ZFC**,[5] which entails, in particular, that all von Neumann ordinals are sets.

Now consider a modified Bohn scenario, consisting of a world of nested universes corresponding to <u>all</u> von Neumann ordinals. The assumption that such a world has a top element - namely, the collection ON of all von Neumann ordinals - immediately leads to a contradiction, that is, the well-known Burali-Forti paradox (cf. Priest 2001, Hellman 2011). Specifically, if ON were a set, it would be a von Neumann ordinal and would therefore occur in the series of all von Neumann ordinals ON. However, there would then be a larger ordinal than ON itself, that is, ON $\cup$ {ON}, which would be a proper subset of ON. This is absurd. In other

[4] Let us ignore the qualms that some philosophers have with the replacement schema (cf. Putnam (2000, 24)).

[5] Zermelo's theory **Z** already included the axiom of choice (cf. Zermelo (1908, 1930)).

words, for this version of Bohn's scenario, Watson's counter-argument flounders because in this world, the principle of universal fusion does not hold.

One can argue against this argument that a world resulting from ON is too large to be a "real possibility". Hence, for the supporter of junky worlds, it would be better to deduce arguments for the possibility of junky worlds that do not depend on the arguably extravagant devices of the "higher infinite". More generally, there is no reason to restrict structural arguments for or against the possibility of worlds to set theory. In a sense, axiomatized set theory has two aspects: on the one hand, it is a universal framework of mathematics, and, on the other hand, it is a mathematical theory among others (cf. Hellman (2002, 340)). Hence, it is an empirical question whether other mathematical theories may provide equally or even more powerful arguments for the possibility of worlds. Because some type of spacetime structure may be considered a natural ingredient for a possible world, it is plausible that topology may provide pertinent arguments for or against the possibility of worlds. Indeed, in the following, we show that there are mathematical models of junky worlds that do not depend on far-fetched set-theoretical constructions. More precisely, we will use sources of elementary topology.[6]

For some time now, some authors have employed models of possible worlds based on topological considerations to argue for the possibility of gunky worlds (i.e., worlds whose parts all have proper parts (cf. Lewis (1991, 20f)). A good Boolean model of a gunky world is provided by the set RO(E) of regular open[7] subsets of the Euclidean plane E, which is endowed with the standard topology such that the parthood relation ≤ is defined as the set-theoretical inclusion restricted to RO(E) (cf. for instance Uzquiano (2006, 152)).

The conceptual sources of topology are not only useful for arguing for the possibility of gunky worlds, but topology also offers some elementary arguments for the possibility of

[6] Other plausible candidates for mathematical theories that may be used to model possible worlds may be "algebraic theories" (as Shapiro (1997) dubbed them) such as the lattice theory and graph theory.

[7] A subset A of E is regular and open if and only if A is the interior of the topological closure operator cl of E, i.e., A = int(cl(A)), with the topological interior operator defined by int(B) := C(cl(CB))) with CB being the set-theoretical complement of B, see below.

junky ones. Recall that a topology on the Euclidean plane E can be defined by a closure operator $PE \xrightarrow{cl} PE$ satisfying the Kuratowski axioms (cf. Kuratowski and Mostowski (1976, 27) or Willard (1998, 25)). Assuming that E is endowed with a Cartesian coordinate system with origin (0, 0), we may consider the punctured plane $E^* := E - \{(0,0)\}$. Now consider the following set J(E) of regular open Euclidean regions:

$$J(E) := \{A;\ A \in RO(E) \text{ and } (0,0) \notin cl(A)\}.$$

Intuitively, J(E) consists of all regular open regions of the Euclidean plane E that have a non-zero distance from the "singular point" (0, 0).[8] When we define a parthood relation ≤ on J(E) as the restriction of the set-theoretical inclusion relation ⊆ defined on E, the system (J(E), ≤) may serve as a mathematical model of a possible world. As is easily verified, the world (J(E), ≤) has <u>all</u> finite (and many infinite) fusions inherited from RO(E), although some fusions of regular open regions do not exist in this world. The following is a simple example. Consider the subset S of J(E) defined by

$$S := \{B_n \mid B_n \text{ is an open ball of E with midpoint } (0, 1) \text{ and radius } (1\text{-}1/n),\ n \geq 2\}.$$

The fusion B of the B_n ($n \geq 2$) is the open ball around (0, 1) with radius 1. Clearly, $(0, 0) \in cl(B)$. Hence, B is <u>not</u> a part of the world J(E); in other words, the family S of open balls B_n has no fusion in J(E). Moreover, J(E) lacks a top element because there is no largest, <u>regular</u> open region L of E whose closure cl(L) does not contain the singular point (0, 0). In sum, J(E) is a junky world that lacks not only a top element but also many other infinite fusions. This example shows that the violation of the principle of general fusion is not a phenomenon that only concerns the existence of top elements.[9] Moreover, J(E) is also gunky. In sum, J(E)

[8] As the standard topology of E is derived from the standard metrical structure of E, the apparently metrical definition of elements of S can be expressed succinctly as $(0, 0) \notin cl(A)$ without explicit reference to the metrical structure of E. More generally, structures such as J(E) can be defined not only for metrical spaces such as E but also for every regular Hausdorff space.
[9] It is not difficult to construct mereological models that do have top elements but lack some non-maximal infinite fusions.

is a mereological model of a "hunky" world (i.e., one that is junky and gunky; cf. Bohn (2009, 193)).

Topological models such as J(E) are mathematically and set-theoretically quite innocent. Hence, for the construction of junky worlds, it is not necessary to go into the depths of the „higher infinite“ of set theory because junkyness appears in rather elementary regions of mathematics.[10]

6. Concluding Remarks

It would be a desperate move to force the validity of the principle of universal fusion by simply stipulating that there is a top element, whatever it may be. This approach would amount to willfully pushing aside the constraints of well-established mathematical theories (e.g., ZF, and other algebraic theories, such as topology) on arguments about possible worlds in favor of a rather obscure metaphysical principle. It would also come dangerously close to the method of "postulating" that, according to Russell, has many "advantages" but that honest philosophers should nevertheless avoid. The resulting clash between mathematics and the metaphysics of possible worlds would be too high of a price to pay.

Another option may be to use another set theory instead of ZF that avoids the Burali-Forti argument. However, Burali-Forti phenomena are "stable" and occur in most other set theories (cf. Priest (2002, 163f)). Moreover, allowing that an abstruse metaphysical principle such as the principle of universal fusion determines the choice of one's set theory seems odd.

A better strategy that is consistent with mathematical approaches to junkyness is to give up the dogmatic principle of universal fusion for all possible worlds and to consider the existence

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[10] At first glance, the definition of the mereological system J(E) may look contrived, particularly because it apparently makes use of some "unmerological" notions such as points and topological closure operators. This impression is misleading. The system (J(E), ≤) can be defined in honest mereological terms without explicit reference to points by conceiving of J(E) as a contraction of the world RO(E), i.e., as the result of deleting some contingent parts of the world defined by RO(E). To be specific, J(E) can be defined as a maximal "round ideal" of the complete Boolean algebra RO(E) with respect to a canonical "interior parthood relation" <<. As Armstrong convincingly argued, contractions of possible worlds should be recognized also as possible worlds (cf. Armstrong (1991, chapter 4, section 4.III, p. 61ff)).

of top elements (and other unrestricted fusions) as contingent on the structure of the worlds considered and on the set-theoretical and mathematical tools one is prepared to use. That is, researchers on the metaphysics of possible worlds should subscribe to a flexible strategy similar to the one that Zermelo proposed for set theory in his visionary paper *Über Grenzzahlen und Mengenbereiche. Neue Untersuchungen über die Grundlagen der Mengenlehre* (Zermelo 1930). In this paper, Zermelo noted that the alleged "antinomies" of set theory (such as the one of Burali-Forti)

> arise solely from a confusion between the non-categorical axioms of set theory and the various particular models of them: What in one model appears as an "ultrafinite un- or superset" ("*ultrafinite Un- oder Übermenge*") is in the next higher domain a perfectly good "set" ... which serves as the foundation stone for the construction of the next domain. The boundless series of Cantor's ordinal numbers gives rise to an equally boundless series of essentially different models of set theory, ... The two polar opposites tendencies of the thinking mind, the idea of creative process and that of an all-embracing completeness that lie at the roots of Kant's ‚antinomies', find their symbolic expression and resolution in the concept of the well-ordered transfinite number-series, whose unrestricted progress comes to no real conclusion, but only to relative stopping-points ("relative Haltepunkte") (Zermelo (1930, 47)).[11]

As Zermelo's project depends on the existence of inaccessible cardinals, its feasibility is far

[11] It may be philosophically interesting to note that Zermelo's "presentation of set theory in an unlimited sequence of well-distinguished models" ("*Darstellung der Mengenlehre in einer unbegrenzten Folge wohlunterschiedener 'Modelle'*") with only "relative stopping-points" (Zermelo (1930, 28)) fits well into the account of philosophy of science of the Marburg school of Neokantianism (Hermann Cohen, Paul Natorp, Ernst Cassirer and others), which flourished in Germany during the first decades of the 20th century. For instance, Natorp, in a quite similar vein as Zermelo, contended that the "transcendental method" - as the true method of all philosophizing - conceptualized philosophical and scientific knowledge as a never-ending conceptual evolution by which the Kantian antinomies of reason could be resolved (cf. Natorp (1912, 200)).

from evident even today (cf. Hellman 2002, Shapiro 2003). Moreover, from the point of view of a philosopher who is interested in questions concerning "really" possible worlds, these large cardinals seem unwieldy (cf. Boolos 2000). Hence, it may be expedient to conclude this paper with a more elementary example of a "*relativer Haltepunkt*" (Zermelo). This example returns to the discussion of the existence of a top element of the natural numbers addressed the beginning of this paper.

Consider the rational numbers **Q**. The set $W := \{x;\ x^2 \leq 2,\ x \in \mathbf{Q}\}$ may be said to be a junky domain relative to **Q** because there is no largest rational number $q \in \mathbf{Q}$ such that $q^2 = 2$. It is well-known that, due to the existence of Dedekind cuts, the set W has a top element relative to the real numbers **R**, to wit, $\sqrt{2}$. More generally, the completion $W^* := \{x;\ x^2 \leq 2,\ x \in \mathbf{R}\}$ of W is clearly a non-junky domain because all fusions of elements of W* exist (they are the suprema in **R** with respect to ≤).

The top element $\sqrt{2}$ of this completion is not, however, simply stipulated. Rather, Dedekind carries out non-trivial conceptual constructions to show that the domain of rationals **Q** could be embedded in an orderly way into the larger complete domain **R** of real numbers. Similarly, to implement Zermelo's project of ever larger models of **ZF**, one should show that the assumption of increasingly large inaccessible cardinals is relatively consistent with standard set theory **ZF**. At the very least, one should offer good arguments why these axioms should be accepted. The mere assertion that these axioms are necessary to insure the validity of the universal fusion principle in mereology does not suffice.[12]

Moreover, from a mathematical point of view, there is no reason to be afraid of junky structures. Mathematicians have developed a variety of methods to address these structures, sometimes they can be extended to reasonable complete structures, but

[12] The paradox of Burali-Forti is not the only issue that causes a conflict between set theory on the one hand and an account of mereology that unconditionally sticks to the principle of unrestricted fusion on the other hand. In his interesting paper *The Price of Universality* (Uzquiano 2006), Uzquiano discusses another incompatibility between set theory and mereology.

sometimes not.[13] In any case, junky structures abound in mathematics and do not cause any insurmountable logical or mathematical problems. Thus, if one subscribes to a structural logico-mathematical notion of possibility, then there are few reasons to deny the possibility of junky worlds.

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[13] To give a trivial example: it seems pointless to extend the junky domain of prime numbers 2, 3, 5, .. by a "top prime" being larger than all ordinary primes and having quite different properties.

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