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Further results on $(\in, \in)$-neutrosophic subalgebras and ideals in BCK/BCI-algebras

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Abstract: Characterizations of an $(\in, \in)$-neutrosophic ideal are considered. Any ideal in a BCK/BCI-algebra will be realized as level neutrosophic ideals of some $(\in, \in)$-neutrosophic ideal. The relation between $(\in, \in)$-neutrosophic ideal and $(\in, \in)$-neutrosophic subalgebra in a BCK-algebra is discussed. Conditions for an $(\in, \in)$-neutrosophic subalgebra to be a $(\in, \in)$-neutrosophic ideal are provided. Using a collection of ideals in a BCK/BCI-algebra, an $(\in, \in)$-neutrosophic ideal is established. Equivalence relations on the family of all $(\in, \in)$-neutrosophic ideals are introduced, and related properties are investigated.

Keywords: $(\in, \in)$-neutrosophic subalgebra, $(\in, \in)$-neutrosophic ideal.

1 Introduction

Neutrosophic set (NS) developed by Smarandache [8, 9, 10] introduced neutrosophic set (NS) as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site

http://fs.gallup.unm.edu/neutrosophy.htm.

Jun et al. studied neutrosophic subalgebras/ideals in BCK/BCI-algebras based on neutrosophic points (see [1], [5] and [7]).

In this paper, we characterize an $(\in, \in)$-neutrosophic ideal in a BCK/BCI-algebra. We show that any ideal in a BCK/BCI-algebra can be realized as level neutrosophic ideals of some $(\in, \in)$-neutrosophic ideal. We investigate the relation between $(\in, \in)$-neutrosophic ideal and $(\in, \in)$-neutrosophic subalgebra in a BCK-algebra. We provide conditions for an $(\in, \in)$-neutrosophic subalgebra to be a $(\in, \in)$-neutrosophic ideal. Using a collection of ideals in a BCK/BCI-algebra, we establish an $(\in, \in)$-neutrosophic ideal. We discuss equivalence relations on the family of all $(\in, \in)$-neutrosophic ideals, and investigate related properties.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [2] and [3]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\left(\begin{array}{l} x \le y \ \Rightarrow\ x * z \le y * z \\ x \le y \ \Rightarrow\ z * y \le z * x \end{array}\right), \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \ \Rightarrow\ x \in I)\,. \tag{2.6}$$

We refer the reader to the books [4, 6] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [9]) is a structure of the form:

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A_\sim = (A_T, A_I, A_F)$ for the neutrosophic set

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$\begin{aligned} T_\in(A_\sim; \alpha) &:= \{x \in X \mid A_T(x) \geq \alpha\}, \\ I_\in(A_\sim; \beta) &:= \{x \in X \mid A_I(x) \geq \beta\}, \\ F_\in(A_\sim; \gamma) &:= \{x \in X \mid A_F(x) \leq \gamma\}. \end{aligned}$$

We say $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are *neutrosophic* $\in$*-subsets*.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic subalgebra* of X (see [5]) if the following assertions are valid.

$$(\forall x, y \in X)\left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \\ \qquad \Rightarrow\ x * y \in T_\in(A_\sim; \alpha_x \wedge \alpha_y), \\ x \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \\ \qquad \Rightarrow\ x * y \in I_\in(A_\sim; \beta_x \wedge \beta_y), \\ x \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \\ \qquad \Rightarrow\ x * y \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.7}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic ideal* of X (see [7]) if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x)\ \Rightarrow\ 0 \in T_\in(A_\sim; \alpha_x) \\ x \in I_\in(A_\sim; \beta_x)\ \Rightarrow\ 0 \in I_\in(A_\sim; \beta_x) \\ x \in F_\in(A_\sim; \gamma_x)\ \Rightarrow\ 0 \in F_\in(A_\sim; \gamma_x) \end{array}\right) \tag{2.8}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \\ \qquad \Rightarrow\ x \in T_\in(A_\sim; \alpha_x \wedge \alpha_y) \\ x * y \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \\ \qquad \Rightarrow\ x \in I_\in(A_\sim; \beta_x \wedge \beta_y) \\ x * y \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \\ \qquad \Rightarrow\ x \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array}\right) \tag{2.9}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

3 $(\in, \in)$-neutrosophic subalgebras and ideals

We first provide characterizations of an $(\in, \in)$-neutrosophic ideal.

Theorem 3.1. *Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.*

(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X)\left(\begin{array}{l} A_T(0) \geq A_T(x), \\ A_I(0) \geq A_I(x), \\ A_F(0) \leq A_F(x) \end{array}\right) \tag{3.1}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \geq A_T(x * y) \wedge A_T(y) \\ A_I(x) \geq A_I(x * y) \wedge A_I(y) \\ A_F(x) \leq A_F(x * y) \vee A_F(y) \end{array}\right) \tag{3.2}$$

Proof. Assume that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Suppose there exist $a, b, c \in X$ be such that $A_T(0) < A_T(a)$, $A_I(0) < A_I(b)$ and $A_F(0) > A_F(c)$. Then $a \in T_\in(A_\sim; A_T(a))$, $b \in I_\in(A_\sim; A_I(b))$ and $c \in F_\in(A_\sim; A_F(c))$. But

$$0 \notin T_\in(A_\sim; A_T(a)) \cap I_\in(A_\sim; A_I(b)) \cap F_\in(A_\sim; A_F(c)).$$

This is a contradiction, and thus $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose that $A_T(x) < A_T(x * y) \wedge A_T(y)$, $A_I(a) < A_I(a * b) \wedge A_I(b)$ and $A_F(c) > A_F(c * d) \vee A_F(d)$ for some $x, y, a, b, c, d \in X$. Taking $\alpha := A_T(x*y) \wedge A_T(y)$, $\beta := A_I(a*b) \wedge A_I(b)$ and $\gamma := A_F(c*d) \vee A_F(d)$ imply that $x*y \in T_\in(A_\sim; \alpha)$, $y \in T_\in(A_\sim; \alpha)$, $a * b \in I_\in(A_\sim; \beta)$, $b \in I_\in(A_\sim; \beta)$, $c * d \in F_\in(A_\sim; \gamma)$ and $d \in F_\in(A_\sim; \gamma)$. But $x \notin T_\in(A_\sim; \alpha)$, $a \notin I_\in(A_\sim; \beta)$ and $c \notin F_\in(A_\sim; \gamma)$. This is impossible, and so (3.2) is valid.

Conversely, suppose $A_\sim = (A_T, A_I, A_F)$ satisfies two conditions (3.1) and (3.2). For any $x, y, z \in X$, let $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_\in(A_\sim; \alpha)$, $y \in I_\in(A_\sim; \beta)$ and

$z \in F_\in(A_\sim;\gamma)$. It follows from (3.1) that $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(y) \geq \beta$ and $A_F(0) \leq A_F(z) \leq \gamma$ and so that $0 \in T_\in(A_\sim;\alpha) \cap I_\in(A_\sim;\beta) \cap F_\in(A_\sim;\gamma)$. Let $a,b,c,d,x,y \in X$ be such that $a*b \in T_\in(A_\sim;\alpha_a)$, $b \in T_\in(A_\sim;\alpha_b)$, $c*d \in I_\in(A_\sim;\beta_c)$, $d \in I_\in(A_\sim;\beta_d)$, $x*y \in F_\in(A_\sim;\gamma_x)$, and $y \in F_\in(A_\sim;\gamma_y)$ for $\alpha_a,\alpha_b,\beta_c,\beta_d \in (0,1]$ and $\gamma_x,\gamma_y \in [0,1)$. Using (3.2), we have

$$\begin{aligned} &A_T(a) \geq A_T(a*b) \wedge A_T(b) \geq \alpha_a \wedge \alpha_b \\ &A_I(c) \geq A_I(c*d) \wedge A_I(d) \geq \beta_c \wedge \beta_d \\ &A_F(x) \leq A_F(x*y) \vee A_F(y) \leq \gamma_x \vee \gamma_y. \end{aligned}$$

Hence $a \in T_\in(A_\sim;\alpha_a \wedge \alpha_b)$, $c \in I_\in(A_\sim;\beta_c \wedge \beta_d)$ and $x \in F_\in(A_\sim;\gamma_x \vee \gamma_y)$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in,\in)$-neutrosophic ideal of X. □

Theorem 3.2. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X. Then the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in,\in)$-neutrosophic ideal of X.*

(2) *The nonempty neutrosophic $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are ideals of X for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in,\in)$-neutrosophic ideal of X and assume that $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty for $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$. Then there exist $x,y,z \in X$ such that $x \in T_\in(A_\sim;\alpha)$, $y \in I_\in(A_\sim;\beta)$ and $z \in F_\in(A_\sim;\gamma)$. It follows from (2.8) that

$$0 \in T_\in(A_\sim;\alpha) \cap I_\in(A_\sim;\beta) \cap F_\in(A_\sim;\gamma).$$

Let $x,y,a,b,u,v \in X$ be such that $x*y \in T_\in(A_\sim;\alpha)$, $y \in T_\in(A_\sim;\alpha)$, $a*b \in I_\in(A_\sim;\beta)$, $b \in I_\in(A_\sim;\beta)$, $u*v \in F_\in(A_\sim;\gamma)$ and $v \in F_\in(A_\sim;\gamma)$. Then

$$\begin{aligned} &A_T(x) \geq A_T(x*y) \wedge A_T(y) \geq \alpha \wedge \alpha = \alpha \\ &A_I(a) \geq A_I(a*b) \wedge A_I(b) \geq \beta \wedge \beta = \beta \\ &A_F(u) \leq A_F(u*v) \vee A_F(v) \leq \gamma \vee \gamma = \gamma \end{aligned}$$

by (3.2), and so $x \in T_\in(A_\sim;\alpha)$, $a \in I_\in(A_\sim;\beta)$ and $u \in F_\in(A_\sim;\gamma)$. Hence the nonempty neutrosophic $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are ideals of X for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$.

Conversely, let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X for which $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty and are ideals of X for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$. Assume that $A_T(0) < A_T(x)$, $A_I(0) < A_I(y)$ and $A_F(0) > A_F(z)$ for some $x,y,z \in X$. Then $x \in T_\in(A_\sim;A_T(x))$, $y \in I_\in(A_\sim;A_I(y))$ and $z \in F_\in(A_\sim;A_F(z))$, that is, $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty. But $0 \notin T_\in(A_\sim;A_T(x)) \cap I_\in(A_\sim;A_I(y)) \cap F_\in(A_\sim;A_F(z))$, which is a contradiction since $T_\in(A_\sim;A_T(x))$, $I_\in(A_\sim;A_I(y))$ and $F_\in(A_\sim;A_F(z))$ are ideals of X. Hence $A_T(0) \geq A_T(x)$, $A_I(0) \geq A_I(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose that

$$\begin{aligned} &A_T(x) < A_T(x*y) \wedge A_T(y), \\ &A_I(a) < A_I(a*b) \wedge A_I(b), \\ &A_F(u) > A_F(u*v) \vee A_F(v) \end{aligned}$$

for some $x,y,a,b,u,v \in X$. Taking $\alpha := A_T(x*y) \wedge A_T(y)$, $\beta := A_I(a*b) \wedge A_I(b)$ and $\gamma := A_F(u*v) \vee A_F(v)$ imply that $\alpha,\beta \in (0,1]$, $\gamma \in [0,1)$, $x*y \in T_\in(A_\sim;\alpha)$, $y \in T_\in(A_\sim;\alpha)$, $a*b \in I_\in(A_\sim;\beta)$, $b \in I_\in(A_\sim;\beta)$, $u*v \in F_\in(A_\sim;\gamma)$ and $v \in F_\in(A_\sim;\gamma)$. But $x \notin T_\in(A_\sim;\alpha)$, $a \notin I_\in(A_\sim;\beta)$ and $u \notin F_\in(A_\sim;\gamma)$. This is a contradiction since $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are ideals of X. Thus

$$\begin{aligned} &A_T(x) \geq A_T(x*y) \wedge A_T(y), \\ &A_I(x) \geq A_I(x*y) \wedge A_I(y), \\ &A_F(x) \leq A_F(x*y) \vee A_F(y) \end{aligned}$$

for all $x,y \in X$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in,\in)$-neutrosophic ideal of X by Theorem 3.1. □

Proposition 3.3. *Every $(\in,\in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the following assertions.*

$$(\forall x,y \in X)\left(x \leq y \;\Rightarrow \begin{cases} A_T(x) \geq A_T(y) \\ A_I(x) \geq A_I(y) \\ A_F(x) \leq A_F(y) \end{cases}\right), \tag{3.3}$$

$$(\forall x,y,z \in X)\left(x*y \leq z \;\Rightarrow \begin{cases} A_T(x) \geq A_T(y) \wedge A_T(z) \\ A_I(x) \geq A_I(y) \wedge A_I(z) \\ A_F(x) \leq A_F(y) \vee A_F(z) \end{cases}\right). \tag{3.4}$$

Proof. Let $x,y \in X$ be such that $x \leq y$. Then $x*y = 0$, and so

$$\begin{aligned} &A_T(x) \geq A_T(x*y) \wedge A_T(y) = A_T(0) \wedge A_T(y) = A_T(y), \\ &A_I(x) \geq A_I(x*y) \wedge A_I(y) = A_I(0) \wedge A_I(y) = A_I(y), \\ &A_F(x) \leq A_F(x*y) \vee A_F(y) = A_F(0) \vee A_F(y) = A_F(y) \end{aligned}$$

by Theorem 3.1. Hence (3.3) is valid. Let $x,y,z \in X$ be such that $x*y \leq z$. Then $(x*y)*z = 0$, and thus

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y) \\ &\geq (A_T((x*y)*z) \wedge A_T(z)) \wedge A_T(y) \\ &\geq (A_T(0) \wedge A_T(z)) \wedge A_T(y) \\ &\geq A_T(z) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_I(x) &\geq A_I(x*y) \wedge A_I(y) \\ &\geq (A_I((x*y)*z) \wedge A_I(z)) \wedge A_I(y) \\ &\geq (A_I(0) \wedge A_I(z)) \wedge A_I(y) \\ &\geq A_I(z) \wedge A_I(y) \end{aligned}$$

and

$$\begin{aligned} A_F(x) &\le A_F(x*y) \vee A_F(y) \\ &\le (A_F((x*y)*z) \vee A_F(z)) \vee A_F(y) \\ &\le (A_F(0) \vee A_F(z)) \vee A_F(y) \\ &\le A_F(z) \vee A_F(y) \end{aligned}$$

by Theorem 3.1. □

Theorem 3.4. *Any ideal of a BCK/BCI-algebra X can be realized as level neutrosophic ideals of some $(\in, \in)$-neutrosophic ideal of X.*

Proof. Let I be an ideal of a BCK/BCI-algebra X and let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X given as follows:

$$A_T : X \to [0,1], \quad x \mapsto \begin{cases} \alpha & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases}$$
$$A_I : X \to [0,1], \quad x \mapsto \begin{cases} \beta & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases}$$
$$A_F : X \to [0,1], \quad x \mapsto \begin{cases} \gamma & \text{if } x \in I, \\ 1 & \text{otherwise} \end{cases}$$

where (α, β, γ) is a fixed ordered triple in $(0,1] \times (0,1] \times [0,1)$. Then $T_\in(A_\sim;\alpha) = I$, $I_\in(A_\sim;\beta) = I$ and $F_\in(A_\sim;\gamma) = I$. Obviously, $A_T(0) \ge A_T(x)$, $A_I(0) \ge A_I(x)$ and $A_F(0) \le A_F(x)$ for all $x \in X$. Let $x, y \in X$. If $x*y \in I$ and $y \in I$, then $x \in I$. Hence

$$\begin{aligned} A_T(x*y) &= A_T(y) = A_T(x) = \alpha, \\ A_I(x*y) &= A_I(y) = A_I(x) = \beta, \\ A_F(x*y) &= A_F(y) = A_F(x) = \gamma, \end{aligned}$$

and so

$$\begin{aligned} A_T(x) &\ge A_T(x*y) \wedge A_T(y), \\ A_I(x) &\ge A_I(x*y) \wedge A_I(y), \\ A_F(x) &\le A_F(x*y) \vee A_F(y). \end{aligned}$$

If $x*y \notin I$ and $y \notin I$, then

$$\begin{aligned} A_T(x*y) &= A_T(y) = 0, \\ A_I(x*y) &= A_I(y) = 0, \\ A_F(x*y) &= A_F(y) = 1. \end{aligned}$$

Thus

$$\begin{aligned} A_T(x) &\ge A_T(x*y) \wedge A_T(y), \\ A_I(x) &\ge A_I(x*y) \wedge A_I(y), \\ A_F(x) &\le A_F(x*y) \vee A_F(y). \end{aligned}$$

If $x*y \in I$ and $y \notin I$, then

$$\begin{aligned} A_T(x*y) &= \alpha \text{ and } A_T(y) = 0, \\ A_I(x*y) &= \beta \text{ and } A_I(y) = 0, \\ A_F(x*y) &= \gamma \text{ and } A_F(y) = 1, \end{aligned}$$

It follows that

$$\begin{aligned} A_T(x) &\ge 0 = A_T(x*y) \wedge A_T(y), \\ A_I(x) &\ge 0 = A_I(x*y) \wedge A_I(y), \\ A_F(x) &\le 1 = A_F(x*y) \vee A_F(y). \end{aligned}$$

Similarly, if $x*y \notin I$ and $y \in I$, then

$$\begin{aligned} A_T(x) &\ge A_T(x*y) \wedge A_T(y), \\ A_I(x) &\ge A_I(x*y) \wedge A_I(y), \\ A_F(x) &\le A_F(x*y) \vee A_F(y). \end{aligned}$$

Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. This completes the proof. □

Lemma 3.5 ([5]). *A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x*y) \ge A_T(x) \wedge A_T(y) \\ A_I(x*y) \ge A_I(x) \wedge A_I(y) \\ A_F(x*y) \le A_F(x) \vee A_F(y) \end{array} \right). \quad (3.5)$$

Theorem 3.6. *In a BCK-algebra, every $(\in, \in)$-neutrosophic ideal is an $(\in, \in)$-neutrosophic subalgebra.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Since $x*y \le x$ for all $x, y \in X$, it follows from Proposition 3.3 and (3.2) that

$$\begin{aligned} A_T(x*y) &\ge A_T(x) \ge A_T(x*y) \wedge A_T(y) \ge A_T(x) \wedge A_T(y), \\ A_I(x*y) &\ge A_I(x) \ge A_I(x*y) \wedge A_I(y) \ge A_I(x) \wedge A_I(y), \\ A_F(x*y) &\le A_F(x) \le A_F(x*y) \vee A_F(y) \le A_F(x) \vee A_F(y). \end{aligned}$$

Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X by Lemma 3.5. □

The following example shows that the converse of Theorem 3.6 is not true in general.

Example 3.7. Consider a set $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 1.
Then $(X; *, 0)$ is a BCK-algebra (see [6]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 2
It is routine to verify that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X. We know that $I_\in(A_\sim;\beta)$ is an ideal of X for all $\beta \in (0,1]$. If $\alpha \in (0.3, 0.7]$, then $T_\in(A_\sim;\alpha) = \{0, 1, 3\}$ is not an ideal of X. Also, if $\gamma \in [0.2, 0.8)$, then $F_\in(A_\sim;\gamma) = \{0, 1, 3\}$ is not an ideal of X. Therefore $A_\sim = (A_T, A_I, A_F)$ is not an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.2.

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Table 2: Tabular representation of $A_\sim = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.9	0.2
1	0.7	0.6	0.2
2	0.3	0.6	0.8
3	0.7	0.4	0.2

We give a condition for an $(\in, \in)$-neutrosophic subalgebra to be an $(\in, \in)$-neutrosophic ideal.

Theorem 3.8. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. If $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X that satisfies the condition (3.4), then it is an $(\in, \in)$-neutrosophic ideal of X.*

Proof. Taking $x = y$ in (3.5) and using (III) induce the condition (3.1). Since $x*(x*y) \leq y$ for all $x, y \in X$, it follows from (3.4) that

$$\begin{aligned} A_T(x) &\geq A_T(x*y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x*y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x*y) \vee A_F(y) \end{aligned}$$

for all $x, y \in X$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. □

Theorem 3.9. *Let $\{D_k \mid k \in \Lambda^T \cup \Lambda^I \cup \Lambda^F\}$ be a collection of ideals of a BCK/BCI-algebra X, where Λ^T, Λ^I and Λ^F are nonempty subsets of $[0, 1]$, such that*

$$X = \{D_\alpha \mid \alpha \in \Lambda^T\} \cup \{D_\beta \mid \beta \in \Lambda^I\} \cup \{D_\gamma \mid \gamma \in \Lambda^F\}, \tag{3.6}$$

$$(\forall i, j \in \Lambda^T \cup \Lambda^I \cup \Lambda^F)\,(i > j \;\Leftrightarrow\; D_i \subset D_j)\,. \tag{3.7}$$

Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined as follows:

$$\begin{aligned} A_T &: X \to [0,1],\ x \mapsto \bigvee\{\alpha \in \Lambda^T \mid x \in D_\alpha\}, \\ A_I &: X \to [0,1],\ x \mapsto \bigvee\{\beta \in \Lambda^I \mid x \in D_\beta\}, \\ A_F &: X \to [0,1],\ x \mapsto \bigwedge\{\gamma \in \Lambda^F \mid x \in D_\gamma\}. \end{aligned} \tag{3.8}$$

Then $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.

Proof. Let $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $T_\in(A_\sim; \alpha) \neq \emptyset$, $I_\in(A_\sim; \beta) \neq \emptyset$ and $F_\in(A_\sim; \gamma) \neq \emptyset$. We consider the following two cases:

$$\alpha = \bigvee\{i \in \Lambda^T \mid i < \alpha\} \text{ and } \alpha \neq \bigvee\{i \in \Lambda^T \mid i < \alpha\}.$$

First case implies that

$$\begin{aligned} x \in T_\in(A_\sim; \alpha) &\Leftrightarrow x \in D_i \text{ for all } i < \alpha \\ &\Leftrightarrow x \in \cap\{D_i \mid i < \alpha\}. \end{aligned} \tag{3.9}$$

Hence $T_\in(A_\sim; \alpha) = \cap\{D_i \mid i < \alpha\}$, which is an ideal of X. For the second case, we claim that $T_\in(A_\sim; \alpha) = \cup\{D_i \mid i \geq \alpha\}$. If $x \in \cup\{D_i \mid i \geq \alpha\}$, then $x \in D_i$ for some $i \geq \alpha$. Thus $A_T(x) \geq i \geq \alpha$, and so $x \in T_\in(A_\sim; \alpha)$. If $x \notin \cup\{D_i \mid i \geq \alpha\}$, then $x \notin D_i$ for all $i \geq \alpha$. Since $\alpha \neq \bigvee\{i \in \Lambda^T \mid i < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha) \cap \Lambda^T = \emptyset$. Hence $x \notin D_i$ for all $i > \alpha - \varepsilon$, which means that if $x \in D_i$ then $i \leq \alpha - \varepsilon$. Thus $A_T(x) \leq \alpha - \varepsilon < \alpha$, and so $x \notin T_\in(A_\sim; \alpha)$. Therefore $T_\in(A_\sim; \alpha) = \cup\{D_i \mid i \geq \alpha\}$ which is an ideal of X since $\{D_k\}$ forms a chain. Similarly, we can verify that $I_\in(A_\sim; \beta)$ is an ideal of X. Finally, we consider the following two cases:

$$\gamma = \bigwedge\{j \in \Lambda^F \mid \gamma < j\} \text{ and } \gamma \neq \bigwedge\{j \in \Lambda^F \mid \gamma < j\}.$$

For the first case, we have

$$\begin{aligned} x \in F_\in(A_\sim; \gamma) &\Leftrightarrow x \in D_j \text{ for all } j > \gamma \\ &\Leftrightarrow x \in \cap\{D_j \mid j > \gamma\}, \end{aligned} \tag{3.10}$$

and thus $F_\in(A_\sim; \gamma) = \cap\{D_j \mid j > \gamma\}$ which is an ideal of X. The second case implies that $F_\in(A_\sim; \gamma) = \cup\{D_j \mid j \leq \gamma\}$. In fact, if $x \in \cup\{D_j \mid j \leq \gamma\}$, then $x \in D_j$ for some $j \leq \gamma$. Thus $A_F(x) \leq j \leq \gamma$, that is, $x \in F_\in(A_\sim; \gamma)$. Hence $\cup\{D_j \mid j \leq \gamma\} \subseteq F_\in(A_\sim; \gamma)$. Now if $x \notin \cup\{D_j \mid j \leq \gamma\}$, then $x \notin D_j$ for all $j \leq \gamma$. Since $\gamma \neq \bigwedge\{j \in \Lambda^F \mid \gamma < j\}$, there exists $\varepsilon > 0$ such that $(\gamma, \gamma+\varepsilon) \cap \Lambda^F$ is empty. Hence $x \notin D_j$ for all $j < \gamma+\varepsilon$, and so if $x \in D_j$, then $j \geq \gamma+\varepsilon$. Thus $A_F(x) \geq \gamma+\varepsilon > \gamma$, and hence $x \notin F_\in(A_\sim; \gamma)$. Thus $F_\in(A_\sim; \gamma) \subseteq \cup\{D_j \mid j \leq \gamma\}$, and therefore $F_\in(A_\sim; \gamma) = \cup\{D_j \mid j \leq \gamma\}$ which is an ideal of X. Consequently, $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.2. □

A mapping $f : X \to Y$ of BCK/BCI-algebras is called a *homomorphism* if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCK/BCI-algebras, then $f(0) = 0$. Given a homomorphism $f : X \to Y$ of BCK/BCI-algebras and a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in Y, we define a neutrosophic set $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ in X, which is called the *induced neutrosophic set*, as follows:

$$\begin{aligned} A^f_T &: X \to [0,1],\ x \mapsto A_T(f(x)), \\ A^f_I &: X \to [0,1],\ x \mapsto A_I(f(x)), \\ A^f_F &: X \to [0,1],\ x \mapsto A_F(f(x)). \end{aligned}$$

Theorem 3.10. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of Y, then the induced neutrosophic set*

$A^f_\sim = (A^f_T, A^f_I, A^f_F)$ *in* X *is an* $(\in, \in)$*-neutrosophic ideal of* X.

Proof. For any $x \in X$, we have

$$A^f_T(x) = A_T(f(x)) \leq A_T(0) = A_T(f(0)) = A^f_T(0),$$
$$A^f_I(x) = A_I(f(x)) \leq A_I(0) = A_I(f(0)) = A^f_I(0),$$
$$A^f_F(x) = A_F(f(x)) \geq A_F(0) = A_F(f(0)) = A^f_F(0).$$

Let $x, y \in X$. Then

$$\begin{aligned} &A^f_T(x*y) \wedge A^f_T(y) = A_T(f(x*y)) \wedge A_T(f(y)) \\ &= A_T(f(x)*f(y)) \wedge A_T(f(y)) \\ &\leq A_T(f(x)) = A^f_T(x), \end{aligned}$$

$$\begin{aligned} &A^f_I(x*y) \wedge A^f_I(y) = A_I(f(x*y)) \wedge A_I(f(y)) \\ &= A_I(f(x)*f(y)) \wedge A_I(f(y)) \\ &\leq A_I(f(x)) = A^f_I(x), \end{aligned}$$

and

$$\begin{aligned} &A^f_F(x*y) \vee A^f_F(y) = A_F(f(x*y)) \vee A_F(f(y)) \\ &= A_F(f(x)*f(y)) \vee A_F(f(y)) \\ &\geq A_F(f(x)) = A^f_F(x). \end{aligned}$$

Therefore $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.1. □

Theorem 3.11. *Let* $f : X \to Y$ *be an onto homomorphism of* BCK/BCI*-algebras and let* $A_\sim = (A_T, A_I, A_F)$ *be a neutrosophic set in* Y. *If the induced neutrosophic set* $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ *in* X *is an* $(\in, \in)$*-neutrosophic ideal of* X*, then* $A_\sim = (A_T, A_I, A_F)$ *is an* $(\in, \in)$*-neutrosophic ideal of* Y.

Proof. Assume that the induced neutrosophic set $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ in X is an $(\in, \in)$-neutrosophic ideal of X. For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$ since f is onto. Using (3.1), we have

$$A_T(x) = A_T(f(a)) = A^f_T(a) \leq A^f_T(0) = A_T(f(0)) = A_T(0),$$
$$A_I(x) = A_I(f(a)) = A^f_I(a) \leq A^f_I(0) = A_I(f(0)) = A_I(0),$$
$$A_F(x) = A_F(f(a)) = A^f_F(a) \geq A^f_F(0) = A_F(f(0)) = A_F(0).$$

Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. It follows from (3.2) that

$$\begin{aligned} A_T(x) &= A_T(f(a)) = A^f_T(a) \\ &\geq A^f_T(a*b) \wedge A^f_T(b) \\ &= A_T(f(a*b)) \wedge A_T(f(b)) \\ &= A_T(f(a)*f(b)) \wedge A_T(f(b)) \\ &= A_T(x*y) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_I(x) &= A_I(f(a)) = A^f_I(a) \\ &\geq A^f_I(a*b) \wedge A^f_I(b) \\ &= A_I(f(a*b)) \wedge A_I(f(b)) \\ &= A_I(f(a)*f(b)) \wedge A_I(f(b)) \\ &= A_I(x*y) \wedge A_I(y), \end{aligned}$$

and

$$\begin{aligned} A_F(x) &= A_F(f(a)) = A^f_F(a) \\ &\leq A^f_F(a*b) \vee A^f_F(b) \\ &= A_F(f(a*b)) \vee A_F(f(b)) \\ &= A_F(f(a)*f(b)) \vee A_F(f(b)) \\ &= A_F(x*y) \vee A_F(y). \end{aligned}$$

Therefore $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of Y by Theorem 3.1. □

Let $\mathcal{N}_{(\in,\in)}(X)$ be the collection of all $(\in, \in)$-neutrosophic ideals of X and let $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Define binary relations $\mathcal{R}^\alpha_T$, $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$ on $\mathcal{N}_{(\in,\in)}(X)$ as follows:

$$\begin{aligned} A_T \mathcal{R}^\alpha_T B_T &\Leftrightarrow T_\in(A_\sim; \alpha) = T_\in(B_\sim; \alpha) \\ A_I \mathcal{R}^\beta_I B_I &\Leftrightarrow I_\in(A_\sim; \beta) = I_\in(B_\sim; \beta) \\ A_F \mathcal{R}^\gamma_F B_F &\Leftrightarrow F_\in(A_\sim; \gamma) = F_\in(B_\sim; \gamma) \end{aligned} \tag{3.11}$$

for all $A_\sim = (A_T, A_I, A_F)$ and $B_\sim = (B_T, B_I, B_F)$ in $\mathcal{N}_{(\in,\in)}(X)$.

Clearly $\mathcal{R}^\alpha_T$, $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$ are equivalence relations on $\mathcal{N}_{(\in,\in)}(X)$. For any $A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X)$, let $[A_\sim]_T$ (resp., $[A_\sim]_I$ and $[A_\sim]_F$) denote the equivalence class of $A_\sim = (A_T, A_I, A_F)$ in $\mathcal{N}_{(\in,\in)}(X)$ under $\mathcal{R}^\alpha_T$ (resp., $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$). Denote by $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\alpha_T$, $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\beta_I$ and $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\gamma_F$ the collection of all equivalence classes under $\mathcal{R}^\alpha_T$, $\mathcal{R}^\beta_I$ and $\mathcal{R}^\gamma_F$, respectively, that is,

$$\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\alpha_T = \{[A_\sim]_T \mid A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X),$$
$$\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\beta_I = \{[A_\sim]_I \mid A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X),$$
$$\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}^\gamma_F = \{[A_\sim]_F \mid A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X).$$

Now let $\mathcal{I}(X)$ denote the family of all ideals of X. Define maps f_α, g_β and h_γ from $\mathcal{N}_{(\in,\in)}(X)$ to $\mathcal{I}(X) \cup \{\emptyset\}$ by
$f_\alpha(A_\sim) = T_\in(A_\sim; \alpha)$, $g_\beta(A_\sim) = I_\in(A_\sim; \beta)$ and
$h_\gamma(A_\sim) = F_\in(A_\sim; \gamma)$,
respectively, for all $A_\sim = (A_T, A_I, A_F)$ in $\mathcal{N}_{(\in,\in)}(X)$. Then f_α, g_β and h_γ are clearly well-defined.

Theorem 3.12. *For any* $\alpha, \beta \in (0, 1]$ *and* $\gamma \in [0, 1)$*, the maps* f_α*,* g_β *and* h_γ *are surjective from* $\mathcal{N}_{(\in,\in)}(X)$ *to* $\mathcal{I}(X) \cup \{\emptyset\}$.

Proof. Let $0_\sim := (0_T, 0_I, 1_F)$ be a neutrosophic set in X where 0_T, 0_I and 1_F are fuzzy sets in X defined by $0_T(x) = 0$, $0_I(x) = 0$ and $1_F(x) = 1$ for all $x \in X$. Obviously, $0_\sim := (0_T, 0_I, 1_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Also, $f_\alpha(0_\sim) = T_\in(0_\sim; \alpha) = \emptyset$, $g_\beta(0_\sim) = I_\in(0_\sim; \beta) = \emptyset$

and $h_\gamma(0_\sim) = F_\in(0_\sim;\gamma) = \emptyset$. For any ideal I of X, let $A_\sim = (A_T, A_I, A_F)$ be the $(\in, \in)$-neutrosophic ideal of X in the proof of Theorem 3.4. Then $f_\alpha(A_\sim) = T_\in(A_\sim;\alpha) = I$, $g_\beta(A_\sim) = I_\in(A_\sim;\beta) = I$ and $h_\gamma(A_\sim) = F_\in(A_\sim;\gamma) = I$. Therefore f_α, g_β and h_γ are surjective. □

Theorem 3.13. *The quotient sets $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_T^\alpha$, $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_I^\beta$ and $\mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_F^\gamma$ are equivalent to $\mathcal{I}(X)\cup\{\emptyset\}$ for any $\alpha,\beta\in(0,1]$ and $\gamma\in[0,1)$.*

Proof. Let $A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X)$. For any $\alpha,\beta\in(0,1]$ and $\gamma\in[0,1)$, define

$$\begin{aligned} f_\alpha^* &: \mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_T^\alpha \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_T \mapsto f_\alpha(A_\sim),\\ g_\beta^* &: \mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_I^\beta \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_I \mapsto g_\beta(A_\sim),\\ h_\gamma^* &: \mathcal{N}_{(\in,\in)}(X)/\mathcal{R}_F^\gamma \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_F \mapsto h_\gamma(A_\sim).\end{aligned}$$

Assume that $f_\alpha(A_\sim) = f_\alpha(B_\sim)$, $g_\beta(A_\sim) = g_\beta(B_\sim)$ and $h_\gamma(A_\sim) = h_\gamma(B_\sim)$ for $B_\sim = (B_T, B_I, B_F) \in \mathcal{N}_{(\in,\in)}(X)$. Then $T_\in(A_\sim;\alpha) = T_\in(B_\sim;\alpha)$, $I_\in(A_\sim;\beta) = I_\in(B_\sim;\beta)$ and $F_\in(A_\sim;\gamma) = F_\in(B_\sim;\gamma)$ which imply that $A_T\mathcal{R}_T^\alpha B_T$, $A_I\mathcal{R}_I^\beta B_I$ and $A_F\mathcal{R}_F^\gamma B_F$. Hence $[A_\sim]_T = [B_\sim]_T$, $[A_\sim]_I = [B_\sim]_I$ and $[A_\sim]_F = [B_\sim]_F$. Therefore f_α^*, g_β^* and h_γ^* are injective. Consider the $(\in, \in)$-neutrosophic ideal $0_\sim := (0_T, 0_I, 1_F)$ of X which is given in the proof of Theorem 3.12. Then $f_\alpha^*([0_\sim]_T) = f_\alpha(0_\sim) = T_\in(0_\sim;\alpha) = \emptyset$, $g_\beta^*([0_\sim]_I) = g_\beta(0_\sim) = I_\in(0_\sim;\beta) = \emptyset$, and $h_\gamma^*([0_\sim]_F) = h_\gamma(0_\sim) = F_\in(0_\sim;\gamma) = \emptyset$. For any ideal I of X, consider the $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of X in the proof of Theorem 3.4. Then $f_\alpha^*([A_\sim]_T) = f_\alpha(A_\sim) = T_\in(A_\sim;\alpha) = I$, $g_\beta^*([A_\sim]_I) = g_\beta(A_\sim) = I_\in(A_\sim;\beta) = I$, and $h_\gamma^*([A_\sim]_F) = h_\gamma(A_\sim) = F_\in(A_\sim;\gamma) = I$. Hence f_α^*, g_β^* and h_γ^* are surjective, and the proof is over. □

For any $\alpha,\beta\in[0,1]$, we define another relations $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ on $\mathcal{N}_{(\in,\in)}(X)$ as follows:

$$\begin{aligned} (A_\sim, B_\sim)\in\mathcal{R}_\alpha \ &\Leftrightarrow\ T_\in(A_\sim;\alpha)\cap F_\in(A_\sim;\alpha)\\ &\qquad = T_\in(B_\sim;\alpha)\cap F_\in(B_\sim;\alpha),\\ (A_\sim, B_\sim)\in\mathcal{R}_\beta \ &\Leftrightarrow\ I_\in(A_\sim;\beta)\cap F_\in(A_\sim;\beta)\\ &\qquad = I_\in(B_\sim;\beta)\cap F_\in(B_\sim;\beta)\end{aligned} \tag{3.12}$$

for all $A_\sim = (A_T, A_I, A_F)$ and $B_\sim = (B_T, B_I, B_F)$ in $\mathcal{N}_{(\in,\in)}(X)$. Then the relations $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are also equivalence relations on $\mathcal{N}_{(\in,\in)}(X)$.

Theorem 3.14. *Given $\alpha,\beta\in(0,1)$, we define two maps*

$$\begin{aligned} \varphi_\alpha &: \mathcal{N}_{(\in,\in)}(X)\to\mathcal{I}(X)\cup\{\emptyset\},\\ &\qquad A_\sim \mapsto f_\alpha(A_\sim)\cap h_\alpha(A_\sim),\\ \varphi_\beta &: \mathcal{N}_{(\in,\in)}(X)\to\mathcal{I}(X)\cup\{\emptyset\},\\ &\qquad A_\sim \mapsto g_\beta(A_\sim)\cap h_\beta(A_\sim)\end{aligned} \tag{3.13}$$

for each $A_\sim = (A_T, A_I, A_F) \in \mathcal{N}_{(\in,\in)}(X)$. Then φ_α and φ_β are surjective.

Proof. Consider the $(\in, \in)$-neutrosophic ideal $0_\sim := (0_T, 0_I, 1_F)$ of X which is given in the proof of Theorem 3.12. Then

$$\begin{aligned}\varphi_\alpha(0_\sim) &= f_\alpha(0_\sim)\cap h_\alpha(0_\sim) = T_\in(0_\sim;\alpha)\cap F_\in(0_\sim;\alpha) = \emptyset,\\ \varphi_\beta(0_\sim) &= g_\beta(0_\sim)\cap h_\beta(0_\sim) = I_\in(0_\sim;\beta)\cap F_\in(0_\sim;\beta) = \emptyset.\end{aligned}$$

For any ideal I of X, consider the $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of X in the proof of Theorem 3.4. Then

$$\begin{aligned}\varphi_\alpha(A_\sim) &= f_\alpha(A_\sim)\cap h_\alpha(A_\sim)\\ &= T_\in(A_\sim;\alpha)\cap F_\in(A_\sim;\alpha) = I\end{aligned}$$

and

$$\begin{aligned}\varphi_\beta(A_\sim) &= g_\beta(A_\sim)\cap h_\beta(A_\sim)\\ &= I_\in(A_\sim;\beta)\cap F_\in(A_\sim;\beta) = I.\end{aligned}$$

Therefore φ_α and φ_β are surjective. □

Theorem 3.15. *For any $\alpha,\beta\in(0,1)$, the quotient sets $\mathcal{N}_{(\in,\in)}(X)/\varphi_\alpha$ and $\mathcal{N}_{(\in,\in)}(X)/\varphi_\beta$ are equivalent to $\mathcal{I}(X)\cup\{\emptyset\}$.*

Proof. Given $\alpha,\beta\in(0,1)$, define two maps φ_α^* and φ_β^* as follows:

$$\begin{aligned}\varphi_\alpha^* &: \mathcal{N}_{(\in,\in)}(X)/\varphi_\alpha \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_{\mathcal{R}_\alpha}\mapsto\varphi_\alpha(A_\sim),\\ \varphi_\beta^* &: \mathcal{N}_{(\in,\in)}(X)/\varphi_\beta \to \mathcal{I}(X)\cup\{\emptyset\},\ [A_\sim]_{\mathcal{R}_\beta}\mapsto\varphi_\beta(A_\sim).\end{aligned}$$

If $\varphi_\alpha^*([A_\sim]_{\mathcal{R}_\alpha}) = \varphi_\alpha^*([B_\sim]_{\mathcal{R}_\alpha})$ and $\varphi_\beta^*([A_\sim]_{\mathcal{R}_\beta}) = \varphi_\beta^*([B_\sim]_{\mathcal{R}_\beta})$ for all $[A_\sim]_{\mathcal{R}_\alpha}, [B_\sim]_{\mathcal{R}_\alpha} \in \mathcal{N}_{(\in,\in)}(X)/\varphi_\alpha$ and $[A_\sim]_{\mathcal{R}_\beta}, [B_\sim]_{\mathcal{R}_\beta} \in \mathcal{N}_{(\in,\in)}(X)/\varphi_\beta$, then

$$f_\alpha(A_\sim)\cap h_\alpha(A_\sim) = f_\alpha(B_\sim)\cap h_\alpha(B_\sim)$$

and

$$g_\beta(A_\sim)\cap h_\beta(A_\sim) = g_\beta(B_\sim)\cap h_\beta(B_\sim),$$

that is,

$$T_\in(A_\sim;\alpha)\cap F_\in(A_\sim;\alpha) = T_\in(B_\sim;\alpha)\cap F_\in(B_\sim;\alpha)$$

and

$$I_\in(A_\sim;\beta)\cap F_\in(A_\sim;\beta) = I_\in(B_\sim;\beta)\cap F_\in(B_\sim;\beta).$$

Hence $(A_\sim, B_\sim)\in\mathcal{R}_\alpha$ and $(A_\sim, B_\sim)\in\mathcal{R}_\beta$. It follows that $[A_\sim]_{\mathcal{R}_\alpha} = [B_\sim]_{\mathcal{R}_\alpha}$ and $[A_\sim]_{\mathcal{R}_\beta} = [B_\sim]_{\mathcal{R}_\beta}$. Thus φ_α^* and φ_β^* are injective. Consider the $(\in, \in)$-neutrosophic ideal $0_\sim := (0_T, 0_I, 1_F)$ of X which is given in the proof of Theorem 3.12. Then

$$\begin{aligned}\varphi_\alpha^*([0_\sim]_{\mathcal{R}_\alpha}) &= \varphi_\alpha(0_\sim) = f_\alpha(0_\sim)\cap h_\alpha(0_\sim)\\ &= T_\in(0_\sim;\alpha)\cap F_\in(0_\sim;\alpha) = \emptyset\end{aligned}$$

and

$$\varphi^*_\beta\left([0_\sim]_{\mathcal{R}_\beta}\right) = \varphi_\beta(0_\sim) = g_\beta(0_\sim) \cap h_\beta(0_\sim) = I_\in(0_\sim;\beta) \cap F_\in(0_\sim;\beta) = \emptyset.$$

For any ideal I of X, consider the $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of X in the proof of Theorem 3.4. Then

$$\varphi^*_\alpha\left([A_\sim]_{\mathcal{R}_\alpha}\right) = \varphi_\alpha(A_\sim) = f_\alpha(A_\sim) \cap h_\alpha(A_\sim) = T_\in(A_\sim;\alpha) \cap F_\in(A_\sim;\alpha) = I$$

and

$$\varphi^*_\beta\left([A_\sim]_{\mathcal{R}_\beta}\right) = \varphi_\beta(A_\sim) = g_\beta(A_\sim) \cap h_\beta(A_\sim) = I_\in(A_\sim;\beta) \cap F_\in(A_\sim;\beta) = I.$$

Therefore φ^*_α and φ^*_β are surjective. This completes the proof. □

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