Foundations of Physics, Vol. 23, No. 2, 1993

Mechanical Models for Lorentz Group Representations

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Received July 17, 1992

Simple classical mechanical models are constructed to help understand the natures of certain unitary representations of the Lorentz group $SO(3, 1)$ associated with its action on spacetime. In particular, different kinds of Principal Series unitary irreducible representations of $SO(3, 1)$ with positive or negative quadratic Casimir invariant are seen to correspond to bounded and unbounded motions, respectively, in the mechanical models.

1. INTRODUCTION

The linear representations of the Lorentz group $SO(3, 1)$, and of its two-fold universal covering group $SL(2, C)$, are of importance in many relativistic problems.(1) The framework of relativistic quantum mechanics motivates a study of the unitary representations of these groups. Every nontrivial unitary representation (UR) of either of these groups is necessarily infinite dimensional, on account of their noncompactness.

The earliest construction and use of unitary irreducible representations (UIR's) of $SL(2, C)$ in a physical context seems to have been in connection with Majorana's infinite component relativistic wave equations, whose principal motivation was to avoid the negative energy solutions of the Dirac equation.(2) Subsequently a systematic analysis of a class of UR's of $SO(3, 1)$ was undertaken by Dirac, leading to his theory of expansors.(3) Soon after, a complete construction of all the UIR's of $SL(2, C)$ was achieved independently by Harish Chandra,(4) on the one hand, and by

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825/23/2-6

0015-9018/93/0200-0245$07.00/0

Gel'fand and Naimark,[5] on the other. In particular, the former introduced the concept of expinors as a half integral spin counterpart to Dirac's expansors.

While the construction of the UIR's of these groups involves a certain amount of nontrivial mathematical analysis, for example at the Lie algebra level, the action of $SO(3, 1)$ on Minkowski spacetime leads in a trivial fashion to certain highly reducible UR's of this group on certain spaces of functions on spacetime.[6] Thus, one can work with the Hilbert space of complex-valued scalar wavefunctions $\psi(x)$ on spacetime with the norm given by

$$\|\psi\|^2 = \int d^4x \, |\psi(x)|^2 \tag{1.1}$$

and in the obvious way set up a unitary action of $SO(3, 1)$ on this space. It is this rather "large" UR of $SO(3, 1)$ that was analyzed by Dirac in terms of his expansor representations.[3]

One can in an equally easy manner set up somewhat "smaller" UR's of $SO(3, 1)$ by working with functions defined, not on all of spacetime, but on one of the Lorentz-invariant three-dimensional hypersurfaces in spacetime. The three possible essentially distinct hypersurfaces will be denoted as Q_ε with $\varepsilon = 1, 0, -1$. (The symbol Q is chosen as these hypersurfaces will be used as classical configuration space manifolds in later sections.) These are, respectively, the unit positive timelike hyperboloid, the positive light cone, and the single-sheeted unit spacelike hyperboloid which also happens to be a ruled surface. Since each of these hypersurfaces Q_ε carries a Lorentz-invariant "volume element," by an obvious modification of Eq. (1.1) we can set up a Hilbert space norm for complex-valued functions ψ defined on Q_ε, and then a UR of $SO(3, 1)$ acting on these functions. It is these UR's of $SO(3, 1)$ that we are interested in; we shall denote them by $\mathscr{U}_\varepsilon(.)$.

Each of the three UR's $\mathscr{U}_\varepsilon(.)$ is still reducible, and the reduction into a direct sum and/or integral of UIR's of $SO(3, 1)$ is a question of natural interest. The results in all three cases are known,[7] and the variations from case to case are very striking. In a sense which will become clear in the next section, the UR $\mathscr{U}_1(.)$ acting on functions on Q_1 is the simplest. It decomposes into a direct integral of certain UIR's of $SO(3, 1)$ of the Principal Series, each one occurring "just once." On the other hand, the UR $\mathscr{U}_0(.)$ acting on functions on the positive light cone Q_0 is exactly "twice as large" as $\mathscr{U}_1(.)$: it appears as the direct sum of two direct integrals of UIR's, which are individually the same direct integral as encountered in the reduction of $\mathscr{U}_1(.)$. Much more surprising is the result of reducing the UR $\mathscr{U}_{-1}(.)$ acting on functions on Q_{-1}. One obtains all the UIR's arising in the reduction of $\mathscr{U}_0(.)$ just as often and, in addition, an infinite discrete direct sum of certain other Principal Series UIR's, which do not occur at all in $\mathscr{U}_1(.)$ and $\mathscr{U}_0(.)$. This last component in the reduction is quite striking, and is intrinsically related to Q_{-1} being a ruled surface.

The purpose of this paper is to provide simple classical dynamical models, in a suitable Lorentz invariant canonical framework, which helps us understand in a qualitative way the representation theoretic results which we have just outlined. At the same time, the manifestly covariant canonical formalisms which we shall set up, while they are quite simple, may be of general interest. We should clarify that our intention is not to give a detailed explanation for the results of reduction of each UR $\mathscr{U}_\varepsilon(.)$ into UIR's. It is limited to providing some suggestive hints and comparisons that may make the differences among the three UR's $\mathscr{U}_\varepsilon(.)$ seem less strange than they may appear at first sight.

The material of this paper is arranged as follows. In Section 2 we recall the description of the UIR's of $SL(2, C)$ and $SO(3, 1)$, and the three UR's $\mathscr{U}_\varepsilon(.)$ of $SO(3, 1)$ associated with the hypersurfaces Q_ε in spacetime. The reduction of the latter UR's into UIR's is then described, and the variations from case to case pointed out. In Section 3 we set up a manifestly covariant canonical formalism based on the Q_ε as possible classical configuration spaces. There are interesting parallels between Q_1 and Q_{-1}, when we attempt to describe their phase spaces T^*Q_1 and T^*Q_{-1} in the most economical terms while maintaining manifest covariance. But the case of Q_0 cannot be brought into the same pattern, and it remains intrinsically more complicated. Section 4 constructs simple mechanical models which can be handled by the formalism of Section 3. The natures of the phase space trajectories in these models help us appreciate in an intuitive way the reasons for the differences among the UR's $\mathscr{U}_\varepsilon(.)$. Here again while Lagrangian methods work satisfactorily for $Q_{\pm 1}$, the space Q_0 needs a separate directly Hamiltonian approach. Section 5 contains concluding remarks.

2. THE UIR's OF $SL(2, C)$, THE UR's $\mathscr{U}_\varepsilon(.)$, AND THEIR REDUCTIONS

Any UR $\mathscr{U}(.)$ of $SL(2, C)$ on a Hilbert space $\mathscr{H}$ is generated by six hermitian operators $M_{\mu\nu} = -M_{\nu\mu}$, $\mu, \nu = 0, 1, 2, 3$ obeying the commutation relations

$$-i[M_{\mu\nu}, M_{\rho\sigma}] = g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\rho\mu} - g_{\mu\sigma}M_{\rho\nu} \qquad (2.1)$$

(We use the diagonal timelike metric $g_{00}=1$, $g_{11}=g_{22}=g_{33}=-1$.) The space–space and the time–space components are

$$\begin{aligned} J_j &= M_{kl}, \qquad jkl=1,2,3 \text{ cyclically} \\ K_j &= M_{0j}, \qquad j=1,2,3 \end{aligned} \tag{2.2}$$

The former generate the subgroup of spatial rotations, while the latter generate pure Lorentz transformations. The two independent Casimir invariants are

$$\begin{aligned} \mathscr{C}_1 &= \tfrac{1}{2}M^{\mu\nu}M_{\mu\nu} = \underline{J}^2 - \underline{K}^2 \\ \mathscr{C}_2 &= \tfrac{1}{8}\varepsilon^{\mu\nu\rho\sigma}M_{\mu\nu}M_{\rho\sigma} = \underline{J}.\underline{K} = \underline{K}.\underline{J}, \qquad \varepsilon^{0123}=1 \end{aligned} \tag{2.3}$$

The UIR's of $SL(2, C)$ come in two families, namely the Principal Series, and the Exceptional or Supplementary Series.[1] Only the former appear in the reduction of the regular representation of $SL(2, C)$. In any UIR, the reduction with respect to the maximal compact subgroup $SU(2)$, or the spectrum of spin values present, is simple and multiplicity free. There is a minimum spin value, j_0 say, which occurs once, and then all the higher spin values j_0+1, j_0+2,... also occur once each.

Each UIR of the Principal Series can be uniquely labelled by two parameters in the form $\{j_0, \rho\}$, where j_0 is the above-mentioned lowest spin value and ρ is a real parameter. The precise ranges so that each distinct UIR is listed just once, and the values of the Casimir invariants, are as follows:

Principal Series

$$\begin{aligned} &j_0=0\colon \{0,\rho\}, \qquad 0\leqslant\rho<\infty \\ &\mathscr{C}_1 = -1-\rho^2, \qquad \mathscr{C}_2=0 \end{aligned} \tag{2.4a}$$

$$\begin{aligned} &j_0=\tfrac{1}{2}, 1, \tfrac{3}{2},\ldots\colon \{j_0,\rho\}, \qquad -\infty<\rho<\infty \\ &\mathscr{C}_1 = -1-\rho^2+j_0^2, \qquad \mathscr{C}_2 = -j_0\rho \end{aligned} \tag{2.4b}$$

On the other hand, each UIR of the Supplementary Series always has $j_0=0$, and is characterized by a real parameter ρ in the open interval (0, 1):

Supplementary Series

$$\begin{aligned} &j_0=0\colon \{0, i\rho\}, \qquad 0<\rho<1 \\ &\mathscr{C}_1 = -1+\rho^2, \qquad \mathscr{C}_2=0 \end{aligned} \tag{2.5}$$

To get a complete catalogue of all UIR's of $SO(3, 1)$, we need only restrict j_0 to integer values. Thus, one has the Principal Series UIR's $\{j_0, \rho\}$

for $j_0 = 0, 1, 2,\ldots$, and the Supplementary Series UIR's$\{0, i\rho\}$, the ranges for ρ being as in Eqs. (2.4) and (2.5).

With these notations, the two UIR's of $SL(2, C)$ occurring in Majorana's pioneering work are the Principal Series UIR$\{1/2, 0\}$ and the Supplementary Series UIR$\{0, i/2\}$.

Now we turn to the UR's of $SO(3, 1)$ acting on suitable functions on invariant hypersurfaces in spacetime. The three hypersurfaces Q_ε are defined as follows:

$$Q_1 = \{x^\mu \,|\, x^\mu x_\mu = 1,\, x_0 \geqslant 1\} \tag{2.6a}$$

$$Q_0 = \{x^\mu \,|\, x^\mu x_\mu = 0,\, x_0 > 0\} \tag{2.6b}$$

$$Q_{-1} = \{x^\mu \,|\, x^\mu x_\mu = -1\} \tag{2.6c}$$

(In passing we note that, strictly speaking, the tip $x^\mu = 0$ is *not* included in Q_0). On each of these hypersurfaces, a Lorentz invariant volume element exists so that we may set up corresponding Hilbert spaces H_ε of square integrable functions. We uniformly write $\psi(x)$ for a general (scalar) wave function in each case, it being understood that x is a point on Q_ε. Then the three Hilbert spaces are

$$\mathscr{H}_1 = \left\{\psi(x),\, x_0 = \sqrt{1 + |\underline{x}|^2} \;\middle|\; \|\psi\|_1^2 = \int_{\mathscr{R}^3} \frac{d^3x}{x_0}\, |\psi(x)|^2 < \infty\right\} \tag{2.7a}$$

$$\mathscr{H}_0 = \left\{\psi(x),\, x_0 = |\underline{x}| \;\middle|\; \|\psi\|_0^2 = \int_{\mathscr{R}^3} \frac{d^3x}{x_0}\, |\psi(x)|^2 < \infty\right\} \tag{2.7b}$$

$$\begin{aligned}\mathscr{H}_{-1} = \Big\{\psi(x),\, |\underline{x}| = \sqrt{1 + x_0^2} \;\Big|\; &\|\psi\|_{-1}^2 \\ &= \int_{-\infty}^{\infty} dx_0\, |\underline{x}| \int_{S^2} d\Omega(\hat{x})\, |\psi(x)|^2 < \infty\Big\}\end{aligned} \tag{2.7c}$$

(In the last line here, $d\Omega(\hat{x})$ is the element of solid angle at the point $\hat{x}$ on the unit sphere S^2). For any element $\Lambda \varepsilon SO(3, 1)$, the corresponding unitary operator $\mathscr{U}_\varepsilon(\Lambda)$ acts on a wave function $\psi(x)\, \varepsilon \mathscr{H}_\varepsilon$ in a standard way:

$$\begin{aligned}&\mathscr{U}_\varepsilon(\Lambda)\, \psi = \psi' : \\ &\quad \psi'(x) = \psi(\Lambda^{-1}x) \\ &\quad \|\psi'\|_\varepsilon = \|\psi\|_\varepsilon\end{aligned} \tag{2.8}$$

Each of these three UR's can be reduced into a direct sum/integral of UIR's of $SO(3, 1)$. When this is done, the Supplementary Series UIR's never appear, while from the Principal Series only those with $\mathscr{C}_2 = 0$ are encountered. The pattern in the three cases can be indicated symbolically as follows[7]:

$$\mathscr{U}_1(.)=\oplus\int_0^{\infty} d\rho\{0,\rho\} \tag{2.9a}$$

$$\mathscr{U}_0(.)=\oplus\, 2\int_0^{\infty} d\rho\{0,\rho\} \tag{2.9b}$$

$$\mathscr{U}_{-1}(.)=\oplus\, 2\int_0^{\infty} d\rho\{0,\rho\}\oplus\sum_{j_0=1,2,\ldots}^{\infty}\{j_0,0\} \tag{2.9c}$$

At this point we may specifically recall that for the UIR's occurring here the first Casimir invariant has these values:

$$\begin{aligned}\{0,\rho\}&:\mathscr{C}_1=-1-\rho^2\leqslant -1\\ \{j_0,0\}&:\mathscr{C}_1=-1+j_0^2\geqslant 0\end{aligned} \tag{2.10}$$

In the case of the UR $\mathscr{U}_{-1}(.)$ acting on functions on the single-sheeted unit spacelike hyperboloid, a further statement can be made. Since spacetime reflection $x^\mu\to -x^\mu$ is defined on Q_{-1} (but not on Q_1 or Q_0), we can consider the two subspaces of $\mathscr{H}_{-1}$ consisting respectively of even and odd functions, $\psi(-x)=\pm\psi(x)$. Each of these subspaces is invariant under $\mathscr{U}_{-1}(.)$, and can be reduced by itself. One finds

$$\begin{aligned}\mathscr{U}_{-1}(.)\,|_{\text{even}\,\psi}&=\oplus\int_0^{\infty} d\rho\{0,\rho\}\oplus\sum_{j_0=2,4,\ldots}^{\infty}\{j_0,0\}\\ \mathscr{U}_{-1}(.)\,|_{\text{odd}\,\psi}&=\oplus\int_0^{\infty} d\rho\{0,\rho\}\oplus\sum_{j_0=1,3,\ldots}^{\infty}\{j_0,0\}\end{aligned} \tag{2.11}$$

It is clear from these statements that there are dramatic differences in the UIR contents of the three natural UR's $\mathscr{U}_\varepsilon(.)$ of $SO(3,1)$. Ultimately, of course, these differences are traceable to the basic geometric differences in the three hypersurfaces Q_ε. Our purpose will be to develop three simple classical mechanical models based on the Q_ε as configuration spaces, such that a qualitative understanding of the above pattern of results can be obtained, though not a detailed one.

3. Q_ε AS CONFIGURATION SPACES, AND THEIR PHASE SPACES

We now view each hypersurface Q_ε as a model of a classical three-dimensional configuration space, admitting $SO(3,1)$ action via point transformations. We then develop a description of the corresponding phase spaces (cotangent bundles) T^*Q_ε in a manifestly covariant form. In keeping with this, we shall hereafter use q^μ rather than x^μ for a general point on Q_ε. It is worth emphasizing that the construction of the phase space T^*Q_ε is properly to be viewed as being completed prior to choice of a Lagrangian to define a mechanical system. We take up Q_1, Q_{-1}, and Q_0 in that order.

Case of Q_1

It is clear that here the space components $\underline{q} \equiv \{q_j\} \varepsilon R^3$ give a globally defined coordinate system over Q_1, with q_0 determined throughout by

$$q_0 = \sqrt{1 + |\underline{q}|^2} \geqslant 1 \tag{3.1}$$

Therefore T^*Q_1 can be described by global canonical variables $(\underline{q}, \underline{p})$ subject to the standard canonical Poisson brackets (PB's)

$$\{q_j, p_k\} = \delta_{jk}, \ \{q_j, q_k\} = \{p_j, p_k\} = 0 \tag{3.2}$$

However, this explicit elimination of q_0 in favor of $\underline{q}$ spoils manifest Lorentz covariance. Moreover, in the case of Q_{-1} (as we can see), even $\underline{q}$ is only locally available as a system of independent generalized coordinates. We therefore look for a description of T^*Q_1 which is manifestly covariant and does not insist on any one of the four q_μ being eliminated.[8] This will then serve as a model to deal with Q_{-1} and Q_0 later. Now the natural infinitesimal motions that q^μ is subject to are the infinitesimal transformations of $SO(3, 1)$, so we seek functions $S_{\mu\nu} = -S_{\nu\mu}$ of $\underline{q}$ and $\underline{p}$ such that

$$\{S_{\mu\nu}, q_\rho\} = g_{\nu\rho} q_\mu - g_{\mu\rho} q_\nu \tag{3.3}$$

In fact, our aim is to give a description of T^*Q_1 using the (overcomplete) system of variables q_μ, $S_{\mu\nu}$. This is indeed possible. We set

$$\begin{aligned} S_{jk} &= \varepsilon_{jkl} J_l = q_j p_k - q_k p_j \\ S_{0j} &= K_j = q_0 p_j \end{aligned} \tag{3.4}$$

Then it is a direct consequence of the basic PB's (3.2) that

$$\begin{aligned} \{q_\mu, q_\nu\} &= 0 \\ \{S_{\mu\nu}, q_\rho\} &= g_{\nu\rho} q_\mu - g_{\mu\rho} q_\nu \\ \{S_{\mu\nu}, S_{\rho\sigma}\} &= g_{\nu\rho} S_{\mu\sigma} - g_{\mu\rho} S_{\nu\sigma} + g_{\nu\sigma} S_{\rho\mu} - g_{\mu\sigma} S_{\rho\nu} \end{aligned} \tag{3.5}$$

At this stage we alter our point of view and regard Eqs. (3.5) as the basic (generalized and singular)[9] PB's characterizing the phase space T^*Q_1. Thus, we regard the $S_{\mu\nu}$ as the entity canonically conjugate to the q_μ, a kind of generalized canonical momentum. The overcompleteness of this description is conveyed through the algebraic relations

$$\begin{aligned} q^\mu q_\mu - 1 &= 0 \\ \tilde{S}_{\mu\nu} q^\nu &= 0 \\ \tilde{S}_{\mu\nu} S^{\mu\nu} &= 0 \end{aligned} \tag{3.6}$$

where

$$\tilde{S}_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} S^{\rho\sigma} \tag{3.7}$$

is the dual to $S_{\mu\nu}$. We may take the relations (3.6) to be identities; they are consistent with the PB's (3.5) in the sense that the PB of any one of the expressions on the left-hand sides of Eqs. (3.6) with q_ρ or $S_{\rho\sigma}$ vanishes modulo the conditions (3.6) themselves. One easily convinces oneself that there are three algebraically independent components among the $S_{\mu\nu}$; and of course $\underline{q}$, $\underline{p}$ are recoverable from q_μ, $S_{\mu\nu}$.

The Casimir invariant $\mathscr{C}_1$ formed from $S_{\mu\nu}$ is

$$\mathscr{C}_1 = \tfrac{1}{2} S^{\mu\nu} S_{\mu\nu} = \underline{J}^2 - \underline{K}^2 = -\underline{p}^2 - (\underline{q}\cdot\underline{p})^2 \tag{3.8}$$

so on T^*Q_1 it is seen to be *nonpositive*:

$$\mathscr{C}_1 = \leqslant 0 \tag{3.9}$$

While we thus have with q_μ, $S_{\mu\nu}$ a global manifestly covariant way of dealing with T^*Q_1, it turns out that we can work with a four-vector b_μ as conjugate to q_μ, rather than the tensor $S_{\mu\nu}$. (Such a simplification is possible also for Q_{-1} but not for Q_0.) As was done above, we initially define b_μ in terms of q_μ, $S_{\mu\nu}$ and obtain all its properties. We then realize that we are free to view q_μ, b_μ as basic, and q_μ, $S_{\mu\nu}$ as derived, objects.

Starting with q_μ, $S_{\mu\nu}$ we set

$$\begin{aligned} & b_\mu = S_{\mu\nu} q^\nu: \\ & \mu = 0\colon\ -\underline{q}\cdot\underline{K} = -q_0 \underline{q}\cdot\underline{p} \\ & \mu = j\colon\ -q_0 K_j - (\underline{q}\wedge\underline{J})_j = -p_j - q_j \underline{q}\cdot\underline{p} \end{aligned} \tag{3.10}$$

The four-vector nature leads to

$$\{S_{\mu\nu}, b_\rho\} = g_{\nu\rho} b_\mu - g_{\mu\rho} b_\nu \tag{3.11}$$

And the complete set of PB's and algebraic relations among q_μ and b_μ is

$$\begin{aligned}&\{q_\mu, q_\nu\} = 0\\ &\{q_\mu, b_\nu\} = g_{\mu\nu} - q_\mu q_\nu\end{aligned} \tag{3.12a}$$

$$\begin{aligned}&\{b_\mu, b_\nu\} = b_\mu q_\nu - b_\nu q_\mu\\ &q^\mu q_\mu - 1 = q^\mu b_\mu = 0\end{aligned} \tag{3.12b}$$

We now switch our viewpoint and take q_μ and b_μ as the basic canonical variables for T^*Q_1. Again the algebraic relations (3.12b) are consistent with the PB's (3.12a) modulo themselves. And we can recover $S_{\mu\nu}$:

$$S_{\mu\nu} = b_\mu q_\nu - b_\nu q_\mu \tag{3.13}$$

The timelike nature of q^μ has as a kinematic consequence that b_μ is either null or spacelike. This agrees with the fact that the Casimir invariant $\mathscr{C}_1$ is the Lorentz square of b_μ:

$$\mathscr{C}_1 = \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} = b_\mu b^\mu \leqslant 0 \tag{3.14}$$

Case of Q_{-1}

We now attempt an analogous treatment of T^*Q_{-1}. A first important difference (apart from the restriction $|\underline{q}| \geqslant |$) is that now $\underline{q}$ is not a globally well-defined coordinate for Q_{-1}, since q_0 is determined by $\underline{q}$ only up to a sign. So q_j as well as its conjugate momentum p_j and the PB's (3.2) are all local in nature. But we can again locally switch to q_μ and $S_{\mu\nu}$ defined by Eqs. (3.4), and then the manifestly covariant PB's (3.5) result. It is easy to check that such a change is possible over each local portion of Q_{-1}, so with q_μ and $S_{\mu\nu}$ one has a global manifestly covariant description of T^*Q_{-1}. The algebraic conditions replacing (3.6) are

$$\begin{aligned}q^\mu q_\mu + 1 &= 0\\ \tilde{S}_{\mu\nu} q^\nu &= 0\\ \tilde{S}_{\mu\nu} S^{\mu\nu} &= 0\end{aligned} \tag{3.15}$$

The Casimir invariant $\mathscr{C}_1$ is (locally) given by

$$\mathscr{C}_1 = \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} = \underline{J}^2 - \underline{K}^2 = \underline{p}^2 - (\underline{q} \cdot \underline{p})^2 \tag{3.16}$$

This shows that on T^*Q_{-1}, *$\mathscr{C}_1$ is not of definite sign.*

To simplify this description and replace $S_{\mu\nu}$ by a vector b_μ, in this case we define

$$\begin{aligned} b_\mu &= -S_{\mu\nu} q^\nu: \\ \mu &= 0: \underline{q} \cdot \underline{K} = q_0 \underline{q} \cdot \underline{p} \\ \mu &= j: q_0 K_j + (\underline{q} A \underline{J})_j = -p_j + q_j \underline{q} \cdot p \end{aligned} \tag{3.17}$$

the last expressions being local. Then, in place of the previous equations (3.12)–(3.14), we now have some differences in sign:

$$\begin{aligned} \{q_\mu, q_\nu\} &= 0 \\ \{q_\mu, b_\nu\} &= g_{\mu\nu} + q_\mu q_\nu \qquad (3.18a) \\ \{b_\mu, b_\nu\} &= b_\nu q_\mu - b_\mu q_\nu \\ q^\mu q_\mu + 1 &= q^\mu b_\mu = 0 \qquad (3.18b) \end{aligned}$$

$$S_{\mu\nu} = b_\mu q_\nu - b_\nu q_\mu \tag{3.18c}$$

$$\mathscr{C}_1 = \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} = -b^2 \tag{3.18d}$$

Consistent with the statement made after Eq. (3.16), we see here that since q^μ is spacelike, there is no kinematic restriction on the nature of b_μ, which is why $\mathscr{C}_1$ is indefinite in sign.

Case of Q_0

This turns out to be qualitatively different from the previous two, in the sense that a $q_\mu - b_\mu$ description of T^*Q_0 is not available. Of course, $\underline{q}$ (strictly speaking, restricted to be nonzero) gives a global coordinate description of Q_0, so the ordinary PB's (3.2) are valid all over T^*Q_0. The definition of $S_{\mu\nu}$ and the q_μ, $S_{\mu\nu}$ PB's are given as before by Eqs. (3.4) and (3.5) with no changes. Consistently with these PB's, the algebraic conditions are

$$\begin{aligned} q^\mu q_\mu &= 0 \\ \tilde{S}_{\mu\nu} q^\nu &= 0 \\ \tilde{S}_{\mu\nu} S^{\mu\nu} &= 0 \end{aligned} \tag{3.19}$$

Turning to the Casimir invariant $\mathscr{C}_1$, we have something new. Its expression is

$$\mathscr{C}_1 = \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} = \underline{J}^2 - \underline{K}^2 = -(\underline{q} \cdot \underline{p})^2 \leqslant 0 \tag{3.20}$$

so, as with T^*Q_1, $\mathscr{C}_1$ *is nonpositive.* But beyond this, we see that $\underset{\sim}{q} \cdot \underset{\sim}{p}$ is a Lorentz invariant variable:

$$\{S_{\mu\nu}, \underset{\sim}{q} \cdot \underset{\sim}{p}\} = 0 \tag{3.21}$$

A similar result does not obtain with either T^*Q_1 or T^*Q_{-1}.

Now suppose we introduce the vector b_μ by the definition

$$b_\mu = S_{\mu\nu} q^\nu \tag{3.22}$$

We see upon simplification that it is a Lorentz-invariant multiple of q^μ:

$$b_\mu = -\underset{\sim}{q} \cdot \underset{\sim}{p}\, q_\mu \tag{3.23}$$

Thus, while it may be convenient to use it in some circumstances, we are not able to reconstruct $S_{\mu\nu}$ in terms of q_μ and b_μ. So the most economical manifestly covariant description of T^*Q_0 uses the vector q_μ and the tensor $S_{\mu\nu}$.

4. MECHANICAL MODELS ON Q_ε

If we wish to define a dynamical system on any one of the three phase spaces T^*Q_ε, we have two choices: we may start with a Lagrangian, and then pass to the Hamiltonian on T^*Q_ε via the Legendre map; or we may directly choose a Hamiltonian to generate the dynamics. In the former case, if the Lagrangian is singular, the result would be a constrained system.[10]

Let us first assemble the covariant equations for the Legendre map if a Lagrangian $\mathscr{L}(q, \dot{q})$ is given. (Here we have q^μ as a function of an invariant evolution parameter τ, and the dot signifies the derivative with respect to τ). For all three spaces Q_ε, $S_{\mu\nu}$ is related to coordinates and velocities by

$$S_{\mu\nu} = q_\nu \frac{\partial \mathscr{L}}{\partial \dot{q}\mu} - q_\mu \frac{\partial \mathscr{L}}{\partial \dot{q}\nu} \tag{4.1}$$

Here and in the following we are permitted to calculate partial derivatives of $\mathscr{L}$ with respect to q^μ and $\dot{q}^\mu$ as though they were independent, that is, temporarily disregarding the kinematic restrictions

$$q^\mu q_\mu = \varepsilon, \qquad q^\mu \dot{q}_\mu = 0 \tag{4.2}$$

This flexibility allows us to choose, if we wish, manifestly $SO(3, 1)$ invariant Lagrangians, and not resort to elimination of variables. For the two cases $\varepsilon = \pm 1$, we have the further relation for b_μ and the Hamiltonian H:

$$\begin{aligned} b_\mu &= \varepsilon S_{\mu\nu} q^\nu \\ &= \frac{\partial \mathscr{L}}{\partial \dot{q}^\mu} - \varepsilon q_\mu q^\nu \frac{\partial \mathscr{L}}{\partial \dot{q}^\nu} \end{aligned} \tag{4.3a}$$

$$\begin{aligned} H &= \varepsilon S_{\mu\nu} q^\nu \dot{q}^\mu - \mathscr{L} \\ &= b^\mu \dot{q}_\mu - \mathscr{L} \\ &= \dot{q}^\mu \frac{\partial \mathscr{L}}{\partial \dot{q}^\mu} - \mathscr{L} \end{aligned} \tag{4.3b}$$

We shall now present two simple Lagrangian dynamical models in the cases $\varepsilon = \pm 1$, where the dynamical trajectories have properties reminiscent of the UR reductions (2.9a, c). (The case $\varepsilon = 0$, the light cone, will be taken up separately later.) Up to the derivation of the Hamiltonian and the equations of motion, we give a combined treatment.

We choose the simplest possible manifestly $SO(3, 1)$ invariant Lagrangian

$$\mathscr{L} = -\tfrac{1}{2} \dot{q}^\mu \dot{q}_\mu \tag{4.4}$$

Then, using Eqs. (4.1)–(4.3) above, we find

$$\begin{aligned} S_{\mu\nu} &= q_\mu \dot{q}_\nu - q_\nu \dot{q}_\mu \\ b_\mu &= -\dot{q}_\mu \\ H &= -\frac{1}{2} b^\mu b_\mu = -\frac{\varepsilon}{2} \mathscr{C}_1 \end{aligned} \tag{4.5}$$

The phase space equations of motion for q^μ and b^μ are

$$\begin{aligned} \dot{q}^\mu &= \{q^\mu, H\} = -b^\mu \\ \dot{b}^\mu &= \{b^\mu, H\} = \varepsilon b^2 q^\mu \end{aligned} \tag{4.6}$$

As a consequence, of course, the $SO(3, 1)$ generators are conserved:

$$\dot{S}_{\mu\nu} = 0 \tag{4.7}$$

This implies the conservation of $\mathscr{C}_1$, i.e., of b^2.

Now we separate the two cases $\varepsilon = \pm 1$. For $\varepsilon = +1$, we have motions on the positive timelike hyperboloid Q_1. Here we know that b^μ is necessarily spacelike (or identically vanishing, a possibility we agree to ignore hereafter). Writing $b^2 = -\alpha^2 < 0$, the solution to Eqs. (4.6) is

$$\begin{aligned} q^\mu(\tau) &= q^\mu(0) \cosh \alpha\tau - b^\mu(0) \frac{\sinh \alpha\tau}{\alpha} \\ b^\mu(\tau) &= b^\mu(0) \cosh \alpha\tau - \alpha q^\mu(0) \sinh \alpha\tau \\ \mathscr{C}_1 &= -\alpha^2 < 0 \end{aligned} \tag{4.8}$$

These are the only kinds of trajectories in this case. We always have *unbounded motion* along a plane curve lying in Q_1; and the Casimir invariant is negative.

Next let us look at $\varepsilon = -1$, the spacelike case. Here b^μ can be timelike, lightlike, or spacelike. The solutions to Eqs. (4.6) in these situations are

b^μ *timelike*, $b^2 = \alpha^2 > 0$:

$$\begin{aligned} q^\mu(\tau) &= q^\mu(0) \cosh \alpha\tau - b^\mu(0) \frac{\sinh \alpha\tau}{\alpha} \\ b^\mu(\tau) &= b^\mu(0) \cosh \alpha\tau - \alpha\, q^\mu(0) \sinh \alpha\tau \\ \mathscr{C}_1 &= -\alpha^2 < 0 \end{aligned} \tag{4.9a}$$

b^μ *lightlike*, $b^2 = 0$:

$$\begin{aligned} q^\mu(\tau) &= q^\mu(0) - \tau b^\mu(0) \\ b^\mu(\tau) &= b^\mu(0) \\ \mathscr{C}_1 &= 0 \end{aligned} \tag{4.9b}$$

b^μ *spacelike*, $b^2 = -\omega^2 < 0$:

$$\begin{aligned} q^\mu(\tau) &= q_\mu(0) \cos \omega\tau - b^\mu(0) \frac{\sin \omega\tau}{\omega} \\ b^\mu(\tau) &= b^\mu(0) \operatorname{Cos} \omega\tau + q^\mu(0)\, \omega \sin \omega\tau \\ \mathscr{C}_1 &= \omega^2 > 0 \end{aligned} \tag{4.9c}$$

The following are evident: (a) for timelike b^μ, we have unbounded motion along a plane curve in Q_{-1}, with $\mathscr{C}_1$ negative; (b) for lightlike b^μ the motion is again unbounded, along a straight-line generator of Q_{-1} (recall Q_{-1} is a ruled surface) with vanishing $\mathscr{C}_1$; (c) for spacelike b^μ we have bounded motion along a plane curve in Q_{-1}, with $\mathscr{C}_1$ positive.

These features bear interesting comparison to Eqs. (2.9a, c). There is a correspondence between the occurrence of the *continuum* of Principal Series UIR's$\{0, \rho\}$ as a *direct integral*, and *unbounded motions* on $Q_{\pm 1}$. In both situations, moreover, $\mathscr{C}_1$ is negative. In the case of Q_1, this is all there is in the UR reduction on the one hand, in the classical model on the other. But with Q_{-1}, we see that we can correlate the occurrence of the discrete sequence of Principal Series UIR's$\{j_0, 0\}$ as a *direct sum* in $\mathscr{U}_{-1}(\cdot)$, with the possibility of *bounded motions* on Q_{-1}; and, moreover, $\mathscr{C}_1$ is positive in both situations. It is in these senses that we have classical mechanical models which give some insight into reductions of group representations.

The case of Q_0 remains, and it is indeed rather singular. If we were to start with a Lagrangian, the natural choice would be as in Eq. (4.4). This implies

$$\begin{aligned} S_{\mu\nu} &= q_\mu \dot{q}_\nu - q_\nu \dot{q}_\mu \\ \mathscr{C}_1 &= \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} = 0 \end{aligned} \tag{4.10}$$

that is, we have a constrained system[10] with the primary Lorentz invariant constraint [cf Eq. (3.20)].

$$\sqrt{-\mathscr{C}_1} = \underset{\sim}{q} \cdot \underset{\sim}{p} \approx 0 \tag{4.11}$$

To avoid having an identically vanishing $\mathscr{C}_1$, we must give up the Lagrangian (4.4), and choose instead to work directly from a suitable Hamiltonian. Guided by the $\varepsilon = \pm 1$ cases, we take

$$\begin{aligned} H &= -\tfrac{1}{2}\mathscr{C}_1 = -\tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} \\ &= \tfrac{1}{2}(\underset{\sim}{q} \cdot \underset{\sim}{p})^2 \geqslant 0 \end{aligned} \tag{4.12}$$

and realize that here (unlike with $Q_{\pm 1}$) this cannot arise from any Lagrangian. The phase space equations of motion are

$$\begin{aligned} \dot{q}^\mu &= \{q^\mu, H\} = \underset{\sim}{q} \cdot \underset{\sim}{p}\, q^\mu \\ \dot{S}_{\mu\nu} &= \{S_{\mu\nu}, H\} = 0 \end{aligned} \tag{4.13}$$

The general solution, better expressed in terms of q^μ and $\underset{\sim}{p}$, is

$$\begin{aligned} q^\mu(\tau) &= e^{\tau \underset{\sim}{q}\cdot\underset{\sim}{p}} q^\mu(0) \\ \underset{\sim}{p}(\tau) &= e^{-\tau \underset{\sim}{q}\cdot\underset{\sim}{p}} \underset{\sim}{p}(0) \\ \underset{\sim}{q}(\tau) \cdot \underset{\sim}{p}(\tau) &= \underset{\sim}{q}(0) \cdot \underset{\sim}{p}(0) \\ \mathscr{C}_1 &= -(\underset{\sim}{q} \cdot \underset{\sim}{p})^2 \leqslant 0 \end{aligned} \tag{4.14}$$

The trajectories are *unbounded* straight lines on Q_0, the generators of the light cone; and as with the UR reduction (2.9b), $\mathscr{C}_1 \leqslant 0$ throughout. Thus, one achieves some understanding of the reduction of the UR$\mathscr{U}_0(\cdot)$ into UIR's, if not of the factor of two in (2.9b), at least of the occurrence only of the Principal Series UIR's$\{0, \rho\}$.

5. CONCLUDING REMARKS

We have constructed simple Lorentz covariant classical mechanical models to mimic features of the three natural UR's $\mathscr{U}_\varepsilon(\cdot)$ of the Lorentz group, especially the pattern of their reductions into irreducibles. While the specific details of the reduction in each case are recognized as basically stemming from the underlying geometrical—and topological—properties of the manifolds Q_ε, we have tried to expose these differences in a formal way by bringing them out at the level of the phase spaces T^*Q_ε. In this way, the intrinsic differences in the spectra of the Casimir invariant $\mathscr{C}_1$ get a mechanical interpretation.

The treatment of the phase spaces T^*Q_ε respecting manifest covariance may be of general interest and of use in other contexts. Here we may note that from the point of view of the range of $\mathscr{C}_1$ it is T^*Q_1 and T^*Q_0 that are similar, while from the point of view of availability of a simple $q_\mu - b_\mu$ description, it is T^*Q_1 and T^*Q_{-1} that go together.

The unusual properties of the phase space T^*Q_0 merit special mention. We see them in two ways—as just noted, the independent vector b_μ cannot be introduced in this case; and the same expression for the Lagrangian which was nonsingular in the other two cases turns out to lead to a constrained system in this case. Thus, we have here another instance of the delicate features associated with descriptions of massless particles, features encountered in many other situations as well.

Felicitation

It is a privilege for me to join the many other contributors to this volume in paying tribute to Professor Asim O. Barut on the occasion of his 65th birthday. While I have unfortunately not had an opportunity to work with and learn directly from him, I have always admired his imaginative and creative uses of symmetry and group theory in many physical contexts. My very best wishes to him on this occasion.

REFERENCES AND FOOTNOTES

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