Branching space-times, general relativity, the Hausdorff property, and modal consistency

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Abstract

The logical theory of branching space-times (BST; Belnap, Synthese 1992), which is intended to provide a framework for studying objective indeterminism, remains at a certain distance from the discussion of space-time theories in the philosophy of physics. In a welcome attempt to clarify the connection, Earman has recently found fault with the branching approach and suggested "pruning some branches from branching space-time" (2008).

The present note identifies the different—order theoretic vs. topological—points of view of both discussion as a reason for certain misunderstandings, and tries to remove them. Most importantly, we give a novel, topological criterion of modal consistency that usefully generalizes the order-theoretic criterion of directedness, and we introduce a differential-geometrical version of BST based on the theory of non-Hausdorff (generalized) manifolds.

Branching space-times (BST; Belnap, 1992) is a logical theory that allows for the representation of objective indeterminism in a space-time setting. It deviates from the mainstream representation of indeterminism in the Lewis tradition, in which wholly separate possible worlds are taken to signal indeterminism if they are partially isomorphic. In BST, the world is allowed to contain different complete possible courses of events, called *histories*, whose past overlap and future branching grounds indeterminism. Arguably this accords better with the notion of objective (rather than epistemic) indeterminism, but various objections have been raised against such a branching conception of indeterminism.

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This note is intended to set some things straight with respect to the interrelation of BST and general relativity, especially as regards the Hausdorff property. We will thereby address a number of worries voiced by Earman (2008). The paper can perhaps function as a sort of companion to Placek and Belnap (2010). That paper approaches the discussion from a philosophical point of view, centering on the interpretation of modality. In the present paper, we try to approach the matter more from the point of view of physics, thus perhaps furthering interaction on that side of the debate.

We keep the discussion as self-contained as possible, also including some rather simple definitions. The reason for this is that the discussion here is situated at an interface of domains, and we want to maintain a high level of mathematical precision, so it seems better to err on the side of being too explicit.

1 BST and the Hausdorff property

This section is mostly a summary of results out there. It also serves to introduce some terminology. Further down, in §3, we will be working with a slightly different framework, which is both stronger and weaker. In §1.1 we comment on some important logical terminology. §1.2 gives the axioms of BST. In §1.3 we briefly comment on the interrelation of BST and GTR, identifying topological considerations on BST as the missing link. §1.4 accordingly gives an overview of topology and BST.

1.1 Terminology: theories and models

We will try to maintain logical rigor in our discussion. We will use the phrase "branching space-times", and the label "BST", for the general class of branching space-time theories. One specific such theory was proposed by Belnap (1992); we will use the label "BST92" for this specific axiomatic logical theory. It is important to keep the following logical terminology straight:[1]

Logical theory A logical theory is a set of axioms in some given formal language, e.g., in the case of BST92, a second-order language with

[1]Here we disagree with Earman (2008, 188n2): if one identifies models and histories, as he does, one has already rejected without argument a conceptual distinction that is crucial for making sense of the notion of branching such as formally defined in the branching time and branching space-times literature.

identity and a single two-place predicate "$\leq$".

Model A model, in the logical sense, is a (set-theoretical) structure that satisfies the axioms of a certain given logical theory. Thus, when we speak of a "model of BST92", we mean a structure w.r.t. which all the axioms of BST92 are true.

Physical theory We try to be realistic in the sense of taking actual practice seriously, and will therefore not require physical theories to be given axiomatically, nor to be specified in a formal language. We will assume that a theory, such as the general theory of relativity (GTR), is given via defining equations, possibly enriched by some local lore about physicality, admissible violations of assumptions, important toy models, idealizations, approximation techniques, and so on—basically, what a good textbook such as Wald (1984) provides.[2]

Solution A solution to the equations of a physical theory is a mathematical structure, e.g., some differential manifold. In some well-behaved cases such a solution is also a model in the logical sense (consider, e.g., Montague's famous paper on deterministic theories (Montague, 1962), and subsequent work in that tradition), but we do not require this.

World We keep "world" as a metaphysical term, and we use David Lewis's sensible criterion: a world has to be unified by "suitable external relations" (Lewis, 1986, 208). It may be, and it is in fact the case for BST92 in the intended interpretation, that each model of a logical theory is a world in this sense. An ensemble of non-overlapping worlds is not a world, as such an ensemble is obviously not unified via suitable external relations.[3] Note that worlds can be entirely alike qualitatively, and yet be different. This is not so for models—models are given purely extensionally.

[2]This attitude appears to be very much in line with Earman et al. (2009, 93).

[3]Note that Lewis's own use of the notion of a world, which Earman (2008, 189) quotes approvingly, may in fact be at variance with his own criterion, as he characterizes branching as positing "overlapping worlds" (Lewis, 1986, 206). We cannot see how two things, each of which is unified by suitable external relations, could overlap without the external relations unifying the whole resulting structure such as to yield a single world.

1.2 The axioms of BST92

BST92 is formulated in second order predicate logic with identity and one single two-place relation symbol, "$\leq$" (written in infix notation). We will also use the relation symbol "$<$", which is defined in the usual way: $x < y$ iff ($x \leq y$ and $x \neq y$).

Let $\langle W, \leq \rangle$ be a nonempty partial order (a nonempty set W together with a transitive, antisymmetric relation $\leq$). Elements of W are called possible point events, or, briefly, events. Let $H \subseteq \wp W$ be the set of maximal upward directed subsets of W. (In a partial order, a set is *upward directed* iff for any two of its elements a and b, there is an element c s.t. $a \leq c$ and $b \leq c$. We often shorten to "directed".) Elements of H, i.e., maximal directed subsets $h \subseteq W$, are called *histories*. A *chain* in W is a linear subset, i.e., a subset $c \subseteq W$ s.t. for any $x, y \in c$ we have either $x \leq y$ or $y < x$.

The axioms of BST92 are as follows (cf., e.g., Belnap, 1992, 2003):

- $\langle W, \leq \rangle$ is a nonempty, dense partial order without maxima.
- Each lower bounded chain $C \subseteq W$ has an infimum in W, written $\inf C$.
- Each upper bounded chain $C \subseteq W$ has a supremum-in-h ($\sup_h C$) for each history $h \in H$ for which $C \subseteq h$.
- (Prior choice principle.) If $C \in h - h'$ is a lower bounded chain in h none of whose elements is an element of h', then there is a choice point $c \in h \cap h'$ such that c is maximal in $h \cap h'$, and $c < C$ (i.e., for all $e \in C$, we have $c < e$).

Note that by the given definition, histories are downward closed: if $e \in h$ and $f \in W$ s.t. $f \leq e$, then also $f \in h$. Accordingly, if c is a lower bounded chain in history h, then $\inf c \in h$ as well.

As a first link with space-time theories, we can give a generic definition of the causal and the chronological past and future of events in a BST92 model, as follows:

Definition 1 *Given a BST92 model $\langle W, \leq \rangle$, an event $e \in W$* lightlike precedes *$f \in W$ (in symbols: $e \lhd f$) iff $e \leq f$ and there is only a single maximal chain that has e as its first and f as its last point. Event e* chronologically precedes *f (in symbols: $e \ll f$) iff $e < f$ and it is not the case that $e \lhd f$.*

Based on these notions we can define the notions of the causal and the chronological future (and analogously, past) of an event $e \in W$, as usual::

Definition 2 *Given a BST92 model* $\langle W, \leq \rangle$ *and some* $e \in W$*, the* causal future of e*,* $J^+(e)$*, and the* chronological future*,* $I^+(e)$*, are defined as follows:*

$$J^+(e) := \{f \in W \mid e \leq f\}; \quad I^+(e) := \{f \in W \mid e \ll f\}.$$

The corresponding past notions are

$$J^-(e) := \{f \in W \mid f \leq e\}; \quad I^-(e) := \{f \in W \mid f \ll e\}.$$

1.3 Some facts about BST92 and general relativity

BST92 is a very general theory, it has extremely many models. Still it is not general enough for full GTR. This can be seen by the following simple fact: There are solutions of GTR that contain closed timelike curves. Let e and f be different events on such a curve. Then, on the intended interpretation of $\leq$ as causal connectibility, we have $e \leq f$, $f \leq e$ and $e \neq f$, violating the order requirement of BST92. On the other hand, BST92 is perhaps too general—it allows for dimension-changing models and many other weird phenomena (see, e.g., Müller (2005); Müller et al. (2008)). Its models are not generally metrizable, and there is no requirement that a BST92 model be a—perhaps generalized—manifold such as presupposed by GTR.

This is not a coincidence, but due to different perspectives taken by BST92, on the one hand, and by GTR, on the other hand. BST92 was conceived in a logical, order-theoretic context familiar from modal semantics. GTR, on the other hand, is based on differential geometry, which in turn is based on topological notions. In order to bring BST92 closer to GTR, it is therefore necessary to look more closely at its topological aspects.

1.4 Topological issues in BST92

We start be giving some standard basic definitions in order to make the presentation self-contained; §1.4.1 and §1.4.2 do not contain any original material. The only point worth noting is that we also define the notion of a *generalized manifold*, which, unlike standard manifolds, is not required to be Hausdorff (see Def. 11 below).

1.4.1 Topological spaces

Definition 3 (Topological space) *A* topological space *is a pair* $\langle X, \mathfrak{T} \rangle$ *where* X *is a nonempty set and the topology* $\mathfrak{T} \subseteq \wp X$ *is a collecion of subsets—the so-called* open *sets—satisfying:*

- $\emptyset \in \mathfrak{T}$ *and* $X \in \mathfrak{T}$.
- *(Finite intersections) If* $a, b \in \mathfrak{T}$, *then* $a \cap b \in \mathfrak{T}$.
- *(Arbitrary unions) If* $\{a_i \mid i \in I\}$ *is a family of open sets (i.e.,* $a_i \in \mathfrak{T}$ *for any* $i \in I$*), then* $\cup_{i \in I} a_i \in \mathfrak{T}$.

A set $c \subseteq X$ *whose complement* $X - c$ *is open, is called* closed. *A set that is both open and closed, is called* clopen. *A topological space* $\langle X, \mathfrak{T} \rangle$ *is* connected *iff* $\emptyset$ *and* X *are the only clopen sets; equivalently, iff* X *is not the disjoint union of two nonempty open sets.*

Any non-empty set can trivially be turned into a topological space in the following two ways: minimally, $\mathfrak{T} = \{\emptyset, X\}$, the "indiscrete" topology, and maximally, $\mathfrak{T} = \wp X$, the "discrete" topology. These are normally not useful for applications.

A topology can usually be given by specifying less than the full collection of open sets. In fact, a topology can be specified via a *basis*, or, even more simply, via a *subbasis*.

Definition 4 (Basis, subbasis) *Given a topological space* $\langle X, \mathfrak{T} \rangle$, *a set* $B \subseteq \mathfrak{T}$ *is called a* basis *iff every open set* $a \in \mathfrak{T}$ *is a (possibly infinite) union of sets from* B. *A* subbasis *is a set* $S \subseteq \mathfrak{T}$ *such that the set of all finitely many intersections of elements of* S *form a basis.*

The real line $\mathbb{R}$ and, more generally, the n-dimensional Euclidean space $\mathbb{R}^n$,[4] have natural topologies according to which they are connected (in fact, even simply connected—see Def. 7). These topologies can be given in many different ways; one is via a metric.

[4]For present purposes it will not be important to distinguish between n-dimensional Euclidean space, which has no origin, and the n-dimensional real vector space $\mathbb{R}^n$, which contains a distinguished origin. Below we will therefore also treat n-dimensional Minkowski space M^n as having a distinguished origin (this amounts to chosing specific coordinates). The existence of an origin *is* a defining feature of tangent spaces, which really are vector spaces.

Definition 5 (Metric) *Let X be a set. A function $d : X \times X \mapsto \mathbb{R}_0^+$ is called a* metric on X *iff*

- $d(x, x) = 0$*,*
- *if* $d(x, y) = 0$*, then* $x = y$ *("non-degeneracy"),*
- $d(x, y) = d(y, x)$*, and*
- $d(x, y) + d(y, z) \geq d(x, z)$ *("triangle inequality").*

There is a natural metric on $\mathbb{R}^n$, given by

$$d(\langle x^1, \dots, x^n \rangle, \langle y^1, \dots, y^n \rangle) := \sqrt{(x^1 - y^1)^2 + \cdots + (x^n - y^n)^2}$$

This metric then induces the natural topology, by using the collection of open balls

$$B(x, r) := \{y \in \mathbb{R}^n \mid d(x, y) < r\}$$

for $x \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$ (the positive reals) as a basis. Note that the countable collection of open balls with rational midpoint coordinates and rational radius also forms a basis.

The most important mappings between topological spaces are the continuous ones, with paths as special mappings of that kind:

Definition 6 (Continuity; homeomorphism; path) *Let $\langle X_1, \mathfrak{T}_1 \rangle$ and $\langle X_2, \mathfrak{T}_2 \rangle$ be topological spaces. A mapping $f : X_1 \mapsto X_2$ is called* continuous *iff the pre-image $f^{-1}(a) \subseteq X_1$ of any open set $a \in \mathfrak{T}_2$ is open. The mapping is a* homeomorphism *iff it is bijective and both it and its inverse are continuous. A* path *is a continuous mapping from the closed unit interval* $[0, 1]$ *(with the usual topology) into some topological space X.*

The notion of a path is central for the definition of two stronger notions of connectedness:

Definition 7 (Path-connected; simply connected) $\langle X, \mathfrak{T} \rangle$ *is* path connected *iff there is a continuous path between any two of its points. Path-connectedness is strictly stronger than connectedness. The space $\langle X, \mathfrak{T} \rangle$ is* simply connected *iff it is path-connected and every continuous mapping of the unit circle into X can be continuously contracted to a point. Again, simple connectedness is strictly stronger than path connectedness.*

Topological spaces that are locally homeomorphic to some Euclidean space are called *locally Euclidean*:

Definition 8 (Locally Euclidean) *A topological space $\langle X, \mathfrak{T} \rangle$ is* locally Euclidean *iff for any $x \in X$ there is some $a \in \mathfrak{T}$ with $x \in a$ and some $n \in \mathbb{N}$ and some open subset $b \subseteq \mathbb{R}^n$ such that there is a homeomorphism f_x between a and b. (We can guarantee, and will normally assume, $f_x(x) = \langle 0, \ldots, 0 \rangle$.)*

1.4.2 Manifolds and the Hausdorff property

Hausdorffness. There is a hierarchy of separation properties for points in topological spaces. For our purposes, the most important property is the T_2 property, or Hausdorffness.

Definition 9 (Hausdorff property) *A topological space $\langle X, \mathfrak{T} \rangle$ is* Hausdorff *iff for any distinct $x, y \in X$ there are $a, b \in \mathfrak{T}$ such that $x \in a$, $y \in b$, and $a \cap b = \emptyset$, i.e., any two points x, y can be separated by two disjoint open sets.*

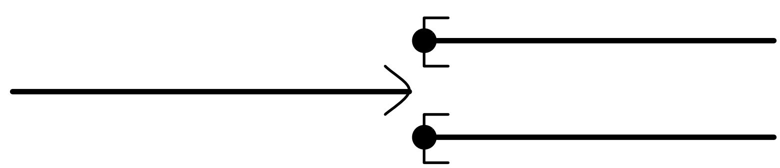

Figure 1: The branching real line as a simple non-Hausdorff space. A basis for the topology is given by the open intervals in both tracks.

Hausdorffness thus forbids, intuitively speaking, the existence of "unseparably close points", or perhaps "doubled points" or "points that occupy the same position". The branching real line pictured in Fig. 1 is a simple example of a non-Hausdorff space (but see Fig. 2 for a different branching line that *is* Hausdorff). Following Hajicek (1971), who credits Geroch for the notation, we will write $x \curlyvee y$ to indicate that the points x and y violate the Hausdorff condition, i.e., that x and y cannot be separated by disjoint open sets. The notation usefully suggests graphically that in such a case, x and y "branch off" from some common trunk, like the left part of the branching line of

Fig. 1. In fact, in our examples below such x and y will always be different limits of a single converging sequence.

In differential geometry, Hausdorffness is a feature of the standard definition of a manifold. Furthermore, any topology induced by a metric is Hausdorff. Deviation from Hausdorffness is therefore a rarity in applied mathematics. Nevertheless, we will need to remain general here so as to link the order-theoretic perspective of BST with the topological perspective of GTR. Accordingly, we define generalized manifolds that allow non-Hausdorffness.

Manifolds. The idea behind the definition of a (generalized) manifold is to capture topological spaces that are locally Euclidean in a useful way.

Definition 10 (Chart, atlas) *Given a topological space $\langle X, \mathfrak{T} \rangle$, a* chart *for $a \in \mathfrak{T}$ is a triple $\langle a, f, b \rangle$ where $b \subseteq \mathbb{R}^n$ for some n is an open set and f is a homeomorphism $f : a \mapsto b$. Such a chart induces* coordinates *on the points of a. An* atlas *for a locally Euclidean space is a collection of charts covering the whole space.*

Here is the official definition of a (generalized) manifold:

Definition 11 (Manifold, generalized manifold) *A locally Euclidean topological space $\langle X, \mathfrak{T} \rangle$ is an n-*dimensional generalized manifold *iff*

- *it has the same dimension n everywhere and*
- *it has at least one countable atlas.*

A generalized manifold is a manifold *iff, additionally, it is Hausdorff.*

Properly generalized manifolds, which are non-Hausdorff, we will call Y-manifolds, again following the terminology of Hajicek (1971).

Differentiability restrictions are important for doing physics on a manifold. A (generalized) manifold is C^k iff on the overlap $a_{ij} \neq \emptyset$ of any two of its charts a_i and a_j, the function $f_i \circ f_j^{-1}$ is a C^k diffeomorphism (a C^k-differentiable homeomorphism) between open subsets of $\mathbb{R}^n$.[5]

[5] C^0 is immediate given the definition of a manifold; in physics contexts it is customary to assume that manifolds are as smooth as needed and often, C^∞.

1.4.3 BST92 and the Hausdorff property

By looking at the axioms in §1.2 above, it becomes clear that BST92 does not provide enough structure to even ask whether its models are Hausdorff—BST92 does not come with a topology. Obviously, one can turn any BST92 model $\langle W, \leq \rangle$ into a Hausdorff topological space by taking the discrete topology. There are however at least three much more natural topologies for BST92, with respect to which the question of Hausdorffness can be sensibly asked. Note that we cannot define a topology that would always turn a BST92 model into a manifold, as the axioms of BST92 do not guarantee local Euclidicity.[6]

Alexandrov topology. Generally, for a partial ordering $\langle W, \leq \rangle$, one can define the so-called *Alexandrov topology* $\mathfrak{T}_A$ based on the *upper sets*, i.e., on sets of the form

$$\uparrow x := \{z \in W \mid x \leq z\}, \quad x \in W.$$

Analogously, one could base a topology on the lower sets (replacing "$\leq$" by "$\geq$"). In some cases taking the upper *and* lower sets as a subbasis gives a useful topology. The name "Alexandrov topology" in physics is mostly used for the topology that takes "chronological diamonds" of the form

$$D(x, y) := I^+(x) \cap I^-(y), \quad x, y \in W$$

as a basis (cf., e.g., Malament, 1977).[7] The Alexandrov topology, both in the mathematicians' sense and in the physicists' sense,[8] is definable on the basis of the BST92 axioms, but it is not normally used in physics. A detailed study in the context of BST92 may be worthwhile, but such a study is not undertaken here. See McWilliams (1981) for conditions under which the Alexandrov topology and the usual manifold topology coincide in a space-time.

The path topology. A topology for space-time manifolds that is finer than the standard manifold topology was introduced by Zeeman (1967) and

[6]As will be laid out in the discussion of §3.1, BST92 in fact almost forbids local Euclidicity, but a small change in the axioms paves the way towards models that *are* manifolds.

[7]In the presence of minima or maxima in the ordering, this definition needs to be patched in a way similar to what is mentioned in note 9 below.

[8]Usage in topological texts on ordered structures thus differs from usage in physics. For some notes on the former, see, e.g., Tholen (2009); for the latter, see also Visser (2009).

simplified by Hawking et al. (1976); see Naber (1992, App. A) for an overview. The basic idea is to take the idea of a *local* neighbourhood seriously physically, so that an environment of x is not required to contain events that are space-like separated from x. The usual definition of the path topology presupposes the manifold topology as a background and is therefore not applicable to BST92 generally. However, a similar idea can be made to work.

Belnap/Bartha topology. Belnap (1992, 432n26), following Bartha, defines a topology taking as basis generalized diamonds (similar to the open sets in the physicists' Alexandrov topology) that are the union of diamonds oriented in all possible causal directions (an idea that triggers associations with the path topology). The formal definition is as follows (Placek and Belnap, 2010):

Definition 12 (causal paths and diamonds) *In the BST92 partial ordering $\langle W, \leq \rangle$, a set $t \subseteq W$ is a* causal path*, $t \in CP$, iff t is a maximal chain in the ordering. Given $t \in CP$,*

$$d_t^{e_1,e_2} := \{y \in W \mid e_1 < e_2 \ \& \ e_1, e_2 \in t \ \& \ e_1 \leq y \leq e_2\}$$

is the diamond oriented by t with vertices e_1 and e_2*. (Note that this definition usefully returns the open set if unsuitable parameters are given, e.g., if $e_1 \notin t$.)*

Thus, if $e_1 < e_2$ and $e_1, e_2 \in t$, we have

$$d_t^{e_1,e_2} := J^+(e_1) \cap J^-(e_2).$$

In the Belnap/Bartha topology $\mathfrak{T}_B$ for $\langle W, \leq \rangle$, open sets are those sets that contain diamonds around all their points, in all directions. Formally, for $A \subseteq W$:[9]

$$A \in \mathfrak{T}_B \quad \text{iff}$$
$$\forall x \in A \, \forall t \in CP \, (x \in t \rightarrow \exists e_1, e_2 \in t \, (e_1 < x < e_2 \ \& \ d_t^{e_1,e_2} \subseteq A)).$$

[9]The definition requires a small fix in the presence of minima, which are allowed by the axioms of BST92: instead of $e_1 < x$, for x a minimum we can obviously require only that there be some $e_1 \leq x$. This patch also makes sure that W itself counts as an open set, which Placek and Belnap (2010) enter as a separate clause. A similar patch would be required in the presence of maxima, which are however forbidden by the BST92 axioms.

This topology is in fact rather natural: it is equivalent to the standard topology on n-dimensional Minkowski space-time, and it is almost the same as our suggested topology for m-fold branching n-dimensional Minkowski space-times. These, to be introduced in §3.1 below, are very simple, well behaved structures that almost fulfill the BST92 axioms and that will form the local material from which we will build generalized manifolds.

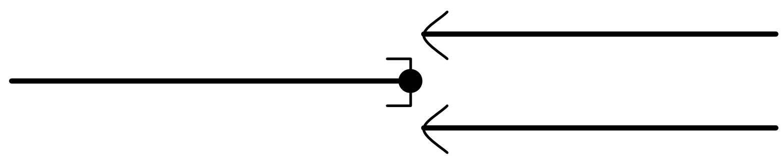

Figure 2: The branching real line as a Hausdorff space. A basis for the topology is given by the open intervals overapping both tracks. The topology is therefore not everywhere locally Euclidean.

The question of Hausdorffness. It would be nice to establish some general results about BST92 with respect to the mentioned topologies, and especially w.r.t. the topology $\mathfrak{T}_B$. E.g., it seems quite plausible to assume that branching into different histories means non-Hausdorffness of the total model. In fact this is true in many cases, but it turns out that there are multiple-history models of BST92 that are Hausdorff: one can simply take the real line and add a disjoint copy of the open set $(0, \infty)$, such that the points in the two copies are not order-related. It is easy to verify that this model, pictured in Fig. 2, satisfies the axioms of BST92 and has two histories, but is also Hausdorff according to $\mathfrak{T}_B$.[10] In the other direction, the question is whether there are non-Hausdorff models of BST92 with only a single history. We have to leave open this question for now; for some further pertinent remarks, see Placek and Belnap (2010). As the topology we will suggest for our version of BST differs slightly from all the mentioned ones and is, in fact, simpler, we leave the details of the topologies mentioned so far to the side and turn to physics.

[10]Note the difference with the non-Hausdorff space of Fig. 1, which in fact violates the Prior Choice Principle of BST92, and therefore is not a model of BST92.

2 Branching manifolds and GTR

The general theory of relativity (GTR) is a physical theory whose solutions are mostly assumed, by definition, to be Hausdorff manifolds. Since various breakdowns of the exact mathematical formalism ("singularities") are crucial for understanding cosmological applications of GTR as well as possible extensions of the theory, some of the mathematical defining features can however be relaxed.

This is important. If GTR was a logical theory with fixed axioms, and models of the theory were required to have the structure of Hausdorff manifolds, it would obviously be silly to consider non-Hausdorff models for GTR. However, GTR is a physical theory, and non-Hausdorff models and other individually branching space-times have actually been researched into. Earman (2008) gives a useful overview of some of the key results.

Note that the overview to follow pertains to a discussion in physics, where the guiding question is whether branching (e.g., non-Hausdorff) manifolds can be useful for describing a *single* solution of GTR, i.e., *a single space-time*. This is *not* the guiding question behind BST: we have already remarked that non-Hausdorffness in BST92 is (generically) a sign of modal separation, i.e., multiple different space-times (multiple histories) in a single model. Nevertheless, it is important to have the physical facts on the table since they may constrain physical applications of BST.

One class of intuitively branching manifolds, so-called trousers worlds, are discussed in §2.1. In §2.2 we give an overview of non-Hausdorff models for single spacetimes.

2.1 Trousers worlds

Some results of branching phenomena within a single spacetime were prompted by research into so-called "trousers worlds". Speaking suggestively, a solution of GTR is a trousers world iff at some time, space is a connected set (think: a slice at the waist), whereas at some later time, space forms two or more disconnected subsets (think: a slice through the two legs). There are suggestive drawings of such worlds, e.g., in Earman (2008, 194).

Early logical research into the causal ordering structure of space-time also took trousers scenarios to be of key importance. Thus, Prior (1967, App. B.5) remarks that a crucial logical difference between the causal ordering of special vs. that of general relativity is that in the former, any two events have a common upper bound, while in the latter, this may not be so due to trousers phenomena. In terms of modal logic, this was interpreted to mean that while a certain modal operator (the so-called Diodorean modality) based on the GTR causal order only satisfies the axioms of the modal system S4, the corresponding operator in the case of special relativity satisfies the stronger confluence property of the modal system S4.2. See Goldblatt (1980) for a key result and Uckelman and Uckelman (2007) for an overview of further relevant logical literature.

While these considerations have become part of the "logical" folklore, Earman (2008) usefully remarks that trousers worlds and similar scenarios come at a very heavy price and may in the end have to be considered unphysical. Generally, topology change in a single space-time is taken to be physically suspect, and important classes of physically reasonable GTR manifolds can in fact be proved to be homeomorphic to $\Sigma \times \mathbb{R}$, where Σ is some 3-dimensional hypersurface, so that topology change over time is excluded. See Earman (2008) for details and further references to the literature.

2.2 Non-Hausdorffness: the point of view of physics

Earman (2008) in his overview of "individual branching" for single space-times concludes that the only viable path to individually branching space-times comes from non-Hausdorff models. We hasten to stress again that BST, at least in the form of BST92 and in the form that we are trying to develop further here, is *not* after individually branching space-times, but after logical models whose overlapping histories are individual *non-branching* and in fact Hausdorff space-times. Still, it is useful to look at the physicists' discussion of non-Hausdorffness.

In mathematical physics, important results about non-Hausdorff space-times come from considerations of a certain form of singularity: there are solutions of the GTR equations whose maximal analytic extensions are non-Hausdorff. Hajicek (1971) proves an important result about this class of solutions: roughly, a non-Hausdorff space-time either fails to be strongly causal, or it admits bifurcating geodesics. He interprets this result as showing that "all such [i.e., non-Hausdorff] space-times must be weakly acausal" (Hajicek, 1971, 75), which would indeed be reason enough for a physicist to shun non-Hausdorff space-times. It is interesting to see how Hajicek supports his interpretation of his theorem. Commenting on bifurcating curves, he writes:

> It is easily seen that such curves can only exist in a non-Hausdorff space. Then, if we have some system of ordinary differential equations which has locally a unique solution [...] it is immediate that this system cannot have two different solutions [...] unless these solutions form a bifurcate curve. Therefore, in view of the classical causality conception coinciding with determinism it is sensible to rule out the bifurcate curves. (Hajicek, 1971, 79)

The dialectics is thus as follows: A result from mathematical physics (Hajicek's Theorem 4) establishes (roughly) that in non-Hausdorff space-time models there is either a violation of strong causality, or there are bifurcating curves. An appeal to determinism rules out the latter; considerations of physicality rule out the former. This amounts to rejecting non-Hausdorff models.

We agree with this argument completely. If BST92 were to give models of a *single* space-time, these models should not contain bifurcating curves, and most probably they shouldn't be weakly acausal either, so that non-Hausdorffness would be ruled out. If one however takes up the issue of non-Hausdorff models in order to build formal models for indeterminism (which is the express aim of Belnap and others working on BST, including the present author), then the above argument obviously pulls no weight.

2.3 Modality in physics

We have seen that the physical discussion of branching manifolds is concerned with *single space-times* allowing for topology change or having a non-Hausdorff topology. BST, on the other hand, is concerned with different alternative, modally incompatible space-times that are still incorporated into a single logical model, which metaphysically speaking should be a single indeterministic world. While a full BST model may be non-Hausdorff, its individual histories (space-times) *are* Hausdorff. Can physics have any use for that?

This is a deep issue, giving rise to very controversial discussions. We will try to remain neutral, but we wish to remark that once the notion of a scientific experiment is taken into consideration, modality plays a crucial and fundamental role in physics. Witness Hawking and Ellis (1973, 189): "a simple notion of free will [...] is not something which can be dropped lightly since the whole of our philosophy of science is based on the assumption that one is free to perform any experiment." Similar considerations can be found, e.g., in discussions of quantum correlation experiments. For a striking example, see the recent debate about a so-called "free will theorem" (Conway and Kochen, 2006).

Whatever the ultimate merits of such discussions, they seem to give us enough motivation for an in-depth development of a *modal* theory of non-Hausdorff manifolds as a generalization of the BST92 theory of branching space-times. Such a development will be attempted now.

3 Locally Minkowskian BST

The discussion above has shown that there are many delicate issues when it comes to considerations of non-Hausdorffness in GTR. It should also have become clear that it is hard to make formally specific contact between BST92 and the GTR discussion. On the one hand, BST92 may be too general: it allows for very weird phenomena to occur. Placek and Belnap (2010), via their definition of Minkowskian BST, have gone a long way to alleviating these worries.[11] On the other hand, BST92 is fundamentally a global theory of a single partial ordering, like special relativity, and not a local one building on differential geometry, like GTR. This provides a fundamental obstacle to any firm connection between BST92 models and GTR solutions.

In this section our aim is to both narrow down and to broaden the BST92 framework, arriving at something we will call LMBST—locally Minkowskian BST, a specific class of generalized manifolds. In §3.1 we will first define very simple structures, simpler than (in fact, almost special cases of) the Minkowskian BST92 models in the mentioned literature. Issues of space-time and modality will be discussed on that basis in §3.2. In the final step, in §3.3, we will then define global Y-manifold structures based on the local structures. These, we will argue, are the natural area of contact between BST and GTR.

3.1 Simple Minkowskian branching: M^n_m

We build up branching structures M^n_m from n-dimensional Minkowski space-time M^n.

[11]For previous work on Minkowskian BST, triggered by remarks of Belnap (1992, 412), see also Müller (2002), Placek and Wroński (2009) and Wroński and Placek (2009).

3.1.1 Minkowski space-time M^n

Minkowskian space-time M^n is the n-dimensional Euclidean space $\mathbb{R}^n$ together with the pseudo-Riemannian metric ds, where (in coordinates in which the speed of light $c = 1$, and assuming $n \geq 2$)

$$ds^2(\langle x^1, \ldots, x^n \rangle, \langle y^1, \ldots, y^n \rangle) := -(x^1 - y^1)^2 + \cdots + (x^n - y^n)^2.$$

(Thus, x^1 is the time coordinate, $x^2, \ldots, x^n$ are spatial coordinates, and we are working with signature $(-, +, +, +)$.) Elements of M^n, which will be denoted by boldface letters $\mathbf{x}$, $\mathbf{y}$, ..., are called *events* (as for the elements of BST92 structures, this is the physicists', not the philosophers' usage). Events for which $ds^2 < 0$ are called *time-like separated*, for $ds^2 = 0$, *light-like separated*, and for $ds^2 > 0$, *space-like separated*. If $ds^2 \leq 0$, we also say that the events are *causally connectible*.

Note that we cannot derive a useful topology from the pseudo-metric ds.[12] If we tried to define open balls like in §1.4.1, as $B(\mathbf{x}, r) = \{\mathbf{y} \mid ds^2(\mathbf{x}, \mathbf{y}) < r\}$, we would be including all timelike and lightlike events in an open environment of an event, and even if we went for a definition via the absolute value of ds^2, we would still be stuck with lightlike separated events. The standard topology $\mathfrak{T}$ of Minkowskian space-time, also called the manifold topology, is therefore simply taken to be that of $\mathbb{R}^n$. As that topology is derived not from a pseudo-metric, but from a metric (the natural metric of $\mathbb{R}^n$), it is guaranteed to be Hausdorff; furthermore, M^n is connected (in fact, simply connected) in that topology.

On M^n we can define the following ("causal") global partial ordering:

$$\langle x^1, \ldots, x^n \rangle \leq \langle y^1, \ldots, y^n \rangle \quad \text{iff} \quad ds^2(\mathbf{x}, \mathbf{y}) \leq 0 \text{ and } x^1 \leq y^1.$$

This ordering captures the light-cone structure of special relativity. We have the following provable results from our Definition 2:

$$J^+(\mathbf{x}) = \{\mathbf{y} \in M^n \mid ds^2(\mathbf{x}, \mathbf{y}) \leq 0 \;\;\&\;\; x^1 \leq y^1\},$$

$$I^+(\mathbf{x}) = \{\mathbf{y} \in M^n \mid ds^2(\mathbf{x}, \mathbf{y}) < 0 \;\;\&\;\; x^1 < y^1\},$$

and similarly for J^- and I^-. See also the definition of a causal path in §1.4.3 above.

[12] As Visser (2009) remarks, we can derive a topology from it, but it is highly non-standard and rarely used.

We thus have n-dimensional Minkowski space-time as an ordered topological space $\langle \mathbb{R}^n, \leq, \mathfrak{T} \rangle$ that is an n-dimensional Hausdorff manifold, where we can take an atlas consisting of a single chart with the identity function as a mapping on $\mathbb{R}^n$. (Obviously, such a manifold is C^∞.)

3.1.2 Branching Minkowski space-time M_m^n

From the ordered topological space $\langle \mathbb{R}^n, \leq, \mathfrak{T} \rangle$ we can now build simple n-dimensional non-Hausdorff manifolds that are also partial orders: we basically replace the forward light-cone of the origin,

$$V := J^+(\mathbf{0}) := \{\mathbf{x} \in M^n \mid \mathbf{0} \leq \mathbf{x}\}$$

by $m \geq 2$ copies

$$V_i := V \times \{i\}, \quad i = 1, \ldots, m$$

and adjust the ordering and the topology accordingly. These structures embody both the order-theoretic aspects of BST and the topological aspects of GTR, and thus are good candidates for bringing the two frameworks together.

Details of the pasting. There are a number of choices for how to proceed, and it is important to be explicit. We certainly want to make copies of the *interior* of the forward light-cone, but there are at least three sensible choices of whether the rim of the forward light-cone,

$$\partial J^+(\mathbf{0}) = \{\mathbf{x} \in M^n \mid \mathbf{0} \leq \mathbf{x} \ \&\ ds^2(\mathbf{0}, \mathbf{x}) = 0\}$$

should be replaced by copies as well:

(1) We can fully replace $\partial J^+(\mathbf{0})$ by m copies, so that indeed $V = J^+(\mathbf{0})$;

(2) we can leave $\partial J^+(\mathbf{0})$ intact and only replace the whole interior of $J^+(\mathbf{0})$ by copies, so that $V = I^+(\mathbf{0}) = J^+(\mathbf{0}) - \partial J^+(\mathbf{0})$, or

(3) we can replace the whole of $\partial J^+(\mathbf{0})$ *except for the origin itself* by m copies, and leave the origin intact, so that $V = J^+(\mathbf{0}) - \{\mathbf{0}\} = \{\mathbf{x} \in M^n \mid \mathbf{0} < \mathbf{x}\}$.

No matter how we proceed, we arrive at a branching structure that can be viewed as consisting of overlapping layers L_i^n, $i = 1, \ldots, m$. Using $\bar{V} := M^n - V$ and $V_i := V \times \{i\}$, the layers are[13]

$$L_i^n := (\bar{V} \times \{1\}) \cup V_i$$

and the global structure is

$$M_m^n := \bigcup_{i=1}^{m} L_i^n.$$

We use normal variables $x, y \ldots$ to range over elements of M_m^n.

The partial ordering on M_m^n is defined as the union of the obvious partial orderings on the m layers:

$$x \leq y \quad \Leftrightarrow_{df} \quad \text{there is } i \in \{1, \ldots, m\} \text{ s.t. } \quad x, y \in L_i^n \quad \text{and} \quad x \leq_i y,$$

where $\leq_i$ is the standard Minkowskian ordering on the i-th layer. Note that by this construction of the ordering, events above the origin in different layers are incomparable. Such events may even occur at the same space-time point, like $\langle \mathbf{x}, i \rangle$ and $\langle \mathbf{x}, j \rangle$ for $i \neq j$, but they are modally incompatible and thus do not occur together in any one space-time. A crucial issue in what follows will be to translate this intuitive verdict, which has a natural order-theoretic explication in terms of BST92's definition of histories as maximal directed sets, into a topological setting. From the point of view of the partial ordering, elements above the origin in different layers are classified as modally incompatible since they do not have a common upper bound (and thus, there is no directed set containing them both); intuitively, there is no perspective available from which one could say that both events have occurred. The question is how to express someting like this without invoking a global ordering, employing instead suitable local, topological notions.

Before we address this issue, however, we need to decide between options (1)–(3) above. How are we to choose the right kind of pasting? Topological considerations play a crucial role here. One important observation can be made by considering certain upward directed chains. Consider M_2^2 according to one of our three options (we choose $n = m = 2$ for simplicity; the example works for any $n \geq 2$ and $m \geq 2$). We define the two upper bounded chains

$$C_0 := \{\langle -1/k, 0, 1 \rangle \mid k \in \mathbb{N}\}$$

[13] The extra label 1 for events outside V is just a convenience—it allows for uniformly addressing elements of M_m^n as $n{+}1$-tuples, and it identifies the first layer L_1^n with $M^n \times \{1\}$.

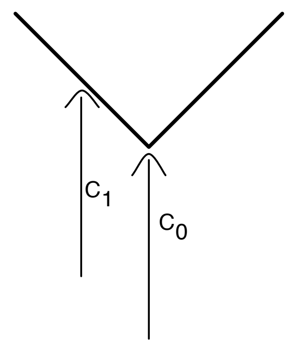

Figure 3: Two chains C_0 and C_1 in layer 1 of M_2^2. See text for details.

and

$$C_1 := \{\langle 1 - 1/k, 1, 1\rangle \mid k \in \mathbb{N}\}.$$

Both chains lie in the intersection of the two layers of M_2^2, in all three pasting options (no element of the chain is above the origin in the Minkowskian ordering); see Fig. 3. Yet, there is a difference when we look at the set of limit points of the chains. According to (1), both chains have *two* minimal upper bounds, one in each layer. According to (2), on the other hand, both chains have a unique minimal upper bound, which is their supremum, lying in the intersection of both layers. According to (3), however, C_0 has a supremum in M_2^2, viz., the origin $\langle 0, 0, 1\rangle$, while C_1 has no supremum, despite being an upper bounded chain in a continuous structure. The reason for this behavior is that on the pasting (3), the origin is not a "doubled point"—there is just one event with coordinates of the origin, $\langle 0, 0, 1\rangle \in \bar{V} \times \{1\}$, but at coordinates $\langle 1, 1\rangle$ there are two events, one in each layer. In layer i, $\langle 1, 1, i\rangle$ is the layer-relative supremum of C_1; in M_2^2, this qualifies these points as minimal upper bounds of the chain, but not as the supremum (which, if it exists, is unique).

Convergent sets having different limit points is typical for non-Hausdorff structures; in fact, according to most topologies (see the discussion below), we have $\langle 1, 1, 1\rangle \curlyvee \langle 1, 1, 2\rangle$. Belnap (1992, 413) has given an extensive discussion of the merits of pasting option (3) based on causal considerations. That option allows one to single out the choice points that are crucial in BST92's prior choice postulate (see §1.2). With respect to pasting option (3), Belnap has called the behavior of C_1 "indeterminism without choice", to be contrasted with the "indeterminism due to choice" that happens at C_0. A more extensive discussion of the appropriate topology for M^n_m reveals reasons that speak against pasting option (3) and in favor of option (1), requiring us to move away from the axiomatic basis of BST92 and to find a replacement for the prior choice principle.

Defining the topology. The fine details about doubling the rim of the light cone that led to the three options (1)–(3) above lead to further considerations of a topological nature. There seems to be no full discussion of these topological issues in the literature, so we will try to be very explicit in what follows.

Earman (2008, 198f.) discusses a simple one-dimensional analogue of options (1) and (2); option (1) is illustrated in Figure 1 above, option (2) in Figure 2.[14] He remarks that option (2) leads to a space that is Hausdorff but not locally Euclidean, while (1) gives a locally Euclidean space that is not Hausdorff. Strictly speaking, the choice of the pasting is a separate issue from the definition of a topology, but indeed, a number of constraints arises.

From the point of view of physics, a very natural constraint is the following:

(E) A topological space that can be useful for GTR has to be locally Euclidean and in fact, a (perhaps generalized) manifold.

We follow Earman (2008, 198f.) in his argument for (E): "topological spaces that are not locally Euclidean cannot be assigned a differentiable structure, and such a structure is essential in formulating the very notion of a Lorentzian metric and in formulating the Einstein field equations" . Thus, if we want to remain close to GTR, we had better arrive at a generalized manifold.

As a precondition for fulfilling (E), we require that the topology on M^n_m be defined on the basis of the open balls in the different layers,

$$B_i(\mathbf{x}, r) := \{y = \langle \mathbf{y}, j \rangle \in L^n_i \mid d(\mathbf{x}, \mathbf{y}) < r\},$$

where $i \in \{1, \ldots, m\}$, $\mathbf{x} \in M^n$, $r \in \mathbb{R}^+$, and d is the n-dimensional Euclidean distance.[15]

[14]Note that in one dimension, case (3) does not arise as a separate option; it coincides with option (2).

[15]Note that we had to write $y = \langle \mathbf{y}, j \rangle \in L^n_i$, allowing for the index $j = 1$ in any layer for events with coordinates in $\bar{V}$, where the layers overlap.

There seem to be at least two options (a) and (b) for proceeding from here.[16]

(a) We can use as a basis the open balls in each layer L_i^n separately, i.e., the basis consists of all the sets $B_i(\mathbf{x}, r)$ defined above.

(b) We can coarse-grain this option and use as a basis the sets

$$B_*(\mathbf{x}, r) := \{y = \langle \mathbf{y}, i \rangle \in M_m^n \mid d(\mathbf{x}, \mathbf{y}) < r\},$$

i.e., the basis consists of balls in all layers simultaneously.

Pasting and choice of topology interact in interesting ways. The following table gives the results, where "LE" stands for "locally Euclidean" and "H" for "Hausdorff":[17]

	pasting (1)	pasting (2)	pasting (3)
topology (a)	$H-/LE+$	not a top.	not a top.
topology (b)	$H+/LE-$	$H+/LE-$	$H+/LE-$

We see that the pasting options (2) and (3), according to which the origin is not doubled, lead to a violation of the conditions on a topology if one tries a definition according to (a): e.g., $B_1(\mathbf{0}, 1/2) \cap B_2(\mathbf{0}, 1/2)$ is not open as it contains no $B_k(\mathbf{0}, r)$ for any k and r (the interior of the forward light cone drops out, so to speak). A definition accordint to (b), on the other hand, leads to "branching" open sets around the origin, which are not homeomorphic to any open set of $\mathbb{R}^n$. Option (1,a) therefore remains as the only sensible way for fulfilling requirement (E), and we will accordingly adopt option (1,a) in what follows.[18]

For the record, here is our official definition of the m-fold branching, n-dimensional Minkowski space-time M_m^n as an ordered topological space:

[16] In fact we could try to create a third option somewhere in between (a) and (b), e.g., by going for (b) only around the origin. However, such an option will inherit the problems of option (b) to be discussed below, so we do not discuss it separately.

[17] We are assuming $m \geq 2$, for otherwise the topology obviously coincides with the standard topology of M^n, and the pasting options (1)–(3) have no effect.

[18] An M_m^n-like construction is also given in Visser (1996, 251–255); the book contains many pointers to relevant literature. Visser calls his construction a "branched spacetime" (252), without however making any connections to the philosophical/logical discussions about branching space-times. Visser opts for topological option (a) as we do, and also his V is option (1), not Belnap's option (3). Penrose (1979, 593) has a suggestive drawing of a branching space-time; while Penrose is not explicit about the topology, and his Fig-

Definition 13 (M^n_m) *The m-fold branching, n-dimensional Minkowski space-time M^n_m is defined from the n-dimensional Minkowski space-time M^n by setting the to-be-multiplied region V to be the future light cone of the origin, including the rim of the light cone and the origin itself:*

$$V := J^+(\boldsymbol{0}) = \{\boldsymbol{x} \in M^n \mid \boldsymbol{0} \leq \boldsymbol{x}\}; \quad \bar{V} := M^n - V; \quad V_i := V \times \{i\};$$

defining the m layers, for $i = 1, \ldots, m$, to be

$$L^n_i := (\bar{V} \times \{1\}) \cup V_i;$$

and pasting them via

$$M^n_m := \bigcup_{i=1}^{m} L^n_i.$$

The ordering $\leq$ is the union of the usual Minkowskian orderings in the layers, and the locally Euclidean topology $\mathfrak{T}$ is given via the countable basis of open balls with rational center coordinates $\boldsymbol{x} \in M^n$ and rational radius $r > 0$ in the finitely many layers $i = 1, \ldots, m$,

$$B_i(\boldsymbol{x}, r) := \{y = \langle \boldsymbol{y}, j\rangle \in L^n_i \mid d(\boldsymbol{x}, \boldsymbol{y}) < r\}.$$

Note that the layers themselves, each of which is homeomorphic to $\mathbb{R}^n$, are open sets in this topology, and that for $i \neq j$, $L^n_i - L^n_j = V_i$. Note also that $\partial V_i = \partial J^+(\boldsymbol{0}) \times \{i\}$.

By our definition we now have a natural and locally Euclidean topology, which qualifies our structures as Y-manifolds.[19] The models M^n_m, considered

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ure 12.3(a) may suggest choice points à la Belnap, he seems to have option (1) in mind as well, since he writes: "on each branch the wavefunction *starts out* as a different eigenvector ..." (Penrose, 1979, 594; italics TM). Deutsch (1991) refers to this discussion; his remarks about "a larger object which has yet to be given a proper geometrical description" (3207) may be read as pointing in the direction of something like our M^n_m structures, or their generalizations discussed in §3.3 below. McCabe (2005) reproduces Penrose's figure. He remarks that such figures themselves are open to different interpretations and do not need to be read as implying non-Hausdorffness; this is in line with our discussion of options above. However, he does not discuss in much detail the price that has to be paid for dropping local Euclidicity in avoiding non-Hausdorffness, remarking that "it is a debate which has not been conducted in the literature" (McCabe, 2005, 670). We agree with Earman that constraint (E) has to be taken very seriously, and we will continue to hold on to it.

[19]We can use an atlas with m charts, each mapping a layer L^n_i of M^n_m to $\mathbb{R}^n$.

as orderings, are almost branching space-times according to BST92, and even almost Minkowskian BST92 models (see note 11 above). From a BST92 point of view, there are m histories: the L_i^n are the maximal directed sets.[20] However, the prior choice postulate, which requires the existence of maxima in the intersection of histories that play a causal role, is violated: for $i \neq j$, $L_i^n \cap L_j^n = \bar{V} \times \{1\}$ has no maximum whatsoever. We will comment on this issue below: this is the price we have to pay for local Euclidicity, and thus, for moving closer to the physics discussion. An alternative version of the prior choice postulate can be built upon a novel, topological definition of the notion of a choice point; §3.3 contains some suggestions on this issue.

In a way, the above discussion should suffice to alleviate Earman's worry that he has "been unable to get a fix on what Belnap branching involves" (Earman, 2008, 192). To be fair, however, we have seen reasons to deviate from BST92, and there are many issues that still need to be addressed. We will start by discussing the question of modality from a topological perspective.

3.2 Modality in M_m^n

3.2.1 Modal consistency and inconsistency

We have already mentioned that the BST92 motivation for constructing branching space-times was to capture (certain forms of) indeterminism. Regions in different layers of M_m^n outside the overlap, i.e., any subsets $R_i \subseteq V_i$ and $R_j \subseteq V_j$, for $i \neq j$, are viewed as *modally incompatible*; they cannot occur together in a single history. (This does not require them to be qualitatively different, though.) On the BST view, it is not that such regions are "worlds apart", like in the "divergence" view on modality championed by Lewis and at least implicitly adhered to in most of the philosophy of science discussion. Rather, R_i and R_j are related via suitable external relations—viz., the causal ordering $\leq$—and can therefore be usefully viewed as parts of one world; hence Belnap's apt name *Our World* for models of BST92.[21]

[20] Obviously the layers are directed sets, being order isomorphic to M^n, which is directed. For maximality, observe that any "new" element to be added to L_i^n has to come from V_j with $j \neq i$; by the definition of the ordering, the resulting superset of L_i^n is not directed. For a detailed proof, see, e.g., Müller et al. (2008). (The fact that we have opted for pasting à la (1), doubling the origin as well, has no influence on this result.)

[21] In BST92 it is provable that the worst case for such an external relation is an M-shaped causal path; in the simpler structures at issue here, the worst case is a V-shaped

Here comes the crucial issue. The notion of modal incompatibility has a formally perspicuous and intuitively satisfactory definition in the order-theoretic framework of BST92: events e and f in a model of BST92 are modally compatible iff they have a common upper bound—in that case, they belong to some directed set, which in turn must be (by Zorn's lemma) a subset of a history. Events that share a history do not need to be order-related, but if they aren't, then we know that they occur together in some one space-time and are space-like separated. Their common upper bound provides a perspective from which one can say that both have occurred.

This logical approach to modal consistency is still applicable to our example structures M^n_m, which can be captured as a single partial ordering as required by BST92. We have however already remarked that this feature of BST92 makes it too narrow for applications to GTR. Thus, we are looking for a purely *topological* definition of modal consistency and inconsistency, which would also apply to such (generalized) manifolds that can no longer be viewed as single partial orders.

The literature on topological issues in GTR that we have been able to consult does not address this question explicitly. Modality is usually absent from physical theorizing, and determinism seems to be viewed as a regulative ideal for physics—despite a number of acknowledgments of the modal presuppositions of, e.g., scientific experiment (see §2.3). This means that we have to do some exploratory work.

3.2.2 From the order-theoretic to a topological characterization of consistency

Intuitively and by the pasting construction, it is clear that the maximal modally consistent subsets of M^n_m are exactly the layers L^n_i, $i = 1, \ldots, m$. These cover the whole of M^n_m without any gaps or holes, and they are also individually such that, intuitively speaking, each space-time point of Minkowski space-time M^n occurs exactly once. As remarked above, these layers are also the histories in the sense of the usual, order-theoretic definition of BST92: each layer is a maximal directed set in M^n_m. Thus, the order theoretic definition of modal consistency remains applicable even though we deviate from the axiomatic basis of BST92. The question before us now is how to capture the intuitive notion of modal consistency not in order theoretic, but

path.

in purely topological terms. A guiding idea is to take seriously the point that non-Hausdorffness should be shunned for individual space-times (histories), and thus to carve up our non-Hausdorff manifolds M^n_m into appropriate Hausdorff submanifolds.

While the GTR literature is generally silent about the issue of modal consistency, relegating, as we have seen, remarks about modality to the non-formal parts of papers, Hajicek (1971) defines the useful notion of an *H-submanifold* of a $\curlyvee$-manifold (where the "H" stands for "Hausdorff"):

Definition 14 (H-manifold) *Given a $\curlyvee$-manifold M, a subset $A \subseteq M$ is an H-*submanifold *iff A is open, connected, Hausdorff, and maximal with respect to these properties. (I.e., every proper superset of A is either not open, not connected, or not Hausdorff.)*

A straightforward application of Zorn's lemma gives us that the set of all H-submanifolds is an open cover of a given $\curlyvee$-manifold (cf. Hajicek, 1971, Theorem 1).

Hajicek (1971) also suggests the notation $\curlyvee^L_M$ for the set of points in M that are non-Hausdorff related to some point in L,

$$\curlyvee^L_M := \{x \in M \mid \exists y \in L \; x \curlyvee y\}.$$

We note some useful facts about the points in M^n_m that are non-Hausdorff related to some other point (obviously there are no such points in case $m = 1$):

Lemma 1 *Let $M := M^n_m$ for some $n \in \mathbb{N}$ and some $m \geq 2$. Then for $x = \langle \boldsymbol{x}, i\rangle, y = \langle \boldsymbol{y}, j\rangle \in M$ we have*

$$x \curlyvee y \quad \textit{iff} \quad \boldsymbol{x} = \boldsymbol{y}, \; i \neq j, \; \textit{and } \boldsymbol{x} \in \partial J^+(\boldsymbol{0}).$$

Accordingly,

$$\curlyvee^M_M = \{\langle \boldsymbol{x}, i\rangle \mid \boldsymbol{x} \in \partial J^+(\boldsymbol{0}) \;\&\; i \in \{1, \ldots, m\}\},$$

and for $L := L^n_i$ a layer ($i \in \{1, \ldots, m\}$), we have

$$\curlyvee^L_M = \{\langle \boldsymbol{x}, j\rangle \mid \boldsymbol{x} \in \partial J^+(\boldsymbol{0}) \;\&\; j \in \{1, \ldots, m\} \;\&\; j \neq i\} = \curlyvee^M_M - \partial V_i.$$

Proof: The second and third assertions follow immediately from the first. For the first, we can argue as follows:

"$\Rightarrow$": Let $\langle \mathbf{x}, i\rangle \curlyvee \langle \mathbf{y}, j\rangle$. If we had $\mathbf{x} \neq \mathbf{y}$ we could obviously separate them via open balls of less than $1/2$ their Euclidean distance. So $\mathbf{x} = \mathbf{y}$, and as $x \curlyvee y$ implies $x \neq y$, we have $i \neq j$. So we have $x \in V_i$ and $y \in V_j$. Now if we had $\mathbf{x} \notin \partial J^+(\mathbf{0})$, x would belong to the interior of V_i and y to the interior of V_j (note again that $\partial V_i = \partial J^+(\mathbf{0}) \times \{i\}$). So, as $V_i \cap V_j = \emptyset$, the points x and y could be separated by disjoint open sets, violating $x \curlyvee y$.
"$\Leftarrow$": Let $\mathbf{x} = \mathbf{y}$, $i \neq j$, and $\mathbf{x} \in \partial J^+(\mathbf{0})$. Any open environment of one of those points contains an open ball of radius r_x and r_y, respectively. These balls overlap in the region $\bar{V}$. So indeed, the points cannot be separated by disjoint open sets. $\square$

We will also need the following Lemma:

Lemma 2 *Let $M := M^n_m$ for some $n \in \mathbb{N}$ and some $m \geq 2$, and let $A \subseteq M$ be open and such that A contains elements of the forward light cone of the origin in different layers, i.e., there are $i, j \in \{1, \ldots, m\}$, $i \neq j$, such that*

$$A_i := A \cap V_i \neq \emptyset, \quad A_j := A \cap V_j \neq \emptyset.$$

Then if A is connected, we have both $A_i \cap \curlyvee^M_M \neq \emptyset$ and $A_j \cap \curlyvee^M_M \neq \emptyset$.

Proof: Assume that $A_i \cap \curlyvee^M_M = \emptyset$ (the case for A_j is symmetrical). Noting that $L^n_i \cap \curlyvee^M_M = \partial V_i$, this means that $A_i \subseteq \operatorname{int} V_i$, and as A is open, $A_i = A \cap \operatorname{int} V_i$, as an intersection of two open sets, is open as well. Now writing $B = A - A_i$, we obviously have $A = A_i \cup B$ and $A_i \cap B = \emptyset$. But we also have

$$B = \bigcup_{j \neq i} A \cap L^n_j,$$

so (noting that the L^n_j are open), B is open as well. A_i is nonempty by assumption, as is A_j, and we have $A_j \subseteq B$, so B is nonempty as well. Thus A, being the disjoint union of nonempty open sets, is not connected. $\square$

We can now prove that the layers of M^n_m are in fact H-submanifolds:

Lemma 3 *Let $M := M^n_m$ for some n, m, and let $L := L^n_i \subseteq M$ be a layer ($i \in \{1, \ldots, m\}$). Then L is an H-submanifold of M.*

Proof: Hausdorffness, openness and connectedness are obvious: L is homeomorphic to M^n (and thus, L is even simply connected). As to maximality, let $A \supsetneq L$ be open and Hausdorff. Note that in the terminology of Lemma 2,

$A_i = A \cap V_i = V_i \neq \emptyset$. As A is a superset of L, A must also contain a nonempty subset A_j of some $V_j = L_j^n - L$, $i \neq j$, so the antecedent of Lemma 2 is satisfied. Note that $A_i \cap \curlyvee_M^M = \curlyvee_M^L = \partial V_i$, so $\curlyvee_M^L \subseteq A$. As A is Hausdorff by assumption, we must have $A_j \cap \curlyvee_M^M = \emptyset$. But then, by Lemma 2, A is not connected, so L is in fact maximal w.r.t. the property of being Hausdorff, open and connected. $\square$

Unfortunately, this Lemma does not hold in the other direction: there are intuitively weird H-submanifolds of M that do not correspond to layers. We will illustrate this by a counterexample for $M = M_2^2$, which also paves the way for our ultimate topological definition of modal consistency (the example easily generalizes to other M_m^n, $m, n \geq 2$).

Fact 1 *$M := M_2^2$ has an H-submanifold that is not equal to one of the layers L_i^n, $i = 1, 2$.*

Proof by example: We divide the rim of the forward light-cone of the origin into a left and a right part, which are allowed to overlap at the origin:

$$J_l := \{\langle t, x\rangle \in J^+(\mathbf{0}) \mid x \leq 0\}, J_r := \{\langle t, x\rangle \in J^+(\mathbf{0}) \mid x \geq 0\}.$$

We have $J_l \cup J_r = J^+(\mathbf{0})$ and $J_l \cap J_r = \{\mathbf{0}\}$. Now consider the set

$$A := M - ((J_l \times \{1\}) \cup (J_r \times \{2\})),$$

i.e., A is the whole of the pasted space M without half of the rim of the forward light-cone in each layer. Note that the origin in both layers is removed in constructing A, which makes it intuitively weird. But as a fact, A is a H-submanifold of M. For a proof, we can cite Hajicek (1971, Theorem 2). More explicitly (since we need the proof to motivate our improved definition of a history), we argue as follows. Hausdorffness is clear: the worrysome double points have been carefully removed. Openness is also quite easy to prove. $\bar{V} \times \{1\}$ is an open set, so any point of A in there has an open environment in A. The same holds for the interior of the two copies of the forward light-cone of the origin. As for the remaining points of A on the rim of that light-cone, is is also straightforward to prove that each point $\mathbf{x}$ on the left-hand side has an open environment of the form $B_2(\mathbf{x}, r)$, and similarly each point $\mathbf{x}$ on the right-hand side has an open environment of the form $B_1(\mathbf{x}, r)$. In order to prove connectedness, we can easily show that A is path-connected; Fig. 4 gives the idea. It turns out that A is even simply connected. This may seem strange, given that two points (at the origin) have been removed from the (two-dimensional) manifold. But there is no closed curve circling the origin: if a curve enters the forward light-cone from the left, into layer 2, then it cannot exit to the right, and the other way round. Finally, as to maximality, we obviously cannot add any point from $(J_l - \{\mathbf{0}\}) \times \{1\}$ or from $(J_r - \{\mathbf{0}\}) \times \{2\}$ without violating Hausdorffness (see Lemma 1). So the only real candidate for adding is the origin in one of the layers, $\langle 0, 0, i\rangle$, $i = 1$ or 2. But any open environment of $\langle 0, 0, i\rangle$ has to contain an open ball $B_i(\mathbf{0}, r)$ for some $r > 0$—and that ball then has to contain additional points from $J_l \times \{1\}$ (for $i = 1$) or from $J_r \times \{2\}$ (for $i = 2$), violating Hausdorffness after all. $\square$

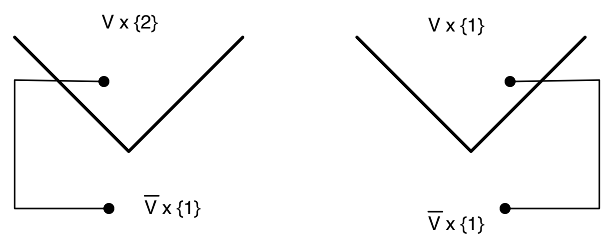

Figure 4: Idea of the proof that the set A exemplifying Fact 1 is path connected.

We reject A as a serious contender to modal consistency: it contains obviously incompatible points, filling in all the space-time points of the interior of $J^+(\mathbf{0})$ twice and just avoiding a failure of Hausdorffness by some trick. This intuitive assessment can be backed up by an order-theoretical argument: as a partial ordering, A is not directed. But what can we say on the topological side?

The more properly local kind of critique of the example structure A seems to be the following: in A, there are convergent sets $C \subseteq M$ wholly contained in A such that C does not converge to any point in A. Note that it would obviously be asking too much to require that A contain all limit points of such C and thus be closed—if some set C has two or more non-Hausdorff-related limit points in M, we obviously cannot require A, which must be Hausdorff, to contain all of those. It does seem reasonable, however, to demand that *at least one limit point* be contained in A. In fact, the example of the chain C_0 given in §3.1.2 above (see Fig. 3) is an appropriate expression of this worry: it is a subset of A with two limit points $\langle 0, 0, 1\rangle$ and $\langle 0, 0, 2\rangle$, but none of these points is contained in or can be added to A.

The two ways of criticizing the example A are complementary: according to the order-theoretic point of view, the set is "too big", containing points that cannot occur together in one (directed) history. According to the topological point of view, the set is rather "too small" in that it does not contain any limit points for sets that converge in the full space, and it cannot be extended to be "big enough" without violating the Hausdorff condition.

We can rephrase our topological considerations as follows: A set A containing two or more non-Hausdorff-related points is obviously, or blatantly, modally inconsistent; it runs together modal alternatives in one set, which therefore cannot represent a possible scenario. A set may however be modally inconsistent without being already blatantly inconsistent: there may be a natural demand on completeness of any candidate for a possible scenario such that the candidate scenario becomes blatantly inconsistent when completed. This is what we diagnosed to be the case for the example A from the proof of Fact 1; the natural demand on completeness, apart from openness and connectedness, was relative hole-freeness as laid out above.[22]

Based on these considerations we venture the following novel *topological* definition of a maximal consistent set, or a history:

Definition 15 *Given $M = M^n_m$ for some n and m, a* history *in M is a subset $h \subseteq M$ that is maximal with respect to the properties of being (i) open, (ii) connected, (iii) Hausdorff, and (iv) for each subset $C \subseteq h$, if $\partial C \neq \emptyset$, then $h \cap \partial C \neq \emptyset$ as well.*

As a first test for the usefulness of this definition, we can now indeed prove both directions of the analogue of Lemma 3:

Lemma 4 *Given $M = M^n_m$ for some n and m, a subset $A \subseteq M$ is a history according to definition 15 iff $A = L^n_i$ for some $i \in \{1, \dots, m\}$.*

Proof. "$\Leftarrow$" Let $A = L^n_i$ for some $i \in \{1, \dots, m\}$. By Lemma 3 we know that A is (i) open, (ii) connected, and (iii) Hausdorff. As A is homeomorphic to M^n, any convergent set in A also contains a limit point in A, so A also

[22]Note that we are only talking about *relative* hole-freeness: histories are not allowed to contain holes relative to the background generalized manifold. This does not amount to demanding that the histories themselves be "hole-free" as individual space-times, e.g., in the sense discussed by Earman et al. (2009). Such conditions can however be added to the definition of the relevant class of generalized manifolds. See §3.3 below for some discussion.

fulfills condition (iv). Again by Lemma 3, A is already maximal w.r.t. (i)–(iii), so it is also maximal w.r.t. (i)–(iv).

"$\Rightarrow$" Let $A \subseteq M$ be a history according to definition 15. If we can show that $A \subseteq L_i^n$ for some $i \in \{1, \dots, m\}$ then we're finished: As the layers are themselves histories, A cannot be a proper subset of any layer, whence $A = L_i^n$. We will use the following abbreviations for subsets of A:

$$A_k := A \cap V_k; \quad R_k := A_k \cap \curlyvee_M^M = A_k \cap (\partial J^+(\mathbf{0}) \times \{k\}).$$

Now assume that there is no $i \in \{1, \dots, m\}$ for which $A \subseteq L_i^n$: This means that there are $i, j \in \{1, \dots, m\}$, $i \neq j$, for which $A_i \neq \emptyset$ and $A_j \neq \emptyset$. By Lemma 2, noting that A is open and connected by assumption, we therefore have $R_i \neq \emptyset$ and $R_j \neq \emptyset$.

We now look at the set R of points on the rim of the forward light cone of the origin occupied in A,

$$R := \{\mathbf{x} \in \partial J^+(\mathbf{0}) \mid \exists i \in \{1, \dots, m\}\ \langle \mathbf{x}, i \rangle \in A\}.$$

As A is open, we have that R is an open subset of $\partial J^+(\mathbf{0})$ (in the subspace topology induced by $\partial J^+(\mathbf{0})$). We now consider $\bar{R} := \partial J^+(\mathbf{0}) - R$, and we will show that $\bar{R}$ is also open (in the subspace topology).

If $\bar{R}$ is empty, there is nothing to be shown. If $\bar{R} \neq \emptyset$, we will show that it contains an open ball (in the subspace topology) around any of its points. Assume not: let $\mathbf{x} \in \bar{R}$ s.t. $\bar{R}$ contains no open ball around $\mathbf{x}$. This means that every open environment of $\mathbf{x}$ contains an element from R, so for every open environment of $\mathbf{x}$ there is some $\langle \mathbf{y}, i \rangle \in R_i \subseteq A$. As there are only finitely many layers, there must be some $k \in \{1, \dots, m\}$ for which every environment of $\mathbf{x}$ contains some $\langle \mathbf{y}, k \rangle \in R_k \subseteq A$. So R_k contains a subset converging on $\langle \mathbf{x}, k \rangle$. By property (iv) of a history, therefore, A must contain some $\langle \mathbf{x}, l \rangle$, whence $\mathbf{x} \in R$, contradicting the assumption that $\mathbf{x} \in \bar{R}$. Thus, on the assumption that there is some $\mathbf{x} \in \bar{R}$, $\bar{R}$ also contains an open ball around $\mathbf{x}$, meaning that it is open. This means that R is closed (in the subspace topology). But the only clopen subsets of $\partial J^+(\mathbf{0})$, which is connected, are $\emptyset$ and $\partial J^+(\mathbf{0})$ itself; by $R_i \neq \emptyset$, we must have $R = \partial J^+(\mathbf{0})$.

By Hausdorffness of A, we cannot have an $\mathbf{x} \in R$ and $i \neq j$ for which both $\langle \mathbf{x}, i \rangle \in A$ and $\langle \mathbf{x}, j \rangle \in A$. So we have a total function

$$f : R = \partial J^+(\mathbf{0}) \mapsto \{1, \dots, m\} \quad \text{s.t.} \quad f(\mathbf{x}) = i \text{ iff } \langle \mathbf{x}, i \rangle \in A.$$

Now chose some $i \in \{1, \ldots, m\}$ and let $\mathbf{x} \in R$ be such that $f(\mathbf{x}) = i$, i.e., $\langle \mathbf{x}, i \rangle \in A$. As A is open, there is some $B_i(\mathbf{x}, r) \subseteq A$, so there is an open subset of R containing $\mathbf{x}$ whose f-image is i. Thus, $f^{-1}(i)$ is an open subset of R, i.e., f is a continuous function from the connected set R to the discrete set $\{1, \ldots, m\}$. Any such function must be constant. This means that there is just *one* $k \in \{1, \ldots, m\}$ for which $R_k \cap \mathsf{Y}_M^M \neq \emptyset$, and so we cannot have, after all, $R_i \neq \emptyset$ and $R_j \neq \emptyset$, which however followed from our assumtion that A overlapped more than one layer. Thus, A is a subset of, and hence equal to, one of the layers of M_m^n. $\square$

3.3 Going global

So far we have achieved two things: (1) we have constructed branching structures that are both partial orders (appropriate for the "logical" approach of BST) and generalized manifolds (appropriate for the topological approach of general relativity): our m-fold branching Minkowski space-times M_m^n; (2) we have given a reasonable, purely topological definition of modal consistency, Definition 15. The main reason for moving from an order-theoretic criterion of modal consistency (BST92's directedness) to a topological definition was the possibility of generalizations to structures that can no longer be viewed as partial orders globally. Such generalizations should therefore be the next step.

In a certain sense, that step is trivial: we can simply take any generalized manifold and identify its histories according Definition 15. Provably, if such a generalized manifold is a properly generalized manifold (i.e., if it is not Hausdorff), it will contain more than one history; in fact, if a point x in a non-Hausdorff manifold M has k points non-Hausdorff related to it, M will harbour at least $k+1$ histories, and each of these histories will be a Hausdorff manifold.

This simple perspective on non-Hausdorff manifolds, however, does not seem to be enough for bringing BST really closer to the physicists' model space-times. There are several issues that need to be addressed:

1. The class of generalized manifolds is very large. It is not clear that carving up a generalized manifold into histories (Hausdorff submanifolds) generally makes intuitive sense. Minimally, one will want the resulting submanifolds (histories) themselves to be good candidates for physical space-times; additionally, one will want to be able to interpret

the branching of the histories in the sense of modal separation. Care also needs to be taken to ensure that the class of histories fully covers the initially given general manifold. This may not hold generally, as a simple appeal to Zorn's lemma, which was possible both in the case of the directed sets of BST92 and in the case of H-manifolds (Definition 14), is not generally possible due to condition (iv) of Definition 15.

2. In BST92, choice points play a crucially important role, e.g., as initials of indeterministic transitions. These choice points were defined to be maximal points in the intersection of histories. In the M^n_m structures defined above, however, the intersection of any two histories (layers) has no maxima. This deviation from BST92 is due to the fact that we needed to tweak the definition of the pasting such as to arrive at locally Euclidean spaces, as laid out in detail above. Thus, an important concept seems to be lost. Intuitively, however, the origin in M^n_m, which is m-fold multiplied (and therefore has $m-1$ non-Hausdorff twins just like any point on the rim of its forward light cone), still seems to be a special point. In terms of the ordering, it is the least point in the structure that has a non-Hausdorff twin. That is however a partially order-theoretic, not a purely topological characterization.[23] If we want to retain the topological outlook of our present approach, we need to capture choice points in a different way.

3. In BST92, the prior choice principle secured a physically reasonable interpretation of modal inconsistency as grounded in causal alternatives represented by inconsistent transitions. That idea has led to a number of applications, e.g., in probability theory. In order to retain, or at least translate, these results, we need to identify transitions in the new framework. This is linked to the previous point, as transitions in BST92 are from a choice point to one of its local possible futures.

4. Relatedly, the order-theoretic background of BST92 secured an intuitive interaction between modality and time: branching into incompatible scenarios is always future-directed, simply due to the demand of upward directedness of histories. This sense of temporal ordering is retained in the M^n_m structures: if a chain in M^n_m has more than one

[23]Curiously, a topological characterization is available in BST92 with the Belnap pasting leading to maxima in the intersection of histories: a choice point is characterized by having no non-Hausdorff twin and no locally Euclidean environment.

endpoint, then these endpoints are not just non-Hausdorff related, but we know that the endpoints are at the future-pointing end of the chain. Certainly, constraints will be needed to secure a similar feature in the general case, both in order to secure temporal orientability of the individual space-times and to secure the proper alignment of the temporal orientation of different histories as well as the mentioned "future point-doubling" feature.

5. If the theory of locally Minkowskian BST developed here is to have physical applications, we need to show how differential equations behave on generalized manifolds. Such considerations may in turn lead to constraints on reasonable global models.

As we can see, a lot of work remains to be done. In a sequel to this paper, we will work out the following points which here are mentioned merely programmatically:

(a) A generalized manifold M will be a locally Minkowskian branching space-time only if additional constraints are met.

  (i) Minimally, we demand that the set of histories fully covers M.

  (ii) We want to be able to define a pseudo-Riemannian metric $g_{\mu\nu}$ on M. (It will be interesting to look at the interrelation of that criterion with the definability of such a metric on each history separately.)

  (iii) It seems reasonable to demand time-orientability of all histories, i.e., the definability of a non-vanishing continuous time-like vector field. (Again, it will be interesting to see whether this is equivalent to demanding time-orientability of M as a whole; if not, it is interesting to find out under which additional conditions it is.)

  (iv) The time-orientation of each history has to be chosen such that branching happens to the future, i.e., if $x \curlyvee y$, then there is a future-directed causal path that has x and y as its endpoints. The possibility of such a choice translates back into a constraint on admissible generalized manifolds.

  (v) Additional constraints on the individual histories may be reasonable, e.g., some criterion of hole-freeness (see, e.g., Earman et al., 2009);[24] again, these translate back into constraints on admissible M. In fact, to be on the safe sinde, we may demand the individual histories (individual space-times) within a generalized manifold to be as physically well-behaved as we like, while still retaining a non-Hausdorff, branching space-time structure globally.

(b) As to the intuitive understanding of the branching, the following seems useful: In BST92, non-directedness was the sign of modal inconsistency; the prior choice principle demands that any case of modal inconsistency be explainable by a prior choice point.[25] In the present setting, modal inconsistency is a two-stage affair. Obviously, any points that are non-Hausdorff related, are modally inconsistent; they cannot occur in a single history simply because a history has to be Hausdorff. Let us call such points, *blatantly inconsistent*: they amount to running together incompatible *local* alternatives and therefore do not demand any additional explanation. (In this sense, two different transitions with the same initial are called "blatantly inconsistent" in Müller et al. (2008).) There is no immediate topological answer to when two points that are *not* non-Hausdorff twins, are modally inconsistent. The definition of a history gives the following answer: such points are modally inconsistent iff they belong to different maximal open, connected, Hausdorff, relatively hole-free sets, i.e., iff no such sets contains both points. Is there a more "local" criterion? One idea is the following: two points x, y are modally inconsistent iff there is a pair x_0, y_0 of blatantly inconsistent points ($x_0 \curlyvee y_0$) and there are two future-directed causal paths, one starting in x_0 ending in x, and one starting in y_0 and ending in y. According to that criterion, two modally inconsistent points are in the future of a modal splitting, which in turn explains their inconsistency.

(c) All of this may be captured by demanding that M be locally homeomor-

[24] Note again that condition (iv) in the definition of a history above only guarantees *relative* hole-freeness: a history may not introduce *additional* holes when compared to the global structure. This says nothing about the hole-freeness of that global structure itself, nor about the absolute hole-freeness of the histories as individual space-times. See note 22 above.

[25] This issue is subtle: in order not to rule out "modal funny business" (Belnap, 2002), one should not demand that the prior choice be in the *common* past of two inconsistent points. So, two choice points may have to be invoked in an explanation of a pair of modally inconsistent points.

phic to some M^n_m. The following general demand on M may be on the right track: if there is $x \in M$ that has exactly k non-Hausdorff twins $y_1, \ldots, y_k$, then there is an open environment a of x containing all the y_i and an open subset b of M^n_{k+1} such that a and b are homeomorphic, and such that the light cone structure is also mapped in the right way (i.e., in such a way that the part of the rim of the light cone in b is mapped onto a set of points all of which have non-Hausdorff twins).

(d) Given the above idea (when suitably worked out), it may be possible to single out the candidate choice points as those points that are mapped onto the origin by suitable homeomorphisms.

(e) Having candidate choice points as specific points with non-Hausdorff twins, one may bundle together classes of pairwise non-Hausdorff related sets of choice point candidates: choice-point candidates at the same space-time location, so to speak. These clusters of points then correspond, more properly, to a *local* set of alternatives, i.e., to a set of basic transitions. In that sense, in M, a *transition* is identified with a point (contrary to BST92's idea that transitions have no "simple location"; see Belnap (2003))—but with a point that carries a multiplicity within in, in the sense of having multiple local alternatives. (A choice point in this sense, then, is a point that already comes with a choice built in.) Given basic transitions in that way, many of the already established results of BST92 will carry over. Thus, e.g., we can import the causality theory of Belnap (2005), and the probability theory of Müller (2005).

(f) A next step would be to look at the behaviour of differential equations—paradigmatically, the Einstein field equations—on such M. It will be interesting to see how additional variables selecting the individual branches can be defined. As a first toy model, one may look at the idea of Pearle (1976) to use a gauge freedom of the theory (in his case, the phase of a quantum state). Facing m-fold splitting, one could e.g. divide up the space of gauge transformations into m parts and propagate the solution appropriately. Of course, a physically more reasonable mechanism, perhaps along the lines of Ghirardi et al. (1986), is needed in the end. In such a way, the philosophically motivated theory developed here could perhaps become a useful model for a very classical theory combining quantum indeterminism and general relativity.

References

Belnap, N. (1992). Branching space-time. *Synthese*, 92(3):385–434. See also the postprint 2003, available on philsci-archive.

Belnap, N. (2002). EPR-like "funny business" in the theory of branching space-times. In Placek, T. and Butterfield, J., editors, *Non-locality and Modality*, pages 293–315. Dordrecht: Kluwer.

Belnap, N. (2003). No-common-cause EPR-like funny business in branching space-times. *Philosophical Studies*, 114:199–221.

Belnap, N. (2005). A theory of causation: *Causae causantes* (originating causes) as inus conditions in branching space-times. *British Journal for the Philosophy of Science*, 56:221–253.

Conway, J. and Kochen, S. (2006). The free will theorem. *Foundations of Physics*, 36(10):1441–1473.

Deutsch, D. (1991). Quantum mechanics near closed timelike lines. *Physical Review D*, 44(10):3197–3217.

Earman, J. (2008). Pruning some branches from branching space-times. In Dieks, D., editor, *The Ontology of Spacetime II*, pages 187–206. Amsterdam: Elsevier.

Earman, J., Smeek, C., and Wüthrich, C. (2009). Do the laws of physics forbid the operation of time machines? *Synthese*, 169:91–124.

Ghirardi, G., Rimini, A., and Weber, T. (1986). Unified dynamics for microscopic and macroscopic systems. *Physical Review D*, 34(2):470.

Goldblatt, R. (1980). Diodorean modality in Minkowski spacetime. *Studia Logica*, 39:219–236.

Hajicek, P. (1971). Causality in non-Hausdorff space-times. *Communications in Mathematical Physics*, 21:75–84.

Hawking, S. W. and Ellis, G. F. R. (1973). *The large scale structure of space-time*. Cambridge: Cambridge University Press.

Hawking, S. W., King, A. R., and McCarthy, P. J. (1976). A new topology for curved space-time which incorporates the causal, differential, and conformal structures. *Journal of Mathematical Physics*, 17(2):174–181.

Lewis, D. K. (1986). *On the plurality of worlds*. Oxford: Blackwell.

Malament, D. (1977). The class of continuous timelike curves determines the topology of spacetime. *Journal of Mathematical Physics*, 18(7):1399–1404.

McCabe, G. (2005). The topology of branching universes. *Foundations of Physics Letters*, 18(7):665–676.

McWilliams, K. J. (1981). Completeness of the Alexandrov topology for space-time. *International Journal of Theoretical Physics*, 20(12):905–910.

Montague, R. (1962). Deterministic theories. In Willner, D., editor, *Decisions, Values and Groups*, pages 325–370. Oxford: Pergamon Press. Reprinted in *Formal Philosophy*, ed. R.H. Thomason, New Haven, CT: Yale University Press 1974, pp. 303–359.

Müller, T. (2002). Branching space-time, modal logic and the counterfactual conditional. In Placek, T. and Butterfield, J., editors, *Non-locality and Modality*, pages 273–291. Dordrecht: Kluwer.

Müller, T. (2005). Probability theory and causation: a branching space-times analysis. *British Journal for the Philosophy of Science*, 56(3):487–520.

Müller, T., Belnap, N., and Kishida, K. (2008). Funny business in branching space-times: infinite modal correlations. *Synthese*, 164:141–159.

Naber, G. L. (1992). *The Geometry of Minkowski spacetime*. Heidelberg: Springer.

Pearle, P. (1976). Reduction of the state vector by a nonlinear Schrödinger equation. *Physical Review D*, 13(4):857–868.

Penrose, R. (1979). Singularities and time-asymmetry. In Hawking, S. W. and Israel, W., editors, *General relativity: an Einstein centenary survey*, pages 581–638. Cambridge: Cambridge University Press.

Placek, T. and Belnap, N. (2010). Indeterminism is a modal notion: branching spacetimes and Earman's pruning. *Synthese*. DOI: 10.1007/s11229-010-9846-8. Online First published 2010.

Placek, T. and Wroński, L. (2009). On infinite EPR-like correlations. *Synthese*, 167(1):1–32.

Prior, A. N. (1967). *Past, Present and Future*. Oxford: Oxford University Press.

Tholen, W. (2009). Ordered topological structures. *Topology and its Applications*, 156:2148–2157.

Uckelman, S. L. and Uckelman, J. (2007). Modal and temporal logics for abstract space-time structures. *Studies in History and Philosophy of Modern Physics*, 38(3):673–681.

Visser, M. (1996). *Lorentzian Wormholes*. New York, NY: American Institute of Physics.

Visser, M. (2009). Notes on general relativity and cosmology (math 465). Lecture notes, School of Mathematics, Statistics, and Computer Science, Victoria University of Wellington, New Zealand.

Wald, R. M. (1984). *General Relativity*. Chicago, IL: University of Chicago Press.

Wroński, L. and Placek, T. (2009). On Minkowskian branching structures. *Studies in History and Philosophy of Modern Physics*, 40:251–258.

Zeeman (1967). The topology of Minkowski space. *Topology*, 6:161–170.