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Inconsistencies in Classical Electrodynamics?

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Abstract. *In a recent issue of this journal, M. Frisch claims to have proven that classical electro-dynamics is an* inconsistent *physical theory. We argue that he has applied classical electro-dynamics inconsistently. Frisch also claims that all other classical theories of electro-magnetic phenomena, when consistent and in some sense an approximation of classical electro-dynamics, are haunted by "serious conceptual problems" that defy resolution. We argue that this claim is based on a partisan if not misleading presentation of theoretical research in classical electro-dynamics.*

Contents

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1 Introduction

In one of the most provocative papers in the philosophy of science of the last twentyfive years or so, M. Frisch [2004, pp. 525, 530, 541] first and foremost claims to prove that four assumptions of Classical Electro-Dynamics (CED) are contradictory. Like 'naive set-theory' say, CED is nothing short of an *inconsistent theory* (Inconsistency Claim). Since CED forms the theoretical pillar of our electrified society ever since the end of the 19th-century and this pillar apparently is logically collapsing, I was relieved to find out that my light-switch still worked.

Frisch [2004, pp. 525, 538–539] his second claim is that all attempts by physicists to revise CED to a theory that is consistent and has in some sense CED as an approximation, give rise to "conceptual problems" that defy resolution or whose solutions ask a price "too high" to pay "in the eyes of most physicists" (Inadequacy Claim). Frisch [2004, pp. 525, 540 ff.] then goes on to propose "conditions for the acceptability of inconsistent theories". We shall ignore this proposal because it will turn out to be ill-motivated.

Is CED indeed logically flawed and should we honor Frisch for having made this sensational discovery, or is the logic of the proof that has led him to this conclusion flawed? In Section 3, we analyse Frisch his alleged proof and argue that he has applied CED inconsistently. This ought to bring the Inconsistency Claim down. Then in the remainder of this paper, we take a look at three conceptual problems of CED, which according to Frisch defy resolution. We attempt to sketch a faithful picture of how these problems have been handled and are being handled in CED, thereby demonstrating that the account Frisch provides of how these problems are handled in CED is partisan if not downright misleading — an account presumably motivated by gathering support for his Inadequacy Claim. In more detail, in Section 4 we distinguish three kinds of application-problems of CED and two classes of CED-models; in Section 5, we describe the three conceptual problems that arise in CED; in Sections 6 and 7 we sketch, from the bird's eye point of view, the two main research programmes in CED and the various ways in which they handle these problems. This ought to bring Frisch his Inadequacy Claim down. But first of all, in Section 2, we define the theory of CED in order to know exactly what is the object of Frisch's his provocative charges.

2 The Postulates of Classical Electro-dynamics

First of all 'the language of CED'. This language consists of, first, a fraction of English that is sufficient to state and develop the theory and that is unambiguously translatable in other natural languages; secondly, some parts of mathematics and its accompanying symbols (numbers, real analysis, tensor calculus, differential geometry); thirdly, elementary predicate logic as its deductive apparatus; and fourthly, it has the following primitive physical concepts: space, time (or space-time), matter, mass, charge-current, force, electro-magnetic field, electro-magnetic force and medium. The postulates of CED are the following ones.

Space-Time Postulate. *Space-time is mathematically represented by a* 3+1*-dimensional, flat, pseudo-metrical Minkowski-manifold* $\langle \mathcal{M}, \eta \rangle$.

Electro-Magnetic Field Postulate. *The electro-magnetic field is mathematically represented by a once-differentiable anti-symmetric tensor-valued function* F *on* $\mathcal{M}$ *of rank* 2*, the charge-current density by a once-differentiable tensor-valued function* J *on* $\mathcal{M}$ *of rank* 1*, and the medium by* $\langle \mu, \epsilon \rangle$*, where* $\mu \in \mathbb{R}$ *is a magnetic permeability and* $\epsilon \in \mathbb{R}$ *is an electric permittivity, such that they obey, for all* $x \in \mathcal{M}$ *and every component* $\alpha, \beta, \gamma \in \{0, 1, 2, 3\}$*, the homogenous Faraday-Maxwell equations*:

$$\partial_\alpha \mathsf{F}_{\beta\gamma}(x) + \partial_\beta \mathsf{F}_{\gamma\alpha}(x) + \partial_\gamma \mathsf{F}_{\alpha\beta}(x) \ = \ 0 \ , \tag{1}$$

and the inhomogeneous Ampère-Maxwell equations:

$$\partial_\alpha \mathsf{F}^{\alpha\beta}(x) \ = \ \mu \mathsf{J}^\beta(x) \ . \tag{2}$$

Electro-Magnetic Force Postulate. *The electro-magnetic force is mathematically represented by a tensor-valued function* f_L *of rank* 1 *that obeys Lorentz's force-law*:

$$\mathsf{f}^\alpha_\mathrm{L}(\mathsf{q}, \mathsf{u}) \ = \ Q \mathsf{F}^{\alpha\beta}(\mathsf{q}) \mathsf{u}_\beta(\tau) \ , \tag{3}$$

where $Q \in \mathbb{R}$ *is the total charge under consideration,* $\mathsf{q} : \tau \mapsto \mathsf{q}(\tau)$ *is the worldline of (the center of mass of) the charge distribution* $\rho_\mathrm{c}(x) \equiv \mathsf{J}_0(x)/c$*, with affine parameter* τ *being the eigentime, and where* $\mathsf{u}(\tau) \equiv d\mathsf{q}(\tau)/d\tau$ *is the four-velocity of the charge-distribution.*

Dynamical Postulate. *Force is mathematically represented by a tensor of rank* 1*; and the total force* f_tot*, which may depend on* $\mathsf{q} : \tau \mapsto \mathsf{q}(\tau)$ *and any of its derivatives, acting on the charge-matter density equals the change in four-momentum* $\mathsf{p}(\tau) = m(\tau)\mathsf{u}(\tau)$*, where* $m(\tau) \in \mathbb{R}$ *is the total mass of the charge-matter distribution (usually constant:* $m(\tau) = m$ *for all* τ*)*:

$$\frac{d\mathsf{p}(\tau)}{d\tau} \ = \ \mathsf{f}_\mathrm{tot} \ , \tag{4}$$

which we call the Newton-Minkowski Equation.

When the theory of CED is to be defined as a set of structures, or models, rather than as a class of sentences — the elementary deductive closure of the Postulates above of CED —, then CED is the set of all set-theoretical structures of the following type:

$$\left\langle\, \mathcal{M},\ \eta,\ \mu,\ \epsilon,\ m,\ Q,\ \mathsf{F},\ \mathsf{J},\ \mathsf{f}_{\rm tot},\ \mathsf{f}_{\rm L} \,\right\rangle\,, \tag{5}$$

such that they obey the set-theoretical predicate that consists of the conjunction of the Postulates of CED when appropriately translated into the language of set-theory; cf. Suppes [2002, pp. 30–33].

When we choose some inertial frame of reference equipped with a Cartesian co-ordinate system, $\mathbb{R}^3 \times \mathbb{R}$, on space-time $\langle \mathcal{M}, \eta \rangle$, equations (1) and (2) obtain the following familiar look, which Frisch uses and we shall also mostly use:

$$\begin{array}{rclcrcl} \nabla \cdot \mathbf{B}(\mathbf{r},t) & = & 0\,, & \qquad & \nabla \times \mathbf{E}(\mathbf{r},t) + \partial_t \mathbf{B}(\mathbf{r},t) & = & \mathbf{0}\,, \\ \epsilon \nabla \cdot \mathbf{E}(\mathbf{r},t) & = & \rho_{\rm c}(\mathbf{r},t)\,, & & \nabla \times \mathbf{B}(\mathbf{r},t) - \epsilon\mu\, \partial_t \mathbf{E}(\mathbf{r},t) & = & \mu \mathbf{J}(\mathbf{r},t)\,, \end{array} \tag{6}$$

where $\mathbf{E}(\mathbf{r},t) \in \mathbb{R}^3$ is the electric field, $\mathbf{B}(\mathbf{r},t) \in \mathbb{R}^3$ the magnetic induction field, $\rho(\mathbf{r},t) \in \mathbb{R}$ the electric charge-density and $\mathbf{J}(\mathbf{r},t) \in \mathbb{R}^3$ the associated current-density, all at space-time point having co-ordinates $(\mathbf{r},t) \in \mathbb{R}^3 \times \mathbb{R}$. The speed of propagation of the electro-magnetic fields in the medium is equal to $(\epsilon\mu)^{-1/2}$; this speed is *in vacuo*, which is medium $\langle \mu_0, \epsilon_0 \rangle$, the speed of light: $c = (\epsilon_0\mu_0)^{-1/2}$. The relations between the tensor fields in equations (1) and (2) and the 3-vector fields in eqs. (6) are, for every $i, k, l \in \{1,2,3\}$:

$$\mathbf{E}_k(\mathbf{r},t) \;=\; -c\mathsf{F}^{0k}(x) \quad \text{and} \quad \mathbf{B}_k(\mathbf{r},t) \;=\; -\tfrac{1}{2}\varepsilon^{kil}\mathsf{F}^{il}(x)\,, \tag{7}$$

where ε^{kil} is the anti-symmetric Lévy-Civita tensor of rank 3, and for the charge- and current-density:

$$c\rho_{\rm c}(\mathbf{r},t) \;=\; \mathsf{J}^0(x) \quad \text{and} \quad c\mathbf{J}_k(\mathbf{r},t) \;=\; \mathsf{J}^0(x)u_k(\tau)\,, \tag{8}$$

where $t = \gamma(u)\tau$, $\gamma(u) \equiv (1 - |\mathbf{u}|/c)^{-1/2}$, and $\mathbf{u}(t) \equiv \dot{\mathbf{q}}(t)$ is the particle's 3-velocity. The integral of the charge-matter density $\rho(\mathbf{r},t)$ over $\mathbb{R}^3$ is required to be equal to 1 for all $t \in \mathbb{R}$, so that the charge-density is $\rho_{\rm c} = Q\rho$ and the matter-density $\rho_{\rm m} = m\rho$.

Frisch considers a point-particle carrying electric charge $Q = e$. Say it has worldline $\mathbf{q} : t \mapsto \mathbf{q}(t)$. Then the charge-matter density at $(\mathbf{r},t)$ is Dirac's delta-functional $\delta(\mathbf{r} - \mathbf{q}(t))$. The charge- and current-density become functionals too:

$$\rho_{\rm c}(\mathbf{r},t) \;=\; e\,\delta\big(\mathbf{r} - \mathbf{q}(t)\big) \quad \text{and} \quad \mathbf{J}(\mathbf{r},t) \;=\; \rho_{\rm c}(\mathbf{r},t)\mathbf{u}(t) \;=\; e\,\delta\big(\mathbf{r} - \mathbf{q}(t)\big)\mathbf{u}(t)\,. \tag{9}$$

The electro-magnetic force 3-vector $\mathbf{F}_{\rm L}$ on the point-charge in an electro-magnetic field $\langle \mathbf{E}, \mathbf{B} \rangle$ is governed by Lorentz's force-law (3):

$$\mathbf{F}_{\rm L}\big(\mathbf{q}(t), \mathbf{u}(t)\big) \;=\; e\mathbf{E}\big(\mathbf{q}(t), t\big) + e\frac{\mathbf{u}(t)}{c} \times \mathbf{B}\big(\mathbf{q}(t), t\big)\,. \tag{10}$$

At this point an announcement is in order. The fact we are dealing with a point-particle is *not* the source of the contradiction that Frisch claims to deduce; his deduction would also have worked for extended charge-densities. In this light it is puzzling why Frisch did not *ab initio* present the argument for an extended charge-density, because it would have saved him the trouble of writing extensively about the point-particles. Of course, point-particles do give rise to a problem (cf. Section 5). But the issue of point-particles is not germane to Frisch his proof. We move on.

Frisch considers Newton's law of motion "in the absence of non-electro-magnetic forces" for a point-particle; in this case $\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{L}}$:

$$\dot{\mathbf{p}}_{\text{nr}}(t) \;=\; m\mathbf{a}(t) \;=\; \mathbf{F}_{\text{L}}\big(\mathbf{q}(t), \mathbf{u}(t)\big) \;, \tag{11}$$

where $\mathbf{p}_{\text{nr}}(t) = m\mathbf{u}(t)$ the particle's non-relativistic linear momentum and $\mathbf{a}(t) \equiv \dot{\mathbf{u}}(t)$ the particle's acceleration 3-vector. Newton's *non-relativistic* equation of motion in *relativistic* Minkowski space-time? Frisch [2004, p. 528] clarifies: "Hence, when I speak of Newton's laws in this paper, I intend this to include their relativistic generalizations". Which is the Newton-Minkowski equation (4); *its* 3-vector part in the situation under consideration is:

$$\dot{\mathbf{p}}(t) \;=\; \gamma(u)m\mathbf{a}(t) \;=\; \mathbf{F}_{\text{L}}\big(\mathbf{q}(t), \mathbf{u}(t)\big) \;, \tag{12}$$

where $\mathbf{p} = \gamma(u)m\mathbf{u}$ is the particle's relativistic momentum 3-vector part of the 4-vector p.

3 Analysis of Frisch's Inconsistency-Proof

The four assumptions of CED that Frisch [2004, pp 525, 530] brandishes "internally inconsistent" are the following ones.

(i) There are discrete, finitely charged accelerating particles.

(ii) Charged particles function as sources of electro-magnetic fields in acccord with the Maxwell-equations (6).

(iii) Charged particles obey Newton's law of motion (11).

(iv) Energy is conserved in particle-field interactions.

How do these four assumptions (i)–(iv) relate to the postulates of CED?

First, if assumption (i) is supposed to express that there are CED-models in the domain of theoretical discourse of CED — which is inside that of set-theory —, having a point-particle charge- and current density (9), then assumption (i) is a provable mathematical theorem, and *not* an assumption of CED. If assumption (i) is supposed to be an existential statement about what there is in physical reality, then (i) is not part of CED and then (i) expresses a realist attitude towards CED at most. Assumption (i) can then be rejected if the conjunction of CED and (i) leads to a contradiction. This sounds a bit cheap, however. Let us therefore adopt the afore-mentioned interpretation of (i), which renders assumption (i) harmless.

Secondly, assumption (ii) is the Electro-Magnetic Field Postulate and (iii) is a consequence of the Dynamical Postulate and the Electro-Magnetic Force Postulate under the assumption that the Lorentz-force is the only force acting on the point-particle ($\mathsf{f}_{\rm tot} = \mathsf{f}_{\rm L}$). The Space-Time Postulate is not mentioned by Frisch but we shall take it to be tacitly assumed.

Thirdly, there is no mentioning of *energy* in the postulates of CED, which raises the question how (iv) can be *an assumption of* CED, as Frisch claims. Standard is to prove conservation of energy and momentum on the basis of the symmetries of the dynamical equations of the theory under consideration. In order to get such a proof going in CED, one needs a definition of the energy-momentum tensor of a combined system of particles and fields. The history of CED teaches us that it can be accomplished, although this is far from a straightforward affair as *prima facie* may seem; cf. Rohrlich [1970], Landau & Lifshitz [1975, pp. 77–80], Schwinger [1983]. If one chooses for a Hamiltonian or for a Lagrangean approach to CED, energy-conservation is automatically guaranteed as a consequence of the time-translation symmetry — an instance of Noether's Theorem. Hence it stands beyond disputation that (iv) is better seen as a *consequence* of the postulates of CED rather than as an *assumption* of CED.

To summarise, we are prepared to accept that the conjunction of CED and the statement '$\mathsf{f}_{\rm tot} = \mathsf{f}_{\rm L}$' is inconsistent if assumptions (i)–(iv) are inconsistent. With this in position, we move on to Frisch's argument.

Frisch [2004, p. 530] begins his proof by asserting that the following expression "follows from the Maxwell-equations in conjunction with the standard way of defining the energy associated ..." He comes up with the *instantaneous* power of the electro-magnetic field emitted by a moving charged particle at $\mathbf{q}(t)$ having acceleration $\mathbf{a}(t)$:

$$P_{\rm Lr}(t) \;=\; \frac{2e^2}{3c^3}|\mathbf{a}(t)|^2 \;> 0\,. \tag{13}$$

What Frisch says is not quite true.

Formula (13) is the so-called Larmor formula; it is the result of a non-relativistic approximation in an adiabatic limit (slowly varying fields), using the Sommerfeld radiation boundary-condition. The formula that does follow from the Maxwell-equations, in the adiabatic limit, by integrating the Poynting-vector $\mathbf{S}(\mathbf{r},t) \equiv (\mathbf{E}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t))/\mu_0$ of the electro-magnetic field (representing the energy-flux of it) over the surface of a retarded sphere surrounding the charge ("the standard way of defining the energy ..."), is Liénard's formula (Jackson [1975, p. 660]):

$$P_{\rm Ln}(t) \;=\; \frac{2e^2}{3c^3}\,\gamma^6\big(|\mathbf{u}(t)|\big)\,\Big(|\mathbf{a}(t)|^2 - c^{-2}|\mathbf{u}(t)\times\mathbf{a}(t)|^2\Big)\; > 0\;. \tag{14}$$

So we have discovered two other tacit assumptions which do not follow from the postulates of CED:

(v) The electro-magnetic fields vary slowly (adiabatic limit).

(vi) Sommerfeld radiation boundary-condition.

Since this adiabatic limit is not important for the purpose of the current Section, and we gladly accept that sources are sources and not sinks of radiation energy, we accept (v) and (vi), and we accept that the conjunction of CED and $\mathsf{f}_{\rm tot} = \mathsf{f}_{\rm L}$ is inconsistent if assumptions (i)–(vi) are inconsistent.

The energy radiated by the moving charge, via its emitted electro-magnetic field, often called the *self-field*, $\mathbf{E}_{\rm self}$ and $\mathbf{B}_{\rm self}$, during time-interval $[t_A, t_B]$, then is positive:

$$E_{\rm rad}(A,B) \;=\; \int_{t_A}^{t_B} P(t)dt \;>\; 0\;, \tag{15}$$

irrespective of whether we take Larmor's formula (13) or Liénard's formula (14) as the integrand $P(t)$ of integral (15). Newton's law of motion (11) and "the definition of external work done one a charge imply that the work on the charge is equal to the change in energy of the charge" [*ibid.*] (iii):

$$W_{\rm ext}(A,B) \;=\; \int_A^B \mathbf{F}_{\rm ext}(\mathbf{r},t)\cdot d\mathbf{l} \;=\; E_{\rm kin}(B) - E_{\rm kin}(A)\;, \tag{16}$$

where in this case $\mathbf{F}_{\rm ext} = \mathbf{F}_{\rm L}$ and the line-integral is taken along the worldline $\mathbf{q}(t)$ of the charge from point A to point B on it. "But for energy to be conserved, that is for assumption (vi) to hold, the energy of the charge at t_B should be less by the amount of the energy radiated $E_{\rm rad}(A,B)$ than the sum of the energy at t_A and the work done on the charge" [*ibid.*]:

$$E_{\rm kin}(B) \;=\; E_{\rm kin}(A) + W_{\rm ext}(A,B) - E_{\rm rad}(A,B)\;. \tag{17}$$

From equalities (16) and (17) it follows that

$$E_{\rm rad}(A,B) = 0\;, \tag{18}$$

in contradiction to inequality (15). *Quod erat demonstrandum?*

Yes, but demonstrated it is not. Here begins our criticism of Frisch's Inconsistency Claim.

When Frisch [2004, p. 528] asserts that "in the absence of non-electromagnetic forces" equation (11) is Newton's law of motion, he is simply in error. Newton's law of motion says that the change in momentum of a physical object having constant mass m equals the *total force* $\mathbf{F}_{\text{tot}}$ exerted on the physical object (Goldstein [1980, p. 1], Suppes [2002, pp. 319–321]):

$$\dot{\mathbf{p}}(t) \;=\; \mathbf{F}_{\text{tot}} \; . \tag{19}$$

The total force $\mathbf{F}_{\text{tot}}$ is usually broken up in two summands: the resulting *external* and the resulting *internal* force acting on the physical object under consideration (Goldstein [1980, p. 5], Suppes [2002, pp. 319–321]):

$$\mathbf{F}_{\text{tot}} \;=\; \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{int}} \; . \tag{20}$$

In the situation under consideration, the emitted self-field $\langle \mathbf{E}_{\text{self}}, \mathbf{B}_{\text{self}} \rangle$ of the charge — earlier assumed by Frisch not to vanish, in order to derive that $E_{\text{rad}}(A, B) > 0$ (15) — exerts an additional, internal force on the charge, the *self-force* ($\mathbf{F}_{\text{int}} = \mathbf{F}_{\text{self}}$), and the external force is the Lorentz-force ($\mathbf{F}_{\text{ext}} = \mathbf{F}_{\text{L}}$); then Newton's law (19) becomes:

$$\dot{\mathbf{p}}(t) \;=\; \mathbf{F}_{\text{L}} + \mathbf{F}_{\text{self}} \; . \tag{21}$$

In other words, "in the absence of non-electromagnetic forces" we have a total electromagnetic force $\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{L}} + \mathbf{F}_{\text{self}}$.

Consequently the total work $W_{\text{tot}}(A, B)$ done on the charge results from this total force:

$$W_{\text{tot}}(A, B) \;=\; \int_A^B \mathbf{F}_{\text{tot}}(\mathbf{r}, t) \cdot d\mathbf{l} \;=\; E_{\text{kin}}(B) - E_{\text{kin}}(A) \; . \tag{22}$$

We can generally subdivide W_{tot} in an amount W_{ext}, performed by the external force $\mathbf{F}_{\text{ext}}$ (which is here the Lorentz-force $\mathbf{F}_{\text{L}}$), and an amount W_{self}, performed by the self-force $\mathbf{F}_{\text{self}}$; in our case:

$$W_{\text{tot}}(A, B) \;=\; W_{\text{ext}}(A, B) + W_{\text{self}}(A, B) \; . \tag{23}$$

From equations (17), (22) and (23), we deduce that

$$E_{\text{rad}}(A, B) \;=\; -W_{\text{self}}(A, B) \; . \tag{24}$$

No inconsistency follows from equation (24) and inequality (15). On the contrary, they are perfectly consistent.

What Frisch takes into account at the beginning, which is the radiated electro-magnetic field by the moving charge — $\mathbf{E}_{\text{self}} \neq \mathbf{0} \neq \mathbf{B}_{\text{self}}$, so that $\mathbf{F}_{\text{self}} \neq \mathbf{0}$, from which it follows that along $\mathbf{q}$ from A to B, $W_{\text{self}}(A, B) \neq 0$, so that $E_{\text{rad}}(A, B) \neq 0$ —, he ignores in the next step — $\mathbf{E}_{\text{self}} = \mathbf{0} = \mathbf{B}_{\text{self}}$, so that $W_{\text{tot}} = W_{\text{ext}}$, from which it follows that $\mathbf{F}_{\text{self}} = \mathbf{0}$ and therefore $W_{\text{self}}(A, B) = E_{\text{rad}}(A, B) = 0$. In this fashion one can prove every (set of assumptions extracted from a) theory to be inconsistent: take something into account in the beginning and ignore it in the next step. This is a recipe for logical disaster. What we are dealing with here is an inconsistent application of a theory, not a proof of the inconsistency of the applied theory.

Once again, the logical structure of Frisch's argument consists in correctly arguing, on the basis of CED, for the following two conditional statements:

$$\begin{array}{ll} \text{(Step 1)} \quad \text{if} \;\; \mathbf{E}_{\text{self}} \neq \mathbf{0} \neq \mathbf{B}_{\text{self}}, \quad \text{then} \;\; E_{\text{rad}}(A, B) \neq 0 \;\; (15); \\ \text{and} \\ \text{(Step 2)} \quad \text{if} \;\; \mathbf{E}_{\text{self}} = \mathbf{0} = \mathbf{B}_{\text{self}}, \quad \text{then} \;\; E_{\text{self}}(A, B) = 0 \;\; (18). \end{array} \tag{25}$$

Then he tacitly assumes both antecedents of Step 1 and Step 2, deduces by *modus ponendo ponens* both consequents, which amounts to deducing a contradiction. But, then, one is inclined to remark, if both contradictory antecedents are assumed, we don't need the arguments establishing Step 1 and Step 2 (25) anymore, because then we already have a contradiction by $\wedge$-introduction! The flaw in concluding the inconsistency of CED does not lie in CED but in assuming two contradictory assumptions: the antecedents of Step 1 and Step 2 (25). No matter in the context of which scientific theory one follows this procedure, one is bound to end up in contradictions. This surely is the easiest recipe for logical disaster: make two assumptions one of which is the negation of the other.

Here ends our criticism of Frisch's Inconsistency Claim. We conclude that Frisch has not proved that assumptions (i)–(iv) — or more precisely (i)–(vi) — are inconsistent, in contradiction to what he has claimed.

A final remark. What does rigorously follow from the postulates of CED is the antecedent of Step 1 (25). But as we shall see in the next Section, in most descriptions or explanations of electro-magnetic phenomena in the domain of CED, the self-fields are ignored, i.e. the antecedent of Step 2 is taken aboard as an approximation or idealisation. Specifically, since only in some ultra-relativistic situations $\mathbf{F}_{\text{self}}$ is of an order of magnitude comparable to that of $\mathbf{F}_{\text{L}}$ and in all other situations $\mathbf{F}_{\text{self}}$ is much smaller (for the synchrotron, cf. Shen [1978], Lieu [1987]), $\mathbf{F}_{\text{self}}$ can be safely neglected when it comes to solving the equation of motion (21): $\mathbf{F}_{\text{self}} \approx \mathbf{0}$. Strictly speaking, one should then write '$W_{\text{tot}} \approx W_{\text{ext}}$', rather than '$W_{\text{tot}} = W_{\text{ext}}$', so that it is immediately clear that no contradiction ensues. In fact, physicist are notoriously sloppy in this respect: a majority of the exact equality signs (=) in most physics papers, articles and books means approximate equality (≈). Specifically, in Frisch his argument, '=' should be replaced with '≈' in formulae (13) — because derived from $\mathbf{F}_{\text{self}} \not\approx \mathbf{0}$ —, and in formulae (14) , (16) and (18) — because derived from $\mathbf{F}_{\text{self}} \approx \mathbf{0}$.

4 Application-Problems

Frisch's argument [2004, p. 529] originates in the classical-electrodynamic description of the behaviour of charged particles in a synchroton accelerator, which he sees as an illustration of the inconsistent descriptions that CED generates. The problem of how to describe the entire behaviour of this physical system correctly does not fall in either one of two classes of application-problems that CED handles (*application-problem* here being to find a description or explanation of any given phenomenon that falls in the domain of CED by means of some CED-model):

A-Problems. The charge-densities are specified (the worldlines of their centers of mass in space-time, and hence their current-densities): the electro-magnetic fields are calculated by solving the Maxwell-equations (6) — mathematically the most general problem is to solve a system of twelve coupled 1st-order partial differential equations.

B-Problems. The electro-magnetic fields are specified (and hence the Lorentz-forces (10) acting on the charge densities): the worldlines of the (centers of mass of the) charge- and current-densities are calculated by solving the Newton-Minkowski equation of motion (4) — mathematically the most general problem is to solve a system of three at least 2nd-order partial differential equations.

In A-Problems, the charges are seen as the sources of electro-magnetic fields, carrying energy and momentum, but their specified worldlines are not corrected for by the self-force that the emitted fields exercise on the charges, whereas in B-Problems the specified electro-magnetic fields are not corrected for the fields emitted by the moving charges. Although it is a blunt fact that every single one of the overwhelming majority of electro-magnetic phenomena can be treated as an A- or a B-Problem "with negligible error", as Jackson puts it in his tome [1975, p. 781], both remain approximations in that the self-force is neglected ($\mathbf{F}_{\text{self}} \approx 0$). In a completely theoretically satisfactory treatment of these phenomena, self-effects should be taken into account. Jackson writes [*ibid.*] that "a completely satisfactory treatment of the reaction effects of radiation does not exist" and provides a two-fold explanation of this state of affairs.

First, self-effects are not needed to describe or explain the phenomena *inside* the domain of CED: with negligible error every problem solved by CED can be classified as being either an A-Problem or a B-Problem.

Secondly, the microscopic behaviour of point-like charges, such as electrons, lies altogether *outside* the domain of CED; it belongs to the realm of quantum physics, where classical physics generally breaks down anyway.

Nevertheless Jackson then goes on to discuss quite a few attempts to deal with self-effects, some of which are as old as CED itself. Let us state the problem in full generality.

Suppose an external electro-magnetic field $\langle \mathbf{E}_{\rm ext}, \mathbf{B}_{\rm ext} \rangle$ and some charge-density $\rho_{\rm c}$ with associated current-density $\mathbf{J}$ are given. These fields exert an *external* force $\mathbf{F}_{\rm L}$ on $\rho_{\rm c}$ via Lorentz's force-law (10). The moving charge-density and current emit an electro-magnetic self-field $\langle \mathbf{E}_{\rm self}, \mathbf{B}_{\rm self} \rangle$, which is a solution of the Maxwell-equations (6) — provided of course that the worldline $\mathbf{q}$ of $\rho_{\rm c}$ does not belong to the class of *radiationless motions* for point-charges, cf. Pearle [1977; 1978]. These self-fields exert an *internal* force on the charge, the self-force $\mathbf{F}_{\rm self}$, again via Lorentz's force-law (10). The motion of the center-of-mass of the charge, having worldline $\mathbf{q}(t)$ and linear momentum $\mathbf{p}(t)$, is the solution of the following equation of motion (19):

$$\dot{\mathbf{p}}(t) \;=\; \mathbf{F}_{\rm L} + \mathbf{F}_{\rm self} \,. \tag{26}$$

These are

> **C-Problems**. Solve the system of fifteen coupled partial differential-equations (6) and (26) for $\mathbf{q}$, $\mathbf{E} = \mathbf{E}_{\rm ext} + \mathbf{E}_{\rm self}$ and $\mathbf{B} = \mathbf{B}_{\rm ext} + \mathbf{B}_{\rm self}$, using Lorentz's force-law (10).

Since this system of coupled differential equations rarely admits a solution 'in closed form', approximations and idealisations are mandatory to get anywhere. There is a variety of ways of *how* to approximate *what* and *how* to idealise *what*. Detailed models are called for. Small wonder there is not a *single* account of self-effects available but there is a multitude of accounts, each of which relying on different approximations and different idealisations — *and this is not quite the same thing as there being no account at all.*

Let $\mathcal{AB}$ be the call the class of CED-models that solve A- or B-Problems, and $\mathcal{C}$ the class of the ones that solve C-Problems. Then models in $\mathcal{AB}$ neglect self-effects whereas models in $\mathcal{C}$ take them into account. Jackson [1975, pp. 781–782] provides conditions in terms of typical energy-values and times of the physical system under consideration in order to decide whether the problem is A, B or C, hence whether the model is in $\mathcal{AB}$ or $\mathcal{C}$. When we take CED to be a class of models (5), then $\mathcal{AB}$ and $\mathcal{C}$ subdivide it, *albeit* vaguely.

The CED-description of the synchroton accelerator, discussed by Frisch [2004, p. 529], is atypical in that the problem to provide a description or explanation of what happens in this device is divided in two separate problems: an A-Problem and a B-Problem. B-Problem: the circular orbit of the accelerated electron is obtained from a specified homogeneous magnetic induction field by solving the equation of motion, neglecting $\mathbf{F}_{\mathrm{self}}$ because it is estimated to be about ten orders of magnitude smaller than the external Lorentz-force $\mathbf{F}_{\mathrm{L}}$; cf. Lieu [1987]. A-Problem: the emitted radiation of the electron is studied by considering its specified circular trajectory and solving the inhomogeneous Maxwell-equations from the Liénard-Wiechert potentials, in order to determine the nature of the emitted 'synchrotron radiation'. In the synchrotron accelerator one compensates for the loss of the energy of the electron by making it periodically pass an electric potential-difference, which changes polarity twice during one revolution. Hence the fact that the accelerated electrons function as sources of radiation is taken fully into account in the design and operation of the synchrotron, because their loss of kinetic energy is appreciable and is compensated for, but $\mathbf{F}_{\mathrm{self}}$ resulting from these emitted fields is ignored because it is far too small in comparison to the external Lorentz-force to have any appreciable effect on the circular orbit of the charges. No relevant C-Problem is considered.

In summary, two application-problems of a different kind (A and B) yet both pertaining to a single situation, i.e. an electron in a synchrotron accelerator, and both making the same assumption ($\mathbf{F} \approx \mathbf{0}$), are solved by CED. This method of making different approximations dependent on which quantitative problem pertaining to a single phenomenon one attempts to solve is generic in physics in particular and in science in general, we submit. If this generic way of doing things in science were sufficient to pronounce the applied theory *inconsistent*, then we ridiculously would have to pronounce every single scientific theory inconsistent.

Frisch admittedly mentions one of the attempts to solve a C-Problem *as a possible way out of his own alleged inconsistency proof*, but rejects this attempt because it suffers from conceptual problems [2004, p. 538], about which more in Section 5. But this means that, on Frisch's own terms, the conclusion that CED is inconsistent already is a *non sequitur*, because what he apparently has established is a Dilemma:

$$\textbf{Inconsistency} \;\vee\; \textbf{Conceptual Problems}\,. \tag{27}$$

Faced with Dilemma (27), we are free to choose what seems to be the horn of lesser evil, namely the right horne of Conceptual Problems. Logic does not compel us to choose for the left horn of Inconsistency (27). On the contrary, logic compels us to choose for Conceptual Problems (27), due to the following elimination-rule for absurdum ($\perp$):

$$\perp \vee \varphi \;\vdash\; \varphi\,, \tag{28}$$

where φ is a sentence-variable. Hence, even if Frisch had proved that the six assumptions (i)–(vi) lead to a contradiction, his claim to have *proved* the inconsistency of CED would still be a *non sequitur*. Rescued from contradictions we could then pay all our attention to the Conceptual Problems, which is what we actually do next.

5 Conceptual Problems

When Frisch [2004, pp. 537–540] draws his "serious conceptual problems" into the limelight, he is reporting problems of CED-models of charges that are extremely well-known in physics.

I. The electro-magnetic potential energy of a point-charge diverges.

II. The occurrence of *pre-accelerations*: a charge accelerating *before* the force begins to act on it.

III. The occurrence of *self-accelerations*: a charge accelerating further *after* the force has stopped acting on it.

These conceptual problems are related to the ABC-classification of application-problems (Section 4) as follows: problem I arises rigorously whenever one uses point-particles, no matter which kind of problem one is trying to solve (A, B or C). Problems II and III arise only in the process of solving some C-Problem by means of some approximation.

The time-scales involved in pre-acceleration effects are extremely small and therefore these effects are generally not seen as a very serious problem (II). Problems I and III are seen as more pressing. Notice that a point-particle that keeps accelerating will keep emitting radiation-energy too, which makes it effectively a source of inifinite energy. Since it has an infinite amount of energy according to problem I, there is consistency even among the infinities! In the current Section we briefly expand on these problems and then give a glimpse of how physicists have handled them and are handling them *wholly within* CED.

Let us begin with problem I. Every physicist at the end of the 19th-century knew that the electrostatic energy of a point-charge diverges. Let us see how this problem arises rigorously from the postulates of CED.

We consider a point-charge *in vacuo* having electric charge e and worldline $\mathbf{q} : t \mapsto \mathbf{q}(t)$ in the Cartesian co-ordinate system of some reference frame on Minkowski space-time. Define the *retarded time* t_{ret} as the time at which the worldline $\mathbf{q}$ crosses the backward light-cone with apex at $(\mathbf{r}, t)$:

$$ct_{\text{ret}} \;=\; ct - |\mathbf{r} - \mathbf{q}(t_{\text{ret}})| \;. \tag{29}$$

Introducing the unit-vector $\mathbf{n} \in \mathbb{R}^3$ such that

$$|\mathbf{r} - \mathbf{q}(t_{\rm ret})|\,\mathbf{n} \;=\; \mathbf{r} - \mathbf{q}(t_{\rm ret}) \;, \tag{30}$$

permits us to write for the retarded electric Lénard-Wiechert field of the moving point-charge (Spohn [2004, p. 12]):

$$\begin{aligned}\mathbf{E}_0(\mathbf{r}, t) \;=\; & \frac{e}{4\pi\epsilon_0}\,\frac{\delta(t - t_{\rm ret})}{(1 - \mathbf{u}(t)\cdot\mathbf{n}/c)^3}\left[\frac{(1 - |\mathbf{u}(t)|^2/c^2)(\mathbf{n} - \mathbf{u}(t)/c)}{|\mathbf{r} - \mathbf{q}(t)|^2}\right. \\ & \left. + \;\frac{\mathbf{n} \times \big((\mathbf{n} - \mathbf{u}(t)/c) \times \mathbf{a}(t)/c\big)}{|\mathbf{r} - \mathbf{q}(t)|}\right] \;. \end{aligned}\tag{31}$$

The first term is the 'near field': it falls off with the square of the distance of the source and always remains with the particle. The second term is the 'far field': it falls off with the distance and is proportional to the acceleration of the point-charge and therefore vanishes when the particle moves uniformly; it dominates at large distances. The field $\mathbf{E}_0$ diverges at $\mathbf{r} = \mathbf{q}(t)$ as $|\mathbf{r} - \mathbf{q}(t)|^{-2}$. The associated electrostatic potential $\phi_0 : \mathbb{R}^3 \to \mathbb{R}$ can be calculated from: $\mathbf{E}_0(\mathbf{r}, t) = -\nabla\phi_0(\mathbf{r})$. The accompanying magnetic induction field $\mathbf{B}_0(\mathbf{r}, t)$ is determined by: $c\mathbf{B}_0(\mathbf{r}, t) = \mathbf{n} \times \mathbf{E}_0(\mathbf{r}, t)$, which is the relativistic generalisation of the magnetic law of Biot & Savart. When the point-charge is at rest, so that the velocity and the acceleration vanish ($\mathbf{u}(t) = \mathbf{0}$ and $\mathbf{a}(t) = \mathbf{0}$), the field reduces to the retarded Coulomb-field $\mathbf{E}_0^{\rm rest}$; it drops out of the 'near field' in (31):

$$\mathbf{E}_0^{\rm rest}(\mathbf{r}, t) \;=\; \frac{e}{4\pi\epsilon_0}\,\frac{\delta(t - t_{\rm ret})\mathbf{n}}{|\mathbf{r} - \mathbf{q}(t)|^2} \qquad \text{and} \qquad \mathbf{B}_0^{\rm rest}(\mathbf{r}, t) = \mathbf{0} \;. \tag{32}$$

The energy-density $u_0 : \mathbb{R}^4 \to \mathbb{R}$ of this electro-magnetic field yields:

$$\begin{aligned} u_0(\mathbf{r}, t) \;&=\; \tfrac{1}{2}\delta(t - t_{\rm ret})\left(\mu_0^{-1}|\mathbf{B}_0^{\rm rest}(\mathbf{r}, t)|^2 + \epsilon_0|\mathbf{E}_0^{\rm rest}(\mathbf{r}, t)|^2\right) \\ &=\; \frac{e^2}{32\pi^2\epsilon_0}\,\frac{\delta(t - t_{\rm ret})}{|\mathbf{r} - \mathbf{q}(t)|^4} \;. \end{aligned}\tag{33}$$

In order to obtain the electrostatic energy of a point-charge at rest at the centre of a ball of radius $R = |\mathbf{r} - \mathbf{q}(t_{\rm ret})|$, one integrates u_0 (33) over it. Going to spherical co-ordinates, $(r, \Omega) \in \mathbb{R}^+ \times [0, 4\pi)$, one obtains for the total electrostatic potential energy for a point-charge at rest:

$$\begin{aligned} U_{\rm pot}(\mathbf{E}_0^{\rm rest}) \;&=\; \lim_{R\uparrow\infty}\lim_{d\downarrow 0}\int_0^{4\pi} d\Omega \int_d^R u_0(r, \Omega)\, r^2 dr \\ &=\; -\frac{e^2}{8\pi\epsilon_0}\left(\lim_{R\uparrow\infty}\frac{1}{R} \;-\; \lim_{d\downarrow 0}\frac{1}{d}\right) \quad \rightsquigarrow \quad +\infty \;. \end{aligned}\tag{34}$$

Notice that when we integrate over all of space save some minute hole of radius $d > 0$ around the point-charge, *no matter how small*, the integral does *not* diverge: it is the contribution of the near field when we take the point-particle limit ($d \downarrow 0$) that blows $U_{\text{pot}}(\mathbf{E}_0^{\text{rest}})$ up. (Something similar happens for the potential energy of Newton's gravity-potential; cf. Rohrlich [1999].)

Mathematically speaking, U_{pot} is a functional, sending every electro-magnetic field $\langle \mathbf{E}, \mathbf{B} \rangle$ to a real number, called its *total potential energy*, $U_{\text{pot}}(\mathbf{E}, \mathbf{B})$. The functional here only depends on the electric field and takes the familiar form of a definite integral (34). When this integral diverges, the conclusion is that $\mathbf{E}_0^{\text{rest}}$ (31) *does not belong to the domain of* U_{pot}, which is some subset of the class of all integrable (and differentiable) functions from $\mathbb{R}^3$ to $\mathbb{R}$. Any conclusion from (34) to the effect that CED is inconsistent would be illicit, just as illicit as when we would conclude the inconsistency of Real Analysis from applying function $x \mapsto 1/x$ to 0 (one is forbidden to apply it to 0 because 0 is not in the domain of $x \mapsto 1/x$). Some physicists have made this illicit move. One example is Feynman [1964, pp. 28–1], who says in this context the following about point-particles:

> The concept of simple charged particles and the electromagnetic field are in some way inconsistent.

Another example forms the celebrated textbook duo Landau & Lifshitz [1975, p. 90], who state more precisely in this context:

> Since the occurrence of the physically meaningless infinite self-energy of the elementary particle is related to the fact that such a particle must be considered as point-like, we can conclude that electrodynamics as a logically closed physical theory presents internal contradictions when we go to sufficiently small distances.

For the sake of emphasis, Feynman and Landau & Lifshitz — illicitly — speak about 'inconsistent' and 'contradictions', respectively, because of the diverging potential energy of a point-particle — this has nothing to do with Frisch his alleged inconsistency-proof.

Physically speaking, if U_{pot} of a point-charge at rest diverges (34), then so does its inertia according to Einstein's universal mass-energy relation $E = mc^2$; this means that a point-charge would have infinite inertia and therefore would never respond to forces acting on it. Point-charges are immovable objects. Since electrons — the very first elementary particles, introduced by J.J. Thompson and H.A. Lorentz, the founding fathers of elementary particle physics — do move and can be accelerated, the proper conclusion to draw is that *according to* CED *electrons are not point-particles*.

Furthermore, when we want to describe the joint evolution of a point-charge and the electro-magnetic fields, the mathematics breaks down because the solution (32) of the

Maxwell-equations is singular at precisely the points where we need to know the Lorentz-force: on the worldline $t \mapsto \mathbf{q}(t)$ of the charge.

The proper general conclusion to draw is that *point-particles fall outside the domain of* CED: the models of point-particles turn out to be *not* models of CED, appearances to the contrary notwithstanding. Hence there is a serious problem for constructing models of (presumably non-existent) point-particles: the argument leading to the diverging potential energy (34) literally is a *reductio ad infinitum* of the statement that CED does contain models of point-particles. Surprisingly, CED does not contain these models. To repeat, this is not to say that CED is an inconsistent theory.

The natural move is to construct models of charged particles having a spatially extended charge-distribution — remember what we remarked just below formula (34). Another move is to change the definition of U_{pot}; an example is the 'Bopp-integral', expounded by Feynman [1964, § 28–5]. An altogether different move is to insist that models of point-particles fit within CED, by pointing out an unwarranted tacit assumption in the argument above that has led us to the conclusion that point-particles fall outside the domain of CED, and then to begin afresh without that assumption. These two moves correspond to a broad classification of theoretical research in CED in two research programmes, which we call the *Extension Programme* and the *Renormalisation Programme*, respectively — where it is to be remarked that syntheses of both programmes are around too.

So much for problem I. Problems II and III are generated by approximate solutions of C-Problems in both research programmes. To these programmes we turn next.

6 The Renormalisation Programme

Historically the *Extension Programme* started around the turn of the 20th-century, with Max Abraham's semi-relativistic and Lorentz's fully relativistic model of the electron by a spherically symmetric extended charge distribution; cf. Rohrlich [1965, pp. 8–25], Spohn [2004, pp. 33–36]. After the rise of quantum physics, these classical models threatened to fade away into oblivion; but the threat never really materialised and, in fact, recently these models have re-entered center stage of theoretical research in CED. The other programme in CED is the *Renormalisation Programme*; it was initiated in 1938 by Dirac, who insisted on working with point-particles. We outline the Renormalisation Programme in the current Section and the Extension Programme in the next Section.

The conclusion that point-charges are immovable objects relies on an attractive assumption that was part and parcel of the so-called Electro-Magnetic Worldview. This

Worldview loomed large among physicists around the turn of the 20th-century and was promulgated in particular by Wilhelm Wien, Lorentz and Abraham. That attractive assumption is that all matter is of electro-magnetic origin and thus somehow inertia is a function of its charge and emitted fields via their potential energy: $m = f(e, U_{\rm pot})$. (Remember that in them days the elementary particles were exhausted by electrons and atoms.) If therefore the potential energy $U_{\rm pot}$ diverges, then so does its mass m — provided one assumes that f is an increasing function of $U_{\rm pot}$ for fixed e without horizontal asymptote. But as Dirac [1938, p. 148] emphasised, since the discovery of the neutron in 1932, which is a massive ($m > 0$) and electrically neutral particle, this assumption has lost its attraction. The neutron deals a considerable blow to the Electro-Magnetic Worldview. About electrons Dirac [1938, p. 155] supposed that "there is an infinite negative mass at its center such that, when subtracted from the infinite positive mass of the surrounding Coulomb-field, the difference is well-defined and is just equal to m". Hence we write:

$$m_{\rm exp} \;=\; m_{\rm f} - m_{\rm b}\ , \tag{35}$$

where $m_{\rm exp} \equiv m$ is the experimentally determined mass, $m_{\rm f}$ the field- or electro-magnetic mass and $m_{\rm b}$ the bare mass. The field-mass $m_{\rm f}$ can in turn be broken up in a longitudinal and transversal component (first done by J.J. Thompson; cf. Lorentz [1909, pp. 38–39]). Anyhow, problem I, then, is solved.

Enter Frisch [2004, p. 538], who dismisses this solution in a single line on the basis of nothing more than a pejorative metaphor: "sweeping the infinity of the self-fields under the rug".

We beg to disagree. Actually two infinities are hauled from under the rug and put on display on top of the rug; then we let them annihilate each other in order to be left with a single finite quantity. We are not saying that these violent clashes of infinities belong to the most endearing spectacles in theoretical physics to watch; but we are saying that no mathematical laws need be broken in such a clash. After all, the difference between two diverging series may very well converge. Dirac [1938, pp. 149, 155] asserted that in the light of quantum physics, his CED-model of the electron "is hardly plausible", but added that as long as "we have a reasonable mathematical schema" and the reasonable schema is "in agreement with well-established principles, such as the principle of relativity and the conservation of energy and momentum", one should not object to this CED-model. Dirac then derived, as an approximate solution of the self-problem, an equation of motion for point-charges that was derived about thirty years earlier by Lorentz and Abraham for spatially extended charge-distributions. This is the celebrated 'Lorentz-Dirac equation' (see below).

Let us first mention what Jackson [1975, p. 784] calls the *Abraham-Lorentz* equation (historically terribly inaccurate but we follow suit for want of terminology):

$$d_\tau \mathsf{p}(\tau) \;\approx\; \mathsf{f}_{\rm L} + \mathsf{f}^{\alpha}_{\rm Schott} \;, \tag{36}$$

where $\mathsf{f}_{\rm L}$ is the external Lorentz-four-force (3), and $\mathsf{f}_{\rm Schott}$ is the 'Schott-term' — after the physicist G.A. Schott; cf. Rohrlich [1965, p. 150], [2000] for their explicit expressions. The Schott-term $\mathsf{f}_{\rm Schott}$ has nothing to do with radiation-reaction but describes a *reversible* process (total time-derivative) and depends neither on the velocity nor on the acceleration; it contains the derivative of the acceleration, $\dot{\mathsf{a}}$.

Due to the occurrence of $\dot{\mathsf{a}}$, the Abraham-Lorentz equation (36) suffers from (II) pre- and (III) self-accelerations. These are *prima facie* serious conceptual problems, but they melt away when the Abraham-Lorentz equation (36) is re-written as an integro-differential equation, so that (III) self-accelerations can be disposed off as artifacts of a particular mathematical way of writing down the equation of motion; cf. Jackson [1975, p. 797] and R.J. Cook [1984; 1986] for tutorial elaborations. This solves problem III. Problem II is solved by imposing appropriate boundary conditions.

An extension of the Abraham-Lorentz equation (36) was derived by Dirac using energy-conservation for point-charges, as we mentioned above; it is generally known as the *Lorentz-Dirac equation* (Lorentz [1909, pp. 48–49], Dirac [1938, p. 155], Spohn [2004, pp. 106–118]):

$$d_\tau \mathsf{p}(\tau) \;\approx\; \mathsf{f}_{\rm L} + \mathsf{f}_{\rm self} \;, \tag{37}$$

where by definition

$$\mathsf{f}_{\rm self} \;\equiv\; \mathsf{f}_{\rm Schott} + \mathsf{f}_{\rm rad} \;, \tag{38}$$

where $\mathsf{f}_{\rm rad}$ is the *radiation-reaction force*, which signals an *irreversible* loss of energy and momentum and is a function of the velocity and the acceleration.

With all dependencies of the different forces in position, let us write down the 3-vector part of the equation of motion:

$$m\mathbf{a}(t) \;\approx\; \mathbf{F}_{\rm L}\big(\mathbf{q}(t), \mathbf{u}(t)\big) + \mathbf{F}_{\rm Schott}\big(\dot{\mathbf{a}}(t)\big) + \mathbf{F}_{\rm rad}\big(\mathbf{u}(t), \mathbf{a}(t)\big) \;. \tag{39}$$

(One may justifiably wonder how Dirac's equation for *point-charges* (37), or (39), can coincide with an equation of Abraham and Lorentz for *extended charges*. Well, Abraham and Lorentz obtained a power-series of the velocity $\mathbf{u}$ in the electron radius, $R_{\rm e}$, the leading terms of which did not contain $R_{\rm e}$; subsequent terms were dropped, as a result of which $R_{\rm e}$ almost disappeared from the equation of motion. We say 'almost', because $R_{\rm e}$ still occurred in the expression for the mass, deemed of electromagetic origin. When this mass is however identified with the mass m in Dirac's equation, *then* the equations coincide. Furter it deserves to be mentioned that Lorentz's derivation, which one must cut and paste from various sections of his book [1909], relied on an *Ansatz* of a uniformly moving charge, as did Abraham's derivation. Schott seems to have been the first to present a fully general derivation not relying on such an *Ansatz*; cf. Yaghijan [1992], Appendix B, for a streamlined presentation of Schott's derivation and historical details.)

In most cases the contribution of the radiation-reaction force $\mathsf{f}_{\mathrm{rad}}$ (38) is many orders of magnitude smaller than $\mathsf{f}_{\mathrm{Schott}}$, which in turn is many orders of magnitude smaller than f_{L} — whence their neglect. The Lorentz-Dirac equation (37) also suffers from (II) pre- and (III) self-accelerations; cf. Dirac [1938, pp. 157, 159], Jackson [1975, p. 784]). Like in the case of the Abraham-Lorentz equation, the problems II and III disappear when the Lorentz-Dirac equation is rewritten in the form of an integro-differential equation; cf. Ibison & Puthoff [2001].

But it also deserves to be mentioned that imposing certain boundary conditions on the original differential equation (39) eliminates all unwanted behaviour; Rohrlich [1965, pp. 168–169] emphasises that these conditions are not *ad hoc* because there are independent physical reasons for imposing them. Remember that the Cauchy-problem for an equation containing $\mathbf{q}(t)$ and its first three partial time-derivatives is only well-posed when these are all given at one instant of time, $\dot{\mathbf{a}}(t)$ included. Dirac [1938, p. 158] pointed out that when we impose the boundary condition that the particle behaves freely when it is far away from the external fields, so that $\mathbf{a}(t) \to 0$ and $\dot{\mathbf{a}}(t) \to 0$ for $t \to \infty$ — as we similarly do in scattering theory —, then not only do we have a well-posed problem but we are also delivered from self-accelerations. The existence of solutions for the resulting Cauchy-like problem under very general conditions was proved by Hale & Stokes [1962].

A similar way to solve problems I, II and III is to renormalise the mass m occurring in the Lorentz-Dirac equation (37) *and* to renormalise the equation itself to the equation of motion of a free particle in the absence of external fields. This line was pursued successfully by A.O. Barut in a series of papers; cf. Barut [1988], [1990], [1992]. This is, to repeat, another way to solve problems I, II and III.

A completely different way to solve problems II and III is to replace $\mathsf{f}_{\mathrm{Schott}}$ in (38) with an entirely different term, so as to remove $\dot{\mathsf{a}}(t)$ from the equation of motion (39) altogether because *it* alone is responsible for (II) pre- and (III) self-accelerations. For example, T.C. Mo and C.H. Papas argued that the only field that can accelerate a particle and therefore make it radiate is the external field, so that $\mathsf{f}_{\mathrm{self}}$ should be expressible in terms of these fields and the particle kinematics. This leads to the *Mo-Papas equation*, which evokes neither (II) pre- nor (III) self-accelerations, and whose solutions for typical problems where self-force effects occur *differ* from those of the Lorentz-Dirac equation.

These differences lie far beyond what is experimentally accessible; cf. Mo & Papas [1971]. Shen [1972, p. 3040] erroneously pronounced these two equations *therefore* "physically indistinguishable" — erroneously, because the correct statement is that all currently available evidence underdetermines the choice between the obviously physically non-equivalent Lorentz-Dirac and Mo-Pappas equations. The Mo-Papas equation has nevertheless not really caught on, because, for one reason, in contrast to the Lorentz-Dirac equation, it is not (approximately) derived from 'first principles', i.e. the postulates of CED, in spite of the fact that it is an impeccable instance of the Newton-Minkowski equation (4).

Recently, M. Marino [2002] has found a way to avoid 'mass-renormalisation' altogether, by re-defining in a systematic way divergent integrals and limits appearing in the basic equations of CED. Marino's procedure leads to a finite expression for the total electromagnetic energy-momentum of the system of point-particles and fields, from which the Lorentz-Dirac equation (37) then is derived. Marion solves problem I and his procedure renders problems II and III harmless in the sense that pre- and self-accelerations surface as artifacts of approximations; they fail to surface exactly.

The past decades have witnessed the rise of several renormalisation programmes in CED; they include scattering theory and the calculation of cross-sections in order to make comparisons between CED and the data gathered in particle accelerators. Some of these programmes are however combined with the Lakatosian core of the Extension Programme, to which we turn next.

7 The Extension Programme

As we mentioned earlier, Abraham and Lorentz were the first to construct models of the electron within CED by means of a spherically symmetric spatially extended charge-density $\rho : \mathbb{R}^3 \to \mathbb{R}$; its extension can be characterised by a single parameter, radius $R_\mathrm{e} > 0$. Then $U_\mathrm{pot} = (1/4\pi\epsilon_0)e^2/R_\mathrm{e}$ of the associated electric field is finite, and by equating it to $m_\mathrm{e}c^2$ one obtains the *classical electron radius* of $r_\mathrm{e} \equiv (1/4\pi\epsilon_0)e^2/m_\mathrm{e}c^2$, which is in the order of magnitude of 10^{-12} mm. This solves problem I. Unlike Abraham (who later however followed suit), Lorentz took *ab initio* into consideration that an extended charge-density is deformed when described in a moving frame (Lorentz-Fitzgerald contraction). H. Poincaré [1906] pioneered the resulting mechanical stress in the extended charge and dealt with the *binding forces* in the charge, necessarily present in order to prevent the charge from exploding due to the repulsive Coulomb-forces in its parts. These binding forces accounted for a violation, in the models of Lorentz and Abraham, of the time-honoured relation between force $\mathbf{F}$ and power P, namely $\mathbf{F} \cdot \mathbf{u} = P$; cf. Yaghjian [1992, pp. 9–29]. Seeming violations of energy- and momentum-conservation were avoided by redefining relativistic energy and momentum, suggestions which go back to E. Fermi in the 1920ies; cf. Rohrlich [1970].

Extended charges rotate even in the absence of external torques (Thomas-precessesion) and this rotational motion is governed by the Fermi-Walker transport equation. But an additional spin-degree of freedom must be taken into account too when we have an extended charge-distribution. All rotational degrees of freedom are jointly governed by the Bargmann-Michel-Telegdi equation, which squares with the Lorentz-Dirac equation for small velocities and gyromagnetic ratio equal to 2; cf. Spohn [2004, pp. 119–129]. Thus far, translational and rotational degrees of freedom decouple, as a result of approximations in the derivation of the equations of motion just mentioned. This changed when J.S. Nodvik [1964] published a landmark paper. Nodvik lifted the Lorentz-covariant electron-model to the next plane of theoretical inquiry by coupling translational and rotational degrees of freedom; this leads to a much more complicated solution of the self-problem. Recently W. Appel & M.K.-H. Kiessling [2001] have taken this programme even further: by treating renormalisation properly, they attain "a mathematically consistent and physically viable Lorentz electrodynamics".

In 1904, Arnold Sommereld also investigated extended charges and showed that a uniformly charged sphere obeys, to a good non-relativistic approximation, a differential-difference equation of motion; this equation can be derived from the Lorentz-Dirac equation (37) by ignoring non-linear terms of the time-derivatives of $\mathbf{u}(t)$. Moniz and Sharp [1977, Section II] demonstrated that this *Sommerfeld-equation* is free from (II) pre- and (III) self-accelerations provided the radius of the sphere is larger than $2r_{\mathrm{e}}/3$, where r_{e} is the classical electron-radius. (Cf. Rorhlich [1997, p. 1053] for a summary.) Problems II and III solved once again for electrons.

The Sommerfeld-equation raises the question whether there is a fully relativistic differential-difference equation of motion that reduces to the Lorentz-Dirac equation in the point-particle limit and to the Sommerfeld-equation in the non-relativistic limit. P. Caldirola [1956, p. 307] answered in the affirmative and conjectured an equation of motion for a charged sphere that demonstrably has the mentioned properties:

$$\mathsf{f}_{\mathrm{L}}^{\alpha} \approx -\frac{m\kappa}{\tau_{\mathrm{e}}}\left(\mathsf{u}^{\alpha}(\tau-\tau_{\mathrm{e}})+\frac{1}{c^2}\mathsf{u}^{\alpha}(\tau)\mathsf{u}^{\beta}(\tau)\mathsf{u}_{\beta}(\tau-\tau_{\mathrm{e}})\right) , \tag{40}$$

where $\tau_{\mathrm{e}} = 2r_{\mathrm{e}}/c$ and κ is some constant to get the units right. But Caldirola also demonstrated that eq. (40) does not give rise to (II) pre- and (III) self-accelerations. The question how to derive eq. (40) remained open for more than thirty years.

The canonical monograph reviewing these and more theoretical investigations into the classical-electrodynamical behaviour of electric charges until the mid-1960ies was

Rohrlich's [1965]. The revised edition of this monograph had hardly appeared in 1990, when another monograph appeared on the subject, written by the electrical engineer A.D. Yaghjian [1992]. Yaghjian derived eq. (40) by ignoring non-linear terms (*ibid.*, Appendix D). Hence models of charged spheres of radius $R_{\rm e} > 2r_{\rm e}/3$ without (II) pre- and (III) self-accelerations are in CED. Since $r_{\rm e} > 2r_{\rm e}/3$, electrons behave decently.

More importantly, by examining the derivations of Lorentz, Abraham, Dirac and others through the looking-glass, Yaghjian [1992] re-discovered the frequently tacitly made assumption of slowly varying electro-magnetic fields in those derivations. This assumption is now known as 'the adiabatic limit'. Yaghjian derived another equation of motion for a charged insulating sphere (in an external field) that does not rely on an adiabatic assumption; the binding forces, that Poincaré had posited *ad hoc* in order to prevent Coulomb-explosion, he derived from first principles. In the resulting *Yaghjian equation* — another instantiation of the Newton-Minkowski equation (4) — neither (II) pre- nor (III) self-accelerations occur. They are artifacts of taking an adiabatic limit and considering a Taylor-expansion of the velocity function $\mathbf{u}(t)$ beyond the domain of analyticity; when analyticity is restored by plugging in an analytic switch-function in the equation of motion that turns the external force on (having a switch-time not smaller than the time needed for light to cross the electron, i.e. $\geqslant 2r_{\rm e}/c$), pre-accelerations do no longer occur where the Taylor-expansion is valid, after the external force is switched on. Hence this is *yet another way* to solve conceptual problems I, II and III. (Cf. Yaghjian [1992, pp. 65–72] for details and Rohrlich [1997] for a summary.)

The *adiabatic limit* has been the subject of rigorous treatments and such treatments go on to appear as we speak. The adiabatic limit really is a 'space-time limit', where the time-limit ensues naturally from the space-limit and the speed of light via $x = ct$. One introduces a dimensionless parameter $\varepsilon > 0$ and re-scales the spatial axes of a Cartesian co-ordinate system by a factor $1/\varepsilon$ by writing $\phi_{\rm ext}(\varepsilon\mathbf{r})$ and $\mathbf{A}_{\rm ext}(\varepsilon\mathbf{r})$ for the potentials of the external electro-magnetic fields. Studying slowly varying fields comes down to studying the limit $\varepsilon \downarrow 0$. In both the Abraham and the Lorentz model, $\mathsf{f}_{\rm self}$ can be Taylor-expanded in ε so as to obtain effective equations of motion, usually to order 2 in ε. The occurrence of (II) pre- and (III) self-accelerations is then understood as an artifact of cutting off the Taylor-series in ε; in the full series they do not occur. This is in line with our tentative claim that, unlike the diverging potential energy for point-charges, pre- and self-accelerations are not rigorous consequences of CED but artifacts of approximative solutions of C-Problems. Familiar results like Larmor's formula (13) appear after having taken the adiabatic limit and the point-particle limit, irrespective of the order in which the limits are taken, as it should be.

Further, the unique existence of a solution of the Cauchy-problem in the semi-relativistic

Abraham model was recently proved by A. Komech & H. Spohn [2000] and independently by G. Bauer & D. Dürr [2001]. The velocity of the charge for $t \to \infty$ is bounded for every extended charge and its acceleration vanishes for $t \to \infty$; cf. Spohn [2004], Chapter 5. Hence (III) self-accelerations do not occur in the semi-relativistic Abraham model, whence the conclusion that problem III is solved. Kiessling [1999] proved that energy and momentum in the Abraham model are conserved iff spin is taken into account. The unique existence of a solution of the Cauchy-problem in the relativistic Lorentz model for electrons moving with uniform velocity was proved by Appel & Kiessling [2002]. To the best of this author's knowledge, there is yet no fully general rigorous proof the unique existence of a solution of the Cauchy-problem for the relativistic Lorentz-electron.

The references mentioned above in our sketches of the Renormalisation and Extension Programme from the bird's eye point of view provide a *far from* exhaustive list. The lists of hundreds of references in Rohrlich [1965], Yaghjian [1992] and notably H. Spohn's recent state-of-the-art monograph [2004] bear testimony to the fact that Frisch's "serious conceptual problems" I, II and III have been solved at various levels of sophistication and rigour that the uninitiated reader could not possibly have suspected to exist when reading Frisch's paper. This is the reason why we judge his presentation of CED to be partisan if not grossly misleading.

Here ends our criticism of Frisch's Inadequacy Claim.

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