Knowledge Representation as Domains

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Abstract

This is a continuing attempt in a series of papers [KM 93, Mur 93, Mur 94] to show how computer-represented knowledge can be arranged as elements of an effectively represented semantic domain in the sense of [GS 90]. We present a direct deductive description of the domain, which was defined semantically in [KM 93], via the Scott's notion of information system. Also, the internal structure of the continuous ampliative operations coordinated with the domain's effective basis is established. Though we always remain in the paradigm of the toleration of contradictory information described in [Bel 75, Bel 76], the approach in question could be extended to include domains for consistency knowledge bases.

Key Words: *semantic domain, effective basis, continuous operation, epistemic state, information system, deduction.*

1 Introduction

The presented approach grounds the notion of approximation for computer-represented knowledge in the same way as it was done in the domain theory for the denotational semantics of programming languages (cf. [Ten 76]). The relation of approximation arises when one imagines the computer as placed into changeable information environment ("information flow"). Thus, the information which is contained in the computer's memory is considered as highly incomplete or partial.

More complete or total ("ideal") information may even be inaccessible to the computer. Considering this, we can distinguish computer-tractable information from a theoretically possible one so that it happens to be possible to attract topological ideas as it also was done in semantic domains (cf. [Sco 71, GHKLMS 80, GS 90]). In particular, continuous functions will be admitted as the only information transformers in the computer. Furthermore, we limit ourselves only to the continuous functions, which are computable (in a sense to be made precise below). Thus, as the structural side of a knowledge-based system in the framework of the approach in question is determined completely by the domain structure, its functional side is determined by computable functions definable on this domain. Moreover, to keep the approach realistic, we admit only effectively presented domains, which are algebraic semilattice with effective basis. Indeed, every element of such a domain is generated by elements of its effective basis (cf. [Sco 71, GHKLMS 80, GS 90]).

The question arises here: Why must the knowledge be structured in that way? Our answer is: It may not be structured at all. But by being inserted in the computer it becomes a data type. And we simply insist that data type be considered as an abstract notion. To ecape confusion, we would like to emphasise that we do not share the view, according to which "a knowledge base ... is treated as an *abstract data type* [in the sense of [LZ 74]] that interacts with a user or system only through a small set of operations" [Lev 84]. We accept rather the concept that represented knowledge is one element of a domain and we may use as many continuous computable functions on this domain as we need for transforming that knowledge. The point is that we do not consider, either theoretically or practically, a current state of the computer's knowledge in isolation from the others, but as one element of a domain.

To realize this plan, we shall from the beginning call our attention to that or another truth theory. We need do that, because we want to limit ourselves to the linguistic interpretation of the knowledge, that is, admit only the knowledge, which can be expressed in a formal language and allows the truth estimation [1]. This interpretation is essentially the first part of the Knowledge Representation Hypothesis (cf. [Smi 82, Lev 86, Isr 93]). The basic language L we employ is the propositional formulas built up of the set Var ($= \{p_1, p_2, \ldots\}$) of propositional variables with help of the connectives: $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The auxiliary language L^* includes also the symbol $\star$ as an "always-true" atomic formula. Thus, we are dealing here with a quite refined representation of information flow which curries information about facts that make propositions of the language L true or false [2].

We would like to emphasise that we are still having a choice here. Our future knowledge-based system is a big deal of choice at this moment. Of course, everything depends on our goals. Therefore, we turn to our purposes, one of which is to take into consideration possible contradictions that may come to the computer's input or appear in a current state of knowledge as a side-effect of an "inoffensive" input. (Remember: the computer is inside of the information flow.) In short, we should have at least four semantic values for propositions: $\boldsymbol{t}$ (truth), $\boldsymbol{f}$ (falsehood), $\perp$ (unknown) and $\top$ (overdetermination, i.e. both truth and falsehood). Another move would be necessary, if we chose, for example, closed-world assumption instead of our notion of information flow (cf. [Rei 78]). Now, we are again standing before choices. Indeed, one can construct a desire knowledge-based system as a domain either semantically or deductively. The first way, call it *semantic*, leads to the notion of Belnap's epistemic state [Bel 75, Bel 76], the second, call it *deductive*, leads to that of Scott's information system [Sco 82, DB 90]. We have to discuss briefly both options.

The convenience of the deductive approach lies, first of all, in that the domain, which is determined by an information system, has the ordinary set inclusion as its partially ordered relation on its elements that look like consistent theories in a considered language within chosen means of inference, where the computer-tractable elements correspond to some finitely axioma-

[1] As far as I know, such understanding of knowledge got aware in modern time due to G.Frege (cf. [Fre 66]).

[2] We are following [Rus 18] in differentiating between the notions of fact and proposition. In connection with this difference, we would like to note that that is probably not easy for the user working with in the information flow to translate a desciption of fact into an appropriate proposition of L. A comprehensive analysis of the models of information flow has been recently developed in [Bar 92].

tizable theories. In our case, holding a paradigm of the toleration of inconsistency, we form a domain $\mathcal{A}$ being come from the formulas of the language L and the formula $\star$ as basic propositions ("tokens"). As a basic means of inference, we take the Anderson-Belnap's calculus E_{fde} of first degree entailment from [AB 75]. In what extent is this choice justified? Any answer within the deductive approach will be hardly satisfied. Therefore, we seek to find an answer to that question from the point of view of the semantic approach.

The semantic approach comes out of our intuitive vision of what a designed knowledge-based system should be and it is certainly more intuitively understandable for the user. Moreover, the intuitively justified definitions such as Belnap's epistemic state in [Bel 75, Bel 76] may precede the theoretical construction of an appropriative domain, and may be turned later into more suitable ones. Intuitive vision is especially important for obtaining definitions of operations as knowledge transformers needed on the domain. An example of such a situation occured, when we employed in [KM 93] the notion of generalized epistemic state to form the elements of the domain AGE and that of minimal epistemic state to define possible states of the computer's knowledge, leaving Belnap's definition as an auxiliary one. Now, we need to make sure that we arrived to the same (up to isomorphism) domain structure using the semantic and deductive approaches. This is described in section 3, establishing an isomorphism between domains $\mathcal{A}$ and AGE. This isomorphism is an exapmle of what we could call *completeness.*

As we said before, we pay attention only to the continuous, and even computable, functions on the effectively presented domains. To hold onto the realistic spirit of the approach in question, we need to add one more condition: knowledge-transformation operations must be closed with respect to the basis, in other words, *coordinated with* it (cf. [KM 93, Mur 93, Mur 94]). Moreover, we narrow down the set of the acceptable operations supposing that the computer itself, being located in the information flow never loses the information it currently has. Thus, those operations have to act *ampliatively*, as Nuel Belnap would say (see the definition below). What has been said, however, does not mean that we cannot correct the computer's behavior in connection, for example, with backtracking [3] or analyse effec-

[3]See on a backtracking strategy on the lattice AFE, an effective basis of the domain AGE, in [Mur 94].

tively development of the computer's knowledge [4]. It only means that the computer itself, as an intelligent system, maintains its knowledge by means of $\boldsymbol{CAC}$-operations (see the definition below). In section 5 we prove that the operations $[A]$ and $[A \rightarrow B]$ introduced in [KM 93] are two examples of the $\boldsymbol{CAC}$-operations of the finite order. We also establish in that section that all the $\boldsymbol{CAC}$-operations possess a definite structure, in which $[A]$-operations play a fundamental role.

Although we consider here only a particular knowledge representation expressed by constructing certain domains, using $\boldsymbol{CAC}$-operations in accordance with our purposes, we maintain the idea that the approach in question has some wide-ranging significance.

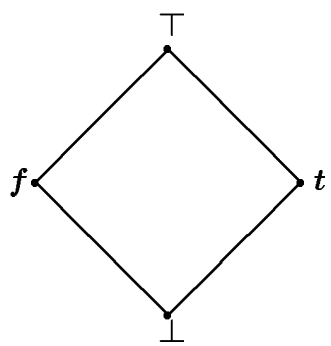
2 Preliminaries

We start with the definitions of the semantic approach. Let us first fix the set $\Im \stackrel{\text{def}}{=} \{\boldsymbol{t}, \boldsymbol{f}, \bot, \top\}$ partially ordered by the relation $\sqsubseteq$ and we will consider it as the lattice $\mathsf{A4}$ $(= (\Im, \sqcap, \sqcup))$ pictured as the left diagram in Figure 1. For the determination of semantic assignments of the formulas, we need another lattice $\mathsf{L4}$ $(= (\Im, \wedge, \vee))$ pictured as the right diagram. Both lattices were first introduced by N. Belnap in [Bel 75, Bel 76]. The interested reader will also find attractive motivations there.

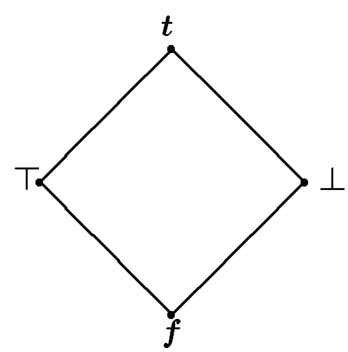
Figure 1: Lattices $\mathsf{A4}$ and $\mathsf{L4}$.

[4] See on a modal epistemic logic on the lattice AFE in [Mur 93].

A4



L4



A *setup* is a mapping $s : \mathsf{Var} \rightarrow \Im$ that is extended to the formulas of the language L^* as follows:

$$s(A \wedge B) = s(A) \wedge s(B),$$
$$s(A \vee B) = s(A) \vee s(B),$$
$$s(\neg A) = \neg s(A),$$
$$s(\star) = \boldsymbol{t},$$

where the operation $\neg$ on $\Im$ is defined by means of the conditions:

$$\neg \boldsymbol{t} = \boldsymbol{f},\ \neg \boldsymbol{f} = \boldsymbol{t} \text{ and } \neg\tau = tau \text{ for } \tau \in \{\bot, \top\}.$$

All the setups form the lattice AS ordered as follows:

$$s \leq s_1 \text{ if and only if } s(p) \sqsubseteq s_1(p) \text{ for every } p \in \mathsf{Var}.$$

A setup s is *finite*, if the set $\{s \,|\, s(p) \neq \bot\}$ is finite. We denote that set via $V(s)$ and do it by means of $V(A)$ the set of variables included in a formula A. Despite being an auxiliary notion, Belnap's concept of *epistemic state* as a nonempty set of setups forms the underlying basis of what follows. An epistemic state is called *finite*, if it consists of a finite set of finite setups. An important example is the state $\mathbf{o}$, which consists of the single setup s_0 such that $s_0(p) = \bot$ for every $p \in \mathsf{Var}$. Other important examples are:

$$Tset\,(A) \stackrel{\text{def}}{=} \{s \,|\, s \in \mathsf{AS}, \boldsymbol{t} \sqsubseteq s\,(A)\,, V(s) \subseteq V\,(A)\},$$

$$Fset(A) \stackrel{\text{def}}{=} \{s \,|\, s \in \mathsf{AS}, \boldsymbol{f} \sqsubseteq s(A), V(s) \subseteq V(A)\}.$$

By definition, we also accept: $Tset(\star) \stackrel{\text{def}}{=} \mathbf{o}$.

Let us denote $m(\varepsilon)$ as meaning the minimal setups in a finite state ε. Because of Descending Chain Condition, $m(\varepsilon)$ is a finite state too. It is obvious we have the equations: $m(m(\varepsilon)) = m(\varepsilon)$ for every finite state ε and $m(\mathbf{o}) = \mathbf{o}$. We call a finite state ε *minimal* whenever $m(\varepsilon) = \varepsilon$. Thus, every minimal state is a finite nonempty set of finite incomparable setups. All the minimal states form the lattice AFE with the partial ordering as follows:

$\varepsilon \leq \varepsilon_1$ if and only if for any $s_1 \in \varepsilon_1$ there is $s \in \varepsilon$ such that $s \leq s_1$.

Belnap's second key notion is that of the *assignment* of a formula A in an epistemic state ε defined as follows:

$$\varepsilon(A) \stackrel{\text{def}}{=} \sqcap\{s(A) \,|\, s \in \varepsilon\}.$$

A *generalized* (*epistemic*) *state* $\overline{\varepsilon}$ (generated by the epistemic state ε) is the set $\{\varepsilon' \,|(\forall \text{ formula } A \text{ of } L)(\varepsilon(A) = \varepsilon'(A)\}$. All the generalized states form the domain AGE with the ordering:

$$\overline{\varepsilon} \leq \overline{\varepsilon_1} \text{ if and only if } \varepsilon(A) \sqsubseteq \varepsilon_1(A) \text{ for every } A;$$

moreover, AFE is an effective basis of AGE(cf. [KM 93]).

Another way to arrive at a domain is via the notion of *information system* (cf. [Sco 82, DB 90]). The information system, which we deal with here, is the quadruple $(\boldsymbol{D}, \star, \boldsymbol{Con}, \vdash)$, where $\boldsymbol{D}$ is the set of all formulas of L and the formula $\star$, $\boldsymbol{Con}$ is all the finite subsets of formulas in $\boldsymbol{D} \setminus \{\star\}$. Furthermore, $\vdash$ means here the relation on $\boldsymbol{Con} \times \boldsymbol{D}$ defined as follows:

$$u \vdash A \text{ if and only if } \vdash_{E^*} \wedge u \to A,$$

where $\wedge\emptyset \stackrel{\text{def}}{=} \star$ and E^* is a conservative extension of E_{fde} by adding one additional axiom scheme $A \to \star$ (cf. [Mur 94]). In what follows, we use expressions like $\vdash_{E^*} A \leftrightarrow B$ for A, B of the language L^* as meaning that both $\vdash_{E^*} A \to B$ and $\vdash_{E^*} B \to A$ hold. Now, for $(\boldsymbol{D}, \star, \boldsymbol{Con}, \vdash)$ to be an information system, we need to check the following properties:

1) $u \vdash \star$;
2) $u \vdash A$, wnenever $A \in u$;
3) if $v \vdash B$ for all $B \in u$ and $u \vdash A$, then $v \vdash A$;

there is meant that $u, v \in \boldsymbol{Con}$ and $A, B \in \boldsymbol{D}$ (cf. [Sco 82]).

Proposition 1 $(\boldsymbol{D}, \star, \boldsymbol{Con}, \vdash)$ *is an information system.*

Proof is obvious.

From now on, we denote via $\mathcal{A}$ the domain determined by the information system $(\boldsymbol{D}, \star, \boldsymbol{Con}, \vdash)$ (see a detailed definition below).

A domain $\mathcal{D}$ (equal, e.g., to $\mathcal{A}$ or AGE) with an order $\leq$ can be turned into a topological space to give the approximation more precise meaning. According to [Sco 71, Sco 72, GHKLMS 80], a set $U \subseteq \mathcal{D}$ is said to be open in the *Scott topology* on $\mathcal{D}$, if

1) $x \in U$ and $x \leq y$ implies $y \in U$;
2) $\sqcup D \in U$ implies $D \cap U \neq \emptyset$
for any directed set $D \subseteq \mathcal{D}$.

For any $x, y \in \mathcal{D}$, define:

$x \ll y$ whenever [5] x is a low bound of some open set U with $y \in U$.

An element $x \in \mathcal{D}$ satisfying $x \ll x$ is said to be *compact* (cf. [GHKLMS 80]). All the elements of AFE are compact with respect to the Scott topology on AGE. Also, AFE is a basis of AGE, because, for every epistemic state ε,

$$\overline{\varepsilon} = \sqcup\{\varepsilon' \,|\, \varepsilon' \in \mathsf{AFE}, \varepsilon' \ll \overline{\varepsilon}\}$$

up to the embedding $(\varepsilon \mapsto \overline{\varepsilon}) : \mathsf{AFE} \to \mathsf{AGE}$ (cf. [KM 93]).

Finally, we will use the following well-known fact: Operation $F : \mathcal{D} \to \mathcal{D}$ is Scott-continuous if and only if for any directed set $\{x_i \,|\, i \in I\} \subseteq \mathcal{D}$, the equation

$$F(\sqcup\{x_i \,|\, i \in I\}) = \sqcup\{F(x_i) \,|\, i \in I\}$$

holds (cf. [Sco 72, GHKLMS 80]).

[5]It is probably more preferable to choose the weaker, though less intuitive, condition from the Definition I-1.1 in [GHKLMS 80] (cf. Notes to Section I-1 and Exercise I-1.24 in [GHKLMS 80]).

3 Isomorphism between $\mathcal{A}$ and AGE

As a preliminary step, we prove that the stractures $(\boldsymbol{Con}, \subseteq)$ and AFE are isomorphic as partially ordered sets.

Lemma 1 *Let A and B be formulas or $\star$. Then $m(Tset\,(B)) \leq m(Tset\,(A))$ if and only if $\vdash_{E^*} A \rightarrow B$.*

Proof. Denote $m(Tset\,(B))$ and $m(Tset\,(A))$ via ε and ε_1, respectively. Applying the Lemmas 13 and 10 from [Mur 94] successively, we receive:

$$\begin{array}{lll} \varepsilon \leq \varepsilon_1 & \text{if and only if} & \vdash_{E^*} A_{\kappa(\varepsilon_1)} \rightarrow A_{\kappa(\varepsilon)} \\ & & \vdash_{E^*} A \rightarrow B. \end{array}$$

Lemma 2 *For any finite sets of formulas u and v, $\overline{u} \subseteq \overline{v}$ if and only if $m(Tset\,(\wedge u)) \leq m(Tset\,(\wedge v))$.*

Proof. It is easy to check that $\overline{u} \subseteq \overline{v}$ if and only if $\vdash_{E^*} \wedge v \rightarrow \wedge u$. Then, in virtue of the Lemma 1, the latter is equivalent to $m(Tset\,(\wedge u)) \leq m(Tset\,(\wedge v))$.

Lemma 3 *The mapping $\phi : \overline{u} \mapsto m(Tset\,(\wedge u))$ is a partially ordered isomorphism between $(\{\overline{u}\,|u \in \boldsymbol{Con}\}, \subseteq)$ and AFE.*

Proof follows immediately from the Lemma 2 and Lemma 12 in [Mur 94].

Now, using this preliminary result, we aim to establish an isomorphism between the lattice AGE and the domain determined by the information system $(\boldsymbol{D}, \star, \boldsymbol{Con}, \vdash)$.

Recall that $x \vdash A$ for any $x \subseteq \boldsymbol{D}$ means the existence $u \in \boldsymbol{Con}$ such that $u \subseteq x$ and $u \vdash A$. Also, $\overline{x}$ means $\{A\,|A \in \boldsymbol{D}, x \vdash A\}$ for any $x \subseteq \boldsymbol{D}$ (cf. [Sco 82]). The following properties are easy to check:

$$\left.\begin{array}{c} 1)\ \overline{\overline{x}} = \overline{x};\quad 2)\ \overline{\emptyset} = \{\star\};\quad 3)\ x \subseteq y \text{ implies } \overline{x} \subseteq \overline{y}; \\ 4)\ \overline{u} \subseteq \overline{x} \text{ if and only if there is } v \in \boldsymbol{Con} \\ \text{such that } v \subseteq x \text{ and } \vdash_{E^*} \wedge v \rightarrow \wedge u. \end{array}\right\} \qquad (1)$$

Denote $\mathcal{A}$ as meaning the domain $(\{\overline{x}\,|x \subseteq \boldsymbol{D}\}, \subseteq)$ corresponding to the information system $(\boldsymbol{D}, \star, \boldsymbol{Con}, \vdash)$.

Lemma 4 *The domain $\mathcal{A}$ is a complete lattice. Moreover,*

$$\sqcup\{\overline{x_i}\,|i \in I\} = \overline{\cup\{x_i\,|i \in I\}} = \overline{\cup\{\overline{x_i}\,|i \in I\}},$$

$$\overline{x} \sqcap \overline{y} = \overline{x} \cap \overline{y}.$$

Proof. The first equation is proved with help of properties (1). Also, (1) implies that $\mathcal{A}$ has the least element $\{\star\}$. Thus, $\mathcal{A}$ is a complete lattice according to a well-known lattice argument. The second equation follows from (1) and the first equation. To establish the equation $\overline{x} \sqcap \overline{y} = \overline{x} \cap \overline{y}$ is enough to show that $u \subseteq \overline{x} \cap \overline{y}$ and $u \vdash A$ implies $A \in \overline{x} \cap \overline{y}$ which, also, follows from (1) and entailments valid in E^* (consult [AB 75]).

Our next step is to establish that $\mathcal{A}$ is a lattice with relative psedo-complement which we will denote as $\overline{x} \Rightarrow \overline{y}$ for any $\overline{x}, \overline{y} \in \mathcal{A}$.

Lemma 5 $\vdash_{E^*} A_1 \wedge \ldots \wedge A_n \rightarrow A$ *implies* $\vdash_{E^*} (A_1 \vee B) \wedge \ldots \wedge (A_n \vee B) \rightarrow A \vee B$.

Proof. Let s be a setup. We come out of the inequality:

$$s(A_1) \wedge \ldots \wedge s(A_n) \leq s(A).$$

Then

$$(s(A_1) \wedge \ldots \wedge s(A_n)) \vee s(B) \leq s(A) \vee s(B).$$

Using distributivity of the lattice $\mathsf{L4}$, we receive:

$$(s(A_1) \vee s(B)) \wedge \ldots \wedge (s(A_n \vee s(B)) \leq s(A) \vee s(B).$$

Lemma 6 *For any $\overline{x}, \overline{y} \in \mathcal{A}$, the equation*

$$\overline{x} \sqcap \overline{y} = \overline{\{A \vee B\,|A \in x, B \in y\}}$$

holds.

Proof. Assume $A \in \overline{\{A \vee B\,|A \in x, B \in y\}}$, that is, there are formulas $A_1, \ldots, A_n \in x$ and $B_1, \ldots, B_n \in y$ such that

$$\vdash_{E^*} (A_1 \vee B_1) \wedge \ldots \wedge (A_n \vee B_n) \rightarrow A.$$

However, we know (cf. [AB 75]) that

$$\vdash_{E^*} A_1 \wedge \ldots \wedge A_n \rightarrow (A_1 \vee B_1) \wedge \ldots \wedge (A_n \vee B_n)$$

and

$$\vdash_{E^*} B_1 \wedge \ldots B_n \rightarrow (A_1 \vee B_1) \wedge \ldots \wedge (A_n \vee B_n)$$

which give

$$\vdash_{E^*} A_1 \wedge \ldots \wedge A_n \rightarrow A \text{ and } \vdash_{E^*} B_1 \wedge \ldots \wedge B_n \rightarrow A.$$

That implies $A \in \overline{x} \cap \overline{y}$ and, in virtue of the Lemma 4, $A \in \overline{x} \sqcap \overline{y}$.

Let now $A \in \overline{x} \sqcap \overline{y}$, that is (the Lemma 4), $A \in \overline{x} \cap \overline{y}$. Then, there are formulas $A_1, \ldots, A_n \in x$ and $B_1, \ldots, B_m \in y$ such that

$$\vdash_{E^*} A_1 \wedge \ldots \wedge A_n \rightarrow A \text{ and } \vdash_{E^*} B_1 \wedge \ldots \wedge B_m \rightarrow A.$$

In virtue of the Lemma 5, we receive:

$$\vdash_{E^*} (A_1 \vee B_1) \wedge \ldots \wedge (A_n \vee B_1) \rightarrow A \vee B_1,$$

$$\vdash_{E^*} (A_1 \vee B_2) \wedge \ldots \wedge (A_n \vee B_2) \rightarrow A \vee B_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\vdash_{E^*} (A_1 \vee B_m) \wedge \ldots \wedge (A_n \vee B_m) \rightarrow A \vee B_m.$$

Denote

$$u \stackrel{\text{def}}{=} \{A_1 \vee B_1, \ldots, A_n \vee B_1, \ldots, A_1 \vee B_m, \ldots, A_n \vee B_m\}.$$

Manipulating with entailments in E_{fde} (consult [AB 75]), we receive:

$$u \vdash (A \vee B_1) \wedge \ldots \wedge (A \vee B_m).$$

However, our premise gives us:

$$\vdash_{E^*} (A \vee B_1) \wedge \ldots \wedge (A \vee B_m) \rightarrow A.$$

Thus, $u \vdash A$.

We will need the following corollary in the section 4.

Corollary 6.1 *For any formulas A and B and $u \subseteq \boldsymbol{Con}$,*

$$\overline{u \cup \{A\}} \sqcap \overline{u \cup \{B\}} = \overline{u \cup \{A \vee B\}}.$$

Proof. Using the Lemma 6 and $\vdash_{E^*} (C \wedge A) \vee (C \wedge B) \leftrightarrow C \wedge (A \vee B)$, we receive:

$$\begin{aligned} \overline{u \cup \{A\}} \sqcap \overline{u \cup \{B\}} &= \overline{\wedge u \wedge A} \sqcap \overline{\wedge u \wedge B} \\ &= \overline{(\wedge u \wedge A) \vee (\wedge u \wedge B)} \\ &= \overline{\wedge u \wedge (A \vee B)} \\ &= \overline{u \cup \{A \vee B\}}. \end{aligned}$$

Lemma 7 *For any $\overline{x}, \overline{y} \in \mathcal{A}$, the relative pseudo-complement $\overline{x} \Rightarrow \overline{y}$ exists and, moreover,*

$$\overline{x} \Rightarrow \overline{y} = \overline{\{B \,|(\forall A \in x)(A \vee B \in \overline{y})\}}.$$

Proof. Denote

$$z \stackrel{\text{def}}{=} \{A \,|(\forall A \in x)(A \vee B \in \overline{y})\}.$$

Assume $A \in \overline{x} \sqcap \overline{z}$. According to the Lemma 6, there are formulas $A_1, \ldots, A_n \in x$ and $B_1, \ldots, B_n \in z$ such that

$$A_1 \vee B_1, \ldots, A_n \vee B_n \vdash A.$$

In virtue of the definition of z,

$$\{A_1 \vee B_1, \ldots, A_n \vee B_n\} \subseteq \overline{y}.$$

And we receive $A \in \overline{y}$ that implies the inclusion $\overline{x} \sqcap \overline{z} \subseteq \overline{y}$.

Now, assume $\overline{x} \sqcap \overline{w} \subseteq \overline{y}$ for a fixed $\overline{w}$ from $\mathcal{A}$. With respect to the Lemma 6, we have

$$\overline{x} \sqcap \overline{w} = \overline{\{A \vee B \,|A \in x, B \in w\}}.$$

It implies the inclusion

$$\{A \vee B \,|A \in x, B \in w\} \subseteq \overline{y}.$$

Therefore, $w \subseteq z$ and, hence, $\overline{w} \subseteq \overline{z}$ (cf. (1)).

Corollary 7.1 *The equation*

$$(\sqcup\{\overline{x_i}\,|i \in I\}) \sqcap \overline{y} = \sqcup\{\overline{x_i} \sqcap \overline{y}\,|i \in I\}$$

holds in $\mathcal{A}$.

Proof follows immediately from the Lemma 7 and Theorem I-11.2 in [RS 63].

Lemma 8 *For any $u \in \boldsymbol{Con}$, the set $\{\overline{x}\,|\overline{u} \subseteq \overline{x}\}$ is open in the Scott topology on $\mathcal{A}$.*

Proof. Let $\{\overline{x_i}\,|i \in I\}$ be any fixed directed set of elements in $\mathcal{A}$. In vitrue of the Corollary 7.1, we receive:

$$\overline{u} = (\sqcup\{\overline{x_i}\,|i \in I\}) \sqcap \overline{u} = \sqcup\{\overline{x_i} \sqcap \overline{u}\,|i \in I\}.$$

And with help of the Lemma 4, we have:

$$\overline{u} = \sqcup\{\overline{x_i} \cap \overline{u}\,|i \in I\}.$$

Notice, first, that the set $\{\overline{x_i} \sqcap \overline{u}\,|i \in I\}$ is directed, because if $\overline{x_i} \sqcup \overline{x_j} \subseteq \overline{x_k}$ then, with respect to the Corollary 7.1, $(\overline{x_i} \sqcap \overline{u}) \sqcup (\overline{x_j} \sqcap \overline{u}) \subseteq \overline{x_k} \sqcap \overline{u}$. Second, the set $\{\overline{x_i} \cap \overline{u}\,|i \in I\}$ is finite. Thus, there is $i_0 \in I$ such that

$$\overline{x_{i_0}} \sqcap \overline{u} = \sqcup\{\overline{x_i} \cap \overline{u}\,|i \in I\}.$$

Consequently, $\overline{u} = \overline{x_{i_0}} \sqcap \overline{u}$ that implies the inclusion $\overline{u} \subseteq \overline{x_{i_0}}$.

Corollary 8.1 *For any $u \in \boldsymbol{Con}$ and $x \subseteq \boldsymbol{D}$, $\overline{u} \ll \overline{x}$ if and only if $\overline{u} \subseteq \overline{x}$; in particular, $\overline{u} \ll \overline{u}$. Hence, $(\{\overline{u}\,|u \in \boldsymbol{Con}\}, \subseteq)$ is a basis of $\mathcal{A}$, that is, for every $\overline{x} \in \mathcal{A}$, the equation $\overline{x} = \sqcup\{\overline{u}\,|u \in \boldsymbol{Con}, \overline{u} \ll \overline{x}\}$ holds.*

Proof. The first part immediately follows from the Lemma 8. The second part follows from the first part, the Lemma 4 and the basic formula in [Sco 82] or the Lemma 3.36 in [DB 90].

Recall that an element $\overline{x} \in \mathcal{A}$ is compact, if $\overline{x} \ll \overline{x}$.

Corollary 8.2 *An element $\overline{x} \in \mathcal{A}$ is compact if and only if there is $u \in \boldsymbol{Con}$ such that $\overline{u} = \overline{x}$.*

Proof. From the Corollary 8.1 follows $\overline{u}$ is compact for any $u \in \boldsymbol{Con}$. Now assume $\overline{x}$ is a compact element. According to the Corollary 8.1, $\overline{x} = \sqcup\{\overline{u}\,|u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\}$. For the set $\{\overline{u}\,|u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\}$ to be directed, there is $u \in \boldsymbol{Con}$ such that $\overline{x} \subseteq \overline{u}$ and $\overline{u} \subseteq \overline{x}$.

Theorem 1 *The mapping $f : \overline{x} \mapsto \sqcup\{\phi(\overline{u})\,|u \in \boldsymbol{Con}, \overline{u} \ll \overline{x}\}$ is an isomorphic extension of ϕ between the domain $\mathcal{A}$ and the lattice AGE.*

Proof. It is clear, because of the Corollary 8.1, that f is an extansion of ϕ. Next we first prove that the mapping f is surjective.

Let $\overline{\varepsilon_0} \in \mathsf{AGE}$. Then, in virtue of the Theorem 4.3, Basic Lemma 3.3 and Theorem 6.4, all in [KM 93], we can write the equation

$$\overline{\varepsilon_0} = \sqcup\{\varepsilon\,|\varepsilon \in \mathsf{AFE}, \varepsilon \leq \varepsilon_0\}.$$

Denote

$$\overline{x} \stackrel{\text{def}}{=} \sqcup\{\phi^{-1}(\varepsilon)\,|\varepsilon \in \mathsf{AFE}, \varepsilon \leq \varepsilon_0\}.$$

Then, prove that

$$\phi^{-1}(\varepsilon) \subseteq \overline{x} \;\text{ if and only if }\; \varepsilon \leq \varepsilon_0 \tag{2}$$

for any $\varepsilon \in \mathsf{AFE}$.

The "if" part of (2) follows from the definition of $\overline{x}$.

To prove the "only if" part we suppose that $\phi^{-1}(\varepsilon') \subseteq \overline{x}$ for some fixed $\varepsilon' \in \mathsf{AFE}$. In virtue of the Lemma 12 in [Mur 94], the equation $\phi^{-1}(\varepsilon) = \overline{\{A_{\kappa(\varepsilon)}\}}$ holds for every $\varepsilon \in \mathsf{AFE}$. Thus, with respect to the Lemma 4, we have the equation

$$\overline{x} = \overline{\cup\left\{A_{\kappa(\varepsilon)}\,|\varepsilon \in \mathsf{AFE}, \varepsilon \leq \varepsilon_0\right\}}.$$

So, our premise implies that there are $\varepsilon_{i_0}, \ldots, \varepsilon_{i_n}$ such that $\varepsilon_{i_0} \sqcup \ldots \sqcup \varepsilon_{i_n} \leq \varepsilon_0$ and $A_{\kappa(\varepsilon_{i_0})}, \ldots, A_{\kappa(\varepsilon_{i_n})} \vdash A_{\kappa(\varepsilon')}$. The latter implies that $\phi^{-1}(\varepsilon') \subseteq \phi^{-1}(\varepsilon_{i_0}) \sqcup \ldots \sqcup \phi^{-1}(\varepsilon_{i_n})$. Then, with help of the Lemma 3, we receive $\phi^{-1}(\varepsilon') \subseteq \phi^{-1}(\varepsilon_{i_0} \sqcup \ldots \sqcup \varepsilon_{i_n})$ and, then, $\varepsilon' \leq \varepsilon_{i_0} \sqcup \ldots \sqcup \varepsilon_{i_n} \leq \varepsilon_0$.

Now, we prove the inquality

$$\sqcup \{\phi(\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\} (A) \sqsubseteq \varepsilon_0 (A) \tag{3}$$

for any fixed formula A.

Let V be the variables occuring in A. Recall from [KM 93] that V-*down-restriction* of setup s is the setup s^{V-} defined as follows:

$$s^{V-} \stackrel{\text{def}}{=} \begin{cases} s(p) & \text{for } p \in V \\ \perp & \text{for } p \notin V; \end{cases}$$

and V-*down-restriction* of epistemic state ε is the state ε^{V-} defined as follows:

$$\varepsilon^{V-} \stackrel{\text{def}}{=} \left\{ s^{V-} \,|s \in \varepsilon \right\}.$$

Also, we mean: $\overline{\varepsilon}^{V-} \stackrel{\text{def}}{=} \overline{\varepsilon^{V-}}$. Then according to the Lemma 6.1 and Theorem 3.1, both in [KM 93], we have the equation:

$$(\sqcup\{\phi(\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\})^{V-} = \sqcup\left\{m\left(\phi(\overline{u})^{V-}\right) |u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\right\}.$$

Notice that, in virtue of the Lemmas 4.2 and 3.3 in [KM 93], the set

$$\left\{m\left(\phi(\overline{u})^{V-}\right) |u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\right\}$$

is finite and directed, because the set $\{\phi(\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\}$ is directed. Then, there is $u_0 \in \boldsymbol{Con}$ and $\overline{u_0} \subseteq \overline{x}$ such that

$$\sqcup\left\{m\left(\phi(\overline{u})^{V-}\right) |u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\right\} = m\left(\phi(\overline{u_0})^{V-}\right).$$

Notice that $m\left(\phi(\overline{u_0})^{V-}\right) \leq \phi(\overline{u_0})^{V-} \leq \phi(\overline{u_0})$. With respect to the Lemma 3, we receive $\phi^{-1}(m\left(\phi(\overline{u_0})^{V-}\right)) \subseteq \overline{u_0} \subseteq \overline{x}$. And according to (2), we have $m\left(\phi(\overline{u_0})^{V-}\right) \leq \varepsilon_0$. Now, (3) follows from the last in virtue of the Proposition 4 and Basic Lemma 3.3 in [KM 93].

Finally, (3) implies the inquality

$$\sqcup\{\phi(\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\} \leq \varepsilon_0$$

which gives the equation $f(\overline{x}) = \overline{\varepsilon_0}$, that is, the mapping f is surjective.

To finish the proof we need to prove the equivalence

$$\overline{x} \subseteq \overline{y} \text{ if and only if } f(\overline{x}) \leq f(\overline{y}) \tag{4}$$

for any $\overline{x}, \overline{y} \in \mathcal{A}$.

The "only if" part of (4) is quite trivial: $\overline{x} \subseteq \overline{y}$ implies the inclusion

$$\{\phi(\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \ll \overline{x}\} \subseteq \{\phi(\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \ll \overline{y}\}$$

which in turn implies the inequality $f(\overline{x}) \leq f(\overline{y})$.

Now, assume $f(\overline{x}) \leq f(\overline{y})$ and denote

$$J_x \stackrel{\text{def}}{=} \{\phi(\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \ll \overline{x}\}$$

for any $x \subseteq \boldsymbol{D}$. Thus, our premise means $\sqcup J_x \leq \sqcup J_y$. Let $\phi(\overline{u}) \in J_x$, where $u \in \boldsymbol{Con}$.

Introduce under consideration two new functions:

$$g \stackrel{\text{def}}{=} (J \mapsto \sqcup J) : IdAFE \to \mathsf{AGE},$$

$$d \stackrel{\text{def}}{=} (\overline{\varepsilon} \mapsto \{\varepsilon' \,|\varepsilon' \in \mathsf{AFE}, \varepsilon' \ll \overline{\varepsilon}\}) : \mathsf{AGE} \to Id\mathsf{AFE},$$

where $Id\mathsf{AFE}$ is the set of the ideals of the lattice AFE (cf. [GHKLMS 80]).

Recall that the following equivalence

$$\overline{\varepsilon} \leq g(J) \text{ if and only if } d(\overline{\varepsilon}) \subseteq J \tag{5}$$

holds [6] according to the Proposition III-4.3 in [GHKLMS 80].

Now, assume $\phi(\overline{u}) \in J_x$, where $u \in \boldsymbol{Con}$. It follows $\phi(\overline{u}) \leq \sqcup J_x$ and, hence, $\phi(\overline{u}) \leq \sqcup J_y$, that is, $\phi(\overline{u}) \leq g(J_y)$. In virtue of (5), $d(\phi(\overline{u})) \subseteq J_y$. However, according to the Lemma 6.2 in [KM 93], $\phi(\overline{u}) \ll \phi(\overline{u})$, that is, $\phi(\overline{u}) \in d(\phi(\overline{u}))$ that implies $\phi(\overline{u}) \in J_y$. That establishes the inclusion $J_x \subseteq J_y$. This inclusion gives:

$$\{\overline{u} \,|u \in \boldsymbol{Con}, \overline{u} \ll \overline{x}\} \subseteq \{\overline{u} \,|u \in \boldsymbol{Con}, \overline{u} \ll \overline{y}\}.$$

Indeed, let $\overline{u} \ll \overline{x}$. Then $\phi(\overline{u}) \in J_x$ and, therefore, $\phi(\overline{u}) \in J_y$. That is, there is $v \in \boldsymbol{Con}$ such that $\phi(\overline{v}) = \phi(\overline{u})$ and $\overline{v} \ll \overline{y}$. In virtue of the Lemma 3, $\overline{v} = \overline{u}$ that implies $\overline{u} \ll \overline{y}$.

Finally, in virtue of the Corollary 8.1, we receive $\overline{x} \leq \overline{y}$.

[6] The pair (g, d) is a Galois connection between $Id\mathsf{AFE}$ and AGE, indeed.

4 Operations $[A]$ and $[A \to B]$ on $\mathcal{A}$

We consider here some operations modifying elements in $\mathcal{A}$. For this, we recall several auxiliary definitions from [Mur 94].

For any epistemic state ε and formula A,

$$\varepsilon_A^- \stackrel{\text{def}}{=} \{s \,|\, s \in \varepsilon, \boldsymbol{t} \not\sqsubseteq s(A)\} \text{ and } \varepsilon_{\overline{A}} \stackrel{\text{def}}{=} \begin{cases} \varepsilon_A^- & \text{if } \varepsilon_A^- \neq \emptyset \\ \text{the unit } \mathbf{1} \text{ in } \mathsf{AGE} & \text{otherwise,} \end{cases}$$

and

$$\kappa(\varepsilon) \text{ means } \{\kappa(\varepsilon) \,|\, s \in \varepsilon\},$$

where

$$p^* \in \kappa(s) \stackrel{\text{def}}{\Longleftrightarrow} \begin{cases} p^* = p \quad \text{and} \quad \boldsymbol{t} \sqsubseteq s(p) \\ \qquad\qquad \text{or} \\ p^* = \neg p \quad \text{and} \quad \boldsymbol{f} \sqsubseteq s(p) \end{cases}$$

for any setup s. Note the following useful inequality:

$$\boldsymbol{t} \sqsubseteq s(\wedge\kappa(s)) \tag{6}$$

for every setup s.

With every nonempty finite collection of sets of literals Σ, we associate the formula A_Σ as follows:

$$A_\Sigma \stackrel{\text{def}}{=} \begin{cases} \vee\{\wedge x \,|\, x \in \Sigma\} & \emptyset \notin \Sigma \\ \qquad \star & \text{otherwise.} \end{cases}$$

Now, for any formula A and $u \in \boldsymbol{Con}$, we define the formula $A_{u_{\overline{A}}}$ as follows:

$$A_{u_{\overline{A}}} \stackrel{\text{def}}{=} \vee\{\wedge\kappa(s) \,|\, s \in \phi(\overline{u}), \wedge\kappa(s) \not\vdash A\}.$$

Similar to operations $[A]$ and $[A \to B]$ from [KM 93] modifying epistemic state of the computer, consider the following operations on $\mathcal{A}$ having the same names:

$$[A](\overline{x}) \stackrel{\text{def}}{=} \overline{x \cup \{A\}},$$

$$[\star](\overline{x}) \stackrel{\text{def}}{=} \overline{x},$$

$$[A \to B](\overline{u}) \stackrel{\text{def}}{=} \overline{\{A_{u_{\overline{A}}}\}} \sqcap [B](\overline{u}),$$

and

$$[A \to B](\overline{x}) \stackrel{\text{def}}{=} \sqcup\{[A \to B](\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\}$$

for any formulas A and B of L, $u \in \boldsymbol{Con}$ and $x \subseteq \boldsymbol{D}$.

Note that for every $u \in \boldsymbol{Con}$, there is $u' \in \boldsymbol{Con}$ such that $[A \to B](\overline{u}) = \overline{u'}$.

We will use at least twice the following simple lattice argument.

Proposition 2 *Let $\{a_s \,|s \in S\}$ and $\{a_t \,|t \in T\}$ be two sets of elements in a complete lattice and $S \subseteq T$. Assume also that for every a_t there is a_s such that $a_t \leq a_s$. Then the equation*

$$\sqcup\{a_s \,|s \in S\} = \sqcup\{a_t \,|t \in T\}$$

holds.

Proof is obvious.

Next we aim to prove that the function f is an isomorphism between $\mathcal{A}$ and AGE with respect to the operations $[A]$ and $[A \to B]$ defined on $\mathcal{A}$ above and those on AGE introduced in [KM 90, KM 93] (also, see [Mur 93, Mur 94]). That is why we used the same names for those operations on $\mathcal{A}$ as on AGE.

Lemma 9 *For every $u \in \boldsymbol{Con}$ such that $\overline{u} \subseteq \overline{x \cup \{A\}}$, there is $u' \in \boldsymbol{Con}$ such that $\overline{u'} \subseteq \overline{x}$ and $\overline{u \cup \{A\}} \subseteq \overline{u' \cup \{A\}}$.*

Proof. Assume $\overline{u} \subseteq \overline{x \cup \{A\}}$. Then, according to (1), there is $v \in \boldsymbol{Con}$ such that $v \subseteq x \cup \{A\}$ and $\vdash_{E^*} \wedge v \to \wedge u$. Denote: $u' \stackrel{\text{def}}{=} v \setminus \{A\}$. Then $u' \subseteq x$ and $\vdash_{E^*} \wedge u' \wedge A \to \wedge u \wedge A$. In virtue of (1), $\overline{u \cup \{A\}} \subseteq \overline{u' \cup \{A\}}$.

Theorem 2 *For any $x \subseteq \boldsymbol{D}$ and formula A, $f([A](\overline{x})) = [A](f(\overline{x}))$.*

Proof. Here is a chain of equations with appropriate references:

$$\begin{array}{rcl}
f([A](\overline{x})) & = & f(\overline{x \cup \{A\}}) \\
\text{[Theorem 1, Corollary 8.1]} & = & \sqcup\left\{\phi(\overline{u})\,\middle|\,\overline{u} \subseteq \overline{x \cup \{A\}}\right\} \\
\text{[Lemma 9, Proposition 2]} & = & \sqcup\left\{\phi(\overline{u})\,\middle|\,A \in u, \overline{u} \subseteq \overline{x \cup \{A\}}\right\} \\
\text{[Lemma 4]} & = & \sqcup\left\{\phi(\overline{u \cup \{A\}})\,\middle|\,\overline{u} \subseteq \overline{x \cup \{A\}}\right\} \\
\text{[Lemma 4]} & = & \sqcup\left\{\phi(\overline{u} \sqcup \overline{\{A\}})\,\middle|\,\overline{u} \subseteq \overline{x \cup \{A\}}\right\} \\
\text{[Lemma 3]} & = & \sqcup\left\{\phi(\overline{u}) \sqcup \phi(\overline{\{A\}})\,\middle|\,\overline{u} \subseteq \overline{x \cup \{A\}}\right\} \\
\text{[Lemma 3]} & = & \sqcup\left\{\phi(\overline{u}) \sqcup m(Tset\,(A))\,\middle|\,\overline{u} \subseteq \overline{x \cup \{A\}}\right\} \\
\text{[Lemma 9, Proposition 2]} & = & \sqcup\{\phi(\overline{u}) \sqcup m(Tset\,(A))\,|\,\overline{u} \subseteq \overline{x}\} \\
\text{[Theorem 4.3 in [KM 93], Theorem 1]} & = & \sqcup\{\phi(\overline{u})\,|\,\phi(\overline{u}) \leq f(\overline{x})\} \sqcup \overline{m(Tset\,(A))} \\
\text{[Theorem 1]} & = & \sqcup\{\varepsilon\,|\,\varepsilon \in \mathsf{AFE}, \varepsilon \leq f(\overline{x})\} \sqcup \overline{m(Tset\,(A))} \\
\text{[Theorem 6.4 in [KM 93]]} & = & f(\overline{x}) \sqcup \overline{m(Tset\,(A))} \\
\text{[Definition in [KM 93]]} & = & [A](f(\overline{x})).
\end{array}$$

Lemma 10 *For any formula A and $u \in \boldsymbol{Con}$, the equation*

$$m\Big(Tset\left(A_{u_{\overline{A}}}\right)\Big) = (m(Tset\,(\wedge u)))_{\overline{A}}$$

holds.

Proof. Assume $s \in m\Big(Tset\left(A_{u_{\overline{A}}}\right)\Big)$. Then $\boldsymbol{t} \sqsubseteq s(A_{u_{\overline{A}}})$ and, hence, there is a setup s' such that $\kappa(s') \in \kappa(m(Tset\,(\wedge u)))$, $\wedge\kappa(s') \not\Vdash\!\wedge A$ and $\boldsymbol{t} \sqsubseteq s(\wedge\kappa(s'))$. From that, we conclude that $\boldsymbol{t} \not\sqsubseteq s(A)$ and $\boldsymbol{t} \sqsubseteq s(A_{\kappa(m(Tset\,(\wedge u)))})$. In virtue of the Lemma 10 in [Mur 94],

$$\vdash_{E^*} A_{\kappa(m(Tset\,(\wedge u)))} \leftrightarrow \wedge\, u$$

and, therefore, $\boldsymbol{t} \sqsubseteq s(\wedge u)$. Thus, we conclude that $s \in Tset\,(\wedge u)$. Let $s'' \leq s$ and $s'' \in m(Tset\,(\wedge u))$, that is, $s'' \in \phi(\overline{u})$ and $\boldsymbol{t} \not\sqsubseteq s''(A)$. Therefore, $\kappa(s'') \in \kappa(\phi(\overline{u}))$ and, in view of (6), $\boldsymbol{t} \sqsubseteq s''(\wedge\kappa(s''))$. Consequently, $\boldsymbol{t} \sqsubseteq s''(A_{u_{\overline{A}}})$. Thus, $s'' = s$ that implies that $s \in (m(Tset\,(\wedge u)))_{\overline{A}}$.

Now assume $s \in (m(Tset\,(\wedge u)))_{\overline{A}}$. That means that $s \in m(Tset\,(\wedge u))$ and $\boldsymbol{t} \not\sqsubseteq s(A)$. Rewrite the former as follows: $\kappa(s) \in \kappa(\phi(\overline{u}))$. In view of (6), we also have $\boldsymbol{t} \sqsubseteq s(\wedge\kappa(s))$ that implies $\wedge\kappa(s) \not\vdash A$. Thus, we conclude that $\boldsymbol{t} \sqsubseteq s(A_{u\overline{A}})$. Then, there is a setup s' such that $s' \leq s$ and $s' \in m\left(Tset\left(A_{u\overline{A}}\right)\right)$. According to the first part of the proof, $s' \in m(Tset(\wedge u))$. Consequently, $s = s'$ and, hence, $s \in m\left(Tset\left(A_{u\overline{A}}\right)\right)$.

Theorem 3 *For any $x \subseteq \boldsymbol{D}$ and formula A, $f([A \rightarrow B](\overline{x})) = [A \rightarrow B](f(\overline{x}))$.*

Proof. We prove previously the following equation:

$$f([A \rightarrow B](\overline{u})) = [A \rightarrow B](f(\overline{u})), \tag{7}$$

where $u \in \boldsymbol{Con}$. To prove that we consider the following chain of equations with appropriate references:

$$\begin{array}{rcl}
f([A \rightarrow B](\overline{u})) & = & f(\overline{\{A_{u\overline{A}}\}} \sqcap [B](\overline{u})) \\
\text{[Theorem 1]} & = & f(\overline{\{A_{u\overline{A}}\}}) \sqcap f([B](\overline{u})) \\
\text{[Theorem 1, Lemma 3]} & = & m\left(Tset\left(A_{u\overline{A}}\right)\right) \sqcap f([B](\overline{u})) \\
\text{[Theorem 2]} & = & m\left(Tset\left(A_{u\overline{A}}\right)\right) \sqcap [B](f(\overline{u})) \\
\text{[Lemma 10]} & = & (m(Tset(\wedge u)))_{\overline{A}} \sqcap [B](f(\overline{u})) \\
\text{[Theorem 1, Lemma 3]} & = & (f(\overline{u}))_{\overline{A}} \sqcap [B](f(\overline{u})) \\
\text{[Theorem 4 in [Mur 94]]} & = & [A \rightarrow B](f(\overline{u})).
\end{array}$$

Now we receive for any $x \subseteq \boldsymbol{D}$:

$$\begin{array}{rcl}
f([A \rightarrow B](\overline{x})) & = & f(\sqcup\{[A \rightarrow B](\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\} \\
\text{[Theorem 1]} & = & \sqcup\{f([A \rightarrow B](\overline{u}) \,|u \in \boldsymbol{Con}, \overline{u} \subseteq \overline{x}\} \\
\text{[Equation (7), Theorem 1]} & = & \sqcup\{[A \rightarrow B](f(\overline{u})) \,|u \in \boldsymbol{Con}, f(\overline{u}) \subseteq f(\overline{x})\} \\
\text{[Theorem 1, Lemma 3]} & = & \sqcup\{[A \rightarrow B](\varepsilon) \,|\varepsilon \in \mathsf{AFE}, \varepsilon \leq f(\overline{u})\} \\
\text{[Theorem 7.8, Theorem 6.4 in [KM 93]]} & = & [A \rightarrow B](f(\overline{x})).
\end{array}$$

Corollary 3.1 *The operations $[A]$ and $[A \rightarrow B]$ are Scott-continuous on $\mathcal{A}$.*

Proof follows immediately from the Theorems 1, 2 and 3 above and the Theorems 7.2 and 7.8 from [KM 93].

Corollary 3.2 *For any fixed $u \in \boldsymbol{Con}$ and formulas A and B, the correlation $[A \rightarrow B](\overline{u}) = \overline{u}$ is effectively decidable* (comp. the Theorem 4 in [Mur 93]).

Proof. Indeed, we have the equivalence:

$$[A \to B](\overline{u}) = \overline{u} \text{ if and only if } \overline{\{A_{u\overline{A}}\}} \sqcap \overline{u \cup \{B\}},$$

the right part of which is equivalent to $\overline{\{A_{u\overline{A}} \vee (\wedge u \wedge B)\}} = \overline{u}$ according to the Lemma 6. The last equation in turn is equivalent to

$$\vdash_{E^*} A_{u\overline{A}} \vee (\wedge u \wedge B) \leftrightarrow \wedge u$$

that is equivalent to that the following entailments

$$\vdash_{E^*} A_{u\overline{A}} \to \wedge u \text{ and } \vdash_{E^*} \wedge u \to A_{u\overline{A}} \vee B$$

hold.

5 Continuous, Ampliative Operations Coordinated with Basis

Denote

$$\mathcal{C} \stackrel{\text{def}}{=} \{\overline{u} \,|u \in \boldsymbol{Con}\},$$

which will consider as a set or a partially ordered set with $\subseteq$ or a lattice with operations as in the Lemma 4.

For any $c \in \mathcal{C}$, we define:

$$\mathcal{D}_c \stackrel{\text{def}}{=} \{[A](c) \,|A \in \boldsymbol{D}\},$$

where $[\star]$ means the identical operation on $\mathcal{A}$.

Furthermore, we denote:

$$\boldsymbol{\mathcal{D}} \stackrel{\text{def}}{=} \prod \{\mathcal{D}_c \,|c \in \mathcal{C}\}.$$

Then, for any operation F on $\mathcal{A}$ and function $G \in \boldsymbol{\mathcal{D}}$, we introduce [7]:

$$G_F \stackrel{\text{def}}{=} \{(\overline{u}, F(\overline{u})) \,|\overline{u} \in \mathcal{C}\} \quad \text{and} \quad F_G(\overline{x}) \stackrel{\text{def}}{=} \sqcup \{c \,|(\overline{u}, c) \in G, \overline{u} \subseteq \overline{x}\}$$

[7] Next two definitions were inspired by [GS 90].

– the restriction of F to $\mathcal{C}$ and an operation on $\mathcal{A}$, respectively.

Following [Bel 75], we call an operation F on $\mathcal{A}$ *ampliative*, if $\overline{x} \subseteq F(\overline{x})$ for every $\overline{x} \in \mathcal{A}$. The operations $[A]$ and $[A \to B]$ considered above both are ampliative. Recall that F is *coordinated* with $\mathcal{C}$ if F is closed on $\mathcal{C}$, that is, $F(c) \in \mathcal{C}$ whenever $c \in \mathcal{C}$. We will especially pay attention to monotone functions G in $\boldsymbol{\mathcal{D}}$, that is, where $\overline{u} \subseteq \overline{u_1}$ implies $c \subseteq c_1$ when the pairs $(\overline{u}, c)$ and $(\overline{u_1}, c_1)$ both are in G.

Theorem 4 *Let a function G from $\boldsymbol{\mathcal{D}}$ be monotone. Then the operation F_G is continuous, ampliative and coordinated with $\mathcal{C}$. Moreover, the equation $G = G_{(F_G)}$ holds.*

Proof. Let $\{\overline{x_i}\,|i \in I\}$ be a directed set and $\overline{x} = \sqcup\{\overline{x_i}\,|i \in I\}$. According to the Corollary 8.1, for any $\overline{u} \in \mathcal{C}$, if $\overline{u} \subseteq \overline{x}$, then there is $i \in I$ such that $\overline{u} \subseteq \overline{x_i}$. Having that, we receive:

$$\begin{aligned} F_G(\overline{x}) &= \sqcup\{c\,|(\overline{u}, c) \in G, \overline{u} \in \mathcal{C}, \overline{u} \subseteq \overline{x}\} \\ &= \sqcup\{c\,|(\overline{u}, c) \in G, \overline{u} \in \mathcal{C}, \overline{u} \subseteq \overline{x_i}, i \in I\} \\ &= \sqcup\{\sqcup\{c\,|(\overline{u}, c) \in G, \overline{u} \in \mathcal{C}, \overline{u} \subseteq \overline{x_i}\}\,|i \in I\} \\ &= \sqcup\{F_G(\overline{x_i})\,|i \in I\}, \text{ that is } F_G \text{ is continuous.} \end{aligned}$$

Notice that $\overline{u} \subseteq c$ whenever $(\overline{u}, c) \in \mathcal{C}$. Thus, F_G is ampliative.

Then, in virtue of monotonicity of the function G, $F_G(\overline{u}) = c$, provided that $(\overline{u}, c) \in G$. Therefore, F_G is coordinated with $\mathcal{C}$.

Again, the monotonicity of G gives us:

$$(\overline{u}, c) \in G_{(F_G)} \iff c = F_G(\overline{u}) \iff (\overline{u}, c) \in G.$$

That means that $G = G_{(F_G)}$.

Theorem 5 *Let F be a continuous, ampliative operation on $\mathcal{A}$ coordinated with $\mathcal{C}$. Then G_F is monotone and belongs to $\boldsymbol{\mathcal{D}}$. Moreover, the equation $F = F_{(G_F)}$ holds.*

Proof. The operation F is continuous and, hence, monotone. It implies the monotonicity of G_F.

Now, assume $(\overline{u}, F(\overline{u})) \in G_F$. Then for some $v \subseteq \boldsymbol{Con}$, the equation $\overline{v} = F(\overline{u})$ holds. It first implies $\overline{u} \subseteq \overline{v}$. That in turn implies that

$\vdash_{E^*} \wedge(u \cup v) \leftrightarrow \wedge v$, that is, the equation $\overline{v} = [\wedge v](\overline{u})$ holds. Thus, we have proved that $G_F \in \boldsymbol{\mathcal{D}}$.

Finally, with help of the Theorem 4, we receive:

$$c = F_{(G_F)}(\overline{u}) \iff (\overline{u}, c) \in G_F \iff c = F(\overline{u}).$$

Thus, the equation $F = F_{(G_F)}$ is proved.

A continuous, ampliative and coordinated with $\mathcal{C}$ operation F on $\mathcal{A}$ is called *computable*, if the corresponding G_F from $\boldsymbol{\mathcal{D}}$ is recursively enumerable (or as a function on $\mathcal{C}$ recursive in view of the Theorem 5-IX in [Rog 67]) after introducing an appropriate enumeration (comp. [GS 90]). Notice that both operations $[A]$ and $[A \rightarrow B]$ are computable. Next we are going to present some classification of the continuous, ampliative and coordinated with $\mathcal{C}$ operations on $\mathcal{A}$. Now on we call them $\boldsymbol{CAC}$*-operations* [8]. We certainly concern of computability of such operations.

As in [Mur 94], we will call every $[A]$-operation (*elementary*) *action*. We will say that a set $\mathcal{O}_F$ *characterizes* a $\boldsymbol{CAC}$-operation F, if for evry $\varepsilon \in \mathsf{AFE}$ there is an action $\pi \in \mathcal{O}_F$ such that $F(\varepsilon) = \pi(\varepsilon)$. The least coordinal number of a set characterizing F among all such sets we call *the order of* F. Thus, we devide all $\boldsymbol{CAC}$-operations on the operations of the *finite* and *infinite* order. It is clear that all $[A]$-operations are operations of the order 1. Our next purpose is to establish that $[A \rightarrow B]$-operations are operations of finite order too. To do that for a fixed $[A \rightarrow B]$-operation is satisfactory to show that there is at least one finite set of actions characterizing $[A \rightarrow B]$.

In what follows, we return to the AFE-notation. We begin with a lemma that could be proved earlier.

Lemma 11 *For any formulas A and B and a minimal state ε ($\varepsilon \in \mathsf{AFE}$),*

$$[A](\varepsilon) \sqcap [B](\varepsilon) = [A \vee B](\varepsilon).$$

[8] Notice that the notion of $\boldsymbol{CAC}$-operation is in accordance with considering knowledge as a competence notion in [Lev 84], because if the computer imagines a current world (as a minimal state) in which p and $p \rightarrow q$ are true and concludes that q will be true in any world imaginable in the current one, then it is only possible, provided that the computer's knowledge in the imaginable world is supposed not to decrease. It should be added that, according to the approach being accepted here (comp. [Mur 93]), an imaginable world is a state accessible from a current one by means of a $\boldsymbol{CAC}$-operation.

Proof follows immediately from the Corollary 6.1 and the definition of $[A]$-operations on $\mathcal{A}$.

Let us denote:

$$Nset\,(A) \stackrel{\text{def}}{=} \{s\,|s \in \mathsf{AS}, \boldsymbol{t} \not\sqsubseteq s\,(A)\,, V(s) \subseteq V\,(A)\}.$$

Notice that the epistemic state $Nset\,(A)$ is finite one for every A. Then recall that we can rewrite the definition from [Mur 94] of the minimal state $\varepsilon_{\overline{A}}$ for every minimal state ε as follows:

$$\varepsilon_{\overline{A}} = \begin{cases} m(\varepsilon \cap Nset\,(A)) & \text{if } \varepsilon \cap Nset\,(A) \neq \emptyset \\ \text{the unit } \mathbf{1} \text{ in } \mathsf{AGE} & \text{otherwise.} \end{cases}$$

We furthermore define:

$$\mathcal{N}_A \stackrel{\text{def}}{=} \{m(\varepsilon)\,|\varepsilon \subseteq Nset\,(A), \varepsilon \neq \emptyset\}.$$

Theorem 6 *Every operation $[A \to B]$ is characterized by the set $\left\{\left[A_{\kappa(\varepsilon)} \vee B\right] \middle| \varepsilon \in \mathcal{N}_A\right\} \cup \{[B]\}$. Hence, every $[A \to B]$-operation is one of the finite order.*

Proof. According to the Theorem 4 in [Mur 94], for every $[A \to B]$-operation, we have the equation:

$$[A \to B](\varepsilon) = \varepsilon_{\overline{A}} \sqcap [B](\varepsilon)$$

holding for every minimal state ε. If $\varepsilon_{\overline{A}} = \mathbf{1}$, then $[A \to B](\varepsilon) = [B](\varepsilon)$. Otherwise, there is $\varepsilon' \in \mathcal{N}_A$ such that $[A \to B](\varepsilon) = \varepsilon' \sqcap [B](\varepsilon)$. However, in virtue of the Lemma 12 and Theorem 4, both in [Mur 94], $\varepsilon' = \left[A_{\kappa(\varepsilon')}\right](\varepsilon)$. Thus, $[A \to B](\varepsilon) = \left[A_{\kappa(\varepsilon')}\right](\varepsilon) \sqcap [B](\varepsilon)$, which, with help of the Lemmas 11 and 3, gives the equation $[A \to B](\varepsilon) = \left[A_{\kappa(\varepsilon')} \vee B\right](\varepsilon)$.

Another interesting example of computable operation on $\mathcal{A}$ (or on AGE, as below) of the finite order is the operation $\left[A \xrightarrow{*} B\right]$ defined as follows:

$$\left[A \xrightarrow{*} B\right](\overline{\varepsilon}) \stackrel{\text{def}}{=} \sqcup\{[A \to B]^n(\overline{\varepsilon})\,|n \geq 0\},$$

where $[A \to B]^0 \stackrel{\text{def}}{=} [\star]$ and $[A \to B]^{n+1} \stackrel{\text{def}}{=} [A \to B] \circ [A \to B]^n$. We do not bring a proof of this fact here. Instead, we will bring a proof of the existence of a computable operation of the infinite order.

Consider the following countable sequence $\{s_i\}_{i<\omega}$ of finite setups:

$$s_i(p_j) = \begin{cases} \boldsymbol{t} & \text{if } j \le i \\ \bot & \text{otherwise,} \end{cases}$$

and correspondent sequence $\mathcal{E}$ $(= \{\varepsilon_i\}_{i<\omega})$ of minimal epistemic states, where $\varepsilon_i = \{s_i\}$. Furthermore, denote:

$$\uparrow \mathcal{E} \stackrel{\text{def}}{=} \{\varepsilon \,|\, \varepsilon \in \mathsf{AFE}, (\exists \varepsilon' \in \mathcal{E})(\varepsilon' \le \varepsilon)\}.$$

Let $U(\mathcal{E})$ be all the upper bounds of $\mathcal{E}$. Thus, $U(\mathcal{E}) \subseteq \uparrow \mathcal{E}$. Also, notice that ε never belongs to $U(\mathcal{E})$ providing $\varepsilon \in \mathsf{AFE}$. Otherwise, we would have in virtue of the Lemma 4 in [Mur 94] that for every $s \in \varepsilon$, $\mathsf{Var} \subseteq \kappa(s)$ which is impossible, because $\kappa(s)$ is a finite set of variables.

Theorem 7 *Let $\mathcal{E}$ be the sequence above and G the set of pairs of minimal states defined as follows:*

$$(\varepsilon, \varepsilon') \in G \stackrel{\text{def}}{\Longleftrightarrow} \varepsilon \in \mathsf{AFE} \textit{ and} \begin{cases} \varepsilon' = [p_1 \wedge \ldots \wedge p_{n+1}](\varepsilon) & \textit{if } \varepsilon_n \le \varepsilon \textit{ and } \varepsilon_{n+1} \not\le \varepsilon \\ \varepsilon' = [\star](\varepsilon) & \textit{if } \varepsilon \notin \uparrow \mathcal{E}. \end{cases}$$

Then G is monotone and F_G is a $\boldsymbol{CAC}$-operation of the infinite order. Moreover, F_G is computable.

Proof. First of all, notice that G is an amplaitive function on AFE. Then, we prove that G is monotone.

According to the Lemma 12 in [Mur 94], we can write:

$$(\varepsilon, \varepsilon') \in G \iff \varepsilon \in \mathsf{AFE} \text{ and} \begin{cases} \varepsilon' = \varepsilon \sqcup \varepsilon_{n+1} & \text{if } \varepsilon_n \le \varepsilon \text{ and } \varepsilon_{n+1} \not\le \varepsilon \\ \varepsilon' = \varepsilon & \text{if } \varepsilon \notin \uparrow \mathcal{E}. \end{cases}$$

Suppose $\varepsilon \le \varepsilon'$. We will show then that $G(\varepsilon) \le G(\varepsilon')$. Consider a number of cases.

Case: $\varepsilon \in \uparrow \mathcal{E} \setminus \mathcal{U}(\mathcal{E})$. Then there is a natural number n such that $G(\varepsilon) = \varepsilon \sqcup \varepsilon_{n+1}$. For ε', it is certainly true that $\varepsilon' \in \uparrow \mathcal{E} \setminus \mathcal{U}(\mathcal{E})$, that is, for some natural p, $\varepsilon_{n+p} \le \varepsilon'$ and $\varepsilon_{n+p+1} \not\le \varepsilon'$. Thus, we have:

$$G(\varepsilon) = \varepsilon \sqcup \varepsilon_{n+1} \le \varepsilon' \sqcup \varepsilon_{n+p+1} = G(\varepsilon').$$

Case: $\varepsilon \not\in \uparrow \mathcal{E}$. It immediately gives: $G(\varepsilon) = \varepsilon \leq \varepsilon' \leq G(\varepsilon')$.

Defining operation F_G, we can conclude with help of the Theorem 4 that F_G is a $\boldsymbol{CAC}$-operation on $\mathcal{A}$.

Now we prove that F_G is an operation of the infinite order. In contrary, suppose there is a finite set $\mathcal{O}$ of actions which characterizes F_G. Then there are two different elements ε_i and ε_j, say $i+1 \leq j$, in $\mathcal{E}$ and an action $[A]$ in $\mathcal{O}$ such that $F_G(\varepsilon_i) = [A](\varepsilon_i)$ and $F_G(\varepsilon_j) = [A](\varepsilon_j)$. Thus, we have $[A](\varepsilon_i) = \varepsilon_{i+1}$ and $[A](\varepsilon_j) = \varepsilon_{j+1}$. It implies, in virtue of the Theorem 2 in [Mur 94] two equations:

$$\varepsilon_{i+1} = \varepsilon_i \sqcup m(Tset\,(A)) \text{ and } \varepsilon_{j+1} = \varepsilon_j \sqcup m(Tset\,(A))$$

which give $m(Tset\,(A)) \leq \varepsilon_{i+1} \leq \varepsilon_j$. Consequently, $\varepsilon_j = \varepsilon_{j+1}$. A contradiction.

To prove the computability of F_G, notice that for any $\varepsilon \in \mathsf{AFE}$, we can effectively find a positive number n such that $\varepsilon_n \leq \varepsilon$ and $\varepsilon_{n+1} \not\leq \varepsilon$, if such a number exists. To check the existence of such a positive number and find it, we have, according to the Lemma 4 in [Mur 94], to find out which among the following conditions:

$$(\forall s \in \varepsilon)(\{p_1, \ldots, p_i\} \subseteq \kappa(s)) \textbf{\textit{ and }} \{p_1, \ldots, p_i, p_{i+1}\} \not\subseteq \kappa(s) \quad (i \geq 1)$$

is satisfied or no one of them is. It is possible to do effectively because of finiteness of ε.

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