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Paradox and Logical Revision. A Short Introduction

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Logical orthodoxy has it that *classical first-order logic*, or some extension thereof, provides the right extension of the logical consequence relation. However, together with naïve but intuitive principles about semantic notions such as truth, denotation, satisfaction, and possibly validity and other naïve logical properties, classical logic quickly leads to inconsistency, and indeed triviality. At least since the publication of Kripke's *Outline of a theory of truth* (Kripke 1975), an increasingly popular diagnosis has been to restore consistency, or at least non-triviality, by restricting some classical rules. Our modest aim in this note is to briefly introduce the main strands of the current debate on paradox and logical revision, and point to some of the potential challenges revisionary approaches might face, with reference to the nine contributions to the present volume.[1]

Our discussion is structured thus. Section 1 reviews the Liar and the Knower paradoxes. Section 2 briefly discusses four revisionary approaches. Section 3 sketches a potential challenge for revisionary approaches to semantic paradox. For reasons of space, we have mostly aimed at presenting the big picture, in broad strokes, thus sacrificing many important details.

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1 Liars & *Co*

Begin with the assumption that truth naïvely plays *capture* and *release* (Beall 2007a, 2009), in the following minimal sense:

$$\textit{Tr}\text{-I}\ \frac{\Gamma \vdash \phi}{\Gamma \vdash Tr(\ulcorner \phi \urcorner)} \qquad \textit{Tr}\text{-E}\ \frac{\Gamma \vdash Tr(\ulcorner \phi \urcorner)}{\Gamma \vdash \phi},$$

where $Tr(x)$ expresses *truth* and $\ulcorner \phi \urcorner$ is a name of ϕ. Somewhat less minimally, $Tr(x)$ may be assumed to further satisfy the T-Scheme

$$\text{(T-Scheme)}\ Tr(\ulcorner \phi \urcorner) \leftrightarrow \phi,$$

or, even less minimally, *transparency*: that $Tr(\ulcorner \phi \urcorner)$ and ϕ are always intersubstitutable *salva veritate* in all non-opaque contexts. Next, assume that our language contains a sentence λ identical to $\neg Tr(\ulcorner \lambda \urcorner)$, so that λ says of itself that it isn't true. Finally, let us further assume that the standard structural rules—rules in which no logical expression essentially figure, governing the *structure* of the consequence relation—are in place:

[1] For a recent introduction to non-classical theories of truth and other semantic notions, see the excellent Beall and Ripley (2014).

$$\text{(Id)}\ \frac{}{\phi \vdash \phi} \qquad \text{(SContr)}\ \frac{\Gamma, \phi, \phi \vdash \psi}{\Gamma, \phi \vdash \psi}$$

$$\text{(Cut)}\ \frac{\Gamma \vdash \phi \qquad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi},$$

and that negation satisfies its standard I- and E-rules:

$$\frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \neg\phi}\ \neg\text{-I} \qquad \frac{\Gamma \vdash \phi \qquad \Delta \vdash \neg\phi}{\Gamma, \Delta \vdash \bot}\ \neg\text{-E},$$

where $\bot$ is a *falsum* constant. We may then reason thus. Let Π be the following derivation of the theorem $\neg Tr(\ulcorner\lambda\urcorner)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{Tr(\ulcorner\lambda\urcorner) \vdash Tr(\ulcorner\lambda\urcorner)}{Tr(\ulcorner\lambda\urcorner) \vdash \lambda}\ Tr\text{-E}}{Tr(\ulcorner\lambda\urcorner) \vdash \neg Tr\ulcorner\lambda\urcorner} \qquad Tr(\ulcorner\lambda\urcorner) \vdash Tr(\ulcorner\lambda\urcorner)}{\dfrac{Tr(\ulcorner\lambda\urcorner), Tr(\ulcorner\lambda\urcorner) \vdash \bot}{Tr(\ulcorner\lambda\urcorner) \vdash \bot}\ \text{SContr}}\ \neg\text{-E}}{\vdash \neg Tr(\ulcorner\lambda\urcorner)}\ \neg\text{-I}$$

Using Π, we can then 'prove' $Tr(\ulcorner\lambda\urcorner)$:

$$\dfrac{\dfrac{\begin{matrix}\Pi \\ \vdash \neg Tr(\ulcorner\lambda\urcorner)\end{matrix}}{\vdash \lambda}}{\vdash Tr(\ulcorner\lambda\urcorner)}\ Tr\text{-I}$$

This is the Liar Paradox.

The paradox can be strengthened using a predicate $\Box(x)$ satisfying the rule of necessitation and the predicate equivalent of the T axiom in modal logic:

$$\text{(NEC)}\ \frac{\vdash \phi}{\vdash \Box(\ulcorner\phi\urcorner)} \qquad \text{(FACT)}\ \vdash \Box(\ulcorner\phi\urcorner) \to \phi.$$

To do so, it is sufficient to define a sentence κ identical to $\neg\Box(\ulcorner\kappa\urcorner)$, interpret $Tr(x)$ as $\Box(x)$ in the above derivation, and replace uses of Tr-I and Tr-E with, respectively, uses of NEC and FACT. The resulting reasoning, a 'proof' of $\Box(\ulcorner\kappa\urcorner)$ and $\neg\Box(\ulcorner\kappa\urcorner)$, is known as the Knower Paradox (Kaplan and Montague 1960; Myhill 1960). $\Box(x)$ may be interpreted in a number of ways: some epistemic, such as *knowledge* or *informal provability*; some non-epistemic, such as *validity* and *necessity*.[2]

Notice the *general form* of the foregoing paradoxical arguments:

$$\begin{matrix}\Pi \\ \neg\Phi(\ulcorner\phi\urcorner)\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\Pi \\ \neg\Phi(\ulcorner\phi\urcorner)\end{matrix}}{\phi}}{\Phi(\ulcorner\phi\urcorner)}$$

One establishes $\neg\Phi(\ulcorner\phi\urcorner)$ by means of an argument Π, here involving a release principle (Tr-E or FACT), $\neg$-I, $\neg$-E, and SContr. Then, the definition of ϕ and a capture principles such as Tr-I and NEC allow one to conclude ϕ, whence $\Phi(\ulcorner\phi\urcorner)$. This striking similarity strongly suggests that the Liar and the Knower paradoxes are little more than notational variants of each other. Notice, too, that Π need not involve $\neg$-I, $\neg$-E, SContr, and Cut; as we'll see in Sect. 3, there may be *other ways* to establish $\neg\Phi(\ulcorner\phi\urcorner)$. Finally, it is worth pointing out that the above paradoxical reasonings also presuppose the validity of Cut since, in a natural deduction format, $\neg$-E effectively codifies a restricted form of transitivity, as the following derivation shows:

$$\dfrac{\Gamma \vdash \phi \qquad \dfrac{\Delta, \phi \vdash \bot}{\Delta \vdash \neg\phi}\ \neg\text{-I}}{\Gamma, \Delta \vdash \bot}\ \neg\text{-E.}$$

The natural way of establishing the premises of an application of $\neg$-E is but an instance of Cut.

It would seem, then, that notions such as truth, necessity, knowledge, validity, and informal provability are all provably inconsistent—indeed trivial, if the logic validates the principle of *ex contraditione quodlibet*, that a contradiction entails any sentence whatsoever:

(ECQ) $\bot \vdash \phi$.

If it is thought that naïve semantic principles are, for some reason or other, non-negotiable, then one *must* blame the logic in order to restore consistency, or at least non-triviality. To be sure, such a revision is not to be taken lightly (see Terzian, THIS VOLUME), and there is no shortage of classical treatments, either hierarchical (Tarski 1936; Parsons 1974b; Burge 1979; Williamson 1998; Glanzberg 2001, 2004a; Schurz 2011), or non-hierarchical (Kripke 1975; McGee 1991; Gupta and Belnap 1993; Maudlin 2004; Leitgeb 2005; Simmons 1993, 2000, THIS VOLUME). But, it has been argued, the alternatives are dire (Kripke 1975; Field 2008), the naïve semantic principles *are* non-negotiable (Field 2008; Beall 2009; for a criticism of their arguments, see Zardini, THIS VOLUME), and there might be independent reasons for putting the blame on the logic in the first place (Ripley, THIS VOLUME; Zardini, THIS VOLUME). So how can logic be revised on the face of semantic paradox?

[2] We will further consider in Sect. 3 paradox-prone non-epistemic predicates such as *determinate* and *stable* truth.

2 Four Revisionary Approaches

Each of $\neg$-I, $\neg$-E, and SContr, and Cut can, and indeed has been, questioned. We very briefly consider the corresponding four revisionary strategies in turn.

2.1 Paracomplete and Paraconsistent

The most popular revisionary approaches to the Liar and Knower paradoxes involve revising the classical (intuitionistic) theory of negation, according to which $\neg$ satisfies both $\neg$-I and $\neg$-E. Thus, so-called *paracomplete* theorists hold that 'paradoxical' sentences such as λ and κ are *gappy*, in the sense of lacking a truth-value, or having an intermediate value in between truth and falsity. Hence, negation fails to be *exhaustive*, i.e. it fails to satisfy the *Law of Excluded Middle*:

(LEM) $\vdash \phi \vee \neg\phi$.[3]

Moreover, $\neg$-I can no longer hold in general either: if ϕ is gappy, the fact that it entails $\bot$ does not yet show that it is *false*. Dually, paraconsistent logicians treat the Liar Paradox as a *proof* of $Tr(\ulcorner\lambda\urcorner)$ and $\neg Tr(\ulcorner\lambda\urcorner)$. In their view, negation fails to be *exclusive*: there is an overlap between truth and falsity, i.e. 'paradoxical' sentences are *glutty*, and ECQ must be given up.[4] But, unless a new conditional is added to the language (about which more in a moment), $\neg$-E, and indeed $\to$-E (*modus ponens*) must be given up too: these rules fail to preserve truth for any ϕ that is both true and false (Priest 2006b; Beall 2009).

Kripke (1975) famously showed how to construct models for languages in which the truth-predicate is fully transparent, provided LEM and $\neg$-I (among other rules) are suitably restricted. More recently, Field (2006, 2008) and Brady (2006) have both defined models for paracomplete theories containing a conditional $\to$ satisfying $\vdash \phi \to \phi$, $\to$-E, and all instances of the T-Scheme. Because of Curry's Paradox, the conditional does not, and cannot, satisfy $\to$-I.[5] Similar (dual) results hold for paraconsistent languages (Priest 2006b; Beall 2009).

Defining a 'suitable' conditional strong enough to sustain ordinary reasoning and weak enough to avoid Curry's Paradox has proved to be no trivial enterprise.[6] Some are more optimistic (Field 2013); others less so (Martin 2011). Partly for this reason, Beall (2011, 2014b, a) has recently advocated, following Goodship (1996), a *detachment-free* glut-theoretic approach to paradox—one that gives up the project of defining a 'suitable' conditional, and decidedly embraces the failure of $\to$-E displayed by basic paraconsistent logics. Horsten (2009) essentially advocates a dual strategy in a paraconsistent setting.

But does revising the logic of $\neg$, $\to$, and more generally *operational rules*, i.e. rules specifically governing the use of logical operators, suffice to solve the semantic paradoxes *in general*? The paradoxes of *naïve logical properties* suggest a negative answer to this question (Beall and Murzi 2013; Zardini 2013a, 2014a).

2.2 Paradoxes of Naïve Logical Properties

Consider the two following principles: that if ψ is a consequence of ϕ, then the argument $\langle \phi \therefore \psi \rangle$ is valid (henceforth, VP), and that one may conditionally assert ψ given the assumptions that ϕ and that the argument $\langle \phi \therefore \psi \rangle$ is valid (henceforth, VD). More formally:

$$\text{(VP)}\ \frac{\phi \vdash \psi}{\varnothing \vdash \mathsf{Val}(\ulcorner\phi\urcorner,\ulcorner\psi\urcorner)} \quad \text{(VD)}\ \frac{\Gamma \vdash \mathsf{Val}(\ulcorner\phi\urcorner,\ulcorner\psi\urcorner) \qquad \Delta \vdash \phi}{\Gamma,\Delta \vdash \psi},$$

where $\mathsf{Val}(x,y)$ expresses validity. Now let π be a sentence identical to $\mathsf{Val}(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$, so that π says of itself that it validly entails absurdity. Then, courtesy of VD, one can easily derive $\bot$ from two occurrences of π and conclude $\mathsf{Val}(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$ by discharging both occurrences via a single application of VP. But, then, $\bot$ follows on no assumptions via VD. This is the Validity Curry Paradox, or v-Curry, for short (Whittle 2004; Shapiro 2011; Beall and Murzi 2013).

Beyond VP and VD, the argument only appeals to the standardly accepted structural rules. The validity of SContr in the above informal reasoning is presupposed by the multiple discharge of π (Negri and von Plato 2001). As for Cut, it is effectively built in our formulation of VD. At a glance, the argument can be presented thus. Let Σ be the following derivation of the theorem $\mathsf{Val}(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$:

$$\cfrac{\cfrac{\cfrac{\cfrac{\pi \vdash \pi}{\pi \vdash \mathsf{Val}(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)} \qquad \pi \vdash \pi}{\pi,\pi \vdash \bot}\ \text{VD}}{\pi \vdash \bot}\ \text{SContr}}{\vdash \mathsf{Val}(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)}\ \text{VP}$$

[3] See e.g. Kripke (1975), Brady (2006), Soames (1999), Field (2008), Horsten (2009).

[4] See e.g. Asenjo (1966), Asenjo and Tamburino (1975), Priest (1979), Goodship (1996) Beall (2009), Beall (2011).

[5] The standard conditional-involving Curry Paradox involves a sentence γ identical to $Tr(\ulcorner\gamma\urcorner) \to \bot$. Given the standard structural rules, $\to$-I, and $\to$-E, a Liar-like argument allows one to 'prove' $\bot$.

[6] The conditional in question would need to serve as a means to express restricted quantification, as in 'Everyone in the room is happy'. It can be shown, however, that a 'suitable' conditional weak enough not to trigger Curry's Paradox is bound not to validate certain

Footnote 6 continued
very intuitive principles about restricted quantification see e.g. Beall et al. (2006), Field (2013), Ripley (2014), Zardini (2014c).

Using Σ, we can then 'prove' $\bot$:

$$\dfrac{\dfrac{\Sigma}{\vdash \mathsf{Val}(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)} \qquad \dfrac{\dfrac{\Sigma}{\vdash Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)}}{\vdash \pi}}{\vdash \bot}\ \mathsf{VD}$$

If paradoxes are to be solved via logical revision, a naïve conception of validity forces the rejection of one of the standardly accepted *structural rules* (Beall and Murzi 2013). Similar paradoxes plague other naïve logical properties, such as consistency and compatibility (Zardini 2013c).

The foregoing paradoxes leave revisionary theorists with two choices. Either blame the naïve semantic principles (see e.g. Ketland 2012; Cook 2013; Zardini 2013c; Field 2014), or restrict one of the structural rules (Shapiro 2011; Beall and Murzi 2013; Zardini 2011; Murzi 2012; Zardini 2014a). Concerning the first possibility, VP and VD may be called into question, respectively, by paracomplete theorists who already reject $\rightarrow$-I and by paraconsistent theorists who already reject $\rightarrow$-E. What is more, Zardini (2013c) and Field (2014) have recently objected that, on the assumption that the validity relation is recursively enumerable, VD contradicts Gödel's Second Incompleteness Theorem.

We briefly mention some possible substructuralist rejoinders. First, it is unclear whether the natural paracomplete and paraconsistent arguments against VP and VD carry over to the paradoxes of consistency and, especially, compatibility. Second, it is hard to see why one should not be in a position to assert that the argument $\langle \phi \therefore \psi \rangle$ is valid if presented with a derivation of ψ from ϕ (see Shapiro, 2010 and Priest, THIS VOLUME). Third, even conceding that VP may be problematic, it is possible to derive a version of the v-Curry Paradox from VD alone and the plausible claim that $\phi \vdash \psi$ iff 'the argument $\langle \phi \therefore \psi \rangle$ is valid' is true (Shapiro 2013). Fourth, Gödel's First Incompleteness Theorem suggests that validity outstrips derivability in any recursively enumerable system to which Gödel's incompleteness theorems apply, and hence may *not* be recursively enumerable (Myhill 1960; Murzi 2014a, c). Let us consider, then, the second choice, viz. to restrict one of the standardly accepted structural rules.

2.3 Substructural Approaches: SContr-free and Cut-free

Substructural approaches are usually met with skepticism. Thus, Field writes:

> I haven't seen sufficient reason to explore this kind of approach (which I find very hard to get my head around), since I believe we can do quite well without it. (Field 2008, pp. 10–11)

While the paradoxes of naïve logical properties don't help one getting one's head around substructural consequence relations (an issue we will briefly touch upon in the next section), these paradoxes put pressure on Field's claim that a substructural revision of classical logic is not *needed* in order to adequately deal with semantic paradoxes in general. Partly for this reason, the literature on substructural approaches, both SContr-free and Cut-free, has recently been burgeoning.

On the SContr-free camp, pioneering efforts by Fitch (1942, 1948, 1950), have been followed by, among others, Mares and Paoli (2014), Shapiro (2011, 2013), Murzi and Shapiro (2014), Weber (2014), Zardini (2011, 2013c, b, 2014a, b), Meadows (2014), Caret and Weber (THIS VOLUME), and Priest (THIS VOLUME). In particular, Zardini (2011) proves syntactic consistency for a transparent theory of truth and naïve validity whose underlying logic is a suitable strengthening of affine linear logic. Zardini (2014b) extends the theory to a SContr-free metatheory which handles the paradoxes of naïve logical properties more generally. Moreover, Zardini sketches a possible independent *motivation* for rejecting SContr and making sense of such a rejection. SContr is standardly invalidated on the grounds that ϕ, ϕ and ϕ are different *resources* and hence, on the assumption that the content of a sentence is to be accounted for in terms of *information*, that they have different *content* (see e.g. Mares and Paoli 2014). Zardini (2011, 2014a) suggests instead a more conservative justification for restricting SContr, one according to which 'paradoxical' sentences are unstable, in the sense of expressing states-of-affairs which may not co-obtain with some of their consequences.

Caret and Weber (THIS VOLUME), Priest (THIS VOLUME), and Shapiro (THIS VOLUME) offer alternative ways of making sense of restrictions of SContr. In particular, Caret and Weber prove soundness and completeness results for a SContr-free logic for languages expressively strong enough to express their validity relation; Priest argues that the fusion connective, and its underlying structural mode of aggregating premises, intensional bunching, allows relevant logicians to provide a coherent treatment of the v-Curry Paradox; Shapiro introduces a naïve conception of sequent structure on which, he claims, SContr is naturally invalid.

On the Cut-free camp, Smiley's and Weir's pioneering efforts (Smiley 1957; Weir 2005) have been recently followed by Cobreros, Egreé, Ripley, and van Rooij, who prove model-theoretic consistency for a non-transitive, Cut-free transparent theory of truth and naïve validity (Cobreros et al. 2012, 2014).[7] Ripley (2012, 2013a, b) has further investigated the logic and developed an *inferentialist* argument against Cut, a version of which is presented in Ripley (THIS VOLUME). An *anti-realist* non-transitive approach is presented in Tennant (2014), on the

[7] The logic they advocate belongs to a family of logics that was first introduced in Zardini (2008a, b).

basis of ideas first introduced in Tennant (1982, 1987, 1995, 1997).

Substructural approaches to paradox are typically consistent, although they need not be (Weber 2014). Indeed, it may be argued that they pose a threat to glut-theoretic diagnoses of the Liar and Knower paradoxes, assuming a minimal principle of uniform solution (Priest 2006b). For while the Liar and the Knower prove, according to glut-theoretic wisdom, a *theorem* of the form $\phi \wedge \neg\phi$, the v-Curry Paradox allows one to 'prove' an *arbitrary sentence* ϕ, and invalidating ECQ won't help avoiding triviality here.[8] If the paradoxes of naïve logical properties are genuine semantic paradoxes (Beall and Murzi 2013; Murzi and Shapiro 2014; Murzi 2014a, c), then they are not uniformly solved on a standard glut-theoretic approach.

While the paradoxes of naïve logical properties seemingly show that substructural approaches are necessary to solve the semantic paradoxes via logical revision, the question naturally arises whether they are also *sufficient*. Zardini (2013c) contends that they are not, unless one is willing to also restrict the structural rules in one's *metatheory*. A certain application of Zardini's argument is criticised in Murzi (2014c). Bacon (THIS VOLUME) argues instead that some paradoxes of *identity* resist substructural treatment.

3 Revenge

Beall (2007a) distinguishes two tasks confronting any revisionary theory of semantic paradox. The *consistency project* must show how languages such as English can non-trivially enjoy naïve semantic predicates, in spite of the semantic paradoxes. The point of the consistency proofs such as the ones given in Kripke (1975), Field (2008), and Zardini (2011) is precisely to prove the consistency of formal theories whose languages resemble English in relevant respects. The task is technically demanding, and by no means philosophically straightforward: for instance, the model-theoretic proofs are reassuring only insofar as the target models, which are standardly described in a *classical* metalanguage, adequately model the real, *non-classical* world.[9] The *expressive characterisation project*, on the other hand, must indicate how it is possible, in languages such as English, to semantically characterise any sentence, as e.g. true, false, gappy, glutty, indeterminate, paradoxical, non-paradoxical, healthy, unhealthy etc. So-called *contextualists* (Parsons 1974a; Glanzberg 2001, 2004b, 2005) take this to be the deep root of the Liar phenomenon: the paradoxes arise because of a perceived need to semantically characterise 'paradoxical' sentences. In what follows, we briefly point to a possible line of argument to the effect that their diagnosis may not be far off the mark.

How, then, to characterise 'paradoxical'? Ideally, paracomplete theorists would find a property of sentences which allows them to (i) characterise 'paradoxical' sentences and (ii) justify restrictions of principles such as LEM and $\neg$-I; the paraconsistent theorists would find a property of sentences which allows them to (i) characterise 'non-paradoxical' sentences and (ii) justify the use of classical principles such as EFQ and $\neg$-E with respect to such sentences.

3.1 Paraconsistent Revenge

Begin with the paraconsistent case. Intuitively, in a glut-theoretic setting in which the base language—the fragment of the language which does not contain semantic predicates—is classical, non-paradoxical sentences are exactly the non-glutty sentences: the sentences that are *true or false only*. But how can such a property be expressed in the glut-theorist's language?

The fact that negation in a glut-theoretic framework fails to be *exclusive* implies a general difficulty to express a notion of *exclusivity*—exclusive truth or exclusive falsehood. It might be thought that a 'just true' predicate $JTr(x)$ expressing truth only might solve the problem. But this will not do. Such a predicate would expresses the property of *being true only* iff the following biconditional holds: $JTr(\ulcorner\phi\urcorner)$ is true iff ϕ really is just true. However, unsurprisingly, it is easy to see that some just true sentences are also false, as shown by a version of the Liar involving a sentence λ identical to $\neg JTr(\ulcorner\lambda\urcorner)$. One first proves by classical means $\neg JTr(\ulcorner\lambda\urcorner)$, as per the standard Liar-Knower recipe. But this is λ, whence via a version of the necessitation rule, $JTr(\ulcorner\lambda\urcorner)$. To treat the paradox by glut-theoretic means, and accept $JTr(\ulcorner\lambda\urcorner) \wedge \neg JTr(\ulcorner\lambda\urcorner)$, would involve allowing for an overlap between truth and falsity only. Thus, either $JTr(x)$ does not express the property of being true only after all, or, if it does, the original Liar and Knower paradoxes have not been solved.

Glut-theorists are all too well aware of the difficulty. Their standard response is that that paraconsistency 'runs deep',[10] and that it should be no surprise if, in a glut-theoretic framework, just true sentences also turn out to be false (Priest 2006b; Beall 2009). Intuitively, however, one would like to be able to express that non-paradoxical sentences are precisely the consistent, non-glutty ones. And it is hard to see how such a thought can be expressed

[8] Likewise, a natural deduction variant of the Compatibility Paradox presented in Zardini (2013c) allows one to prove $\neg\phi$, for arbitrary ϕ's (Murzi 2014c).

[9] For more discussion, see e.g. Beall (2007b) and Bacon (2013).

[10] As Field (2008, p. 72) puts it in a perfectly dual context.

without resorting to the classical notions of truth and falsity only.

Beall has recently proposed a different solution to the problem—one inspired by Priest (2006b, pp. 105–106). According to this, the thought that ϕ is just true is conveyed by the addition of a suitable *shrieking rule* of the form $\phi \wedge \neg\phi \vdash \bot$ to one's theory Beall (2013, 2014a). Murzi and Carrara (THIS VOLUME) raise some potential challenges for Beall's proposal; Simmons (THIS VOLUME) raises even more potential problems for paraconsistent theories.[11]

3.2 Revenge of the Knower

Now to paracomplete, and, more generally, consistent, approaches. Here the difficulty is, dually, that we'd intuitively like to find a property Ψ such that sentences such as κ, λ, and π are $\neg\Psi$. Thus, in a paracomplete setting, Field identifies such a Ψ with *determinate truth* (Field 2006, 2008); in a SContr-free setting, Zardini identifies it with *stability* (Zardini 2011, 2014a); in a Cut-free setting, Ripley identifies it with *strict assertibility or deniability* (Ripley 2013a, §§4.1-2). However, trouble begins to arise as soon as we notice that the predicate expressing the property of being truly Ψ satisfies versions of both NEC and FACT. Then, letting Ψ be an operator, in each of these cases, one can define a sentence κ identical to $\neg\Psi Tr(\ulcorner\kappa\urcorner)$, and run a version of the Knower Paradox. One establishes $\neg\Psi Tr(\ulcorner\kappa\urcorner)$ via a sub-argument Π involving, by the the target theorist's lights, some illicit logical moves (one assumes $\Psi Tr(\ulcorner\kappa\urcorner)$ twice and 'proves' $\neg\Psi Tr(\ulcorner\kappa\urcorner)$ using *inter alia* $\neg$-I, SContr, and Cut). One then concludes $\Psi Tr(\ulcorner\kappa\urcorner)$ via NEC and the definition of κ.

To be sure, the foregoing version of the Knower Paradox—call it the *truly* Ψ *Knower*—is invalid in the target theories: after all, it is just a variant of the original Liar and Knower paradoxes, and the target theories provably solve those. However, it can be shown that standard paracomplete, SContr-free, and Cut-free theorists are still nevertheless committed to asserting $\neg\Psi Tr(\ulcorner\kappa\urcorner)$ on grounds which *do not themselves make use of* $\neg$-I, SContr, *or* Cut (Murzi 2014b, d). The Liar version of the foregoing argument, where $\Psi =$ 'true', is usually referred to as the Strengthened Liar Paradox (see e.g. Sainsbury 2009). We call the more general argument the *Strengthened Knower.*

Let us look at an example. In the paracomplete theory defended in Field (2006, 2008), Ψ is a determinacy operator and the relevant κ is equivalent to a sentence saying that κ is not determinately true, where ϕ is determinately true iff both it and $\Psi(\phi)$ are true. One can then easily generate a version of the Knower Paradox and conclude that, *because of the paradox*, κ must itself be not determinately true. But this is to assert κ itself, whence, courtesy of NEC, κ is determinately true—contradiction. Field suggests that κ should be deemed *indeterminately determinately true* instead, and that the notion of determinacy is, accordingly, intrinsically hierarchical (one can now define a sentence γ identical to 'γ is indeterminately determinately true', run a new version of the Knower Paradox, classify γ as indeterminately determinately determinately true, and so on). But, one would like to argue, surely we possess a general, unified notion of indeterminacy. If so, consistency has only been restored, once more, at the price of expressive incompleteness. Field dedicates three chapters of his book *Saving Truth from Paradox* to the problem (Field 2008, Chaps. 22, 23, 27), arguing that there is no coherent unified notion of indeterminacy to be expressed in the first place. However, his critics are not convinced (see e.g. Simmons, THIS VOLUME and Priest 2007).[12]

Matters are a little more complicated in the case of SContr-free and Cut-free theories: here the truly Ψ Knower doesn't immediately give grounds for asserting that κ is *itself* not truly Ψ. All the same, in both cases it can be shown that SContr-free and Cut-free theorists are nevertheless both committed to asserting, for the relevant Ψ, $\neg\Psi Tr(\ulcorner\kappa\urcorner)$ (Murzi 2014b, d). Contradiction now looms again, courtesy of NEC (recall, κ and $\neg\Psi Tr(\ulcorner\kappa\urcorner)$ are *ex hypothesi* the same sentence).

To be sure, it may be insisted that this is just a fact of life: we should have learned by now that *something* must be given up in light of the semantic paradoxes. Perhaps so. However, notice that the Strengthened Knower reasoning has, once again, the familiar general form of the Liar and Knower paradoxes:

[11] More generally, Murzi and Carrara (THIS VOLUME) focus on whether glut-theorists are able to express *disagreement*, a topic also investigated in Ripley (2014). Objections to paraconsistent theories are legion; see e.g. Field (2008, Part V) and Carrara and Martino (2014).

[12] The problem had already been identified by Kripke, who famously fell the pull towards resurrecting, because of the Strengthened Liar, 'the ghost of Tarski's hierarchy' (Kripke 1975, p. 714). For an approach that, like Field's, resorts to infinitely many notions of defectiveness, see Cook (2007). The main difference between Field and Cook, as we see it, is that, while Field rejects (implausibly, in our view) the existence of a unified notion of indeterminacy, or, more generally, defectiveness, following Dummett (1991), Cook rejects absolute generality instead—the possibility of quantifying over absolutely everything. As a result, one cannot quantify over absolutely all the semantic categories other than true, and would-be revenge sentences such as $\kappa = \kappa$ falls in one of the semantic categories other than true actually don't breed revenge. Our main concern about Cook's strategy is that (as in Field's case) it solves the Liar and the Strengthened Liar in different ways. Yet, as we argue below, there are reasons for thinking that they are essentially the same paradox and that, for this reason, they should receive a uniform treatment.

$$\dfrac{\Pi}{\neg\Phi(\ulcorner\phi\urcorner)} \qquad \dfrac{\dfrac{\dfrac{\Pi}{\neg\Phi(\ulcorner\phi\urcorner)}}{\phi}}{\Phi(\ulcorner\phi\urcorner)}$$

The only difference is that, in the case of the Strengthened Knower, Π is no longer a subproof from two occurrences of $\Psi Tr(\ulcorner\kappa\urcorner)$) to $\bot$ involving uses of $\neg$-I, SContr, and Cut. Thus, it might be thought, one cannot in general solve the Strengthened Knower by blaming such rules. But if the Liar Paradox, the Knower Paradox, and the Strengthened Knower are all essentially the same paradox, in virtue of sharing the same general form, it is tempting to conclude that one cannot in general solve the semantic paradoxes by weakening the logic. Or so one might argue.[13]

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[13] Priest (2006a, Ch. 1; 2007) and Scharp (2013, Ch. 4) both describe a similar revenge argument. For a general revenge argument against certain (widespread) classical approaches to semantic paradox, see Bacon (2014).

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